

Analytic Pricing of Employee Stock Options

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We introduce a model that captures the main properties that characterize employee stock options (ESO). We discuss the likelihood of early voluntary ESO exercise, and the obligation to exercise immediately if the employee leaves the firm, except if this happens before options are vested, in which case the options are forfeited. We derive an analytic formula for the price of the ESO and in a case study compare it to alternative methods.

Since the mid-1980s, stock options have been a substantial component of compensation packages for employees. For example, in 1999, 94% of companies in the S&P 500 offered stock options to their top employees (see Murphy, 1999; Hall and Murphy, 2002).

In 1995, the Financial Accounting Standards Board (FASB) (with FAS 123) set a standard that required firms to expend stock-based compensation at the moment the compensation was granted (see FASB, 1995). Firms were encouraged to use the “fair value” of the stock option to compute the value of the compensation, but were allowed to use the “intrinsic value”—market price of the stock minus strike price. Since employee stock options (ESOs) are typically granted at the money, the intrinsic value is zero, which results in no expense recorded at the time of the grant, and this is probably one of the reasons that helped their popularity.

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FAS 123 was amended and replaced by FAS 123R (see FASB, 2004), which became mandatory toward the end of 2005. FAS 123R requires (instead of simply encouraging, as FAS 123 does) the use of a fair value method of accounting to compute the value of option compensation. A similar approach is followed by international standards: Recently issued, International Financial Reporting Standards (IFRS 2) (see International Accounting Standards Board, 2004) states the same principle. In 2005, the SEC issued SAB 107 (see Securities and Exchange Commission, 2005) which, although not legally binding, provides guidance about the interpretation of FAS 123R.

In particular, in order to meet the fair value criterion, any accounting practice must satisfy the following three conditions (FAS 123R, paragraph A8, and Staff Accounting Bulletin (SAB) No. 107, p. 13):

1. It “is applied in a manner consistent with fair value measurement objective and the other requirements of this Statement” (that is, FAS 123R).
2. It “is based on established principles of financial economic theory and generally applied in that field.”
3. “Reflects all substantive characteristics of the instrument,” except those explicitly excluded in FAS 123R.

SAB 107 emphasizes the importance of meeting all these three requirements, in particular item 3. We now mention some of the characteristics of a typical option-grant package for employees (see Rubinstein (1995) for a detailed discussion about the differences between standard options and option grants):

- First, the options are usually long-term (up to 10 years).
- They have a vesting period of up to 4 years.
- They are American type, so that the employees can exercise them at any time after vesting.
- If the employee leaves the firm before maturity of the options:
 - Before vesting, the options are forfeited.
 - After vesting, the employee has a short time (typically up to 3 months) to exercise the options, or they are also forfeited.
- Employees cannot transfer the options and are seriously restricted about hedging them (cannot shortsell the stock of the firm or buy puts on the firm, for example).

As a consequence of the last point, the employee faces an “incomplete markets” setting and has incentives to exercise options early, often way before maturity, even if she/he stays with the firm.

Neither FAS 123R nor SAB 107 requires a specific valuation model beyond the three requirements just mentioned. After FAS 123 was issued, the Black and Scholes (BS) formula became an obvious candidate for those firms that chose to apply fair value. However, given the long maturity of the options, valuations resulting from the BS formula are too high, if we take into account the fact that most options are exercised long before maturity. In fact, the lack

of obvious alternative valuation models was a deterrent to the implementation of fair value.¹ FAS 123R acknowledges this fact and, although it does not completely rule out the Black and Scholes (1973) model, it emphasizes that a lattice model “more fully reflects the substantive characteristics of a particular employee share option or similar instrument.” In the same spirit, IFRS 2 requires the use of a generally accepted valuation method that incorporates relevant parameters. Nevertheless, if the BS framework is applied, FAS 123R requires the use of the “expected term,” instead of the contractual maturity, and provides some guidance for its computation (FAS 123R, paragraphs A26–A30). SAB 107 recognizes the difficulty of obtaining information to compute the expected term and temporarily allows the use of a “simplified Black and Scholes formula,” which uses the average of the vesting period and contractual maturity as expected term. However, SAB 107 goes on to state that this method will probably not be appropriate after 2007, because firms will have enough information to properly estimate expected term by then (SAB 107, p. 37).²

In this paper, we provide a closed-form expression for pricing that takes into consideration the characteristics of ESOs described above. The spirit of our approach is similar to Hull and White (2004), who use a binomial method. First, we take into account that a proportion of the options in the grant will mature early, as a result of the employee leaving the firm or being terminated. The rate of exit (as Hull and White (2004) call it) is easy to estimate in practice (Carpenter (1998) has estimates of this parameter). In order to incorporate it into our pricing formula, we mimic the approach used in the default bonds literature (e.g., Duffie and Singleton, 1999).³ Second, we take into consideration that employees tend to exercise early, even if they do not leave the firm.⁴ To capture the effect of early exercise, we include as a feature of our pricing approach a barrier, such that when the price of the stock hits the barrier, the option is exercised. The implicit assumption is that when the option is deep enough in-the-money, the employee will collect its value and avoid the risk of a possible subsequent drop in price, maybe associated with a termination of her/his contract. The barrier represents the point at which the employee decides to collect the payoff and forfeit the remaining time-value of the option.

Obviously, it would be ideal to derive the exercise barrier of the employees endogenously, as an optimal policy. However, there are some problems that make the assumption of an exogenous barrier preferable for practical purposes (that is, for a model that can be easily implemented by the industry):

¹ For a thorough discussion of the arguments in favor of expensing versus not expensing—plus a broad literary review of existing pricing methods at the time—see Chance (2004).

² The use of such criterion is justified by the findings of Carpenter (1998) and Bettis, Bizjak, and Lemmon (2005).

³ Jennergren and Naslund (1993) suggest to use a Poisson process to account for the rate of exit. They show how to adjust BS to incorporate this rate.

⁴ Huddart (1994) discusses the optimal early exercise policy of a risk-averse, utility-maximizing option holder. Huddart and Lang (1996) study this issue empirically and point out that, although it is pervasive, the early exercise rule is not uniform.

- A simple tractable criterion that would allow the derivation of the optimal exercise policy and yet yield a closed form expression for the option does not seem feasible.⁵
- In practice, the exercise policy will depend on variables such as what proportion of the compensation package is in options, the proportion of the total wealth of the employees represented by the options package, employees' risk-aversion, employees' beliefs about stock volatility . . . As we have just pointed out, such an optimization problem does not appear sufficiently tractable for practical purposes. Equally important is the fact that such information is unlikely to be available in most of the cases.
- Preference heterogeneity among employees is likely to add another dimension to the difficulty of such an approach.

On the other hand, from the knowledge about the early exercise practice of the employees of a firm, it is easy to estimate a barrier that represents their optimal "desired" level of exercise. This approach can provide an approximation to the early exercise policy that is likely to be at least as reliable as the solution resulting from a highly parametric model.

The advantage of our approach with respect to Hull and White (2004) is that we provide an analytic expression that can be computed directly, after the rate of exit and barrier are estimated. Hull and White rely on numerical methods: they use a binomial tree (as in Cox, Ross, and Rubinstein, 1979) to compute the price of the option after the parameter values are estimated. The binomial tree approach converges very slowly and nonmonotonically (which also creates problems for hedging computations). As an example of the differences among the different approaches, consider an option grant with price of the stock 100, strike price 100, time to maturity 10 years, vesting period 3 years, interest rate 6%, and volatility 50%. The BS price of this option with maturity at 10 years is 69.21; the BS price with maturity at 3.25 years (that is, shortly after vesting) is 41.21; the price using a binomial tree as in Hull and White with 50 steps is 33.24; the price with the model presented in this paper is 32.29.⁶ We also mention the paper by Raupach (2003): he considers a model very similar to the one we study in this paper but solves the integrals numerically. He calibrates the model to the data.

There are several features of option grants (some of them discussed in the literature) that we ignore in this paper (also ignored in Hull and White, 2004). These are not general features; plus, they greatly complicate computations. First, we do not consider the possibility of resetting. This is the practice of exchanging the terms of the options grant at some point before maturity, typ-

⁵ Sircar and Xiong (2007) derive the optimal exercise policy, but they have to assume that the employee is risk-neutral and has infinite-horizon; furthermore, their solution is not fully analytical. Ingersoll (2006) considers a risk-averse employee (also with infinite horizon) and derives the optimal exercise policy numerically.

⁶ For the binomial tree and the model presented in this paper, we need three additional parameter values that we will describe later in detail. For this example, these parameter values are $L = 200$; $\alpha = 0$; and $\lambda = 0.15$.

ically when the stock has dropped in price and options are out-of-the-money. This is not part of the compensation contract, but it often takes place, mostly as a way to keep disgruntled employees in the company. Its practice has become so general that some authors point out that it should be an element of the pricing of options (see Acharya, John, and Sundaram (2000) for a discussion of this issue). Another feature that is sometimes incorporated into the option grant but we do not consider in this paper is reloading. This is the provision by which more options will be granted when the options of the initial package are exercised. We also ignore the dilution effects of option grants which, although probably not very important in general, are clearly a factor. Similarly, we do not consider the possibility of default of the company, which will have a negative effect on the price of the options, since they will have zero payoff if the firm defaults before the ESO is vested. Finally, we ignore the agency issues related to the use of options as compensation.⁷

In addition to these limitations, the model presented in this paper is based on the Black and Scholes framework and suffers from similar shortcomings. In particular, our model does not allow lump-sum dividend payments and it is necessary to assume that dividends are continuously paid; furthermore, it assumes that volatility is constant, ruling out more realistic stochastic volatility dynamics; finally, the interest is also assumed to be constant, also ruling out stochastic dynamics of the term structure.

Among other pricing models, we mention here Brenner, Sundaram, and Yermack (2000), who discuss the effects of the possibility of resetting. Carr and Linetsky (2000) develop a pricing formula exclusively based on the rate of exit. They consider two possible models, depending on whether the rate of exit is given by a constant intensity parameter (which is larger when the option is in-the-money) or whether it depends on how deep in-the-money the option is. They provide some numerical examples, but only for the case in which the option is already vested. Unlike theirs, our formula does not require numerical integration, even when vesting is included. Dybvig and Lowenstein (2003) focus on the feature of reloading. Stoughton and Wong (2003) study the pricing and resetting of stock options in a labor-competitive environment. Bulow and Shoven (2005) propose the use of BS, but with only 90 days to maturity, so as to reflect the period during which the option does not expire with certainty. They suggest to upgrade values quarterly: if at the end of the quarter the employee is still with the company, the time-value of the option corresponding to another 90 days will be expensed. Finally, we mention Sircar and Xiong (2007), who use a dynamic programming approach to find the price of ESOs in a similar setting, both with resetting and reloading features, but in a model with infinite maturity; their solution is not fully analytical.

⁷ There is a large body of literature that focuses on the incentive effects of ESOs, but we ignore it in this paper. See, for example, Jensen and Murphy (1990). Palmon, Bar-Yosef, Chen, and Venezia (2004) study the optimality of option grants (with choice of the strike price) versus stock grants.

We also mention a strand of related literature that discusses models to price option grants from the point of view of the employee, and in particular, the point of view of the top executives. As we have noted, the employee faces incomplete markets; therefore, this is a substantially different problem—trying to assess the incentives of options as compensation. Lambert, Larcker, and Verrecchia (1991) price ESOs by computing the “certainty equivalent,” or cash amount that will leave the employee indifferent in terms of the expected utility between a guaranteed cash payment and the option grant. This approach is also used by Detemple and Sundaresan (1999), who compute numerically the optimal exercise policy and certainty equivalent. Hall and Murphy (2002) also study the incentive effects of option-based compensation. Ingersoll (2006) uses the risk-neutral probability that corresponds to the constrained optimization problem of the employee who cannot trade options at will. Cao and Wei (2007) show that a hedging index brings the private value of the option closer to its market value. Finally, Grasselli (2005) uses a numerical algorithm that allows the computation of the price, both for the firm and the employee, as well as the optimal exercise policy.

We structure this paper as follows. In Section 1 we explain the assumptions of our model, introduce the pricing formula, and analyze its components and properties. In Section 2 we discuss a particular example and compare the results from our formula with the alternatives. We close the paper with some conclusions.

1. Pricing Model

First, we discuss the assumptions of the model and its economic foundations. Second, we present the model and discuss its features. Finally, we provide some examples.

1.1 Foundations

Our approach attempts to capture the following stylized facts:

- Typical options granted as compensation have a long maturity and include a long vesting period during which the option cannot be exercised and it is forfeited if the employee leaves the firm (whether the employee decides to leave or is fired).
- If the employee leaves the firm after vesting, the employee must exercise the option quickly (typically within 90 days) and after that the option is forfeited. This is the case whether the employee decides to leave or is fired.
- Employees tend to exercise options well before maturity. Often, as soon as options rights are vested, the employee exercises the option, even when there are still several years left until maturity. Of course, this is only the case if the option is in the money and, arguably, it is more likely to happen

the deeper in the money the option is and the shorter the time left until maturity.

Our objective is to price the option from the point of view of the firm. As we have pointed out in the literature review, the firm is considerably less constrained than the option holder with regard to risk diversification. While the employee cannot hedge in general the risk represented by holding the options (there are some companies that are willing to buy option grants under some conditions), the firm is mostly unconstrained with respect to the diversification of the risk represented by having a short position in the options. Therefore, it is reasonable to argue that, while the employee is very risk-averse concerning the expected payoff implicit in the long position in options, it is safe to assume that the firm is risk-neutral concerning the potential liability.

Our model intends to compute the expected (risk-neutral) payoff of a call option that can only be exercised after a vesting period. Then, we assume that there is a barrier such that, if the barrier is crossed, the option is exercised at that point. This barrier captures the fact that options are exercised early (see our discussion of this approach in the introduction). We allow the barrier to decrease. That would capture the fact that the employee is more likely to exercise the option (that is, for a lower price of the stock), the closer the exogenous maturity. Additionally, we assume that there is an exogenous exit rate of expiry of the option, which captures the possibility that the employee will leave the firm (willingly or not). Thus, the maturity of the option is one of the parameters of our formula, but it is possible that the option will expire before that final maturity date.

1.2 The model

As in the BS (1973) setting, we assume that the stock price follows a lognormal process:

$$dS_t/S_t = \mu dt + \sigma dW_t; \quad S_0 = s \quad (1)$$

which, under the risk-neutral pricing measure, becomes

$$dS_t/S_t = r dt + \sigma dW_t \quad (2)$$

with constant parameters σ , r , and μ (we only use this at the end of this section in order to compute the probabilities of early exercise for different reasons). We denote by s the current price of the stock. There is another source of uncertainty in the model: a Poisson process that characterizes the exit rate with a parameter that we explain below. Additionally, in our model we need the parameters that characterize the option grant: time left until vesting $T_0 \leq T$, maturity of the option T , and strike price K (typically, $K = s$). Finally, we have the parameters that capture:

- The usual exercise patterns of ESOs. We need two parameters: the level L of the fictitious barrier at which the employee exercises the option, and the rate α of decay of that barrier as maturity approaches.⁸
- The constant attrition or exit rate of employees, which we denote by λ and is the intensity of the Poisson process we mentioned above.

In fact, we allow the exit rate λ_0 before the vesting period to be different from the exit rate λ after the vesting.

In order to get more intuition about the effect of the different parameters, we price the option under four scenarios (that we call cases), denoted A, B, C, and D. Case A represents the situation in which the ESO is immediately vested and is exercised only when the underlying hits the desired level, or at maturity. In Case B, we assume that the ESO is immediately vested but it is only exercised at the random arrival time whose intensity we denote by λ , or at maturity. Case C is a combination of the two previous cases: the ESO is immediately vested, and will be exercised under the conditions of Case A or the conditions of Case B, whichever comes earlier. Case D is like C but with a vesting period. We now present the price of the ESO in each of the four cases, as an expectation. The analytic version of the formula and the proof are presented in Appendixes A, B, and C. A simple implementation of our formulas can be found at: <http://pluto.mscc.huji.ac.il/~mswiener/research/ESO.htm>.

