Agency and Optimal Investment Dynamics

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Agency problems limit firms' access to capital markets, curbing investment. Firms and investors seek contractual ways to mitigate these problems. What are the implications for investment? We present a theory of a firm's investment dynamics in the presence of agency problems and optimal long-term financial contracts. We derive results relating firms' investment decisions, current and past cash flows, firm size, capital structure, and dividends. Among the results, optimal investment is increasing in current and past cash flow; and optimal investment is positively serially correlated over time (after controlling for investment opportunities). These results hold for a range of agency problems. *(JEL* G30, G31, G32, G35, D82, D86, D92)

Agency problems limit the extent of borrowing and consequently limit firms' investments. These limitations on credit and investment tighten or relax over time depending on a firm's performance. This article analyzes the implied investment dynamics, given optimal long-term contracts. We derive a number of results regarding firms' investment decisions, current and past cash flows, firm size, capital structure, and dividends.

Among our results, controlling for a firm's current investment opportunities:

- Current investment is higher if current cash flow is high.
- Current investment is increasing in past cash flows and past investment. Hence, investment is positively serially correlated over time.
- The sensitivity of current investment to cash flow is higher for small firms.
- The agent receives cash payments only if the firm achieves its target growth rate (which occurs if cash flow exceeds a threshold).

We show that these results follow from the general nature of optimal long-term contracts in the presence of agency problems, and hence these results will hold for a variety of different agency problems—for example,

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the problem of inducing an agent to exert costly unobservable effort or the problem associated with an agent's ability to divert a firm's cash flow to himself. Of course, the extent to which the data are consistent with the above results is a much-debated subject; see Hubbard (1998) for a discussion of the literature on investment and imperfect capital markets. But regarding the question of whether agency problems lead to such investment dynamics, we show that these results are rather robust to the variations in agency problems faced by different firms. Specifically, the above results hold for any agency model for which a static version of the model (described in detail in Section 2.4) satisfies the following two conditions:

(i) the agent's payoff is increasing in the cash flow generated; and
(ii) the agent's payoff is increasing in his reservation utility.

These are natural conditions that hold for a variety of agency problems.

Our analysis takes investment to be contractible. So an optimal contract includes a specification of the level of investment conditional on the history of observable variables. We show that the optimal contract specifies additional investment following good agent performance and disinvestment following bad performance and that these effects are persistent. The intuition is as follows. In a static agency model, incentives are provided through a cash reward for good performance. In a dynamic setting, good performance is also rewarded by giving the agent a higher share of future cash flows. But increasing the agent's share of future cash flows mitigates the agency problem (e.g., if the agent is given the entire firm, agency problems are eliminated). Because losses due to agency problems are diminished, the return to investment increases. Consequently, it is optimal to also provide for additional investment following good performance. The opposite is true following bad performance. Hence, we have a positive correlation between investment and agent performance.

It is worth stressing that the intuition for the positive relation between cash flow and investment in our model is quite different from the usual explanation based on costs of raising external funds. There, an extra \$1 of internal cash raises investment because internal cash is less expensive than external cash. Thus, higher internal cash flow lowers the firm's cost of capital. In our model, it is not the firm's cost of capital that changes with internal cash flow but rather the return on investment. The return on investment rises with the agent's share of the rents, and the agent's share of the rents rises as a reward for current performance, for which cash flows are a natural indicator.

Given this logic, condition (i) above implies that current cash flow is a performance metric that should be rewarded to provide incentives.

Therefore, investment will increase with current cash flow. The static relation between pay and performance translates into a dynamic relation between cash flow and investment. Condition (ii) above states that in a static context, the agent's ex ante and ex post payoffs are positively correlated. In a dynamic setting, this linkage implies that an agent who expects to receive a high payoff today will receive a higher share of future cash flows. This persistence in the agent's share translates into persistence in investment.

The above results are about investment. With further structure on the agency problem, we can also derive results on the relation between investment, capital structure, and dividends. So after presenting the general analysis of agency and investment, we apply the solution to three agency problems. In one, a firm's cash flow is observed only by the agent who runs the firm, and the agent can divert the cash flow for private benefit. The second considers the problem when the agent can divert a publicly observed cash flow. In the third, the case of unobservable costly agent effort is considered. In these applications, the agency problem can be seen as one between inside equity holders and outside debt and equity holders, and we have the following additional results:

- The sensitivity of investment to cash flow is higher for firms that are not paying dividends.
- Dividend payouts are less likely for smaller firms. Dividends are paid only if cash flow exceeds a threshold.
- Firms with lower leverage invest more and are more likely to pay dividends.
- Survival rates are higher for firms with higher net worth.

If instead, the agency problem is seen as one between managers and outside security holders (debt holders and equity holders), the above results on dividends are replaced by comparable results on managers' compensation/bonus payments.

Our article belongs to the literature on optimal long-term financial contracting in the presence of agency problems. We incorporate investment into such a model. There are a number of interesting theoretical analyses of investment based on agency problems and optimal long-term contracts. In the multiperiod models of Clementi and Hopenhayn (2006) and Quadrini (2004) and the two-period model of Gertler (1992), an agent privately observes the firm's risky cash flow and can divert it for private benefit. Our model covers this agency problem as well as all others that share properties (i) and (ii) stated above. And we demonstrate that these basic properties—which commonly appear in static corporate finance models—have predictable consequences for the dynamics of firm investment and growth. Our analysis also differs in the way investment is modeled. As in Abel and Eberly (1994), we allow for the possibility that the purchase and sale prices of capital are different (partial or complete irreversibility), and we allow for adjustment costs that may include a fixed cost component. Consequently, the firm's optimal capital stock, or size, in a given period depends on its size in the prior period. Even aside from any underlying agency problem, investment is a dynamic problem for the firm. By contrast, in the analyses cited above, the firm's current capital stock has no direct effect on the firm's future size. In Quadrini (2004), each period the nondepreciated capital can be resold at book value (no irreversibility), and there are no adjustment costs. In Clementi and Hopenhayn (2006) and Gertler (1992), all capital depreciates after one period. So in these models, investment is effectively a static problem in the absence of an agency problem—growth is present only because the agency problem is relaxed over time. On the empirical side, evidence suggests that an investment model with convex and nonconvex adjustment costs and irreversibilities (like ours) better matches the data (see, for example, Cooper and Haltiwanger, 2005).

Our analysis and that of the others cited feature risk-neutral agents, so there are no risk-sharing considerations. Atkeson and Cole (2005) analyze a multiperiod financial contracting model with a risk-averse agent. Their focus is on the optimal capital structure and agent compensation. Like the papers cited above, they also analyze a particular agency problem, one based on the costly state verification model. And their investment model, like those cited above, has no adjustment costs or irreversibilities. Green (1987) and Wang (2005) also analyze risk sharing in multiperiod settings in which agents can misreport cash flows, though there is no investment in these models.

Albuquerque and Hopenhayn (2004), Cooley, Marimon, and Quadrini (2004), and Quintin (2003) also study multiperiod models of investment in which an agent can divert resources. In these models, the payoff to investment is riskless. Thus, these analyses do not generate implications regarding the relation between cash flows and investment. Langberg (2005) focuses on the relation between the level of investor protection and the dynamics of a firm's capital structure and investment. His short-term contracting model constrains the agent and investor to only direct the split of the current cash flow, not the allocation of future cash flows. By contrast, in our model an investor's payoff is not limited to the cash available today; for example, unpaid debt could be rolled over into the next period.

Numerous articles analyze multiperiod contracting in the presence of agency problems in which there is no ongoing investment/growth. For example, Allen (1983), Hart and Moore (1994), and Hart (1995) analyze settings with deterministic cash flows that can be diverted by an agent. Diamond (1984), Bolton and Scharfstein (1990), Harris and Raviv (1995), Hart and Moore (1998), Gromb (1999), and DeMarzo and Fishman (2004) analyze settings with stochastic privately observed cash flows that can be diverted by an agent. In these analyses, the choice at any time is whether to allow the agent to continue operating the firm at an exogenously given scale or to "liquidate" the firm (where liquidation accommodates a variety of interpretations). These analyses focus on security design. There is no analysis of firms' investment and growth.

Section 1 presents the model. Section 2 contains the general analysis of the optimal contract, characterizing the composition of the agent's payoff, cash or continuation utility, in a given period; the optimal investment as a function of the agent's continuation utility; and the relation between investment and cash payments for the agent. Section 2 presents the two main results: Theorem 1 relates investment and agent cash payments to business cash flows, and Theorem 2 characterizes the intertemporal properties of investment. These results deliver the implications in the first four bullet points above. Section 3 applies the optimal contract to three moral hazard settings—two cash flow diversion models and a costly unobservable effort model. In the context of these agency models, we derive results on the relation between capital structure, dividends, and investment, the second four bullet points above. Section 4 has concluding remarks, including a discussion of important extensions to the model.

1. The Model

Risk-neutral investors finance a firm operated by a risk-neutral agent. A consumption stream $\{y_t\}_{t\in T}$ is valued as $\sum_{t\in T} e^{-\gamma t}E[y_t]$ by the agent and as $\sum_{t\in T} e^{-rt}E[y_t]$ by investors, where r is the riskless interest rate and $\gamma \geq r$. We will consider settings in which the agent does not save across periods—rather the agent consumes all that he is paid every period.[1]

An initial investment of K is required to start the firm. The firm generates cash flows for T periods (the infinite horizon case is discussed in Appendix B3). The agent has initial wealth of $Y_0 \geq 0$, and the investor has unlimited capital. The agent also has limited liability (i.e., consumption in any period is bounded below by zero). At any time, the agent can quit the firm and take an outside option that we normalize to zero.