The discounted call option payoff is given by

$$C_t = e^{-rt}(S_t - K)^+, \quad (3)$$

for t after the vesting period, and zero prior to vesting. Let $\min(\tau, T)$ be the time when the option is exercised or expires, where τ is a random time and T is the maturity. Introduce the conditional distribution of τ :

$$F_t := P_t(\tau \leq t). \quad (4)$$

Here, $P_t(\cdot)$ is the probability conditional on the information available from the stock prices up to time t . The formula for the expectation of random variable $C_{\min(\tau, T)}$ is:

$$E[C_{\min(\tau, T)}] = E \left[\int_0^T C_u dF_u + C_T(1 - F_T) \right]. \quad (5)$$

Intuitively, the first term corresponds to conditioning on $\tau = u$ and integrating over u , while the second term corresponds to $\tau = T$, which happens with conditional probability $1 - F_T$.

We assume that the option is either exercised by the employee when the stock reaches level $Le^{\alpha t}$ after the vesting period,⁹ or it is exercised/expires due to the

⁸ We are grateful to Kevin Murphy for suggesting this rate of decay.

⁹ Although our results do not depend on it, throughout the paper we will assume that $\alpha < 0$, consistent with our interpretation of α as a rate of decay.

employee quitting or being fired, with intensity λ . If quitting/firing happens before the vesting period, the employee gets nothing. We now list the option price in all four cases, A, B, C, and D, as defined above. As mentioned above, the explicit formulas for all the cases are given in Appendixes A, B, and C, together with the proofs.

Case A: Exercise time as a hitting time of desired level, no vesting period, i.e., $\lambda = 0, T_0 = 0$

Consider the case in which the option is exercised by the employee the first time the stock price hits the desired level $L_t = Le^{\alpha t}$ before maturity T , where α is a constant number such that $L_t > K$ for $t \leq T$:

$$\tau = T_L := \inf\{t > 0, S_t \geq L_t\} = \inf\{t > 0, S_t e^{-\alpha t} \geq L\}. \quad (6)$$

Then the option price is equal to, for $s < L$,

$$P_1 + P_2 := E[e^{-rT}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}] + E[(Le^{-r\alpha\tau} - K)e^{-r\tau} \mathbf{1}_{\{\tau \leq T\}}] \quad (7)$$

where $r_\alpha = r - \alpha$. Here, the term P_1 corresponds to the case in which the stock never reaches the desired level $L_t = Le^{\alpha t}$, while P_2 corresponds to the option being exercised when the stock reaches the desired level.

Case B: Intensity-based model for exercise time, no vesting period, i.e., $L = \infty, T_0 = 0$

Here we assume, as in Carr and Linetsky (2000), that the option is either exercised according to the arrival of a process with a given intensity, or expires at maturity. More precisely, suppose now that the conditional distribution of the exercise time is

$$F(t) = 1 - e^{-\int_0^t \lambda_s ds} \quad (8)$$

and that

$$\lambda_t = g(t, S_t), \quad (9)$$

for some function g . In other words, conditionally on knowing λ , the exercise time is the first arrival of a Poisson process with the mean arrival rate $\frac{1}{t} \int_0^t \lambda_s ds$ per unit time.

Then the price can be written as

$$E \left[\int_0^T (S_t - K)^+ g(t, S_t) e^{-\int_0^t (r + g(u, S_u)) du} dt + (S_T - K)^+ e^{-\int_0^T (r + g(t, S_t)) dt} \right]. \quad (10)$$

The PDE for this is

$$V_t(t, s) + \frac{1}{2}\sigma^2 V_{ss}(t, s) + rV_s(t, s) - (r + g(t, s))V(t, s) + g(t, s)(s - K)^+ = 0, \quad V(T, s) = (s - K)^+. \quad (11)$$

In the case where the arrival rate λ is constant, the price is

$$E \left[\int_0^T \lambda (S_t - K)^+ e^{-(r+\lambda)t} dt + (S_T - K)^+ e^{-(r+\lambda)T} \right]. \quad (12)$$

The first term corresponds to expiration/exercise before maturity (at time t , with “probabilities” $\lambda e^{-\lambda t}$), and the second term to expiration/exercise at maturity.

Case C: A combination of the intensity and hitting-time models for exercise time, no vesting period, i.e., $T_0 = 0$

We now assume that the exercise time is:

$$\tau = \min(T_L, T_\lambda), \quad (13)$$

where T_L is the first time the stock hits the level $L_t = Le^{at}$, and T_λ is a time having intensity λ . When $\alpha = 0$, this is also the model of Hull and White (2004), but in a binomial tree model. Assume that T_L and T_λ are conditionally independent. Then, we have

$$\begin{aligned} F(t) &= P_t(\tau \leq t) = P_t(T_L \leq t) + P_t(T_\lambda \leq t) - P_t(T_L \leq t)P_t(T_\lambda \leq t) \\ &= \mathbf{1}_{\{T_L \leq t\}} + P_t(T_\lambda \leq t) - \mathbf{1}_{\{T_L \leq t\}}P_t(T_\lambda \leq t) = \mathbf{1}_{\{T_L \leq t\}} + P_t(T_\lambda \leq t)\mathbf{1}_{\{T_L > t\}} \\ &= 1 - e^{-\lambda t}\mathbf{1}_{\{T_L > t\}}. \end{aligned} \quad (14)$$

Therefore, by Equation (5), the price is equal to:

$$\begin{aligned} J_1 + J_2 + J_3 &= E[(Le^{-(r+\lambda)T_L} - Ke^{-(r+\lambda)T_L})\mathbf{1}_{\{T_L \leq T\}}] \\ &\quad + E\left[\int_0^T \lambda e^{-(r+\lambda)t}(S_t - K)^+\mathbf{1}_{\{T_L > t\}}dt\right] \\ &\quad + E\left[e^{-(r+\lambda)T}(S_T - K)^+\mathbf{1}_{\{T_L > T\}}\right]. \end{aligned} \quad (15)$$

Here, J_1 corresponds to exercising at the desired level, J_2 , to being fired/quitting at intensity λ , and J_3 to exercise/expiry at maturity.

Case D: Combined model with a vesting period

Suppose now that there is a vesting period $[0, T_0]$, $T_0 < T$, in which the employee may quit or be fired with intensity λ_0 , and gets nothing from the option.

After the vesting period, the intensity of quitting/being fired is λ ,¹⁰ and the employee will exercise when the stock reaches the desired level $L e^{\alpha(t-T_0)}$. We denote by T_λ^0 the time of quitting/being fired and

$$T_L^0 = \min\{t \in [T_0, T], S_t \geq L e^{\alpha(t-T_0)}\}, \quad (16)$$

so that the time of exercise/expiry is

$$\tau = \min\{T_L^0, T_\lambda^0, T\}. \quad (17)$$

As before, we find that

$$F(t) = 1 - e^{-\lambda_0 t} \mathbf{1}_{\{T_L^0 > t\}}, \quad t \leq T_0 \quad (18)$$

and

$$F(t) = 1 - e^{-\lambda_0 T_0 - \lambda(t-T_0)} \mathbf{1}_{\{T_L^0 > t\}}, \quad t > T_0. \quad (19)$$

Therefore, similarly as in Equation (15), we get that the price is equal to:

$$\begin{aligned} & K_{11} + K_{12} + K_2 + K_3 \\ &= e^{(\lambda-\lambda_0)T_0} E \left[\left(L e^{-\alpha T_0} e^{-(r+\lambda)T_L^0} - K e^{-(r+\lambda)T_L^0} \right) \mathbf{1}_{\{T_L^0 \leq T, S_{T_0} < L T_0\}} \right] \\ &+ e^{(\lambda-\lambda_0)T_0} E \left[e^{-(r+\lambda)T_0} (S_{T_0} - K)^+ \mathbf{1}_{\{S_{T_0} \geq L T_0\}} \right] \\ &+ e^{(\lambda-\lambda_0)T_0} E \left[\int_{T_0}^T \lambda e^{-(r+\lambda)t} (S_t - K)^+ \mathbf{1}_{\{T_L^0 > t\}} dt \right] \\ &+ e^{(\lambda-\lambda_0)T_0} E \left[e^{-(r+\lambda)T} (S_T - K)^+ \mathbf{1}_{\{T_L^0 > T\}} \right]. \end{aligned} \quad (20)$$

We interpret the previous parameters using Figure 1, which explains the different exercise possibilities in the model. In segment B of Figure 1, K_{11} corresponds to exercising at the desired level after the vesting period (in segment B of Figure 1); K_{12} corresponds to exercising right after the vesting period (in segment C); K_2 corresponds to being fired/quitting at intensity λ after the vesting period (in region D); and K_3 corresponds to exercise/expiry at maturity (in segment E).

The PDE and the boundary conditions for the price, when the intensity is $\lambda(t, s)$, are given by the following:

For $t \geq T_0$ and $s < L e^{\alpha(t-T_0)}$, we have

$$V_t(t, s) + \frac{1}{2} \sigma^2 V_{ss}(t, s) + r V_s(t, s) - (r + \lambda(t, s)) V(t, s) + \lambda(t, s) (s - K)^+ = 0, \quad (21)$$

¹⁰ Since, in general, leaving the firm before the ESO vests results in forfeiture, while that is not the case after vesting, it is reasonable to allow two different rates of exit. In addition, accounting regulations require the use of two different rates: before and after vesting.

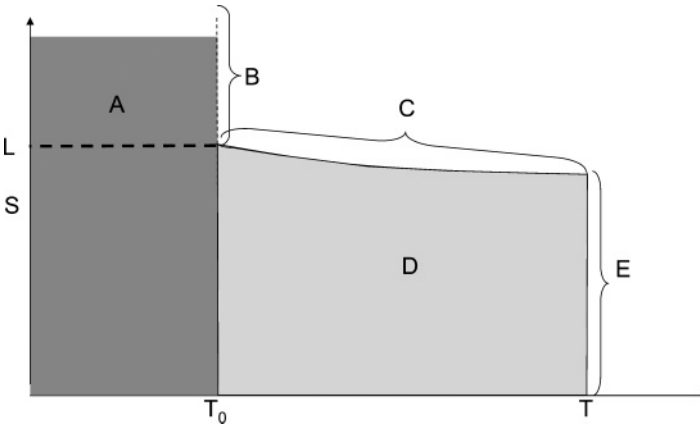


Figure 1
Expiration of ESO—the probable scenarios

This figure illustrates all possible scenarios of expiration of the ESO. The ESO expires in region A if the employee is fired/quits before vesting. We denote the possibility of this event by P_1 . Then, if at T_0 the price of the underlying is in segment B, the ESO is exercised. We denote the probability of this event as P_2 . If the ESO is unexercised after T_0 , the ESO is exercised if it hits the barrier, corresponding to segment C, with probability P_3 , or if the employee is fired or leaves the firm in region D, with probability P_4 . If none of the above takes place, the option reaches maturity and is exercised in segment E, with probability P_5 . Obviously, $P_1 + P_2 + P_3 + P_4 + P_5 = 1$.

with boundary conditions

$$V(t, Le^{\alpha(t-T_0)}) = Le^{\alpha(t-T_0)} - K, \quad (22)$$

$$V(T, s) = (s - K)^+; \quad (23)$$

for $t < T_0$, we have

$$\begin{aligned} V_t(t, s) + \frac{1}{2}\sigma^2 V_{ss}(t, s) + rV_s(t, s) - (r + \lambda(t, s))V(t, s) \\ + \lambda(t, s)(s - K)^+ = 0, \end{aligned} \quad (24)$$

$$V(T_0, s) = s - K, \quad s \geq L. \quad (25)$$

In addition, we require smooth pasting of the boundary conditions, as $t \rightarrow T_0$. In Appendix B, we also explain how to modify our formulas in case there are dividends paid at a constant rate. Moreover, we also explain how to modify for the case where there is a possibility that the employee will leave the company and have to forfeit the options, due perhaps to being hired by competing firms, or other breach of contract.

Additionally, as we show in Appendix D, we can compute the probability that the option will be exercised before maturity. Overall, we compute the probabilities of five mutually exclusive scenarios that can characterize the life of the ESO. Figure 1 illustrates these scenarios.

1. First, it is possible that the employee will leave, at rate λ_0 , the firm before the ESO is vested, in which case she gets nothing. We denote this probability by P_1 and it corresponds to region A of Figure 1.
2. Then, the employee might decide to exercise the option right after vesting. We denote the probability of this event by P_2 , corresponding to segment B at time T_0 in Figure 1.
3. If the option is still unexercised after T_0 , it is possible that the employee might decide to exercise early if the underlying hits the barrier, corresponding to segment C in Figure 1. We denote the probability of this event by P_3 .
4. Alternatively, if the option is still unexercised after T_0 , it is possible that the employee might leave the firm or be fired before the ESO expires or the barrier is reached. That can happen at the exogenous rate λ in region D. We denote by P_4 the probability of that event.
5. Finally, if none of the above has happened, the option will attain maturity and be exercised then, corresponding to segment E in Figure 1. We denote by P_5 the probability of this event.

Obviously, it has to be the case that:

$$P_1 + P_2 + P_3 + P_4 + P_5 = 1. \quad (26)$$

The importance of probabilities P_1 , P_2 , P_3 , P_4 , and P_5 lies in the fact that they can be easily estimated in practice and provide the grounds to calibrate the parameter models L , α , and λ for pricing computations. One of the problems brought up in debates about the fair-price approach to expensing ESO's (see FASB, 2004) is the lack of a uniform criterion. The formula introduced in this paper only requires three "subjective" parameters: L , α , and λ (we do not include volatility, which is a parameter even if the BS formula is used). It would be easy to introduce a criterion for computation of the probabilities, which would pin down the values for L , α , and λ .

In addition, we also derive analytic formulas for the expected life of the option, as well as for the expected stock price at expiry/exercise. These can also be used for calibration purposes.

1.3 Numerical examples

We now apply the formula derived in Appendixes A, B, and C to price the ESO, for different parameter values. Although the formula involves several terms, it can easily be computed in standard commercial software.¹¹ A significant advantage of the formula is that it is differentiable with respect to all parameters of the model. This provides the firm with a powerful tool for hedging purposes, as well as to study the sensitivity of the price of the option, both to the parameters of the underlying stock price and to the parameters that characterize

¹¹ We used Mathematica 5.0.

Table 1
Convergence of the binomial tree approach

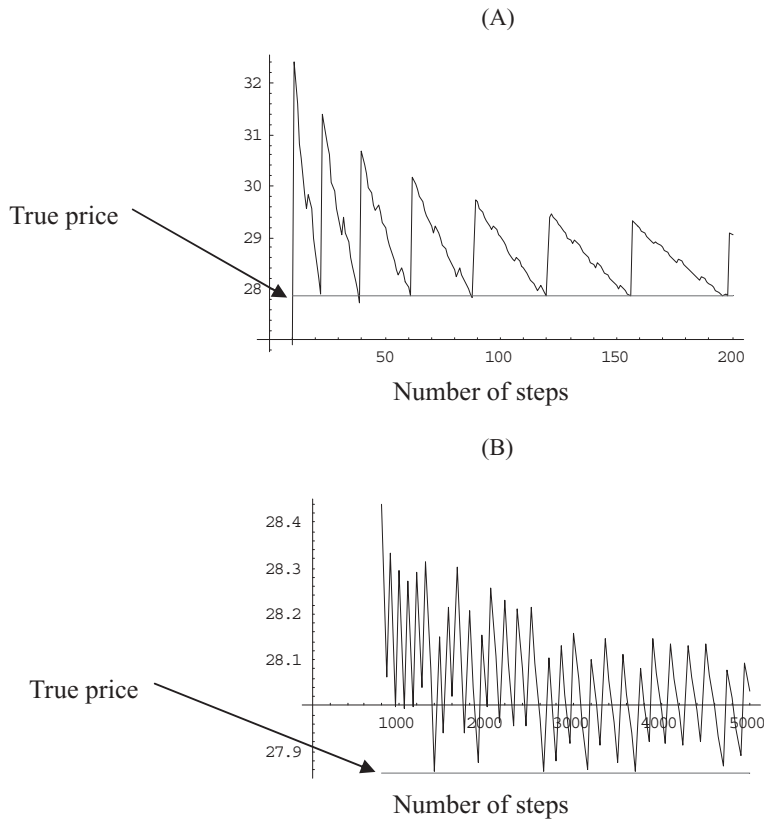
“True” price is 27.8551	
<i>N</i>	<i>P</i>
50	29.1894
100	29.0063
250	28.8949
500	28.4249
750	28.1550
1000	28.2934
1250	28.0380
1500	27.9424
1750	27.9404
2000	27.9973
2250	28.0925
2500	28.2135
3000	28.1587
4000	27.9921
5000	28.0327
7500	27.9592
10000	28.0239
20000	27.9003
40000	27.9291

We compute the price of the ESO using a binomial tree for different numbers of time steps. N represents the number of steps and P the price of the ESO according to the binomial tree approach. Parameter values are $s = 100$; $K = 100$; $T = 10$; $T_0 = 2$; $\sigma = 0.2$; and $r = 0.06$. Additionally, for the exercise barrier, we take $L = 150$; for the rate of increase of the barrier, we take $\alpha = 0$; and for the probability of leaving the firm, we take $\lambda = 0.04$. The price of the ESO (as obtained using the analytical formula discussed in the paper) is 27.8551.

the exercise policy and attrition rate. On the other hand, binomial pricing converges very slowly at a speed that depends on the parameter values considered. Additionally, the convergence is not uniform. This might be an important obstacle when computing hedging portfolios. Table 1 shows the convergence of a tree for different numbers of steps for an example with reasonable parameter values. Figure 2 shows the convergence to the option price for different numbers of steps. We observe that, for all our exercises, the binomial tree always overestimates the price of the ESO. We point out the following:

Remark 1. The price that we get for the employee option is the limit of the binomial tree procedure of Hull and White (2004), if we use the usual parameterization that results in the convergence of the Cox-Ross-Rubinstein (1979) price to the Black-Scholes price. The only addition is that, at each step in the tree, the employee might quit/get fired with probability $\lambda \Delta t$, where λ is the exit rate.

We present the examples in Table 2. In Case A, options are automatically vested and they would be exercised if the underlying hits the barrier $L > K$.

**Figure 2****Computing the price of ESO—the binomial tree approach**

The plots show the convergence of pricing using a binomial tree to the price of the ESO. Parameter values are $s = 100$; $K = 100$; $L = 150$; $T = 10$; $T_0 = 2$; $\sigma = 0.2$; $r = 0.06$; and $\lambda = 0.04$. In A we plot convergence to the price of the ESO for up to 200 steps in the tree. In B we plot convergence to the price of the ESO for up to 5000 steps in the tree. The price of the ESO is 27.8551.

In Case B, options are also automatically vested but they are exercised at some exogenously determined time that happens randomly with intensity λ , the exit rate. Case C combines A and B, so that the exercise happens either when the underlying hits the barrier or when the randomly determined time arrives, whatever comes first. Case D is the most complete and is equivalent to Case C but with a vesting period T_0 . In all tables, we also include the corresponding Black and Scholes prices (that we denote by BS) for comparison purposes.

In Table 2, as expected, the Black and Scholes formula overestimates the price of the ESO greatly, even for cases of a high-barrier L (the employee is less likely to exercise early) and low λ (low probability of early departure). We also see that the vesting period affects the price of the ESO in two simultaneous, opposite ways. It has a negative effect on the price of the option because the

Table 2
Prices of ESOs for different parameter values

$T_0 = 3; \lambda = 0.04$						
	s	A	B	C	D	BS
$L = 125$	100	16.1088	38.9753	15.3372	22.7792	45.1930
	120	23.1375	56.4807	22.9921	35.7948	63.0836
$L = 150$	100	26.0510	38.9753	23.9052	26.8375	45.1930
	120	35.3827	56.4807	34.0925	39.2417	63.0836
$T_0 = 3; \lambda = 0.2$						
	s	A	B	C	D	BS
$L = 125$	100	16.1088	24.4350	12.9962	13.5253	45.1930
	120	23.1375	41.1104	22.5402	21.7856	63.0836
$L = 150$	100	26.0510	24.4350	18.1005	15.2048	45.1930
	120	35.3827	41.1104	30.5150	23.2637	63.0836
$s = 120; L = 125$						
	T_0	A	B	C	D	BS
$\lambda = 0.04$	1	23.1375	56.4807	22.9921	29.2254	63.0836
	3	23.1375	56.4807	22.9921	35.7948	63.0836
$\lambda = 0.2$	1	23.1375	41.1104	22.5402	24.2800	63.0836
	3	23.1375	41.1104	22.5402	21.7856	63.0836
$s = 100; L = 150$						
	T_0	A	B	C	D	BS
$\lambda = 0.04$	1	26.0510	38.9753	23.9052	24.5668	45.1930
	3	26.0510	38.9753	23.9052	26.8375	45.1930
$\lambda = 0.2$	1	26.0510	24.4350	18.1005	17.4525	45.1930
	3	26.0510	24.4350	18.1005	15.2048	45.1930

We compute prices of ESOs and study the effect of the distance between the price of underlying (s) and the barrier L , as well as the impact of different combinations of the rate of exit λ and the vesting period T_0 . The other parameter values of the model are $K = 100$; $T = 10$; $T_0 = 3$; $\sigma = 0.2$; $r = 0.05$; and $\alpha = -0.02$. The table presents the results for cases A, B, C, and D as explained in Section 2, plus the BS price.

employee can be fired before the option is vested and get nothing (regardless of whether the option is in- or out-of-the-money). However, it has a positive effect on the price of the ESO because it prevents the employee from exercising the option early. When λ is large, the negative effect prevails because there is a high probability that the employee will leave the firm before having a chance to exercise the ESO.

2. Case Study

The objective of this section is to apply the formula derived in this paper to a real grant program, and compare resulting prices with prices from alternative formulas.

We use data from TEVA, an Israeli firm. It has a market capitalization of more than \$20B. Its shares trade on Nasdaq (as ADRs) with average daily volume of

Table 3
Characteristics of option grants of TEVA

# of grant	Date	# of options	$s = K$	T	% T_{2y}	% T_{3y}	% T_{4y}	δ
1	7/23/2001	845,000	65.33	5	1/4	1/4	1/2	0.1%
2	2/14/2002	800,000	60.41	7	1/4	1/4	1/2	0.14%
3	3/24/2003	2,000,000	40.40	7	1/3	1/3	1/3	0.35%
4	7/12/2004	2,019,000	33.27	7	1/3	1/3	1/3	0.53%

We present the option grants of TEVA. There are four option grants. The stock price s and strike price K are equal (options are granted at the money). # of options is the number of options in the plan; T is the maturity of the options; and % T_{iy} , $i = 2, 3$, and 4 represents the proportion of the total grant vesting in i years. δ is the annual dividend yield, computed over the year preceding the grant.

about 7.5M shares. The company is one of the biggest in its industry (generic drugs) and has activity in several countries worldwide. Table 3 presents the details of the option plans of this firm.¹²

We compare four possible ways to price the ESO:

1. The BS formula, with maturity equal to the full maturity of the options. We compute it just for benchmarking, since, as we have discussed, it is not a good representation of the true price of the option.
2. The simplified BS formula, which consists in applying the BS formula with a maturity equal to the average of the vesting period and the maturity of the options. SAB 107 permits its use temporarily, not beyond 2007.
3. The binomial method (BM) introduced by Hull and White (2004).
4. The analytical formula (AF) derived in the current paper.

All four methods need the five BS option pricing parameters: price of stock s , strike price K , interest rate r , time to maturity T and stock return volatility σ . We also consider the dividend yield, for which the BS formula can be adjusted; AF allows for dividends, as discussed in Appendix B. In addition, in all four cases we will take into account the possibility that some options will be forfeited as a result of the employee leaving the firm before vesting. For that purpose, we need the rate of exit λ , and the time to vesting T_0 . The rate of exit also triggers early exercise after vesting in both BM and AF. Finally, both BM and AF rely on an early exercise barrier L , as well as the rate of decay of the barrier, α .

We do not have specific information about the actual exercise policy of TEVA employees (in fact, executives). However, we did several statistical studies of other Israeli companies, and in most cases found values of L in the range of 2–2.5 times the strike price. Thus, we provide two sets of data for these values. Similarly, we estimate the rate of exit for this company to be 0.1. This seems to be a good estimate of the turnover rate at TEVA, although it is relatively low with respect to the average numbers. For robustness purposes, we also consider a rate of exit of 0.15, more in line with overall employee turnover.

¹² We used only publicly available information and, therefore, we might have missed some relevant characteristics of the option, not publicly disclosed.

Table 4
Grant prices

$\lambda = 0.1$						
	#	$\sigma(\%)$	BS	SBS	BM	AF
$L = 2.5$	1	42.18	23.9921	15.6037	16.5567	16.6647
	2	41.00	24.4120	14.8436	15.8963	15.9173
	3	39.00	36.2617	22.1947	24.4876	24.7658
	4	37.20	29.5931	18.2149	19.9808	20.3841
$L = 2.0$	1	42.18	23.9921	15.6037	16.2487	16.3575
	2	41.00	24.4120	14.8436	15.3715	15.3461
	3	39.00	36.2617	22.1947	23.7564	23.9036
	4	37.20	29.5931	18.2149	19.2769	19.6506
$\lambda = 0.15$						
$L = 2.5$	1	42.18	23.9921	13.2898	13.9910	14.0878
	2	41.00	24.4120	12.6476	13.2744	13.3024
	3	39.00	36.2617	19.1473	20.6329	20.8796
	4	37.20	29.5931	15.7139	16.8303	17.1775
$L = 2.0$	1	42.18	23.9921	13.2898	13.7382	13.8354
	2	41.00	24.4120	12.6476	12.8664	12.8570
	3	39.00	36.2617	19.1473	20.0645	20.2074
	4	37.20	29.5931	15.7139	16.2833	16.6049

We compute the prices of the option grants of Table 3. # corresponds to the number of grant of Table 3. Parameter values are as in Table 3. In addition, we compute volatility using returns for a period of time equal to the mean between the vesting period and the maturity of the grant. Volatility is represented as σ . The four prices computed correspond to the following methods: Black and Scholes (BS); the simplified Black and Scholes (SBS); the binomial method of Hull and White (2004) (BM); and the analytic formula derived in the current paper (AF). For all methods we need the rate of forfeiture λ . For BM and AF, we also need the barrier L . The level of the barrier is given as a multiple of the strike price. Prices are given in millions of dollars.

Table 4 shows the prices of the option grants for all four methods, using the previous parameter values. In the comparison, we only consider the options that are expected to survive the vesting periods. This is automatic in BM and AF, while we subtract options that are expected to be forfeited in the BS and simplified BS methods. As expected, BS always gives a very high price, substantially above all the others. In addition, we observe that simplified BS underestimates the price of the options with respect to AF. This is always the case, although obviously the difference is smaller the lower the barrier and the higher the exit rate, both of which reduce the price of the option.

Overall, then, it appears that SBS will tend to underprice option plans, except for firms in which the turnover is very high and employees tend to exercise their options at relatively low prices. As we have discussed, SAB 107 states that simplified BS should not be used beyond 2007. The argument is that by then firms will have enough of a track record to be able to get good estimates of the expected time to exercise.

We also observe that, as discussed, the binomial method produces estimates that deviate from the true value of the option grants in a nonuniform way.

This case study illustrates that the formula derived in this paper fulfills the requirements of FAS 123R but is superior to the two alternatives contemplated by SAB 107:

1. SAB 107 seems to favor lattice methods; they are imprecise. Another problem is the difficulty of computing hedging ratios through lattice methods, if the firm wishes to hedge the risk-exposure to the options. This computation is straightforward, from our explicit formula.
2. In general, simplified BS does not provided a good approximation to the cost of the options. Beyond 2007, simplified BS has to be replaced by another BS formula with expected maturity corresponding to the observed maturity in existing option grants. However, historic average time to exercise can also lead to large deviations: arguably, the key parameter that drives the exercise policy of the manager is the price of the underlying, maybe as a function of the time left to maturity, or the volatility of the underlying. Our formula shares the advantage of a completely explicit expression, trivial to compute, but it uses as input parameters with more economic significance than the expected maturity necessary for simplified BS (or BS with adjusted maturity, in general).

3. Conclusions

In this paper, we derive an analytic expression for the price of an employee stock option (ESO). The option has a vesting period. Also, the employee exercises when the stock price hits a given level; this way our model can capture early exercise patterns observed in practice. Additionally, it is possible that the employee will be fired or quit, which happens randomly. Prices of ESOs are considerably lower than the equivalent BS prices and less sensitive to changes in parameter values, since it is likely that the option will not reach maturity. We present a case study and we find that the simplified BS method, temporarily allowed by the SEC (SAB 107), underprices the options in general. Furthermore, in Appendix F we compute the expected times until the employee exercises the option, is fired, or quits the firm. Given sample information, it would be straightforward to calibrate the value of the parameters so that expected times match empirical observations. Additionally, the model we describe could be applied to a firm by considering a ladder of different L values. This characterizes the desired voluntary early exercise of the employee. By using a ladder of barriers, we could account for the diversity of risk-aversion/liquidity needs across different employees. Finally, since options often do not reach maturity, we could simplify some of the previous expressions by considering infinite maturity (as in Sircar and Xiong, 2007).

Appendix

A. List of formulas

We list here the explicit formulas for all the cases. The notation is explained in Appendix B.

Case A: The price when stopping at the hitting time:

If $s < Ln$, then the price is

$$P_1^\alpha(s) + LP(s, T, y_-^\alpha, y_+^\alpha) - KP(s, T, y_-^\alpha, \bar{y}). \quad (\text{A.1})$$

If $s \geq L$, then the price is $s - L$.

Case B: The price when stopping at the random time:

$$\begin{aligned} & I_1(s, T, K, r_0, b_0) + I_2(s, T, K, r_0, r_0, c_0) + e^{-\lambda T} s N \left(\frac{\tilde{K}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} y_+^0 \right) \\ & - K e^{-(r_0+\lambda)T} N \left(\frac{\tilde{K}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} y_-^0 \right). \end{aligned} \quad (\text{A.2})$$

Case C: The price in the combined case, no vesting period:

$$\begin{aligned} & e^{-\lambda T} P_1(s) + LP(s, T, y_-^\alpha, c_\alpha) - KP(s, T, y_-^\alpha, \bar{c}) \\ & + I_1(s, T, K, r_0, b_0) + I_2(s, T, K, r_0, r_0, c_0) - I_1(s, T, L, r_\alpha, b_\alpha) \\ & - \frac{K}{L} I_2(s, T, L, r_0, r_\alpha, \bar{c}) - \left(\frac{L}{s} \right)^{\frac{2r_\alpha}{\sigma^2} - 1} \left[I_1 \left(\frac{L^2}{s}, T, K, r_0, b_0 \right) \right. \\ & + I_2 \left(\frac{L^2}{s}, T, K, r_0, r_0, c_0 \right) - I_1 \left(\frac{L^2}{s}, T, L, r_\alpha, b_\alpha \right) \\ & \left. - \frac{K}{L} I_2 \left(\frac{L^2}{s}, T, L, r_0, r_\alpha, \bar{c} \right) \right]. \end{aligned} \quad (\text{A.3})$$