The agent's incentive to raise external financing is twofold. First, Y_0 may not be sufficient to cover the initial investment K required to start the firm and/or sufficient to cover the investment required for the subsequent growth of the firm. Second, if the agent discounts consumption at a rate

[1] With risk-neutrality, this result follows if we assume (i) the agent earns a (continuously compounded) return of $\rho \leq r$ on savings; and (ii) savings can only be used for future consumption—the agent cannot use savings to boost the firm's future cash flow. In our application of the DeMarzo and Fishman (2004) model, we can relax assumption (ii) and the agent would still choose not to save (see Section 3.1).

above the riskless rate ($\gamma > r$), then the agent would like external financing for the purpose of funding early consumption.

1.1 Moral hazard

The agent is subject to an agency problem. In each period t, the agent takes some unobservable action z_t. This action affects the cash flow Y_t per-unit scale (scale is defined below) received by investors in period t, as well as a private benefit (or cost) W_t to the agent. In addition, z_t affects the outcome of a set of variables Q_t that can be contracted on.

Because of the agency problem, the agent may know more about the firm's current cash flow than investors—for example, the agent may privately observe the current cash flow or the agent may privately observe his current effort choice. We assume that except for the agency problem, there is no issue of asymmetric information. That is, in each period t, the agent and investors have symmetric information regarding all investment opportunities, agent and investor payoffs, etc. in all periods after t. This rules out settings in which the agent knows more than investors about the future cash flows of the firm, for example, because the agent has private information about future investment opportunities or cash flows or because the agent uses hidden savings to alter future cash flows.

Our analysis covers a variety of agency problems. Moreover, the agency problem need not be the same each period—as the firm ages, the agency problem may change. Here we mention three applications that we will consider (these applications are fully described in Section 3). Our model encompasses each of these agency problems, with the possibility that in some periods there is one agency problem while in other periods there is another.

1.1.1 Privately observed cash flows. Suppose the agent privately observes the firm's cash flow and the agent can divert some or all of the cash flow for private benefit. In this case, the agent's action z_t is a choice of how much of the firm's unobservable cash flow to divert. The investor's cash flow Y_t is the cash flow that remains after the agent's diversion choice, and W_t is the private benefit to the agent. In this case, Q_t simply includes the cash flows received by investors.

1.1.2 Observable, noncollectible cash flows. Now suppose that the firm's cash flows are directly observable by investors but that the agent can still divert them. Again, the agent's action z_t is a choice of how much of the firm's cash flow to divert, the investors' cash flow Y_t is the cash flow that remains after the agent's diversion choice, and W_t is the private benefit to the agent. In this case, however, Q_t includes both the firm's cash flow and the cash flows received by investors.

1.1.3 Hidden effort. Suppose the firm's cash flow is directly observed by investors and cannot be diverted by the agent but that the cash flow realization depends on the agent's costly unobservable effort. In this case, the agent's action z_t is an effort choice. The investors' cash flow Y_t is the firm's cash flow, and W_t is the agent's personal cost of effort. In this case, Q_t simply includes the firm's cash flow.[2]

1.2 Investment

We model investment as a decision to expand or contract the firm each period. We measure the size of the firm by its scale ϕ, which we think of as a measure of physical size (e.g., number of stores or number of employees). We assume that rescaling the firm rescales all of the (primitive) cash flows associated with the firm, including any costs associated with the agency problem. That is, if a firm with a scale of 1 in period t produces cash flow Y_t, then with scale ϕ the firm produces ϕY_t. Similarly, if a firm with a scale of 1 in period t implies that action z_t entails a personal cost (or private benefit) of W_t for the agent, then scale ϕ implies a personal cost of ϕW_t. Any other (primitive) cash flows or effort costs or other costs scale as well. This rescaling of all cash flows implies that the payoff possibility set rescales proportionally.

The scale ϕ_t of the firm in period t is determined as follows. At the start of period t, there is an initial scale, denoted ϕ_{t-}. This is the scale absent any additional investment. In period t, an investment decision is made. With growth of θ_t, the new firm scale is given by

$$\phi_t = \phi_{t-}\theta_t,$$

where we set $\phi_{0-} = 1$. Thus, $\theta_t - 1$ can be interpreted as the growth rate of the firm, with $\theta_t > 1$ corresponding to an expansion of the firm and $\theta_t < 1$ corresponding to a contraction of the firm.

Growth requires a total investment in period t of $\hat{c}_t(\theta_t)\phi_{t-}$, where we interpret $\hat{c}_t(\theta_t)$ as the cost per-unit size of the firm. We assume without loss of generality (w.l.o.g.) that $\hat{c}_t(1) = 0$, so that retaining the firm's current scale does not require additional investment.[3]

As in Abel and Eberly (1994), we allow for the possibility that the purchase and sale prices of capital are different (partial or complete irreversibility), and we allow for adjustment costs that may include a fixed cost component. That is, we do not restrict the form of $\hat{c}_t$. We

[2] We could also consider a costly state verification model. There, Y_t is investors' cash flow net of auditing costs; W_t is cash flow not reported by the agent; and Q_t consists of investors' cash flow and the outcome of any audits.

[3] This normalization defines the notion of scale: the scale of the firm is constant absent any investment. It does not rule out depreciation since the expected cash flows of the firm at a constant scale could decline over time (we do not assume stationarity), and investment would be required to maintain a constant level of expected cash flows.

allow $\hat{c}_t$ to be nonlinear and so do not assume constant returns to scale. Indeed, in many applications, $\hat{c}_t$ may not even be continuous—some resources may not be perfectly divisible (e.g., retail outlets). In such cases, it may be optimal to randomize over scale changes. Allowing for stochastic scale changes, it is natural to define the implied cost function c_t as the "convex hull" of $\hat{c}_t$ as follows:[4]

$$c_t(\theta_t) \equiv \min_{\tilde{\theta}} E\left[\hat{c}_t(\tilde{\theta})\right] \quad \text{s.t. } E\left[\tilde{\theta}\right] = \theta_t, \quad \tilde{\theta} \geq 0. \tag{1}$$

The function $c_t(\theta_t)$ is the lowest cost of achieving expected growth θ_t. In analyzing the model, rather than consider stochastic scale choices explicitly, we simply specify expected growth θ_t and use the convexified function $c_t(\theta_t)$. In Appendix A we show that this is w.l.o.g. Thus, for the remainder of the article, when we refer to growth θ_t, we mean expected growth in the event that $c_t(\theta_t) \neq \hat{c}_t(\theta_t)$.

By construction, c_t is convex and so exhibits an increasing marginal cost of expansion. Note that $c_t'(1)$ can be interpreted as the marginal cost of expanding (or contracting) the firm. There is no requirement that c_t be differentiable at 1, so the marginal cost of expansion may differ from the marginal liquidation proceeds associated with contraction. Also, c_t need not be increasing—a slow liquidation may be preferable to a fire sale. To avoid infinite scale choices, we make the weak assumption that

$$\lim_{\theta_t \to \infty} c_t'(\theta_t) = \infty.$$

For simplicity, at this stage we assume that the investment cost function, (as well as other determinants of the firm's production function and the agent's payoffs) is a deterministic function of time. In Appendix B1 we show that our results continue to hold if we introduce a publicly observed state variable that may affect future payoff possibilities. For example, the state of the business cycle may determine current or future investment opportunities.

1.3 Contracts

Investment and disinvestment is observable and contractible. We assume that investors pay for any investment, $c_t(\theta_t)\phi_{t-}$, or simply $c_t(\theta_t)$, on a per-unit scale basis (investors receive the funds if this investment is negative). This assumption is w.l.o.g., since contributions by the agent are equivalent to reductions in transfers to the agent. Hence a contract between the investors and agent specifies, as a function of the history of observable

[4] In the minimization in Equation (1) and in optimization problems throughout the article, we signify choice variables that are random variables with a tilde, and the optimization is with respect to their probability distribution.

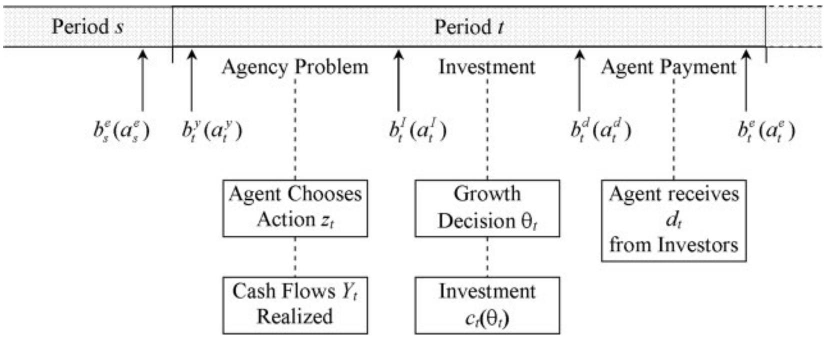

Figure 1
The timeline
In each period t, the agent chooses action z_t; then the cash flow Y_t is realized; then the investment $c_t(\theta_t)$ is invested and firm growth is θ_t; and then investors pay the agent d_t.

variables, the current growth/investment decision θ_t and the investors' payment to the agent d_t.