Case D: The price in the combined case with vesting period T_0 :

$$K_{11} + K_{12} + K_2 + K_3. \quad (\text{A.4})$$

B. Notation for the formulas

The notation used in the foregoing is explained in the following: First, we denote by N the standard normal distribution function, and by $n = N'$ its density. Moreover, we introduce:

$$K_\alpha(T) = K e^{-\alpha T}, \quad r_\alpha = r_0 - \alpha; \quad (\text{B.1})$$

$$x_Y = \frac{\log(Y/s)}{\sigma\sqrt{T_0}} - (r_0 - \sigma^2/2) \frac{\sqrt{T_0}}{\sigma}; \quad (\text{B.2})$$

$$y_+^\alpha = r_\alpha + \frac{\sigma^2}{2}, \quad y_-^\alpha = r_\alpha - \frac{\sigma^2}{2}, \quad \bar{y} = \sqrt{(y_-^\alpha)^2 + 2\sigma^2 r_0}; \quad (\text{B.3})$$

$$\tilde{K}_\alpha(T) = \log(s/K_\alpha(T)), \quad \tilde{L} = \log(s/L); \quad (\text{B.4})$$

$$b_\alpha := \sqrt{(r_\alpha + \sigma^2/2)^2 + 2\sigma^2\lambda}; \quad (\text{B.5})$$

$$c_\alpha := \sqrt{(r_\alpha - \sigma^2/2)^2 + 2\sigma^2(\lambda + r_\alpha)}, \quad \bar{c} := \sqrt{(r_\alpha - \sigma^2/2)^2 + 2\sigma^2(\lambda + r_0)}. \quad (\text{B.6})$$

$$\begin{aligned} P_1^\alpha(s) = & sN \left(\frac{\tilde{K}_\alpha(T)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha + \frac{\sigma^2}{2} \right) \right) \\ & - K_\alpha(T)e^{-r_\alpha T} N \left(\frac{\tilde{K}_\alpha(T)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha - \frac{\sigma^2}{2} \right) \right) \\ & - sN \left(\frac{\tilde{L}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha + \frac{\sigma^2}{2} \right) \right) \\ & + K_\alpha(T)e^{-r_\alpha T} N \left(\frac{\tilde{L}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha - \frac{\sigma^2}{2} \right) \right) \\ & - \left(\frac{L}{s} \right)^{\frac{2g}{\sigma^2}-1} \left[\frac{L^2}{s} N \left(\frac{1}{\sigma\sqrt{T}} \log \left(\frac{L^2}{sK_\alpha(T)} \right) + \frac{\sqrt{T}}{\sigma} \left(r_\alpha + \frac{\sigma^2}{2} \right) \right) \right. \\ & - K_\alpha(T)e^{-r_\alpha T} N \left(\frac{1}{\sigma\sqrt{T}} \log \left(\frac{L^2}{sK_\alpha(T)} \right) + \frac{\sqrt{T}}{\sigma} \left(r_\alpha - \frac{\sigma^2}{2} \right) \right) \\ & \left. - \frac{L^2}{s} N \left(-\frac{\tilde{L}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha + \frac{\sigma^2}{2} \right) \right) \right]. \end{aligned} \quad (\text{B.7})$$

$$K_\alpha(T)e^{-r_\alpha T} N \left(-\frac{\tilde{L}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} \left(r_\alpha - \frac{\sigma^2}{2} \right) \right) \Bigg]. \quad (\text{B.8})$$

$$\begin{aligned} P(s, T, \mu, y) = & \left(\frac{L}{s} \right)^{\frac{\mu-y}{\sigma^2}} N \left(\frac{\log(s/L)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} y \right) \\ & + \left(\frac{L}{s} \right)^{\frac{\mu+y}{\sigma^2}} N \left(\frac{\log(s/L)}{\sigma\sqrt{T}} - \frac{\sqrt{T}}{\sigma} y \right). \end{aligned} \quad (\text{B.9})$$

$$I_1(x, T, z, r, b)$$

$$\begin{aligned} = & x \left[\mathbf{1}_{\{x>z\}} + \frac{1}{2} \mathbf{1}_{\{x=z\}} - e^{-\lambda T} N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} (r + \sigma^2/2) \right) \right] \\ & + x \left(\frac{z}{x} \right)^{\frac{r-b}{\sigma^2} + \frac{1}{2}} \left\{ \frac{r + \sigma^2/2}{b} \left[N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} b \right) - \mathbf{1}_{\{x>z\}} - \frac{1}{2} \mathbf{1}_{\{x=z\}} \right] \right. \\ & + \frac{1}{2} \left[1 - \frac{r + \sigma^2/2}{b} \right] \left[N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} b \right) \right. \\ & \left. \left. + \left(\frac{z}{x} \right)^{\frac{2b}{\sigma^2}} N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} - \frac{\sqrt{T}}{\sigma} b \right) - \left[1 + \left(\frac{z}{x} \right)^{\frac{2b}{\sigma^2}} \right] \mathbf{1}_{\{x>z\}} - \mathbf{1}_{\{x=z\}} \right] \right\}. \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned}
 I_2(x, T, z, R, r, c) &= -\frac{\lambda z}{\lambda + R} \left[\mathbf{1}_{\{x > z\}} + \frac{1}{2} \mathbf{1}_{\{x = z\}} - e^{-(\lambda + R)T} N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} (r - \sigma^2/2) \right) \right] \\
 &\quad - \frac{\lambda z}{\lambda + R} \left(\frac{z}{x} \right)^{\frac{r - \sigma^2}{\sigma^2} - \frac{1}{2}} \left\{ \frac{r - \sigma^2/2}{c} \left[N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} c \right) \right. \right. \\
 &\quad \left. \left. - \mathbf{1}_{\{x > z\}} - \frac{1}{2} \mathbf{1}_{\{x = z\}} \right] + \frac{1}{2} \left[1 - \frac{r - \sigma^2/2}{c} \right] \left[N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma} c \right) \right. \right. \\
 &\quad \left. \left. + \left(\frac{z}{x} \right)^{\frac{2c}{\sigma^2}} N \left(\frac{\log(x/z)}{\sigma\sqrt{T}} - \frac{\sqrt{T}}{\sigma} b \right) - \left[1 + \left(\frac{z}{x} \right)^{\frac{2c}{\sigma^2}} \right] \mathbf{1}_{\{x > z\}} - \mathbf{1}_{\{x = z\}} \right] \right\}.
 \end{aligned} \tag{B.11}$$

$$B(a, b, c) := \int_{-\infty}^{x_L} e^{ax} N(bx + c) n(x) dx = e^{\frac{a^2}{2}} P(X \leq x_L, Y \leq c) \tag{B.12}$$

where (X, Y) has a bivariate normal distribution with

$$\mu_X = a, \quad \mu_Y = -ab, \quad \sigma_X^2 = 1, \quad \sigma_Y^2 = 1 + b^2, \quad \rho = -\frac{b}{\sqrt{1 + b^2}} \tag{B.13}$$

$$Q = \sqrt{\frac{T_0}{T - T_0}}. \tag{B.14}$$

$$d_0(Y) = \frac{1}{\sigma\sqrt{T - T_0}} \left[\log(s/Y) + (r_0 - \sigma^2/2)T_0 \right]. \tag{B.15}$$

$$d_1(Y) = \frac{1}{\sigma\sqrt{T - T_0}} \left[\log\left(\frac{L^2}{sY}\right) - (r_0 - \sigma^2/2)T_0 \right]. \tag{B.16}$$

$$\begin{aligned}
 K_{11} &= L e^{-\alpha T_0} e^{-(r_\alpha + \lambda_0)T_0} \left(\frac{L}{s} \right)^{\frac{y_\alpha^\alpha - c_\alpha}{\sigma^2}} e^{\frac{c_\alpha - y_\alpha^\alpha}{\sigma^2} (r_0 - \sigma^2/2)T_0} \\
 &\quad \times B \left(\frac{(c_\alpha - y_\alpha^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T - T_0}}{\sigma} c_\alpha \right) \\
 &\quad - K e^{-(r_0 + \lambda_0)T_0} \left(\frac{L}{s} \right)^{\frac{y_\alpha^\alpha - \tilde{c}}{\sigma^2}} e^{\frac{\tilde{c} - y_\alpha^\alpha}{\sigma^2} (r_0 - \sigma^2/2)T_0} \\
 &\quad \times B \left(\frac{(\tilde{c} - y_\alpha^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T - T_0}}{\sigma} \tilde{c} \right) \\
 &\quad + L e^{-\alpha T_0} e^{-(r_\alpha + \lambda_0)T_0} \left(\frac{L}{s} \right)^{\frac{y_\alpha^\alpha + c_\alpha}{\sigma^2}} e^{-\frac{c_\alpha + y_\alpha^\alpha}{\sigma^2} (r_0 - \sigma^2/2)T_0} \\
 &\quad \times B \left(-\frac{(c_\alpha + y_\alpha^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T - T_0}}{\sigma} c_\alpha \right) \\
 &\quad - K e^{-(r_0 + \lambda_0)T_0} \left(\frac{L}{s} \right)^{\frac{y_\alpha^\alpha + \tilde{c}}{\sigma^2}} e^{-\frac{\tilde{c} + y_\alpha^\alpha}{\sigma^2} (r_0 - \sigma^2/2)T_0}
 \end{aligned}$$

$$\times B\left(-\frac{(\bar{c} + y_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T-T_0}}{\sigma}\bar{c}\right) \quad (\text{B.17})$$

$$\begin{aligned} K_{12} = & e^{(\lambda-\lambda_0)T_0} e^{-\lambda T_0} \left[sN\left(\frac{\sqrt{T_0}}{\sigma}\left(r_0 + \frac{\sigma^2}{2}\right) + \frac{\log(s/L)}{\sigma\sqrt{T_0}}\right) \right. \\ & \left. - Ke^{-r_0 T_0} N\left(\frac{\sqrt{T_0}}{\sigma}\left(r_0 - \frac{\sigma^2}{2}\right) + \frac{\log(s/L)}{\sigma\sqrt{T_0}}\right) \right]. \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} C_0(Y) = & se^{(r_0-\sigma^2/2)T_0} B\left(\sigma\sqrt{T_0}, Q, d_0(Y) + \frac{\sqrt{T-T_0}}{\sigma}(r_\alpha + \sigma^2/2)\right) \\ & - Ke^{-\alpha(T-T_0)} e^{-r_\alpha(T-T_0)} B\left(0, Q, d_0(Y) + \frac{\sqrt{T-T_0}}{\sigma}(r_\alpha - \sigma^2/2)\right). \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} C_1(X) = & L\left(\frac{L}{s}\right)^{\frac{2r_\alpha}{\sigma^2}} e^{-\frac{2r_\alpha}{\sigma^2}(r_0-\sigma^2/2)T_0} B\left(-\frac{2r_\alpha}{\sigma}\sqrt{T_0}, -Q, -d_0(L^2/X) \right. \\ & \left. + \frac{\sqrt{T-T_0}}{\sigma}(r_\alpha + \sigma^2/2)\right) - Ke^{-\alpha(T-T_0)} \left(\frac{L}{s}\right)^{\frac{2r_\alpha}{\sigma^2}-1} \\ & \times e^{-r_\alpha(T-T_0)-(\frac{2r_\alpha}{\sigma^2}-1)(r_0-\sigma^2/2)T_0} B\left(\frac{\sigma^2-2r_\alpha}{\sigma}\sqrt{T_0}, -Q, -d_0(L^2/X) \right. \\ & \left. + \frac{\sqrt{T-T_0}}{\sigma}y^\alpha\right). \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} K_3 = & e^{(\lambda-\lambda_0)T_0} e^{-\lambda T-r_0 T_0} [C_0(K_\alpha(T-T_0)) \\ & -C_0(L) - C_1(K_\alpha(T-T_0)) + C_1(L)]. \end{aligned} \quad (\text{B.21})$$

$$\begin{aligned} D_1(Y, R, r, b) = & \int_{-\infty}^{x_L} I_1(p(T_0, s, x), T-T_0, Y, r, b)n(x)dx \\ = & -se^{(R-\sigma^2/2)T_0-\lambda(T-T_0)} B\left(\sigma\sqrt{T_0}, Q, d_0(Y) + (r + \sigma^2/2)\frac{\sqrt{T-T_0}}{\sigma}\right) \\ & + \frac{1}{2}\left[1 + \frac{1}{b}(r + \sigma^2/2)\right] s\left(\frac{Y}{s}\right)^{\frac{r-b}{\sigma^2}+\frac{1}{2}} e^{[\frac{b-r}{\sigma^2}+\frac{1}{2}](R-\sigma^2/2)T_0} \\ & \times B\left(\left[\frac{b-r}{\sigma} + \frac{\sigma}{2}\right]\sqrt{T_0}, Q, d_0(Y) + b\frac{\sqrt{T-T_0}}{\sigma}\right) \\ & + \frac{1}{2}\left[1 - \frac{1}{b}(r + \sigma^2/2)\right] s\left(\frac{Y}{s}\right)^{\frac{r+b}{\sigma^2}+\frac{1}{2}} e^{[-\frac{b+r}{\sigma^2}+\frac{1}{2}](R-\sigma^2/2)T_0} \\ & \times B\left(\left[-\frac{b+r}{\sigma} + \frac{\sigma}{2}\right]\sqrt{T_0}, Q, d_0(Y) - b\frac{\sqrt{T-T_0}}{\sigma}\right) \\ & + se^{RT_0} [N(x_L - \sigma\sqrt{T_0}) - N(x_Y - \sigma\sqrt{T_0})] \\ & - \frac{1}{2}\left[1 + \frac{1}{b}(r + \sigma^2/2)\right] s\left(\frac{Y}{s}\right)^{\frac{r-b}{\sigma^2}+\frac{1}{2}} e^{[\frac{b-r}{\sigma^2}+\frac{1}{2}](R-\sigma^2/2)T_0 + [\frac{b-r}{\sigma^2}+\frac{1}{2}]^2 \frac{\sigma^2}{2} T_0} \\ & \times \left[N\left(x_L - \left[\frac{b-r}{\sigma} + \frac{\sigma}{2}\right]\sqrt{T_0}\right) - N\left(x_Y - \left[\frac{b-r}{\sigma} + \frac{\sigma}{2}\right]\sqrt{T_0}\right)\right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \left[1 - \frac{1}{b} (r + \sigma^2/2) \right] s \left(\frac{Y}{s} \right)^{\frac{r+b}{\sigma^2} + \frac{1}{2}} e^{\left[-\frac{b+r}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0 + \left[-\frac{b+r}{\sigma^2} + \frac{1}{2} \right]^2 \frac{\sigma^2}{2} T_0} \\
 & \times \left[N \left(x_L - \left[-\frac{b+r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) - N \left(x_Y - \left[-\frac{b+r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) \right].
 \end{aligned}
 \tag{B.22}$$

$$\begin{aligned}
 D_2(Y, R, r, c) &= \int_{-\infty}^{x_L} I_2(p(T_0, s, x), T - T_0, Y, R, r, c) n(x) dx \\
 &= \frac{\lambda Y}{\lambda + R} \left\{ e^{-(\lambda+R)(T-T_0)} B \left(0, Q, d_0(Y) + (r - \sigma^2/2) \frac{\sqrt{T-T_0}}{\sigma} \right) \right. \\
 & \quad - \frac{1}{2} \left[1 + \frac{1}{c} (r - \sigma^2/2) \right] \left(\frac{Y}{s} \right)^{\frac{r+c}{\sigma^2} - \frac{1}{2}} e^{\left[\frac{c-r}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0} \\
 & \quad \times B \left(\left[\frac{c-r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0}, Q, d_0(Y) + c \frac{\sqrt{T-T_0}}{\sigma} \right) \\
 & \quad - \frac{1}{2} \left[1 - \frac{1}{c} (r - \sigma^2/2) \right] \left(\frac{Y}{s} \right)^{\frac{r+c}{\sigma^2} - \frac{1}{2}} e^{\left[-\frac{c+r}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0} \\
 & \quad \times B \left(\left[-\frac{c+r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0}, Q, d_0(Y) - c \frac{\sqrt{T-T_0}}{\sigma} \right) \\
 & \quad - [N(x_L) - N(x_Y)] \\
 & \quad + \frac{1}{2} \left[1 + \frac{1}{c} (r - \sigma^2/2) \right] \left(\frac{Y}{s} \right)^{\frac{r+c}{\sigma^2} - \frac{1}{2}} e^{\left[\frac{c-r}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0 + \left[\frac{c-r}{\sigma^2} + \frac{1}{2} \right]^2 \frac{\sigma^2}{2} T_0} \\
 & \quad \times \left[N \left(x_L - \left[\frac{c-r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) - N \left(x_Y - \left[\frac{c-r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) \right] \\
 & \quad + \frac{1}{2} \left[1 - \frac{1}{c} (r - \sigma^2/2) \right] \left(\frac{Y}{s} \right)^{\frac{r+c}{\sigma^2} - \frac{1}{2}} e^{\left[-\frac{c+r}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0 + \left[-\frac{c+r}{\sigma^2} + \frac{1}{2} \right]^2 \frac{\sigma^2}{2} T_0} \\
 & \quad \times \left[N \left(x_L - \left[-\frac{c+r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) - N \left(x_Y - \left[-\frac{c+r}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0} \right) \right] \Big\}.
 \end{aligned}
 \tag{B.23}$$

$$\begin{aligned}
 G_1(Y, R, r, b) &= \int_{-\infty}^{x_L} \left(\frac{L}{p(T_0, s, x)} \right)^{\frac{2r_\alpha}{\sigma^2} - 1} I_1(L^2/p(T_0, s, x), T - T_0, Y, r, b) n(x) dx \\
 &= -L \left(\frac{L}{s} \right)^{\frac{2r_\alpha}{\sigma^2}} e^{-\frac{2r_\alpha}{\sigma^2} (R - \sigma^2/2) T_0 - \lambda (T - T_0)} \\
 & \quad \times B \left(-\frac{2r_\alpha}{\sigma} \sqrt{T_0}, -Q, d_1(Y) + (r + \sigma^2/2) \frac{\sqrt{T-T_0}}{\sigma} \right) \\
 & \quad + \frac{L}{2} \frac{\frac{2(r_\alpha+b-r)}{\sigma^2}}{\sigma^2} \left[1 + \frac{1}{b} (r + \sigma^2/2) \right] Y^{\frac{r-b}{\sigma^2} + \frac{1}{2}} s^{\frac{r-b-2r_\alpha}{\sigma^2} + \frac{1}{2}} e^{\left[\frac{1}{2} + \frac{r-b-2r_\alpha}{\sigma^2} \right] (R - \sigma^2/2) T_0} \\
 & \quad \times B \left(\left[\frac{\sigma}{2} + \frac{r-b-2r_\alpha}{\sigma} \right] \sqrt{T_0}, -Q, d_1(Y) + b \frac{\sqrt{T-T_0}}{\sigma} \right) \\
 & \quad + \frac{L}{2} \frac{\frac{2(r_\alpha-b-r)}{\sigma^2}}{\sigma^2} \left[1 - \frac{1}{b} (r + \sigma^2/2) \right] Y^{\frac{r+b}{\sigma^2} + \frac{1}{2}} s^{\frac{r+b-2r_\alpha}{\sigma^2} + \frac{1}{2}} e^{\left[\frac{r+b-2r_\alpha}{\sigma^2} + \frac{1}{2} \right] (R - \sigma^2/2) T_0} \\
 & \quad \times B \left(\left[\frac{r+b-2r_\alpha}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0}, -Q, d_1(Y) - b \frac{\sqrt{T-T_0}}{\sigma} \right)
 \end{aligned}$$

$$\begin{aligned}
& + L \left(\frac{L}{s} \right)^{\frac{2r_\alpha}{\sigma^2}} e^{-\frac{2r_\alpha}{\sigma^2}(R-\sigma^2/2)T_0 + \frac{2r_\alpha^2}{\sigma^2}T_0} N \left(x_{\min[L, L^2/Y]} + \frac{2r_\alpha}{\sigma} \sqrt{T_0} \right) \\
& - \frac{L}{2} \frac{\frac{2(r_\alpha+b-r)}{\sigma^2}}{\sigma^2} \left[1 + \frac{1}{b}(r + \sigma^2/2) \right] Y^{\frac{r-b}{\sigma^2} + \frac{1}{2}} s^{\frac{r-b-2r_\alpha}{\sigma^2} + \frac{1}{2}} \\
& \times e^{\left[\frac{1}{2} + \frac{r-b-2r_\alpha}{\sigma^2} \right] (R-\sigma^2/2)T_0 + \left[\frac{1}{2} + \frac{r-b-2r_\alpha}{\sigma^2} \right]^2 \frac{\sigma^2}{2} T_0} \\
& \times N \left(x_{\min[L, L^2/Y]} - \left[\frac{\sigma}{2} + \frac{r-b-2r_\alpha}{\sigma} \right] \sqrt{T_0} \right) \\
& - \frac{L}{2} \frac{\frac{2(r_\alpha+b-r)}{\sigma^2}}{\sigma^2} \left[1 - \frac{1}{b}(r + \sigma^2/2) \right] Y^{\frac{r+b}{\sigma^2} + \frac{1}{2}} s^{\frac{r+b-2r_\alpha}{\sigma^2} + \frac{1}{2}} \\
& \times e^{\left[\frac{r+b-2r_\alpha}{\sigma^2} + \frac{1}{2} \right] (R-\sigma^2/2)T_0 + \left[\frac{r+b-2r_\alpha}{\sigma^2} + \frac{1}{2} \right]^2 \frac{\sigma^2}{2} T_0} \\
& \times N \left(x_{\min[L, L^2/Y]} - \left[\frac{\sigma}{2} + \frac{r+b-2r_\alpha}{\sigma} \right] \sqrt{T_0} \right). \tag{B.24}
\end{aligned}$$

$$G_2(Y, R, r, c)$$

$$\begin{aligned}
& = \int_{-\infty}^{x_L} \left(\frac{L}{p(T_0, s, x)} \right)^{\frac{2r_\alpha}{\sigma^2} - 1} I_2(L^2/p(T_0, s, x), T - T_0, Y, R, r, c) n(x) dx \\
& = \frac{\lambda Y}{\lambda + R} \left\{ \left(\frac{L}{s} \right)^{\frac{2r_\alpha}{\sigma^2} - 1} e^{(1 - \frac{2r_\alpha}{\sigma^2})(R-\sigma^2/2)T_0 - (\lambda + R)(T - T_0)} \right. \\
& \quad \times B \left(\left(\sigma - \frac{2r_\alpha}{\sigma} \right) \sqrt{T_0}, -Q, d_1(Y) + (r - \sigma^2/2) \frac{\sqrt{T - T_0}}{\sigma} \right) \\
& \quad - \frac{L}{2} \frac{\frac{2(r_\alpha+c-r)}{\sigma^2}}{\sigma^2} \left[1 + \frac{1}{c}(r - \sigma^2/2) \right] Y^{\frac{r-c}{\sigma^2} - \frac{1}{2}} s^{\frac{r-c-2r_\alpha}{\sigma^2} + \frac{1}{2}} e^{\left[\frac{1}{2} + \frac{r-c-2r_\alpha}{\sigma^2} \right] (R-\sigma^2/2)T_0} \\
& \quad \times B \left(\left[\frac{\sigma}{2} + \frac{r-c-2r_\alpha}{\sigma} \right] \sqrt{T_0}, -Q, d_1(Y) + c \frac{\sqrt{T - T_0}}{\sigma} \right) \\
& \quad - \frac{L}{2} \frac{\frac{2(r_\alpha+c-r)}{\sigma^2}}{\sigma^2} \left[1 - \frac{1}{c}(r - \sigma^2/2) \right] Y^{\frac{r+c}{\sigma^2} - \frac{1}{2}} s^{\frac{r+c-2r_\alpha}{\sigma^2} + \frac{1}{2}} e^{\left[\frac{r+c-2r_\alpha}{\sigma^2} + \frac{1}{2} \right] (R-\sigma^2/2)T_0} \\
& \quad \times B \left(\left[\frac{r+c-2r_\alpha}{\sigma} + \frac{\sigma}{2} \right] \sqrt{T_0}, -Q, d_1(Y) - c \frac{\sqrt{T - T_0}}{\sigma} \right) \\
& \quad - \left(\frac{L}{s} \right)^{\frac{2r_\alpha}{\sigma^2} - 1} e^{\left[1 - \frac{2r_\alpha}{\sigma^2} \right] (R-\sigma^2/2)T_0 + \left[1 - \frac{2r_\alpha}{\sigma^2} \right]^2 \frac{\sigma^2}{2} T_0} N \left(x_{\min[L, L^2/Y]} \right. \\
& \quad \left. - (\sigma - 2r_\alpha/\sigma) \sqrt{T_0} \right) + \frac{L}{2} \frac{\frac{2(r_\alpha+c-r)}{\sigma^2}}{\sigma^2} \left[1 + \frac{1}{c}(r - \sigma^2/2) \right] Y^{\frac{r-c}{\sigma^2} - \frac{1}{2}} s^{\frac{r-c-2r_\alpha}{\sigma^2} + \frac{1}{2}} \\
& \quad \times e^{\left[\frac{1}{2} + \frac{r-c-2r_\alpha}{\sigma^2} \right] (R-\sigma^2/2)T_0 + \left[\frac{1}{2} + \frac{r-c-2r_\alpha}{\sigma^2} \right]^2 \frac{\sigma^2}{2} T_0} \\
& \quad \times N \left(x_{\min[L, L^2/Y]} - \left[\frac{\sigma}{2} + \frac{r-c-2r_\alpha}{\sigma} \right] \sqrt{T_0} \right) \\
& \quad + \frac{L}{2} \frac{\frac{2(r_\alpha+c-r)}{\sigma^2}}{\sigma^2} \left[1 - \frac{1}{c}(r - \sigma^2/2) \right] Y^{\frac{r+c}{\sigma^2} - \frac{1}{2}} s^{\frac{r+c-2r_\alpha}{\sigma^2} + \frac{1}{2}}
\end{aligned}$$

$$\begin{aligned} & \times e^{\left[\frac{r+c-2r_\alpha}{\sigma^2} + \frac{1}{2}\right](R-\sigma^2/2)T_0 + \left[\frac{r+c-2r_\alpha}{\sigma^2} + \frac{1}{2}\right]^2 \frac{\sigma^2}{2} T_0} \\ & \times N\left(x_{\min[L, L^2/Y]} - \left[\frac{\sigma}{2} + \frac{r+c-2r_\alpha}{\sigma}\right] \sqrt{T_0}\right) \Bigg\}. \end{aligned} \quad (\text{B.25})$$

For computational purposes, note that the last five lines disappear when $Y = L$ in D_1, D_2 .

$$\begin{aligned} K_2 = & e^{(\lambda-\lambda_0)T_0} e^{-(r_0+\lambda)T_0} \Bigg[D_1(K, r_0, r_0, b_0) + D_2(K, r_0, r_0, c_0) \\ & - D_1(L, r_0, r_\alpha, b_\alpha) - \frac{K}{L} D_2(L, r_0, r_\alpha, \bar{c}) - G_1(K, r_0, r_0, b_0) \\ & - G_2(K, r_0, r_0, c_0) + G_1(L, r_0, r_\alpha, b_\alpha) + \frac{K}{L} G_2(L, r_0, r_\alpha, \bar{c}) \Bigg]. \end{aligned} \quad (\text{B.26})$$

B.1 Dividends and forfeitures We now consider two straightforward extensions of our formulas. In the first place, dividends, paid at rate q_0 before T_0 , and at rate q after T_0 . For this, it is useful to note that the process

$$\tilde{S}_t := S_t e^{-q(T-t)}, \quad t > T_0 \quad (\text{B.27})$$

$$\tilde{S}_t := S_t e^{-q(T-T_0)-q_0(T_0-t)}, \quad t \leq T_0 \quad (\text{B.28})$$

satisfies the same SDE under the risk-neutral measure, as the stock price process S_t^0 in the case with zero dividends. In addition, $\tilde{S}_T = S_T$. Only the initial condition is different:

$$\tilde{S}_0 = S_0 e^{-q(T-T_0)-q_0 T_0}. \quad (\text{B.29})$$

Also, we have, for $t > T_0$, $S_t > L e^{\alpha(t-T_0)}$ if and only if $\tilde{S}_t > L e^{-q(T-T_0)+(\alpha+q)(t-T_0)}$. These facts show that in our proofs we should be replacing S_0 with $\tilde{S}_0$ as defined above, and also replacing L with $L e^{-q(T-T_0)}$ and α with $\alpha + q$. We give precise formulas below.

Second, forfeitures: Assume that there is also a possibility that the employee will quit the firm, and not be able to exercise the option. For example, the individual might join a competing company, in which case the option compensation is forfeited. If this event is also modeled by an exponential distribution with intensity rate λ^f , and if it is independent of other random variables in the model, as it is in the credit-risk literature, we get the result that we simply have to add λ^f to the discount rate r .

We now describe more precisely how to change the formulas. Consider the case in which forfeitures occur with intensity λ_0^f before T_0 and with intensity λ^f after T_0 . In this case, we can think of the payoff as being zero before T_0 , and $C_t = e^{-rt-\lambda^f(t-T_0)}(S_t - K)^+$ for $t > T_0$. Moreover, assume that the dividends are paid at the rate q_0 before T_0 , and at the rate q for $t > T_0$.

We change the definitions of I_1, I_2, D_1, D_2, G_1 , and G_2 as follows:

$$I_1 = \frac{\lambda}{\lambda + q + \lambda^f} I_1^{\text{old}}. \quad (\text{B.30})$$

$$I_2 = \frac{\lambda + R}{\lambda + \lambda^f + R} I_2^{\text{old}}. \quad (\text{B.31})$$

$$D_1 = \frac{\lambda}{\lambda + q + \lambda^f} D_1^{\text{old}}. \quad (\text{B.32})$$

$$D_2 = \frac{\lambda + R}{\lambda + \lambda^f + R} D_2^{\text{old}}. \quad (\text{B.33})$$

$$G_1 = \frac{\lambda}{\lambda + q + \lambda^f} G_1^{\text{old}}. \quad (\text{B.34})$$

$$G_2 = \frac{\lambda + R}{\lambda + \lambda^f + R} G_2^{\text{old}}. \quad (\text{B.35})$$

We also change these notations:

$$x_Y = \frac{\log(Y/s)}{\sigma\sqrt{T_0}} - (r_0 - q_0 - \sigma^2/2) \frac{\sqrt{T_0}}{\sigma}. \quad (\text{B.36})$$

$$\begin{aligned} y_+^\alpha &= \sqrt{(r_\alpha - q - \sigma^2/2)^2 + 2\sigma^2(r_\alpha + \lambda^f)}, \quad y_-^\alpha = r_\alpha - q - \sigma^2/2, \\ \bar{y} &= \sqrt{(r_\alpha - q - \sigma^2/2)^2 + 2\sigma^2(r_0 + \lambda^f)}. \end{aligned} \quad (\text{B.37})$$

$$b_\alpha := \sqrt{(r_\alpha - q + \sigma^2/2)^2 + 2\sigma^2(\lambda + q + \lambda^f)}. \quad (\text{B.38})$$

$$\begin{aligned} c_\alpha &:= \sqrt{(r_\alpha - q - \sigma^2/2)^2 + 2\sigma^2(\lambda + \lambda^f + r_\alpha)}, \\ \bar{c} &:= \sqrt{(r_\alpha - q - \sigma^2/2)^2 + 2\sigma^2(\lambda + \lambda^f + r_0)}. \end{aligned} \quad (\text{B.39})$$

$$d_0(Y) = \frac{1}{\sigma\sqrt{T - T_0}} \left[\log(s/Y) + (r_0 - q_0 - \sigma^2/2)T_0 \right]. \quad (\text{B.40})$$

$$d_1(Y) = \frac{1}{\sigma\sqrt{T - T_0}} \left[\log\left(\frac{L^2}{sY}\right) - (r_0 - q_0 - \sigma^2/2)T_0 \right]. \quad (\text{B.41})$$

In the following, when we mention about replacing one variable with another everywhere, we mean virtually everywhere, *except* in the new definitions above. The formulas are then modified as follows:

Case A: If $s < L$, then the price is of the form:

$$e^{-\lambda^f T} P_1^\alpha(s) + LP(s, T, y_-^\alpha, y_+^\alpha) - KP(s, T, y_-^\alpha, \bar{y}) \quad (\text{B.42})$$

where y_+^α and $\bar{y}$ have new definitions above. In addition, with the exception of new definitions in Equations (B.36)–(B.41), we replace α with $\alpha + q$ everywhere, we replace s with se^{-qT} everywhere, and we replace L with Le^{-qT} everywhere except the one multiplying $P(\cdot)$ in the middle term above.