The timeline within each period is summarized in Figure 1 (for now ignore the b functions): first the agent chooses an action; then a cash flow for investors is realized; then as specified by the contract, the investment decision is made; and finally the agent is paid by investors. So in period t, the agent receives cash of d_t. The investor receives the cash flow Y_t, pays the investment cost $c_t(\theta_t)$, and pays the agent d_t (all cash flows are on a per-unit scale basis at the time they are paid).

In solving for an optimal long-term contract, we will not restrict the contract form, but instead solve for an optimal mechanism given the incentive constraints. The contract can depend on all observable variables, including possible public randomizations. The contract governs the investment and growth of the firm and the payments to the agent. Given a contract, the agent chooses his actions optimally.

For now, we assume that the agent and investors can commit to a contract. We extend our analysis to the case of renegotiation-proof contracts in Appendix B2.

2. Analysis

Solving for the optimal contract involves dynamic programming. At any stage, a contract provides the agent and investors with some expected discounted payoffs, which we refer to as their *continuation payoffs*. For any given continuation payoff for the agent, the optimal contract maximizes the investors' continuation payoff.[5] Thus, the payoffs that can be

[5] The optimal contract need not maximize the agent's payoff holding fixed the payoff of investors. Giving the agent a low payoff can improve incentives in the agency problem. The same is not true for investors since they are not subject to moral hazard.

achieved at any point in time can be described by a *continuation function* that specifies the best possible continuation payoff of investors as a function of the agent's continuation payoff. Using the continuation function, the behavior of the optimal contract can be written as a function of a state variable representing the agent's promised future payoff (rather than keep track of the entire history).

As illustrated in Figure 1, b_t^e is the continuation function as of the end of period t. That is, if the agent expects to receive a_t^e by continuing at the end of period t, the highest possible investor continuation payoff is $b_t^e(a_t^e)$. Similarly $b_t^d(a_t^d)$ is the continuation function immediately before the period t payment to the agent; if the agent expects to receive a_t^d by continuing from this stage of period t, the highest possible investor continuation payoff is $b_t^d(a_t^d)$. Working back, $b_t^I(a_t^I)$ is the continuation function immediately before the period t growth decision, and $b_t^y(a_t^y)$ is the continuation function immediately before the period t agency problem.

Our analysis proceeds by characterizing these continuation functions, working from the end of period t to the beginning. Section 2.1 formally describes the continuation function b_t^e. This section makes clear the relation between contracts and continuation payoffs. Section 2.2 examines the effect of transfers and randomization on the continuation function and solves for b_t^d. Section 2.3 characterizes the determination of investment levels in the optimal contract, deriving b_t^I. Finally, Section 2.4 presents our main results, establishing how various general properties of agency problems lead to general patterns in investment dynamics.

2.1 Incentive compatibility and end-of-period continuation payoffs

A contract specifies all actions of investors. Specifically, a contract in place at the end of period t consists of a pair (θ^t, d^t), specifying growth θ_τ and the payment to the agent d_τ as a function of the history of contractual variables, for all $\tau > t$. Given the contract, the agent chooses actions z_τ, for $\tau > t$, strategically to maximize his payoff. We denote the agent's strategy regarding these actions by z^t. The space of feasible contract-strategy pairs (θ^t, d^t, z^t) is represented by Γ_t.

Given $(\theta^t, d^t, z^t) \in \Gamma_t$, we denote the payoff to the agent by $A_t(\theta^t, d^t, z^t)$ and the payoff to investors by $B_t(\theta^t, d^t, z^t)$. Incentive compatibility requires that the agent's strategy is optimal. That is, the agent's strategy choice z^t must satisfy

$$A_t(\theta^t, d^t, z^t) \geq A_t(\theta^t, d^t, \hat{z}^t) \quad \text{for all } \hat{z}^t \text{ such that } (\theta^t, d^t, \hat{z}^t) \in \Gamma_t. \tag{2}$$

A contract-strategy pair (θ^t, d^t, z^t) that satisfies the condition (2) is called incentive compatible, and we denote the set of incentive-compatible pairs by Γ_t^*.

As stated earlier, we abstract in this model from asymmetric information regarding the future prospects of the firm. Thus, we assume the payoff functions A_t, B_t, and the set of incentive-compatible contract-strategy pairs Γ_t^* are common knowledge in period t. From them, we define the payoff possibility set

$$\beta_t \equiv \left\{(a,b)|a = A_t(\theta^t, d^t, z^t),\, b = B_t(\theta^t, d^t, z^t) \quad \text{for some } (\theta^t, d^t, z^t) \in \Gamma_t^*\right\}.$$

This set describes the payoff combinations that can be achieved by operating the firm beyond period t. Of course, many of these combinations are inferior. The relevant combinations are on the frontier of this set and can be described by the continuation function

$$b_t^e(a) \equiv \max\{\, b \mid (a,b) \in \beta_t\}.$$

This continuation function gives the highest possible payoff attainable by investors given the payoff a for the agent. If the payoff a is unattainable in β_t, we define $b_t^e(a) = -\infty$. Due to limited liability, and the fact that the agent can quit and receive a payoff of 0, there is a lower bound to the agent's continuation payoff. We denote this lower bound by

$$a_t^0 \equiv \min{}_{(\theta^t, d^t, z^t) \in \Gamma_t^*} A_t(\theta^t, d^t, z^t) = \min\left\{a : b_t^e(a) > -\infty\right\} \geq 0. \qquad (3)$$

Due to the agency problem, providing incentives may require that the agent receive positive rents, so that $a_t^0 > 0$ is possible.

To summarize, any feasible contract specifies, as a function of the history up to period t, some continuation of the contract (θ^t, d^t). The agent's incentive-compatible response corresponds to z^t such that (θ^t, d^t, z^t) $\in \Gamma_t^*$. Thus, any feasible contract leads to some continuation payoff pair in the set β_t, where the pair that is chosen depends on the history. An optimal contract chooses points from the frontier of β_t, defined by the continuation function b_t^e. If the continuation payoffs were not on this frontier, it would be possible to raise the investors' payoff without altering the agent's payoff or incentives. This gain could be split upfront between the agent and investors to provide a strictly Pareto-superior initial payoff. Thus, the continuation function b_t^e fully characterizes the payoff-relevant attributes of the firm beyond period t.
The final period. In the final period T of the firm, there are no future interactions between the agent and investors. In this case, we simply define $\beta_T = \{(0,0)\}$, and the continuation function $b_T^e(0) = 0$ and $-\infty$ elsewhere.

2.2 Cash payments to the agent and randomization

Given the continuation function b_t^e available at the end of the period, we now construct the continuation function b_t^d that applies before investors make the payment d_t to the agent (see Figure 1). Because the agent has limited liability, we require $d_t \geq 0$.

In general, the payoff possibility set β_t need not be convex, and thus the continuation function b_t^e need not be concave. However, by randomizing over elements of Γ_t^*, it is possible to achieve payoffs in the convex hull of β_t and thus "concavify" the continuation function. We allow for randomization by letting the contract specify a random continuation payoff $\tilde{a}_t^e$ for the agent.[6] This leads to the following definition of b_t^d:

$$\begin{aligned} b_t^d\left(a_t^d\right) = \max{}_{d_t, \tilde{a}_t^e} \quad & E\left[b_t^e\left(\tilde{a}_t^e\right)\right] - d_t \\ \text{s.t.} \quad & E\left[\tilde{a}_t^e\right] + d_t = a_t^d \\ & d_t \geq 0. \end{aligned} \tag{4}$$

That is, subject to the requirement that the agent's total payoff is a_t^d, the optimal contract maximizes the total payoff for the investor, $E\left[b_t^e(\tilde{a}_t^e)\right]$ less the cash payment of d_t.

When will the optimal contract provide a payment $d_t > 0$ to the agent? At this stage, there are two ways to compensate the agent, by paying him cash today (d_t) and by promising a continuation payoff (meaning potential cash payments in the future). The optimal contract will use whichever form of compensation is least expensive for investors. Since paying the agent \$1 in cash costs the investor \$1, cash payments will be used if the slope of the continuation function is below –1; that is, if paying the agent in the future would cost more than \$1. Given the concavity of the continuation function, there is a threshold level of the agent's payoff, a_t^1, such that cash payments are used above this threshold. That is,

$$d_t\left(a_t^d\right) \; = \; \max\left(a_t^d - a_t^1, 0\right) \tag{5}$$

where[7]

$$a_t^1 \; = \; \inf\left\{a \;\middle|\; b_t^{d\prime}(a) = -1\right\}. \tag{6}$$

Given this payment schedule, the agent's expected continuation payoff after the cash payment but prior to randomization is $a_t^e \equiv E[\tilde{a}_t^e] = \min\left(a_t^d, a_t^1\right)$.

The continuation function b_t^d encapsulates the payoff-relevant information regarding the firm in period t, allowing for both cash payments to the agent and randomization (see Figure 2).

[6] The randomization can be contracted on and is independent of all other variables in the model.

[7] Throughout the article, for a concave (convex) function f, we interpret the statement "$f'(x) = y$" as "y is a super(sub)-gradient of f at x." Thus, these statements are well defined even for f nondifferentiable.

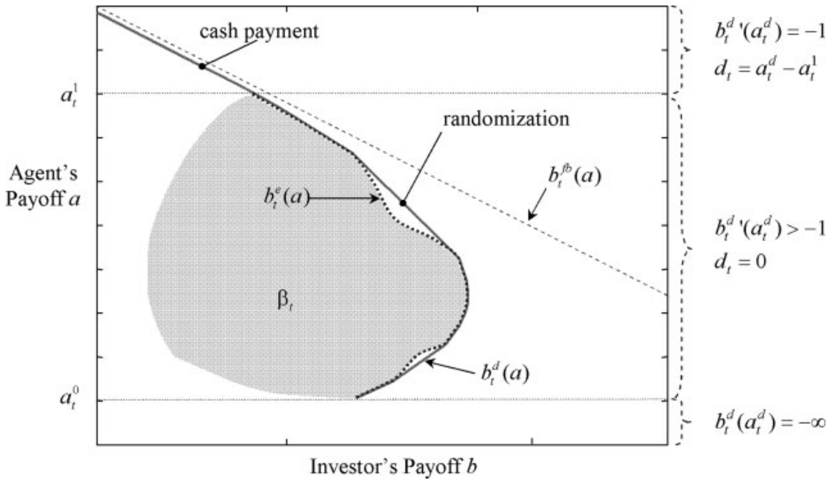

Figure 2
Constructing b_t^d from b_t^e
The optimal cash payment to the agent and (possibly random) agent continuation utility determine the continuation function b_t^d. Cash payments are used if the agent's continuation utility is above a_t^1, randomization is used to make the function $b_t^d(a)$ concave below a_t^1.

The first best. It is useful to compare the continuation function b_t^d with the first-best continuation function. This is the set of payoff possibilities that can be achieved if the agent can commit to a strategy; that is, if we drop the incentive-compatibility condition (2). In this case, it is optimal to maximize the cash flows and pay the agent entirely with cash. The first-best value of the firm is

$$V_t^{fb} \equiv \max_{(\theta^t, d^t, z^t) \in \Gamma_t} A_t(\theta^t, d^t, z^t) + B_t(\theta^t, d^t, z^t).$$

The first-best continuation function is then given by,

$$b_t^{fb}(a) \equiv V_t^{fb} - a \geq b_t^d(a). \tag{7}$$

Importantly, this first-best continuation function is linear, whereas the continuation function b_t^d, which embodies future incentive constraints, is generally concave. Note also that the total value of the firm, $b_t^d(a) + a$, is strictly increasing in a up to the payment boundary a_t^1.

The final period. In the final period T of the firm, $\beta_T = \{(0,0)\}$ and so $b_T^d(a) = -a$, with $a_T^1 = a_T^0 = 0$. That is, any agent payoff $a \geq 0$ is feasible as a transfer from investors. In this case the continuation function is linear, coinciding with the first best, since there is no remaining agency problem once the firm terminates.

Summary of key properties. We conclude with a summary of the key properties of the continuation function b_t^d, which in particular specify when the agent is paid in cash.

Proposition 1. *Given the end-of-period continuation function b_t^e, by allowing for contract randomization and transfers, the continuation function b_t^d is concave and satisfies*

1. *If $a_t^d < a_t^1$, then $d_t = 0$, and $\tilde{a}_t^e$ is increasing in a_t^d in the sense of strict MLRP.*
2. *If $a_t^d \geq a_t^1$, then $d_t = a_t^d - a_t^1$ and $\tilde{a}_t^e = a_t^1$.*

These properties are intuitive in the context of a general agency problem. The concavity of b_t^d means that the lower we push the agent's payoff, the more onerous the incentive constraints become, reducing investors' ability to extract money from the firm. Property 1 states that for low agent payoffs, raising the agent's payoff relaxes incentive constraints, and so deferred compensation, rather than an immediate cash payment, is preferred. From Property 2, once the agent's payoff is high enough, incentive problems are sufficiently relaxed that it is optimal to pay the agent with cash in the current period. Not only are these properties natural in the context of an agency model, it is important to emphasize that we derived them without any specific assumptions on the underlying agency problem.

2.3 Investment and growth

In this section we examine the properties of optimal investment decisions, and we characterize the continuation payoff b_t^I (given the continuation function b_t^d).

2.3.1 Preliminaries. Given the assumption that rescaling the firm rescales all of the cash flows associated with the firm, rescaling expands the payoff possibility set β_t proportionally, so that the new payoff possibility set is given by

$$\phi_t \beta_t.$$

That is, we can interpret β_t as the payoff possibilities *per-unit size* of the firm.

How is the payoff possibility set prior to the investment decision determined? That is, what is the continuation function prior to investment, b_t^I, representing the highest per-unit payoffs of investors as a function of the per-unit payoff of the agent?

Let θ_t be the choice of growth in period t. Let a_t^d be the per-unit payoff of the agent postinvestment, and a_t^I be the per-unit payoff of the agent prior to investment. Since no transfers to or from the agent happen during the investment stage, these payoffs are related according to

$$\phi_{t\_}a_t^I = \phi_t a_t^d \quad \text{or} \quad a_t^I = \theta_t a_t^d.$$

What is the payoff to investors? Given that the agent gets a_t^d postinvestment, investors receive $b_t^d(a_t^d)$ postinvestment on a per-unit basis. Adjusting for the investment cost, this implies an investor payoff of

$$\phi_t b_t^d\left(a_t^d\right) - \phi_{t\_}c_t(\theta_t).$$

Dividing by ϕ_{t-} to put this on a preinvestment per-unit basis and substituting $a_t^d = a_t^I/\theta_t$ yields the preinvestment continuation function

$$b_t^I\left(a_t^I\right) = \max_{\theta_t \geq 0} \quad \theta_t b_t^d\left(a_t^I/\theta_t\right) - c_t(\theta_t), \tag{8}$$

where we define $\theta_t b_t^d(a/\theta_t) = -a$ for $\theta_t = 0$.[8]

We can interpret Equation (8) as follows. In choosing growth θ_t, investors must pay the cost $c_t(\theta_t)$, per-unit size. However, investors benefit in two ways from this investment. First, their payoffs grow directly by θ_t. Second, they can reduce the per-unit compensation paid to the agent by the factor θ_t, without changing the agent's overall payoff.

Note also that $\theta_t = 0$ can be interpreted as liquidation of the firm. Choosing $\theta_t = 0$ implies that $\phi_\tau = 0$ for all $\tau \geq t$. In this case the liquidation proceeds are $-c_t(0)$ per unit. Sharing some amount a of these liquidation proceeds with the agent leaves investors with $-c_t(0)-a$.

The solution θ_t to Equation (8) need not be unique. In fact, the set of optimal θ_t will be an interval if both b_t^d and c_t are linear in a neighborhood of the solution.[9] This indeterminacy has no economic consequence but makes statements regarding formal properties of the solution more cumbersome. To avoid this, we arbitrarily (and w.l.o.g.) assume that if the set of optimal θ_t is nondegenerate, the firm chooses the smallest optimal scale. That is, we define

$$\theta_t\left(a_t^I\right) = \min\{\theta \mid \theta \text{ solves } (8)\}.$$

We analyze the properties of the optimal growth decision next.

2.3.2 Optimal growth. Before analyzing properties of θ_t, we first characterize the optimal scale decision in a first-best setting. From Definition (7),

[8] In Appendix A we show that the same characterization holds for nonconvex cost functions and randomized investment.

[9] This corresponds to a range over which investment is zero NPV at the margin.

$\theta\, b_t^{fb}(a/\theta) \;=\; \theta\, V_t^{fb} - a$. Thus, the first-best optimal growth path is given by θ_t^{fb} such that

$$V_t^{fb} \;=\; c_t'\left(\theta_t^{fb}\right). \tag{9}$$

First-best growth does not depend on the payment to the agent, a_t^I. In contrast, optimal growth subject to the incentive constraints captured by b_t^d will depend on a_t^I, as we now show.

Proposition 2. *The solution to Equation (8) has the following properties:*

1. *θ_t is increasing in a_t^I and bounded above by* $\min\left(a_t^I/a_t^0, \theta_t^{fb}\right)$.
2. *For $\theta_t > 0$, $a_t^d \;=\; a_t^I/\theta_t$, and is increasing in a_t^I.*
3. *$b_t^I(a_t^I)$ is concave and $b_t^{I\prime}(a_t^I) \geq -1$.*

Also, if $\theta_t = 0$ for any $a_t^I > 0$, then $\theta_t = 0$ for all a_t^I, and liquidation of the firm is optimal.

See Figure 3 for an example of an optimal θ_t and a_t^d. The growth of the firm is increasing in the payoff to the agent a_t^I. The intuition for this result is the following. Recall that the total value of the firm, $b_t^d(a_t^d) + a_t^d$, is increasing in the agent's payoff for $a_t^d \leq a_t^1$. Thus, the return to investment is higher when the payoff to the agent is higher. Note also the general concave shape of $\theta_t(a_t^I)$. When the agent's payoff is sufficiently high, growth approaches the first best, and the rate of increase must slow.