Case B:

$$I_1(s, T, K, r_0 - q, b_0) + I_2(s, T, K, r_0, r_0 - q, c_0)$$

$$\begin{aligned}
 & + e^{-(\lambda+q+\lambda^f)T} sN\left(\frac{\tilde{K}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma}(r_0 - q + \sigma^2/2)\right) \\
 & - K e^{-(r_0+\lambda+\lambda^f)T} N\left(\frac{\tilde{K}}{\sigma\sqrt{T}} + \frac{\sqrt{T}}{\sigma}(r_0 - q - \sigma^2/2)\right), \quad (\text{B.43})
 \end{aligned}$$

where we use the new definitions of I_1 , I_2 , y_+^0 , b_0 , c_0 . Also, in the definition of I_1 , in the factor $e^{\lambda T}$, we have to add q to λ to get $e^{(\lambda+q)T}$.

Note that as far as forfeitures go, we simply replaced λ by $\lambda + \lambda^f$ everywhere.

Case C:

$$\begin{aligned}
 & e^{-(\lambda+\lambda^f)T} P_1^\alpha(s) + LP(s, T, y_-^\alpha, c_\alpha) - KP(s, T, y_-^\alpha, \bar{c}) \\
 & + I_1(s, T, K, r_0 - q, b_0) + I_2(s, T, K, r_0, r_0 - q, c_0) \\
 & - I_1(s, T, L, r_\alpha - q, b_\alpha) - \frac{K}{L} I_2(s, T, L, r_0, r_\alpha - q, \bar{c}) \\
 & - \left(\frac{L}{s}\right)^{\frac{2(r_\alpha-q)}{\sigma^2}-1} \left[I_1\left(\frac{L^2}{s}, T, K, r_0 - q, b_0\right) \right. \\
 & + I_2\left(\frac{L^2}{s}, T, K, r_0, r_0 - q, c_0\right) - I_1\left(\frac{L^2}{s}, T, L, r_\alpha - q, b_\alpha\right) \\
 & \left. - \frac{K}{L} I_2\left(\frac{L^2}{s}, T, L, r_0, r_\alpha - q, \bar{c}\right) \right]. \quad (\text{B.44})
 \end{aligned}$$

Here, y_+^α and $\bar{y}$ have new definitions as above.

Also, in the definition of I_1 , in the factor $e^{\lambda T}$, we have to add q to λ to get $e^{(\lambda+q)T}$.

Case D: The price is still:

$$K_{11} + K_{12} + K_2 + K_3 \quad (\text{B.45})$$

but we have to do the following modifications.

In all the terms, we replace λ_0 by $\lambda_0 + \lambda_0^f$ and λ by $\lambda + \lambda^f$ (already done in some of the new definitions, but there are other places). This, together with the new definition of D_i , G_i is the only change involving forfeitures.

Moreover, if there are dividends, we also need to do the following:

- K_{11} : In addition to using new definitions and modifying λ and λ_0 , change the exponent $(r_0 - \sigma^2/2)T_0$ to $(r_0 - q_0 - \sigma^2/2)T_0$.
- K_{12} : In addition to modifying λ and λ_0 , change r_0 to $r_0 - q_0$ inside the two normal distribution functions $N(\cdot)$, and multiply the first term with $e^{-q_0 T_0}$.
- K_3 : In addition to modifying λ and λ_0 , do the following everywhere, with the exception of new definitions in Equations (B.36)–(B.41): replace S_0 with $S_0 e^{-q(T-T_0)-q_0 T_0}$, L with $L e^{-q(T-T_0)}$, and α with $(\alpha + q)$.
- K_2 : In addition to using all the new definitions (including D_i , G_i) and modifying λ and λ_0 , change the following: In the third argument of functions D_1 , D_2 , G_1 , G_2 , replace r_0 with $(r_0 - q)$ (q , not q_0 !), and r_α with $(r_\alpha - q)$. In the second argument of functions D_1 and G_1 , replace r_0 with $(r_0 - q_0)$. In the formulas for D_2 and G_2 , replace the exponent term $(R - \sigma^2/2)T_0$ by $(R - q_0 - \sigma^2/2)T_0$, at four places in D_2 and six places in G_2 . In addition, replace r_α with $(r_\alpha - q)$ everywhere in the formulas for G_1 and G_2 . Moreover, in the definitions of D_1 and G_1 , we replace λ by $(\lambda + q)$ in the exponent of the first term.

C Proofs

Case A: Exercise time as a hitting time

The option price for $s < L$ can be written as:

$$P_1 + P_2 := E[e^{-r_\alpha T} (S_T e^{-\alpha T} - K_\alpha(T))^+ \mathbf{1}_{\{\tau > T\}}] + E[(L e^{-r_\alpha \tau} - K e^{-r \tau}) \mathbf{1}_{\{\tau \leq T\}}]. \quad (\text{C.1})$$

The first term corresponds to the price of the up-and-out call option with strike $K_\alpha(T)$ and barrier L , of the stock with drift r_α . Using known formulas (e.g., Bjork, 1999), we get, denoting by $C(s, K)$ the call option price with strike K , maturity T , and stock price s :

$$P_1 = P_1(s, T, K) = C(s, T, K_\alpha(T)) - D(s, T, K_\alpha(T), L) - \left(\frac{L}{s}\right)^{\frac{2r_\alpha}{\sigma^2}-1} \left[C\left(\frac{L^2}{s}, T, K_\alpha(T)\right) - D\left(\frac{L^2}{s}, T, K_\alpha(T), L\right) \right]. \quad (\text{C.2})$$

Here,

$$C(s, T, K) = sN\left(\frac{\sqrt{T}}{\sigma}(r_\alpha + \sigma^2/2) + \frac{\log(s/K)}{\sigma\sqrt{T}}\right) - K e^{-r_\alpha T} N\left(\frac{\sqrt{T}}{\sigma}(r_\alpha - \sigma^2/2) + \frac{\log(s/K)}{\sigma\sqrt{T}}\right). \quad (\text{C.3})$$

$$D(s, T, K, L) = sN\left(\frac{\sqrt{T}}{\sigma}(r_\alpha + \sigma^2/2) + \frac{\log(s/L)}{\sigma\sqrt{T}}\right) - K e^{-r_\alpha T} N\left(\frac{\sqrt{T}}{\sigma}(r_\alpha - \sigma^2/2) + \frac{\log(s/L)}{\sigma\sqrt{T}}\right). \quad (\text{C.4})$$

As for P_2 , from Karatzas-Shreve (1991), the density of τ is given by:

$$f_\tau(t) = \frac{-\tilde{L}}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(\tilde{L} + y_-^\alpha t)^2}{2\sigma^2 t}} \quad (\text{C.5})$$

where

$$\tilde{L} = \log(s/L), \quad y_-^\alpha = r_\alpha - \frac{\sigma^2}{2}. \quad (\text{C.6})$$

After some computations, we can write

$$\int_0^T e^{-xt} f_\tau(t) dt = e^{\frac{L(\sqrt{(y_-^\alpha)^2 + 2\sigma^2 x} - y_-^\alpha)}{\sigma^2}} \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(L + t\sqrt{(y_-^\alpha)^2 + 2\sigma^2 x})^2}{2\sigma^2 t}} dt. \quad (\text{C.7})$$

Using the fact that

$$e^{-\tilde{L} \frac{2y}{\sigma^2}} n\left(\frac{1}{\sigma}\left(\sqrt{t}y - \frac{\tilde{L}}{\sqrt{t}}\right)\right) = n\left(\frac{1}{\sigma}\left(\sqrt{t}y + \frac{\tilde{L}}{\sqrt{t}}\right)\right), \quad (\text{C.8})$$

we can easily check that:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[N\left(\frac{1}{\sigma}\left(\sqrt{t}y + \frac{\tilde{L}}{\sqrt{t}}\right)\right) + e^{-\tilde{L} \frac{2y}{\sigma^2}} N\left(\frac{1}{\sigma}\left(-\sqrt{t}y + \frac{\tilde{L}}{\sqrt{t}}\right)\right) \right] \\ &= \frac{-\tilde{L}}{\sigma t^{1.5}} n\left(\frac{1}{\sigma}\left(\sqrt{t}y + \frac{\tilde{L}}{\sqrt{t}}\right)\right). \end{aligned} \quad (\text{C.9})$$

Therefore, we get:

$$\begin{aligned}
 & \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi}t^3} e^{-\frac{(\tilde{L}+ty)^2}{2\sigma^2t}} dt \\
 &= \int_0^T \frac{-\tilde{L}}{\sigma t^{1.5}} n\left(\frac{1}{\sigma}\left(\sqrt{t}y + \frac{\tilde{L}}{\sqrt{t}}\right)\right) dt \\
 &= N\left(\frac{1}{\sigma}\left(\sqrt{T}y + \frac{\tilde{L}}{\sqrt{T}}\right)\right) + e^{-\tilde{L}\frac{2y}{\sigma^2}} N\left(\frac{1}{\sigma}\left(-\sqrt{T}y + \frac{\tilde{L}}{\sqrt{T}}\right)\right) \\
 &\quad - \left(1 + e^{-\tilde{L}\frac{2y}{\sigma^2}}\right) \mathbf{1}_{\{s>L\}} - \mathbf{1}_{\{s=L\}}.
 \end{aligned} \tag{C.10}$$

Then, we get, for $s < L$, and noting that $e^{-\tilde{L}} = L/s$,

$$P_2 = LP(s, T, y_-^\alpha, y_+^\alpha) - KP(s, T, y_-^\alpha, \bar{y}) \tag{C.11}$$

where P is defined in Equation (B.9).

Case B: Intensity-based model for exercise time

The price is

$$E\left[\int_0^T \lambda(S_t - K)^+ e^{-(r+\lambda)t} dt + (S_T - K)^+ e^{-(r+\lambda)T}\right]. \tag{C.12}$$

Expected value of the second term is equal to the product of the BS formula with $e^{-\lambda T}$. The first term can be written, after taking the expectation inside the integral and using the BS formula, as:

$$\lambda \int_0^T e^{-\lambda t} [sN(d_1(t)) - Ke^{-rt}N(d_2(t))] dt = I_1(s, T, K, r) + I_2(s, T, K, r) \tag{C.13}$$

where $N(d_i(t))$ is the usual BS notation with time to maturity equal to t . Let us compute this as two integrals I_1, I_2 . For this we will need the following obvious fact:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{a^2+b^2t^2}{2\sigma^2t}} = \frac{1}{\sqrt{2\pi}} e^{\frac{ab}{\sigma^2}} e^{-\frac{b^2}{2\sigma^2t}(t+\frac{a}{b})^2} = e^{\frac{ab}{\sigma^2}} n\left(\frac{1}{\sigma}\left(b\sqrt{t} + \frac{a}{\sqrt{t}}\right)\right). \tag{C.14}$$

Integrating by parts we get:

$$\begin{aligned}
 I_1(s, T, K, r) &= \lambda \int_0^T e^{-\lambda t} sN(d_1(t)) dt \\
 &= -s \int_0^T N(d_1(t)) d(e^{-\lambda t}) \\
 &= s \left[\mathbf{1}_{\{s>K\}} + \frac{1}{2} \mathbf{1}_{\{s=K\}} - e^{-\lambda T} N(d_1(T)) \right. \\
 &\quad \left. + \int_0^T e^{-\lambda t} n(d_1(t)) \frac{t(r + \sigma^2/2) - \log(s/K)}{2\sigma t^{3/2}} dt \right] \\
 &= s \left[\mathbf{1}_{\{s>K\}} + \frac{1}{2} \mathbf{1}_{\{s=K\}} - e^{-\lambda T} N\left(\frac{\tilde{K}}{\sigma\sqrt{T}} + \left(r + \frac{\sigma^2}{2}\right) \frac{\sqrt{T}}{\sigma}\right) \right. \\
 &\quad \left. + I_3(s, T, K, r) \right].
 \end{aligned} \tag{C.15}$$

The last integral, $I_3(s, T, K, r)$, can be written as:

$$I_3(s, T, K, r) = \frac{1}{2\sigma\sqrt{2\pi}} \left(\frac{K}{s}\right)^{\frac{r}{\sigma^2} + \frac{1}{2}} \int_0^T \left[\frac{r + \sigma^2/2}{t^{0.5}} - \frac{\log(s/K)}{t^{1.5}} \right] \times \exp \left\{ -\frac{1}{2\sigma^2} \left(\frac{\log^2(s/K)}{t} + ((r + \sigma^2/2)^2 + 2\sigma^2\lambda)t \right) \right\} dt. \quad (C.16)$$

Using Equation (C.14), we can write this as:

$$I_3(s, T, K) = \frac{1}{2\sigma} \left(\frac{K}{s}\right)^{\frac{r}{\sigma^2} + \frac{1}{2}} \int_0^T \left[\frac{r + \sigma^2/2}{t^{0.5}} - \frac{\log(s/K)}{t^{1.5}} \right] e^{\frac{\tilde{K}b}{\sigma^2} n} \times \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) dt, \quad (C.17)$$

where $b = b_0$ and

$$\tilde{K} := \log(s/K), \quad b_\alpha := \sqrt{(r_\alpha + \sigma^2/2)^2 + 2\sigma^2\lambda}. \quad (C.18)$$

We have the following useful observation:

$$\frac{d}{dt} N \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) = n \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) \left[\frac{1}{2\sigma} \left(\frac{b}{\sqrt{t}} - \frac{\tilde{K}}{t^{1.5}} \right) \right]. \quad (C.19)$$

Thus,

$$\begin{aligned} & \int_0^T \frac{1}{2\sigma\sqrt{t}} n \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) dt \\ &= \frac{N \left(\frac{1}{\sigma} \left(b\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) - \mathbf{1}_{\{\tilde{K} > 0\}} + \frac{1}{2} \mathbf{1}_{\{\tilde{K} = 0\}}}{b} \\ &+ \int_0^T \frac{\tilde{K}}{2b\sigma t^{1.5}} n \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) dt. \end{aligned} \quad (C.20)$$

Using this in Equation (C.17), we get:

$$\begin{aligned} I_3(s, T, K, r, b) &= \left(\frac{K}{s}\right)^{\frac{r}{\sigma^2} + \frac{1}{2}} e^{\frac{\tilde{K}b}{\sigma^2}} \left\{ \frac{r + \sigma^2/2}{b} \left[N \left(\frac{1}{\sigma} \left(b\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) \right. \right. \\ &\quad \left. \left. - \mathbf{1}_{\{s > K\}} - \frac{1}{2} \mathbf{1}_{\{s = K\}} \right] + \left[\frac{\tilde{K}(r + \sigma^2/2)}{b} - \log(s/K) \right] \right. \\ &\quad \left. \times \int_0^T \frac{1}{2\sigma t^{1.5}} n \left(\frac{1}{\sigma} \left(b\sqrt{t} + \frac{\tilde{K}}{\sqrt{t}} \right) \right) dt \right\}. \end{aligned} \quad (C.21)$$

This is nice, because we can compute the last integral from Equation (C.10) to get:

$$\begin{aligned} I_3(s, T, K, r, b) &= \left(\frac{K}{s}\right)^{\frac{r}{\sigma^2} + \frac{1}{2}} \left\{ \frac{r + \sigma^2/2}{b} \left[N \left(\frac{1}{\sigma} \left(b\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) - \mathbf{1}_{\{s > K\}} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \mathbf{1}_{\{s = K\}} \right] + \frac{1}{2} \left[1 - \frac{r + \sigma^2/2}{b} \right] \right\} \end{aligned}$$

$$\begin{aligned} & \times \left[N \left(\frac{1}{\sigma} \left(b\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) \right. \\ & + \left(\frac{K}{s} \right)^{\frac{2b}{\sigma^2}} N \left(\frac{1}{\sigma} \left(-b\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) \\ & \left. - \left[1 + \left(\frac{K}{s} \right)^{\frac{2b}{\sigma^2}} \right] \mathbf{1}_{\{s > K\}} - \mathbf{1}_{\{s = K\}} \right] \Bigg\}. \end{aligned} \quad (\text{C.22})$$