The postinvestment per-unit payoff of the agent, a_t^d, is constant and then increasing in a_t^I. This is due to the convexity of the investment cost function. Since the marginal cost of investment is increasing, it is optimal to increase the agent's payoff both by increasing the scale of the firm and also by increasing the per-unit payoff.

Next, Figure 4 illustrates several properties of the continuation function b_t^I. First, the ability to invest and grow expands the payoff frontier; for all a, $b_t^I(a) \geq b_t^d(a)$. Second, if the cost function, c_t, were linear so that there were constant returns to scale in investment, b_t^I would coincide with the dashed line through $c_t'(1)$. The curvature of b_t^I comes from the fact that the cost function is convex, and thus returns are diminishing. Third, while the agent's per-unit payoff postinvestment is bounded below by a_t^0, the preinvestment payoff can be as low as zero. This is because we can always reduce the agent's rents by shrinking the size of the firm. In the limit, the agent can be given zero rents by liquidating the firm.

Cash payments versus growth. Recall from Proposition 1 that if the agent's postinvestment payoff is high enough, he may be rewarded with a cash payment. In this section we have shown that investment and growth are used to provide the agent with a high preinvestment payoff. Our next result

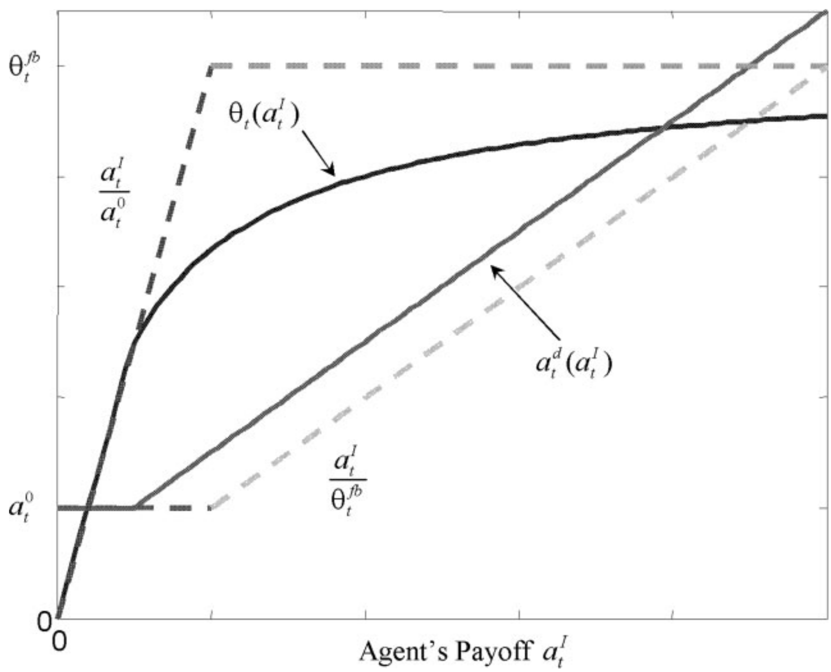

Figure 3
Dependence of growth θ and payoff a_t^d on the agent's preinvestment payoff a_t^I
A higher preinvestment payoff a_t^I for the agent increases the return to investment (due to reduced inefficiencies from agency costs) and thus increases the optimal growth rate, θ. It also raises the agent's post investment payoff a_t^d due to the convex adjustment costs of growth.

establishes that there is a strict priority over these reward systems: if cash payments to the agent are observed, then target growth must already have been achieved. To show this, first define the target growth rate,

$$\bar{\theta}_t = \lim_{a_t^I \to \infty} \theta_t\left(a_t^I\right). \tag{10}$$

This is the firm's maximal growth rate.[10] Then we have the following result.

Proposition 3. *Cash payments to the agent are observed, $d_t(a_t^d) > 0$, only if target growth is achieved, $\theta_t(a_t^I) = \bar{\theta}_t$.*

The final period. As discussed earlier, in the final period of the firm, $b_T^d(a_T^d) = -a_T^d$. And while c_t for $t < T$ need not be increasing, w.l.o.g. we can take c_T to be increasing (since it can represent the present value of

[10] It can be shown that maximal growth is nonstochastic; that is, $\hat{c}_t(\bar{\theta}_t) = c_t(\bar{\theta}_t)$.

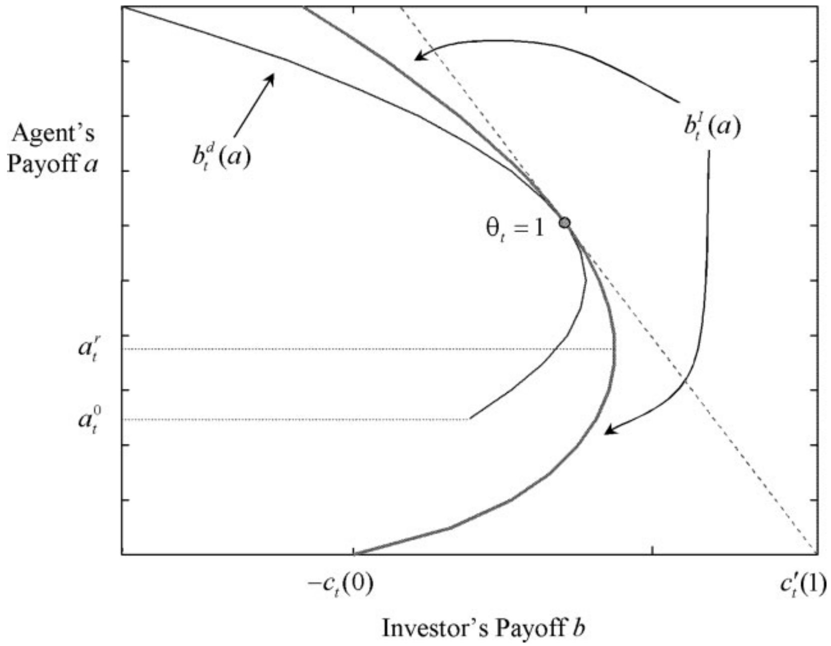

Figure 4
The preinvestment continuation function $b_t^I(a)$
The preinvestment continuation function exceeds $b_t^d(a)$ and is strictly concave when the investment cost function $c_t(\theta_t)$ is strictly convex.

an optimal liquidation). Thus optimal growth is $\theta_T = 0$. Then $b_T^I(a_T^I) = -a_T^I - c_T(0)$; investors collect the liquidation proceeds, net of any payments to the agent.

2.4 The agency stage

This section presents our main results, establishing how various general properties of agency problems lead to general patterns in investment dynamics.

2.4.1 Providing incentives for the agent. Based on the history up to the agency stage of period t, the agent will be due some continuation payoff a_t^y. The agent will receive this reward through the private benefit (or cost) W_t received during the agency stage of period t, together with a promised continuation payoff $a_t^I(Q_t)$ that depends upon the outcome of the contractual variables.

This leads to the general agency problem,

$$
\begin{aligned}
b_t^y\ (a_t^y) = \max_{z_t, a_t^I}\ & E\left[Y_t + b_t^I\left(a_t^I(Q_t)\right) \mid z_t\right] \\
\text{s.t.}\quad & (\text{IC}) \quad z_t \in \operatorname{argmax}_{\hat{z}_t} E\left[W_t + a_t^I(Q_t) \mid \hat{z}_t\right], \\
& (\text{PK}) \quad E\left[W_t + a_t^I(Q_t) \mid z_t\right] = a_t^y, \\
& (\text{AF}) \quad a_t^I(\cdot) \geq 0.
\end{aligned}
\tag{11}
$$

The objective function in Equation (11) is the expected payoff to investors. This includes the net cash flow Y_t, plus the continuation value they receive, $b_t^I(a_t^I)$, if the agent receives the payoff a_t^I. The incentive-compatibility constraint (IC) captures the fact that the agent's action choice must maximize the agent's payoff, which is given by the continuation utility a_t^I plus private benefit (or cost) W_t. The "promise-keeping" constraint (PK) states that the agent's expected payoff must equal what the agent was expecting to receive at the start of period t, a_t^y. This is a generalization of the standard participation constraint to this dynamic context; as we remarked above, a_t^y is the continuation utility promised to the agent, given the current history. Finally, the constraint (AF) restricts the agent's continuation utility to the feasible range.

The solution to Equation (11) describes the continuation payoffs available to the agent and investors prior to the period t agency problem and is summarized by the continuation function b_t^y. It is important to recognize that the agency problem in Equation (11) is a standard, static agency problem. The dynamics of investment and growth under the optimal dynamic contract will follow from standard comparative static results of this one-period problem. For example, in many standard static agency models, the agent's reward is increasing in the principal's cash flows, Y_t. This yields, as an immediate implication, the following main result below:

Theorem 1. *Suppose the optimal static reward function, a_t^I, is increasing in the cash flow Y_t. Then*

1. *Growth in period t, θ_t, is increasing in the period t cash flow, Y_t,*
2. *The agent receives a cash payment only if the cash flow exceeds a threshold. This event corresponds to the firm achieving its target growth ($\theta_t = \bar{\theta}_t$).*

The first result in Theorem 1 has received much empirical attention. Beginning with Fazzari, Hubbard, and Petersen (1988), researchers have attempted to identify firms as financially constrained by providing evidence of a positive correlation between current cash flow and investment. As we have shown, such a relation is quite robust for financing constraints that arise from agency problems. Any agency problem that leads to a positive relation between the agent's payoff and cash flow delivers such a result. The intuition is that if the agent would be promised a higher future payoff for delivering a higher current cash flow, then the agency problem is relaxed and investment is more profitable.

There is a target growth rate, $\bar{\theta}_t$; see Equation (10). Growth increases in the cash flow up until this target is reached. After that, investment will show no further sensitivity to cash flow. In addition, Figure 3 suggests that there may also be a stronger result—even before achieving growth of $\bar{\theta}_t$, the sensitivity of investment to cash flow may be decreasing in cash flow.

The agent's payoff is comprised of current and future cash payments. The second result in Theorem 1 establishes that a positive relation between the agent's payoff and cash flow further implies that for low cash flows, the agent is compensated solely with future payments, while for high cash flows the agent is compensated with current payments too. Moreover, current cash payments are only used once the target growth rate, $\bar{\theta}_t$, is achieved. All current cash flow is devoted to either investment or the repayment of investors until this target growth is achieved. After that, some cash may be paid to the agent. Hence the agent's payments resemble a levered equity claim. In our applications in Section 3, we will further relate the agent's cash payments to dividends and derive results regarding dividends and investment.

The final period. In the final period $b^I_T(a^I_T) = -a^I_T - c_T(0)$. In this case, Equation (11) is identical to a standard one-period model: the principal receives the firm cash flows plus liquidation proceeds, less any transfers to the agent. In the multiperiod setting, the problem changes since $b^I_t \geq -1$ implies that it is cheaper to compensate the agent using future rewards rather than an immediate transfer. Because incentive provision becomes cheaper, the agency problem can be mitigated with a longer horizon.

2.4.2 Intertemporal linkage. Once we have solved Equation (11), we can use the resulting continuation function b^y_t to solve for the continuation function at the end of the prior period, s. To provide the agent with payoff a^e_s at the end of period s, investors must give the agent a continuation payoff of $a^y_t = e^{\gamma(t-s)} a^e_s$ at the start of period t. Thus we have,

$$b^e_s\left(a^e_s\right) = e^{-r(t-s)} b^y_t\left(e^{\gamma(t-s)} a^e_s\right). \tag{12}$$

Given b^e_s, we can begin the analysis again starting from Equation (4). Thus, this completes our specification of the dynamic programming problem for the optimal contract design, which we summarize below:

Proposition 4. *The system of Equations (1), (4), (8), (11), and (12) recursively defines the payoff frontier available from an optimal contract,* *starting from the boundary condition* $b_T^d(a) = -a$, *where* T *is the life of the firm.*

Proposition 4 allows us to calculate the optimal continuation function recursively. The dynamics of the optimal contract are determined by the evolution of the state variable a representing the promised continuation payoff for the agent at any stage in the contract. In Figure 5, we summarize the evolution of the agent's continuation payoff in period t given a promised payoff of a_s^e at the end of the prior period s.

With the possible exception of $a_t^I(Q_t)$, all steps of the dynamics of the agent's continuation payoff are monotonic. That is, the agent's continuation payoff at any stage is weakly increasing in the agent's continuation payoff at an earlier stage. If this is also true for the agency stage of the contract, then this monotonicity will be preserved across periods, which has consequences for the dynamics of growth. For example, suppose growth is high in period s. Because θ_s and a_s^I are positively correlated, this implies that the agent's promised reward at the end of period s is also high, which implies that the promised a_t^y will also be high. If the agent's optimal static reward a_t^I is increasing in the reservation utility a_t^y, then growth in period t will also be high. This intuition is formalized in our second main result below:

Theorem 2. *Suppose the optimal static reward function,* a_t^I, *is increasing in the promised utility* a_t^y. *Then growth in period t,* θ_t, *is increasing in growth in the prior period s,* θ_s, *in the sense of first-order stochastic dominance.*

The result in Theorem 2 implies that investment over time will be positively serially correlated, even after controlling for investment opportunities. If current cash flow is high, both current and future investment will be high. This persistence in investment follows from the optimal contract in the presence of an agency problem. Evidence consistent with persistence in investment is provided by, for example, Devereux and

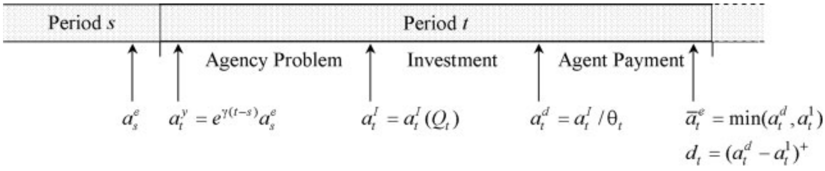

Figure 5
Dynamics of the agent's continuation payoff
This figure summarizes the evolution of the agent's continuation payoff: a promised payoff a_s^e at the end of period s becomes a payoff a_t^y at the beginning of period t, reflecting the agent's discount rate; which becomes a payoff a_t^I depending on the outcome of the agent's actions; which becomes a payoff a_t^d accounting for the firm's investment; which at the end of period t becomes a_t^e, after the agent's cash payment.

Schiantarelli (1990) and Himmelberg (2000), and evidence consistent with a positive relation between current investment and past cash flow is found by Devereux and Schiantarelli (1990) and Gilchrist and Himmelberg (1998).

If cash flow is high in the current period, both current and future investment will be higher. So other things equal, larger firms will be ones with better cash flow histories and ones in which the growth rate target, $\bar{\theta}_t$, is more likely to be reached. Since the sensitivity of investment to cash flow disappears once the target $\bar{\theta}_t$ is reached, investment at larger firms should show less sensitivity to cash flows. Evidence consistent with a lower investment/cash flow sensitivity at large firms is found by Gilchrist and Himmelberg (1998). Growth of $\theta_t = 0$ corresponds to liquidating the business. Hence the model also predicts higher survival rates for larger firms.

The initial period. By recursively computing the payoff frontier as outlined in Proposition 4, we can compute the initial payoff frontier associated with the start of the firm. The period 0 cash flow of the firm is simply the agent's initial wealth Y_0, which is paid to investors as part of the period 0 agency problem (which may require that the agent be provided with incentives to report Y_0 truthfully). Investors contribute initial capital K, which is the cost of setting up the firm at an initial scale of $\phi_{0-} = 1$. We assume that $c_0(0) \leq -K$, so that if the firm is shut down immediately (or never started), the agent and investors can recover their initial capital.

The initial contract and payoffs will depend upon the competitive environment at the time the contract is signed. For example, if the agent is an entrepreneur with sole rights to the firm and investors are competitive, the initial contract will maximize the agent's payoff subject to a zero profit condition for investors:

$$a_0^y = \max \left\{ a \,\middle|\, b_0^y(a) \geq K \right\}.$$

On the other hand, if the investors initially own the rights to the firm and recruit an agent from a competitive pool, then the agent's initial payoff is determined by

$$a_0^y = \operatorname{argmax}_a b_0^y(a) \quad \text{subject to} \quad a \geq R_0$$

where $R_0 \geq Y_0$ is the reservation utility of the agent (including the option to consume his initial wealth).

3. Applications to Specific Agency Models

In this section we examine three agency models that fit our general setting. Each of these models places additional structure on the agent's action z_t and private benefit (or cost) W_t, the investors' cash flow Y_t, and the contractible variables Q_t.

3.1 Privately observed cash flows

Consider a model in which the agent privately observes the business cash flows, and the contract must induce the agent to reveal and share the cash flows with investors. Here we follow DeMarzo and Fishman (2004) (DF).

The cash flows Y_t are independent and bounded below by Y_t^0. The agency problem is that the agent privately observes and can underreport the realization of Y_t, consuming the residual. Specifically, given the cash flow Y_t, the agent can report $\hat{y}_t \in [Y_t^0, Y_t]$ and receive private benefit of $\lambda(Y_t - \hat{y}_t)$. The constant $\lambda \in (0,1]$ and is strictly less than 1 if there are costs of diverting the cash flows (e.g., the agent cannot consume the cash directly but instead consumes inefficient perks). Because the agent's consumption is hidden from investors, the agent may save rather than consume cash. We assume the agent earns the continuously compounded return $\rho \leq r$ on hidden savings. For this application, the agent's action z_t is the choice of how much cash to report, $\hat{y}_t$; the agent's private benefit W_t is $\lambda(Y_t - \hat{y}_t)$; and the set of contractible variables Q_t consists of the history of the agent's reports (as well as any public randomizations). But since in our dynamic programming formulation the state variable a_t^y is a sufficient statistic for the contract's history prior to date t, we can write $Q_t = (\hat{y}_t, a_t^y)$.

Since the agent's savings rate $\rho \leq r$, saving by the agent is not necessary for efficiency.[11] Thus, we characterize the optimal contract under the assumption that the agent consumes all of his cash in each period, saving nothing. Then we show that with this contract, the agent indeed has no incentive to save, and the assumption of no saving by the agent is w.l.o.g. Using the revelation principle, we can restrict attention to mechanisms in which the agent truthfully reports the cash flows. By Proposition 1 of DF, this is equivalent to having the agent pay Y_t directly to investors in period t, who then pay the agent according to the optimal contract.