We now similarly compute I_2 :

$$\begin{aligned} I_2(s, T, K, R, r) &= -\lambda K \int_0^T e^{-(\lambda+R)t} N(d_2(t)) dt \\ &= \frac{\lambda K}{\lambda + R} \int_0^T N(d_2(t)) d(e^{-(\lambda+R)t}) \\ &= -\frac{\lambda K}{\lambda + R} \left[\mathbf{1}_{\{s > K\}} + \frac{1}{2} \mathbf{1}_{\{s = K\}} - e^{-(\lambda+R)T} N(d_2(T)) \right. \\ & \quad \left. + \int_0^T e^{-(\lambda+R)t} n(d_2(t)) \frac{t(r - \sigma^2/2) - \log(s/K)}{2\sigma t^{3/2}} dt \right] \\ &= -\frac{\lambda K}{\lambda + R} \left[\mathbf{1}_{\{s > K\}} + \frac{1}{2} \mathbf{1}_{\{s = K\}} - e^{-(\lambda+R)T} \right. \\ & \quad \left. \times N \left(\frac{\tilde{K}}{\sigma\sqrt{T}} + (r - \sigma^2/2) \frac{\sqrt{T}}{\sigma} \right) + I_4(s, T, K, R, r) \right]. \end{aligned} \quad (\text{C.23})$$

The last integral, $I_4(s, T, K, R, r)$, can be written as:

$$\begin{aligned} I_4(s, T, K, R, r) &= \frac{1}{2\sigma\sqrt{2\pi}} \left(\frac{K}{s} \right)^{\frac{r}{\sigma^2} - \frac{1}{2}} \int_0^T \left[\frac{r - \sigma^2/2}{t^{0.5}} - \frac{\log(s/K)}{t^{1.5}} \right] \\ & \quad \times \exp \left\{ -\frac{1}{2\sigma^2} \left(\frac{\log^2(s/K)}{t} + ((r - \sigma^2/2)^2 \right. \right. \\ & \quad \left. \left. + 2\sigma^2(\lambda + R)t) \right) \right\} dt. \end{aligned} \quad (\text{C.24})$$

As above for I_3 , we get that:

$$\begin{aligned} I_4(s, T, K, R, r, c) &= \left(\frac{K}{s} \right)^{\frac{r}{\sigma^2} - \frac{1}{2}} \left\{ \frac{r - \sigma^2/2}{c} \left[N \left(\frac{1}{\sigma} \left(c\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) \right. \right. \\ & \quad \left. \left. - \mathbf{1}_{\{s > K\}} - \frac{1}{2} \mathbf{1}_{\{s = K\}} \right] + \frac{1}{2} \left[1 - \frac{r - \sigma^2/2}{c} \right] \right. \\ & \quad \left. \times \left[N \left(\frac{1}{\sigma} \left(c\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}} \right) \right) + \left(\frac{K}{s} \right)^{\frac{2c}{\sigma^2}} \right] \right\} \end{aligned}$$

$$N\left(\frac{1}{\sigma}\left(-c\sqrt{T} + \frac{\tilde{K}}{\sqrt{T}}\right)\right) - \left[1 + \left(\frac{K}{s}\right)^{\frac{2c}{\sigma^2}}\right] \times \mathbf{1}_{\{s>K\}} - \mathbf{1}_{\{s=K\}} \Bigg\} \quad (\text{C.25})$$

where

$$c := \sqrt{(r - \sigma^2/2)^2 + 2\sigma^2(\lambda + R)}. \quad (\text{C.26})$$

This, together with Equation (C.23), gives us the formula for I_2 .

Case C: The combined case, no vesting

Recall the notation for c_α and $\bar{c}$. Similarly as above, we get

$$J_1 = LP(s, T, y_-^\alpha, c_\alpha) - KP(s, T, y_-^\alpha, \bar{c}) \quad (\text{C.27})$$

where P is given as in Equation (B.9). Similarly, we have

$$J_3 = e^{-\lambda T} P_1(s, T, K_\alpha(T)) \quad (\text{C.28})$$

where P_1 is as given in Equation (B.7). In order to compute J_2 , we need to integrate P_1 , but this reduces to integrating call option formulas, which we have done in the foregoing. Doing this, we get, denoting by $P_1(t)$ the value of P_1 when time to maturity is t ,

$$\begin{aligned} J_2 &= J_2(s, T) = \lambda \int_0^T e^{-\lambda t} P_1(s, t, K e^{-\alpha t}) dt \\ &= I_1(s, T, K, r, b_0) + I_2(s, T, K, r, c_0) - I_1(s, T, L, r_\alpha, b_\alpha) \\ &\quad - \frac{K}{L} I_2(s, T, L, r, r_\alpha, \bar{c}) - \left(\frac{L}{s}\right)^{\frac{2r_\alpha}{\sigma^2}-1} \left\{ I_1\left(\frac{L^2}{s}, T, K, r, b_0\right) \right. \\ &\quad \left. + I_2\left(\frac{L^2}{s}, T, K, r, c_0\right) - I_1\left(\frac{L^2}{s}, T, L, r_\alpha, b_\alpha\right) \right. \\ &\quad \left. - \frac{K}{L} I_2\left(\frac{L^2}{s}, T, L, r, r_\alpha, \bar{c}\right) \right\}. \end{aligned} \quad (\text{C.29})$$

Case D: Combined model with a vesting period

Introduce the notation

$$p(t, s, x) = s e^{(r - \sigma^2/2)t + \sigma\sqrt{t}x} \quad (\text{C.30})$$

and recall that

$$x_Y = \frac{\log(Y/s)}{\sigma\sqrt{T_0}} - (r - \sigma^2/2) \frac{\sqrt{T_0}}{\sigma}. \quad (\text{C.31})$$

By conditioning on the stock price history up to time T_0 and comparing with Case C, we get:

$$\begin{aligned} K_3 &= e^{(\lambda - \lambda_0)T_0} e^{-\lambda T - rT_0} E[P_1(S_{T_0}, T - T_0, K_\alpha(T - T_0)) \mathbf{1}_{\{S_{T_0} < L_{T_0}\}}] \\ &= e^{(\lambda - \lambda_0)T_0} e^{-\lambda T - rT_0} \int_{-\infty}^{x_L} P_1(p(T_0, s, x), T - T_0, K_\alpha(T - T_0)) n(x) dx. \end{aligned} \quad (\text{C.32})$$

$$\begin{aligned}
 K_{11} &= L e^{-\alpha T_0} e^{(\lambda - \lambda_0)T_0} e^{-(r + \lambda)T_0} \int_{-\infty}^{x_L} P(p(T_0, s, x), T - T_0, y_{-}^{\alpha}, c_{\alpha}) n(x) dx \\
 &\quad - K e^{(\lambda - \lambda_0)T_0} e^{-(r + \lambda)T_0} \int_{-\infty}^{x_L} P(p(T_0, s, x), T - T_0, y_{-}^{\alpha}, \bar{c}) n(x) dx.
 \end{aligned} \tag{C.33}$$

$$\begin{aligned}
 K_{12} &= e^{-\lambda_0 T_0} \left[s N \left(\frac{\sqrt{T_0}}{\sigma} \left(r + \sigma^2/2 \right) + \frac{\log(s/L)}{\sigma \sqrt{T_0}} \right) \right. \\
 &\quad \left. - K e^{-r T_0} N \left(\frac{\sqrt{T_0}}{\sigma} \left(r - \sigma^2/2 \right) + \frac{\log(s/L)}{\sigma \sqrt{T_0}} \right) \right].
 \end{aligned} \tag{C.34}$$

$$K_2 = e^{(\lambda - \lambda_0)T_0} e^{-(r + \lambda)T_0} \int_{-\infty}^{x_L} J_2(p(T_0, s, x), T - T_0) n(x) dx. \tag{C.35}$$

Note that all these integrals are linear combinations of integrals of the form:

$$\int_{-\infty}^{x_L} e^{ax} N(bx + c) n(x) dx \tag{C.36}$$

for some constants a, b, c . Let us express this integral in terms of a bivariate normal distribution function. We have:

$$\begin{aligned}
 \int_{-\infty}^{x_L} e^{ax} N(bx + c) n(x) dx &= \int_{-\infty}^{x_L} \int_{-\infty}^c e^{ax} n(bx + y) n(x) dy dx \\
 &= \frac{1}{2\pi} e^{\frac{a^2}{2}} \int_{-\infty}^{x_L} \int_{-\infty}^c \exp \left\{ -\frac{1}{2} (1 + b^2) (x - a)^2 \right. \\
 &\quad \left. - \frac{1}{2} (y + ab)^2 - b(y + ab)(x - a) \right\} dy dx.
 \end{aligned} \tag{C.37}$$

This can be related to the bivariate normal distribution as follows:

$$B(a, b, c) := \int_{-\infty}^{x_L} e^{ax} N(bx + c) n(x) dx = e^{\frac{a^2}{2}} P(X \leq x_L, Y \leq c) \tag{C.38}$$

where (X, Y) has a bivariate normal distribution with

$$\mu_X = a, \quad \mu_Y = -ab, \quad \sigma_X^2 = 1, \quad \sigma_Y^2 = 1 + b^2, \quad \rho = -\frac{b}{\sqrt{1 + b^2}}. \tag{C.39}$$

We get K_{11} from Equations (C.38) and (B.9). It is straightforward to get K_{12} . We get K_3 from Equations (C.38) and (B.7). We get K_2 from Equations (C.29) and (C.38).

D Probabilities

Denote by μ the drift of the stock price process and:

$$f_{\alpha} := \sqrt{(\mu_{-}^{\alpha})^2 + 2\lambda\sigma^2} \tag{D.1}$$

$$\mu_{-}^{\alpha} = \mu - \alpha - \sigma^2/2, \quad \mu_{\alpha} = \mu - \alpha. \tag{D.2}$$

P_1

Probability of being fired before the end of the vesting period:

$$P_1 = 1 - e^{-\lambda_0 T_0}. \tag{D.3}$$

P_2

Probability of voluntarily exercising at the end of the vesting period (corresponds to K_{12} in Case D):

$$P_2 = P[S_{T_0} > L, T_\lambda^0 > T_0] = e^{-\lambda_0 T_0} [1 - N(x_L)]. \quad (D.4)$$

P_3^0

Probability of the option being exercised at the desired level, no vesting period (corresponds to J_1 in Case C):

$$\begin{aligned} P_3^0(\lambda, T, s) &= P[T_L < T, T_L < T_\lambda] \\ &= \int_0^T \int_y^\infty \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L}+\mu_-^\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \\ &= \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{\tilde{L} \frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} - \mu_-^\alpha}{\sigma^2}} e^{-\frac{(\mu_-^\alpha)^2 + 2\lambda\sigma^2}{2\sigma^2 y}} \left(y + \frac{\tilde{L}}{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2}}\right)^2 dy \\ &= \left(\frac{s}{L}\right)^{\frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} - \mu_-^\alpha}{\sigma^2}} N\left(\frac{\sqrt{T}}{\sigma} \sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \frac{\tilde{L}}{\sigma\sqrt{T}}\right) \\ &\quad + \left(\frac{s}{L}\right)^{-\frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \mu_-^\alpha}{\sigma^2}} N\left(-\frac{\sqrt{T}}{\sigma} \sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \frac{\tilde{L}}{\sigma\sqrt{T}}\right). \end{aligned} \quad (D.5)$$

P_3

Probability of the option being exercised at the desired level, after the vesting period (corresponds to K_{11} in Case D):

$$\begin{aligned} P_3(\lambda, T_0, T) &= P[T_0 < T_L^0 < T, T_L^0 < T_\lambda^0] = E[E_{T_0}(\mathbf{1}_{\{T_L^0 < T, T_L^0 < T_\lambda^0\}})] \\ &= e^{-\lambda_0 T_0} \left(\frac{s}{L}\right)^{\frac{f_\alpha - \mu_-^\alpha}{\sigma^2}} e^{\frac{f_\alpha - \mu_-^\alpha}{\sigma^2} (\mu_0 - \sigma^2/2) T_0} \\ &\quad \times B\left(\frac{(f_\alpha - \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T - T_0}}{\sigma} f_\alpha\right) \\ &\quad + e^{-\lambda_0 T_0} \left(\frac{s}{L}\right)^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2}} e^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2} (\mu_0 - \sigma^2/2) T_0} \\ &\quad \times B\left(-\frac{(f_\alpha + \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T - T_0}}{\sigma} f_\alpha\right). \end{aligned} \quad (D.6)$$

P_4^0

Probability of being fired between $T_0 = 0$ and T , no vesting period (corresponds to J_2 in Case C):

$$\begin{aligned} P_4^0(\lambda, T, s) &= P[0 < T_\lambda < \min\{T, T_L\}] \\ &= \int_0^T \int_0^y \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L}+\mu_-^\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \end{aligned}$$

$$\begin{aligned}
 & + \int_T^\infty \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L}+\mu_\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \\
 & = \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L}+\mu_\alpha y)^2}{2\sigma^2 y}} (1 - e^{-\lambda y}) dy \\
 & \quad + \int_T^\infty \frac{-\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L}+\mu_\alpha y)^2}{2\sigma^2 y}} (1 - e^{-\lambda T}) dy \\
 & = P_3^0(0, T, s) - P_3^0(\lambda, T, s) + [1 - e^{-\lambda T}] [1 - P_3^0(0, T, s)] \\
 & = 1 - e^{-\lambda T} - P_3^0(\lambda, T, s) + e^{-\lambda T} P_3^0(0, T, s). \tag{D.7}
 \end{aligned}$$

P_4

Probability of being fired between T_0 and T , with vesting period (corresponds to K_2 in Case D; note that in the last term we have T , not $T - T_0$; this is because the boundaries of the first integrals in the double integrals above are then T_0 to T , and still T , not $T - T_0$, to infinity, respectively):

$$\begin{aligned}
 P_4 & = P[T_0 < T_\lambda^0 < \min\{T, T_L^0\}] = E[P_{T_0}[T_0 < T_\lambda^0 < \min\{T, T_L^0\}]] \\
 & = e^{-\lambda_0 T_0} E[P_3^0(0, T - T_0, S_{T_0}) - P_3^0(\lambda, T - T_0, S_{T_0}) \\
 & \quad + \mathbf{1}_{\{S_{T_0} < L\}} [1 - e^{-\lambda T}] [1 - P_3^0(0, T, S_{T_0})]] \\
 & = e^{-\lambda_0 T_0} P_3(0, 0, T - T_0) - e^{-\lambda_0 T_0} P_3(\lambda, 0, T - T_0) [e^{-\lambda_0 T_0} \\
 & \quad - e^{-\lambda(T-T_0)-\lambda_0 T_0}] [N(x_L) - P_3(0, 0, T)]. \tag{D.8}
 \end{aligned}$$

P_5^0

Probability of attaining maturity, no vesting period (corresponds to J_3 in Case C):

$$P_5^0 = P[T_\lambda > T, T_L > T] = e^{-\lambda T} [1 - P_3^0(0, T, s)]. \tag{D.9}$$

P_5

Probability of attaining maturity, with vesting period (corresponds to K_3 in Case D):

$$\begin{aligned}
 P_5 & = P[T_\lambda^0 > T, T_L^0 > T] = e^{(\lambda-\lambda_0)T_0} e^{-\lambda T} E[\mathbf{1}_{\{S_{T_0} < L\}} (1 - P_3^0(0, T - T_0, S_{T_0}))] \\
 & = e^{(\lambda-\lambda_0)T_0} e^{-\lambda T} [N(x_L) - P_3(0, 0, T - T_0)]. \tag{D.10}
 \end{aligned}$$

E Expected value of stock at expiry/exercise

We compute the expected value of the stock at expiry/exercise, $E[S_\tau]$. This is simply the value of the option without discounting and with $K = 0$. We need new notation for this value. First, *in all the formulas from elsewhere that we use, we replace r by μ , the actual drift of the stock*. The corresponding values in this case are:

$$I_1^S(T) = \frac{\lambda s}{\mu - \lambda} [e^{(\mu-\lambda)T} - 1]. \tag{E.1}$$

$$P_1^S = e^{\mu T} \left[s - D(s, T, 0, L) - \left(\frac{L}{s}\right)^{\frac{2\mu_\alpha}{\sigma^2}-1} \left(s - D\left(\frac{L^2}{s}, T, 0, L\right)\right) \right]. \tag{E.2}$$

$$y_-^\mu = \mu_\alpha - \frac{\sigma^2}{2}, \quad y_\alpha^S = \sqrt{(y_-^\mu)^2 - 2\sigma^2\alpha}. \quad (\text{E.3})$$

$$c_\alpha^S = \sqrt{(y_-^\mu)^2 + 2\sigma^2(\lambda - \alpha)}. \quad (\text{E.4})$$

$$P_2^S = LP(s, T, y_-^\mu, y_\alpha^S). \quad (\text{E.5})$$

$$J_1^S = LP(s, T, y_-^\mu, c_\alpha^S). \quad (\text{E.6})$$

$$J_3^S = e^{-\lambda T} P_1^S. \quad (\text{E.7})$$

Also denote:

$$I_1^{S,\mu}(s, T) = I_1(s, T, L, \mu_\alpha, b_\alpha(\lambda = \lambda - \mu)), \quad (\text{E.8})$$

the value of the old function I_1 , but with r replaced by μ everywhere and λ replaced by $\lambda - \mu$ in the computation of b_α . Similarly, introduce:

$$D_1^{S,\mu}(s, T) = D_1(L, \mu_0, \mu_\alpha, b_\alpha(\lambda = \lambda - \mu)). \quad (\text{E.9})$$

$$G_1^{S,\mu}(s, T) = G_1(L, \mu_0, \mu_\alpha, b_\alpha(\lambda = \lambda - \mu)). \quad (\text{E.10})$$

$$\begin{aligned} J_2^S &= E \int_0^T \lambda e^{(\mu-\lambda)t} e^{-\mu t} S_t \mathbf{1}_{\{T_\lambda > t\}} dt \\ &= \lambda \int_0^T e^{-\lambda t} P_1^S(t) dt \\ &= I_1^S(T) - \left(\frac{L}{s}\right)^{\frac{2\mu_\alpha}{\sigma^2}-1} I_1^S(T) - \frac{\lambda}{\lambda-\mu} I_1^{S,\mu}(s, T) + \left(\frac{L}{s}\right)^{\frac{2\mu_\alpha}{\sigma^2}-1} \\ &\quad \times \frac{\lambda}{\lambda-\mu} I_1^{S,\mu}\left(\frac{L^2}{s}, T\right). \end{aligned} \quad (\text{E.11})$$

$$K_{12}^S = e^{(\lambda-\lambda_0)T_0} e^{-\lambda T_0} s N\left(\frac{\sqrt{T_0}}{\sigma} \left(\mu_0 + \frac{\sigma^2}{2}\right) + \frac{\log(s/L)}{\sigma\sqrt{T_0}}\right). \quad (\text{E.12})$$

$$\begin{aligned} K_{11}^S e^{-(\lambda-\lambda_0)T_0} &= L e^{-\lambda T_0} \left(\frac{L}{s}\right)^{\frac{y_-^\mu - c_\alpha^S}{\sigma^2}} e^{\frac{c_\alpha^S - y_-^\mu}{\sigma^2}(\mu_0 - \sigma^2/2)T_0} \\ &\quad \times B\left(\frac{(c_\alpha^S - y_-^\mu)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T-T_0}}{\sigma} c_\alpha^S\right) \\ &\quad + L e^{-\lambda T_0} \left(\frac{L}{s}\right)^{\frac{y_-^\mu + c_\alpha^S}{\sigma^2}} e^{-\frac{c_\alpha^S + y_-^\mu}{\sigma^2}(\mu_0 - \sigma^2/2)T_0} \\ &\quad \times B\left(-\frac{(c_\alpha^S + y_-^\mu)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T-T_0}}{\sigma} c_\alpha^S\right). \end{aligned} \quad (\text{E.13})$$

$$\begin{aligned}
 K_2^S &= e^{(\lambda-\lambda_0)T_0} e^{-\lambda T_0} \int_0^{x_L} J_2^S dx \\
 &= e^{(\lambda-\lambda_0)T_0} e^{-\lambda T_0} \left[\left(e^{(\mu-\lambda)(T-T_0)} - 1 \right) \frac{\lambda}{\mu-\lambda} s e^{(\mu-\sigma^2/2)T_0} N\left(x_L - \sigma\sqrt{T_0}\right) \right. \\
 &\quad \left. - L \frac{\frac{2\mu\alpha}{\sigma^2}-1}{\sigma^2} \left(e^{(\mu-\lambda)(T-T_0)} - 1 \right) \frac{\lambda}{\mu-\lambda} s^{-\frac{2\mu\alpha}{\sigma^2}} e^{-\frac{2\mu\alpha}{\sigma^2}(\mu-\sigma^2/2)T_0} \right. \\
 &\quad \left. \times N\left(x_L + \frac{2\mu\alpha}{\sigma}\sqrt{T_0}\right) - \frac{\lambda}{\lambda-\mu} D_1^{S,\mu}(s, T) + \frac{\lambda}{\lambda-\mu} G_1^{S,\mu}(s, T) \right].
 \end{aligned} \tag{E.14}$$

$$\begin{aligned}
 K_3 &= e^{(\lambda-\lambda_0)T_0} e^{-\lambda T + \mu(T-T_0)} \left[s e^{(\mu-\sigma^2/2)T_0} N(x_L - \sigma\sqrt{T_0}) \right. \\
 &\quad \left. - L \frac{\frac{2\mu\alpha}{\sigma^2}-1}{\sigma^2} s^{-\frac{2\mu\alpha}{\sigma^2}} e^{-\frac{2\mu\alpha}{\sigma^2}(\mu-\sigma^2/2)T_0} N\left(x_L + \frac{2\mu\alpha}{\sigma}\sqrt{T_0}\right) \right. \\
 &\quad \left. - C_0(L; K=0, r=\mu) + C_1(L, K=0, r=\mu) \right].
 \end{aligned} \tag{E.15}$$

The expected value of stock at expiry/exercise is:

$$K_{11}^S + K_{12}^S + K_2^S + K_3^S. \tag{E.16}$$

F Expected value of the time to expiry/exercise

For this computation we will need to calculate:

$$B^S(a, b, c) := \int_{-\infty}^{x_L} x e^{ax} N(bx + c) n(x) dx. \tag{F.1}$$

Denote by:

$$B(a, b, c, x_L = bx_L + c), \tag{F.2}$$

the value of the function $B(a, b, c)$ defined in Equation (C.38), but with r replaced by μ and with x_L replaced by $bx_L + c$. We find:

$$\begin{aligned}
 B^S(a, b, c) &= -e^{a^2/2} \int_{-\infty}^{x_L} N(bx + c) d\left(\frac{e^{-(x-a)^2/2}}{\sqrt{2\pi}}\right) \\
 &\quad + a \int_{-\infty}^{x_L} e^{ax} N(bx + c) n(x) dx \\
 &= -e^{\frac{a^2}{2}} N(bx_L + c) N(x_L - a) \\
 &\quad + e^{\frac{a^2}{2}} \int_{-\infty}^{x_L} b N(x - a) n(bx + c) dx + a B(a, b, c) \\
 &= -e^{\frac{a^2}{2}} N(bx_L + c) N(x_L - a) \\
 &\quad + e^{\frac{a^2}{2}} B\left(0, \frac{1}{b}, -\frac{c}{b} - a, x_L = bx_L + c\right) + a B(a, b, c).
 \end{aligned} \tag{F.3}$$

Let us compute different components of the expected value of the time to exercise, $E[\tau]$.

E_1

The expiry occurs before the vesting period:

$$\begin{aligned}
E_1 &= E[\tau \mathbf{1}_{\{T_\lambda < T_0\}}] = E[T_\lambda \mathbf{1}_{\{T_\lambda < T_0\}}] = \int_0^{T_0} \lambda_0 e^{-\lambda_0 t} dt \\
&= -e^{-\lambda_0 T_0} \left(T_0 + \frac{1}{\lambda_0} \right) + \frac{1}{\lambda_0}.
\end{aligned} \tag{F.4}$$

 E_2

Voluntarily exercising at the end of the vesting period:

$$\begin{aligned}
E_2 &= E[\tau \mathbf{1}_{\{S_{T_0} > L, T_\lambda^0 > T_0\}}] = T_0 P(S_{T_0} > L, T_\lambda^0 > T_0) \\
&= T_0 e^{-\lambda_0 T_0} [1 - N(x_L)].
\end{aligned} \tag{F.5}$$

 E_3^0

Exercising at the desired level, no vesting period (we use Equation (C.19) in the computation):

$$\begin{aligned}
E_3^0(\lambda, T, s) &= E[T_L \mathbf{1}_{\{0 < T_L < T, T_L < T_\lambda\}}] \\
&= \int_0^T \int_y^\infty \frac{-y \tilde{L}}{\sigma \sqrt{2\pi y^3}} e^{-\frac{(\tilde{L} + \mu_-^\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \\
&= \int_0^T \frac{-\tilde{L}}{\sigma \sqrt{2\pi y}} e^{\tilde{L} \frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} - \mu_-^\alpha}{\sigma^2}} e^{-\frac{(\mu_-^\alpha)^2 + 2\lambda\sigma^2}{2\sigma^2 y}} \left(y + \frac{\tilde{L}}{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2}} \right)^2 dy \\
&= -\frac{\tilde{L}}{f_\alpha} \left(\frac{s}{L} \right)^{\frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} - \mu_-^\alpha}{\sigma^2}} N \left(\frac{\sqrt{T}}{\sigma} \sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \frac{\tilde{L}}{\sigma \sqrt{T}} \right) \\
&\quad + \frac{\tilde{L}}{f_\alpha} \left(\frac{s}{L} \right)^{-\frac{\sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \mu_-^\alpha}{\sigma^2}} N \left(-\frac{\sqrt{T}}{\sigma} \sqrt{(\mu_-^\alpha)^2 + 2\lambda\sigma^2} + \frac{\tilde{L}}{\sigma \sqrt{T}} \right).
\end{aligned} \tag{F.6}$$

 E_3

Exercising at the desired level, after the vesting period:

$$\begin{aligned}
E_3 &= E_3(\lambda, T_0, T) = E \left[T_L^0 \mathbf{1}_{\{T_0 < T_L^0 < T, T_L^0 < T_\lambda^0\}} \right] \\
&= E \left[E_{T_0}(T_0 < T_L^0 \mathbf{1}_{\{T_L^0 < T, T_L^0 < T_\lambda^0\}}) \right] \\
&= T_0 P_3(\lambda, T_0) + e^{-\lambda_0 T_0} E \left[E_3^0(\lambda, T - T_0, S_{T_0}) \right] \\
&= T_0 P_3(\lambda, T_0) + e^{-\lambda_0 T_0} \left\{ \frac{\tilde{L} + (\mu_0 - \sigma^2/2)T_0}{f_\alpha} \left(\frac{s}{L} \right)^{\frac{f_\alpha - \mu_-^\alpha}{\sigma^2}} e^{-\frac{f_\alpha - \mu_-^\alpha}{\sigma^2} (\mu_0 - \sigma^2/2)T_0} \right. \\
&\quad \times B \left(\frac{(f_\alpha - \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T - T_0}}{\sigma} f_\alpha \right) \\
&\quad \left. + \frac{\tilde{L} + (\mu_0 - \frac{\sigma^2}{2})T_0}{f_\alpha} \left(\frac{s}{L} \right)^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2}} e^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2} (\mu_0 - \frac{\sigma^2}{2})T_0} \right\}
\end{aligned}$$

$$\begin{aligned}
 & \times B \left(-\frac{(f_\alpha + \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T-T_0}}{\sigma} f_\alpha \right) \\
 & - \frac{\sigma\sqrt{T-T_0}}{f_\alpha} \left(\frac{s}{L} \right)^{\frac{f_\alpha - \mu_-^\alpha}{\sigma^2}} e^{\frac{f_\alpha - \mu_-^\alpha}{\sigma^2}(\mu_0 - \sigma^2/2)T_0} \\
 & \times B^S \left(\frac{(f_\alpha - \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) + \frac{\sqrt{T-T_0}}{\sigma} f_\alpha \right) \\
 & + \frac{\sqrt{T-T_0}}{f_\alpha} \left(\frac{s}{L} \right)^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2}} e^{-\frac{f_\alpha + \mu_-^\alpha}{\sigma^2}(\mu_0 - \sigma^2/2)T_0} \\
 & \times B^S \left(-\frac{(f_\alpha + \mu_-^\alpha)\sqrt{T_0}}{\sigma}, Q, d_0(L) - \frac{\sqrt{T-T_0}}{\sigma} f_\alpha \right) \Big\} \tag{F.7}
 \end{aligned}$$

where in the last two terms B^S is as defined in Equation (F.3).

E_4^0

Being fired/quitting between $T_0 = 0$ and T , no vesting period:

$$\begin{aligned}
 E_4^0(\lambda, T, s) &= E[T_\lambda \mathbf{1}_{\{0 < T_\lambda < \min\{T, T_L\}\}}] \tag{F.8} \\
 &= \int_0^T \int_0^y \frac{-y\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L} + \mu_-^\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \\
 &\quad + \int_T^\infty \int_0^T \frac{-y\tilde{L}}{\sigma\sqrt{2\pi y^3}} e^{-\frac{(\tilde{L} + \mu_-^\alpha y)^2}{2\sigma^2 y}} \lambda e^{-\lambda x} dx dy \\
 &= \int_0^T \frac{-\tilde{L}}{\sigma\sqrt{2\pi y}} e^{-\frac{(\tilde{L} + \mu_-^\alpha y)^2}{2\sigma^2 y}} (1 - e^{-\lambda y}) dy \\
 &\quad + \int_T^\infty \frac{-\tilde{L}}{\sigma\sqrt{2\pi y}} e^{-\frac{(\tilde{L} + \mu_-^\alpha y)^2}{2\sigma^2 y}} (1 - e^{-\lambda T}) dy \\
 &= E_3^0(0, T, s) - E_3^0(\lambda, T, s) + [1 - e^{-\lambda T}] \left[-\frac{\tilde{L}}{|\mu_-^\alpha|} - E_3^0(0, T, s) \right] \\
 &= -(1 - e^{-\lambda T}) \frac{\tilde{L}}{|\mu_-^\alpha|} - E_3^0(\lambda, T, s) + e^{-\lambda T} E_3^0(0, T, s).
 \end{aligned}$$

E_4

Being fired/quitting between $T_0 > 0$ and T , with vesting period (note that in the last term we have T , not $T - T_0$; this is because the boundaries of first integrals in the double integrals above are then T_0 to T , and still T , not $T - T_0$, to infinity, respectively):

$$\begin{aligned}
 E_4 &= E[T_\lambda^0 \mathbf{1}_{\{T_0 < T_\lambda^0 < \min\{T, T_L^0\}\}}] \tag{F.9} \\
 &= E[E_{T_0}[T_\lambda^0 \mathbf{1}_{\{T_0 < T_\lambda^0 < \min\{T, T_L^0\}\}}]] \\
 &= T_0 P_4(\lambda, T_0) + E \left[e^{-\lambda_0 T_0} E_3^0(0, T - T_0, S_{T_0}) \right. \\
 &\quad \left. - e^{-\lambda_0 T_0} E_3^0(\lambda, T - T_0, S_{T_0}) + (e^{-\lambda(T-T_0) - \lambda_0 T_0} - e^{-\lambda_0 T_0}) \right. \\
 &\quad \left. \times \left[\frac{\tilde{L}}{|\mu_-^\alpha|} + E_3^0(0, T, S_{T_0}) \right] \right]
 \end{aligned}$$

$$\begin{aligned}
 &= T_0 P_4(\lambda, T_0) + e^{-\lambda_0 T_0} E_3(0, 0, T - T_0) \\
 &\quad - e^{-\lambda_0 T_0} E_3(\lambda, 0, T - T_0) + (e^{-\lambda(T-T_0)-\lambda_0 T_0} - e^{-\lambda_0 T_0}) \\
 &\quad \times \left[\frac{\tilde{L} + (\mu_0 - \sigma^2/2)T_0}{|\mu_-^\alpha|} + E_3(0, 0, T) \right].
 \end{aligned}$$

E_5^0

Arriving to maturity, no vesting period:

$$E_5^0 = T P_5^0. \quad (\text{F.10})$$

E_5

Arriving to maturity, with vesting period:

$$E_5 = T P_5. \quad (\text{F.11})$$

The expected time to expiry/exercise is

$$E[\tau] = E_1 + E_2 + E_3 + E_4 + E_5. \quad (\text{F.12})$$

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