Given these simplifying results, we can consider a contract that specifies a period t payment from investors to the agent d_t (which the agent will consume), and a period t investment decision θ_t, as a function of the history of payments from the agent to the investors $(Y_s)_{s \leq t}$. Given this contract, in each period t, the agent pays Y_t to investors, who then pay the agent d_t and invest $c_t(\theta_t)$ to achieve growth of θ_t.

In this setting the only contractual variable learned in the agency stage is the reported cash flow. Thus, the agency problem can be written as[12]

[11] See DeMarzo and Fishman (2004) Proposition 1.

[12] Not all values of a_t^y are necessarily feasible. In this case we define $b_t^y(a_t^y) = -\infty$.

$$
\begin{aligned}
b_t^y(a_t^y) \;=\; & \max_{a_t^I(\cdot)} E\left[Y_t + b_t^I\left(a_t^I(Y_t)\right)\right] \\
\text{s.t.} \qquad & \text{(IC-1)} \quad Y_t \in \operatorname{argmax}_{Y_t^0 \le \hat{y} \le Y_t} \lambda(Y_t - \hat{y}) + a_t^I(\hat{y}) \\
& \text{(PK-1)} \quad E\left[a_t^I(Y_t)\right] = a_t^y \\
& \text{(AF-1)} \quad a_t^I(Y_t) \ge 0.
\end{aligned}
\tag{13}
$$

The incentive-compatibility constraint (IC-1) ensures that the agent pays the investor all of the cash flow. This problem is a special case of Equation (11). We now characterize the optimal reward schedule, a_t^I.

Proposition 5. *Let $\mu_t = E[Y_t]$. Then at a solution to Equation (13), $a_t^I(Y_t) = a_t^y + \lambda(Y_t - \mu_t)$. Also, b_t^y is concave.*

From Proposition 5, the agent's marginal benefit from reporting an extra dollar of cash flow is constant over time. Therefore there is no incentive for the agent to engage in hidden savings, implying that our no-agent-saving assumption is w.l.o.g. Note also that since b_t^y is concave, so is b_s^e, and therefore no randomization will be required at the end of each period. Thus, if the investment cost function $\hat{c}_t$ is also convex, no public randomization is necessary for an optimal contract.

By Proposition 5, the conditions of Theorem 1 and Theorem 2 are satisfied, and we have the following results regarding the dynamics of an optimal contract in this setting:

Corollary. *In the DF model, current investment θ_t is increasing in current cash flow Y_t, and current investment θ_t is also increasing in past investment, $\{\theta_s\}_{s<t}$ (and past cash flow).*

Theorem 1 also implies that cash payments are made to the agent only if the cash flow exceeds a threshold. In this case, given the linear form of the reward a_t^I in Proposition 5, we can interpret these payments as a shareholder dividend. In particular, let D_t be the debt per-unit size of the firm at the start of period t. Suppose that each period, the agent uses the firm's cash flow to repay the debt, and once the debt is repaid uses any remaining cash to pay a dividend to shareholders. That is, on a per-unit size basis,

$$
\text{Div}_t = (Y_t - D_t)^+ \equiv \max(Y_t - D_t, 0).
$$

We can implement the optimal contract with debt and equity as follows:

Proposition 6. *The optimal contract can be implemented by giving the agent the share λ of the firm's equity and having the debt of the firm evolve according to*

$$\theta_t D_{t+1} \;=\; e^{\gamma}\left[(D_t - Y_t)^+ + (m_t\theta_t - n_t)\right] \tag{14}$$

where $n_t = \bar{\theta}_t a_t^1/\lambda$ and $m_t = e^{-\gamma}(\mu_{t+1} + n_{t+1})$. Firm growth θ_t, determined by Equation (8), is decreasing in $(D_t - Y_t)^+$.

By Equation (14), the firm is effectively rolling over unpaid debt and it is issuing or retiring debt, where issuance/retirement depends on investment. Consistent with the findings of Devereux and Schiantarelli (1990), Whited (1992), Lang, Ofek, and Stulz (1996), and Himmelberg (2000), firms with lower leverage invest more. The preceding result also allows us to relate some of our earlier results to the use of dividend payments by the firm:

Corollary. *In the DF model, shareholder dividends are paid only if target growth has been achieved, at which point investment is insensitive to cash flows.*

This result that investment is not sensitive to cash flow for dividend payers is consistent with the evidence of Fazzari, Hubbard, and Petersen (1988). Since dividends are only paid once the target growth $\bar{\theta}_t$ is achieved and since (other things equal) larger firms and firms with higher current cash flows are more likely to achieve $\bar{\theta}_t$, we have that dividend payouts are more likely for larger and more profitable firms. This is a natural result and one that is consistent with the evidence (see, for example, Fama and French, 2001).

The results of Proposition 6 describe the dynamics of an optimal capital structure in this setting. From them we can determine the mix of debt and equity that is used to finance new investment. We are currently exploring the implications of this model on the co-movement of investment and capital structure in separate work.

3.2 Publicly observed, noncollectible cash flows

Consider the model of the previous section but suppose the cash flows are publicly observed. Because the agent can divert cash flows rather than pay investors, the contract must still provide appropriate incentives. For this application, again the agent's action z_t is the choice of how much cash to report, $\hat{y}_t$, and the agent's private benefit W_t is $\lambda(Y_t - \hat{y}_t)$. The difference is that the set of contractible variables Q_t includes the agent's payment to investors, $\hat{y}_t$, and the cash flow Y_t.[13]

Again, we can restrict attention to mechanisms in which the agent reports truthfully and does not divert the cash flows. The strongest incentives are provided by setting $a_t^I = 0$ if the agent diverts any amount. In that case, the agent will either pay the cash flows Y_t to the investors in their entirety or divert them completely and consume λY_t. Using these observations, we can simplify the incentive-compatibility constraint in this setting to

$$\text{(IC-2)} \qquad a_t^I(Y_t) \geq \lambda Y_t$$

where we write $a_t^I(Y_t)$ for the agent's reward conditional on not diverting the cash flows Y_t. Except for this change to the incentive constraint, the contracting problem coincides with Equation (13). In this case, we have the following characterization.

Proposition 7. *At a solution to Equation (13), with (IC-2) replacing (IC-1),* $a_t^I(Y_t) = \max(\underline{a}_t^I, \lambda Y_t)$, *where* $\underline{a}_t^I \geq 0$ *is determined by (PK-1) and is increasing in* a_t^y.

The agent's reward a_t^I is increasing in both Y_t and a_t^y. Moreover, if λY_t is sufficiently low, a_t^I depends only on a_t^y (through $\underline{a}_t^I$), whereas for λY_t high, a_t^I depends only on Y_t. Also, given the linearity of the agent's payoff for high cash flows, we can again interpret the agent's payment as an equity share and implement the optimal contract through a combination of inside equity and outside equity and debt. By Theorem 1 and Theorem 2, these static properties imply the following dynamic properties:

Corollary. *With publicly observable, noncollectible cash flows, current investment* θ_t *is increasing in current cash flow* Y_t *and current investment* θ_t *is also increasing in past investment,* $\{\theta_s\}_{s<t}$ *(and past cash flow). When the current cash flow is low, current investment depends only on past investment; otherwise, current investment depends only on the current cash flow. The sensitivity of investment to cash flow is decreasing in past growth.*

In this model the sensitivity of investment to cash flow takes a more subtle form—depending on the current cash flow, there is either (local) sensitivity to current cash flow or to past cash flow but not both.

We can again implement the optimal contract with debt and equity so that the agent holds a levered equity claim. Then shareholder dividends are paid only if target growth is achieved and therefore insensitive to cash flows.

[13] This is effectively the agency problem of Allen (1983), Hart and Moore (1994), Hart (1995), and Albuquerque and Hopenhayn (2004) who analyze settings with deterministic cash flows. Our model allows for risky cash flows.

3.3 Hidden effort

As a third example, we consider a standard principal-agent setting. We make two changes to the model in the previous application. First, the business cash flow is publicly observed and is collectible by the investor. Second, the cash flow in period t depends on the agent's period t privately observed effort z_t. The agent's cost of effort in period t is $k_t(z_t)$. Otherwise, the environment is the same. The incentive problem here involves inducing the agent to exert effort. For this application, z_t is the agent's effort choice; W_t is the agent's cost of effort, k_t; and the contractible variables Q_t includes the cash flow, Y_t.

In this application, the agent's consumption-saving decision is of no importance to the optimal contract. Since cash flows are observable, there is no scope for the agent to save with the intent of influencing future compensation. And since the agent is risk-neutral, there is no precautionary demand for saving.

Again letting the agent's continuation utility a_t^I include the investors' period t cash payment to the agent, we can write the period t "static" agency problem as

$$\begin{aligned} b_t^y(a_t^y) \; = \; & \max_{z_t, a_t^I(\cdot)} \; E\left[Y_t + b_t^I\,\left(a_t^I\,(Y_t)\right) \mid z_t\right] \\ \text{s.t.} \quad & (\text{IC-3}) \;\; z_t \in \operatorname{argmax}_z E\left[a_t^I\,(Y_t) - k_t(z) \mid z\right] \\ & (\text{PK-3}) \quad E\left[a_t^I\,(Y_t) - k_t(z_t) \mid z_t\right] \; = \; a_t^y \\ & (\text{AF-3}) \quad a_t^I\,(Y_t) \geq 0. \end{aligned} \tag{15}$$

The incentive-compatibility constraint (IC-3) ensures that the agent chooses the designated effort. Note that this problem is also a special case of Equation (11).

The formulation in Equation (15) is quite similar to that used by Grossman and Hart (1983) to consider the static principal-agent problem: the choice variable a^I represents the agent's utility, and $-b^I$ is the cost to the principal of implementing that utility. The standard assumption that the first-order approach is justified leads to the optimality condition

$$-b_t^{I\prime}\left(a_t^I(y)\right) \; = \; \eta_{PK} + \eta_{IC}\left(\frac{\partial}{\partial z}\ln f_t(y, z)\right) \tag{16}$$

where η_{PK} and η_{IC} are the multipliers on the promise-keeping and incentive-compatibility constraints and $f_t(\cdot, z)$ is the density of the cash flows given action z. Condition (16) implies that a_t^I is increasing in the cash flow Y_t if effort increases output in the sense of the monotone likelihood ratio property ($\frac{\partial}{\partial y \partial z} \ln f_t \geq 0$). Also, since $b_t^{I\prime}(a) = -1$ once target growth is achieved, the following result follows from Theorem 1:

Corollary. *Suppose the first-order approach is valid and effort increases cash flow in the sense of the monotone likelihood ratio property. Then in the dynamic principal-agent model, current investment θ_t is increasing in current cash flow Y_t. Moreover, either target growth is achieved in period t ($\theta_t = \bar{\theta}_t$) with probability 1, or only after the highest possible cash flow Y_t.*

Consider the special case in which the cash flows are binary, $Y_t \in \{y_t^L, y_t^H\}$. The agent's effort increases the likelihood of the high cash flow; hence, we can parameterize effort by the mean of the cash flow

$$\mu_t = E[Y_t \,|\, z_t] \in \left[y_t^L, y_t^H\right].$$

Assume the cost of effort $k_t(\mu)$ is convex with $k_t'(y_t^L) = 0$ and $k_t'(y_t^H) = \infty$. The agent's incentives are determined by the difference in his continuation payoff after a high versus low cash flow, $\Delta a_t = a_t^I(y_t^H) - a_t^I(y_t^L)$. Letting $\Delta Y_t = y_t^H - y_t^L$, then implementing effort level μ requires:[14]

$$\frac{\Delta a_t}{\Delta Y_t} = k_t'(\mu). \tag{17}$$

This leads to the following result:

Proposition 8. *In the dynamic principal-agent model with binary cash flows,*

$$a_t^I(Y_t) = a_t^y + k_t(\mu_t) + k_t'(\mu_t)(Y_t - \mu_t). \tag{18}$$

Thus, current investment θ_t is increasing in current cash flow Y_t. Current investment θ_t is increasing in past investment, $\{\theta_s\}_{s<t}$ as long as

$$\frac{-1}{y_t^H - \mu_t} \leq \frac{\partial}{\partial a_t^y} k_t'(\mu_t) \leq \frac{1}{\mu_t - y_t^L}. \tag{19}$$

By Proposition 8, the conditions of Theorem 1 are satisfied. Because the principal's choice of the optimal effort level to implement, μ_t, may vary with a_t^y, condition (19) is required to guarantee that the conditions of Theorem 2 are satisfied. Condition (19) bounds the variation in the degree of incentives $k_t'(\mu_t)$ provided to the agent.

[14] If k_t is kinked or effort is discrete, we interpret $k_t(\mu)$ as the smallest sub-gradient at μ.

While condition (19) is restrictive and depends upon the model parameters, two special cases are worthy of note. First, the bounds in condition (19) grow to $[-\infty, +\infty]$ as the volatility of the cash flow *per period* decreases to zero, as would be the case if we consider a binary approximation to a continuous-time model.[15] That is, in the continuous-time setting, the positive relation of current investment to past investment would follow solely from monotonicity of the agent's payoff to the cash flows. Second, in many models the agent's effort choice is also binary (work or shirk). Assuming effort is sufficiently productive, it will be optimal to implement high effort at all times, and condition (19) is automatically satisfied since k_t' does not depend on a_t^y.

Finally, we remark that given the linearity of the agent's payoff with respect to the cash flows, we can again implement the optimal contract in terms of debt and equity-type securities as in Section 3.1. In this case, in addition to the firm's leverage, the agent's share of the firm (given by $k_t'(\mu_t)$) may vary over time. The relation between the agent's equity stake and past performance is a topic of future research.

4. Concluding Remarks

Agency problems bring financing constraints and prevent firms from undertaking first-best investment plans. Our results show that some common properties of agency problems lead to similar qualitative features for investment dynamics. Among them, we find that after controlling for a firm's investment opportunities, there is a positive relation between current and past cash flow and investment. This is of particular interest since determining where and when these relations exist has been a keen focus of the recent empirical literature on investment.

In the analysis here, the positive relation between investment and cash flow follows from the optimal long-term incentive contract. Investment is tied to the agent's rents, which fluctuate for incentive reasons. Consequently, the theory predicts that investment should only depend on cash flows that are subject to incentive problems (though in many settings these are perhaps the only kind of cash flows). This is distinct from the argument for an investment/cash flow sensitivity based on (deadweight) transaction costs—external funding is more expensive than internal funding because of underwriting fees, administrative expenses, etc. The transaction cost explanation implies a positive relation between investment and *any* cash flow. In practice, we might expect both reasons to apply. If so, we would expect to observe greater investment sensitivity to cash flows that are subject to incentive problems.

[15] In that case, the agent would control the drift μ_t of the Brownian motion. See DeMarzo and Sannikov (2004) for a contracting model of this type.

In our model, cash flows are independent over time. In Appendix B1 we consider the case in which the firm's prospects are influenced by a publicly observable Markov state variable that summarizes the available information regarding the future characteristics of the firm. For example, it might contain information about the return to investment, such as the state of the business cycle and/or information on the parameters of next period's agency problem, such as the distribution of the firm's cash flows. We show that all of our earlier results continue to hold.

What about renegotiation of the contract in period t? Note from Figure 4 that for $a < a_t^r$, the payoff pair $(a, b_t^I(a))$ is Pareto-inferior to other feasible outcomes. If the parties have the opportunity to renegotiate costlessly at this stage, then they will both agree to some superior outcome. Thus, allowing the parties to renegotiate restricts the payoffs that may occur to points on the frontier with $a \geq a_t^r$. In Appendix B2 we consider the implications of the possibility of renegotiation. We show that in the agency stage, the feasibility constraint (AF) should be replaced by

$$\text{(AF-R)} \quad a_t^I(\cdot) \geq a_t^r.$$

Also, depending on the agency problem, renegotiation may restrict the lowest feasible payoff for b_t^y. Aside from these changes, the underlying dynamics of the optimal contract are unaltered, and our main conclusions continue to apply. There we also note that with a renegotiation-proof contract, long-term securities are not needed to implement the optimal long-term contract. Short-term securities, refinanced over time, will also suffice.

Our model considered a finite horizon. This was for simplicity; it allows us to use backward induction to solve for the optimal contract recursively. As described in Appendix B3, all of our analysis can be extended to the infinite horizon case.

Our analysis considers the class of agency problems in which there is symmetric information (between the agent and investors) regarding the payoff possibility set. An important extension would be to cover agency problems with asymmetric information. Another important extension would be to consider agent risk aversion.

Appendix A: Using the convexified cost function

Define $\tilde{\theta}_t$ to be the (possibly stochastic) growth decision in period t. Let a_t^d be the (possibly stochastic) per-unit payoff of the agent postinvestment and a_t^I be the per-unit payoff of the agent prior to investment. Since no transfers to or from the agent happen during the investment stage, these payoffs are related according toy

$$\phi_{t-} a_t^I = E\left[\phi_t a_t^d\right] \quad \text{or} \quad a_t^I = E\left[\tilde{\theta}_t a_t^d\right].$$

What is the payoff to investors? Given that the agent gets a_t^d postinvestment, investors receive $b_t^d(a_t^d)$ postinvestment on a per-unit basis. Adjusting for the investment cost, this implies an investor payoff of

$$E\left[\phi_t b_t^d\left(a_t^d\right) - \phi_{t-}\hat{c}_t\left(\tilde{\theta}_t\right)\right].$$

Dividing by ϕ_{t-} to put this on a preinvestment per-unit basis yields the preinvestment continuation function,

$$b_t^I\left(a_t^I\right) = \max_{\tilde{\theta}, a_t^d} E\left[\tilde{\theta}\, b_t^d\left(a_t^d\right) - \hat{c}_t(\tilde{\theta})\right] \qquad \text{s.t. } E\left[\tilde{\theta}\, a_t^d\right] = a_t^I, \ \tilde{\theta} \geq 0. \tag{20}$$

We now show that rather than considering stochastic scale choices explicitly, we can simplify the problem by using the convexified cost function c_t:

Lemma 1. *Let c_t be defined by Equation (1). Then $b_t^I(a_t^I)$ is given by Equation (8). Also, if θ solves Equation (8) and $\tilde{\theta}$ solves Equation (1), then Equation (20) is solved by $\tilde{\theta}$ and $a_t^d = a_t^I/\theta$.*

Proof of Lemma 1. Consider any $\tilde{\theta}$ feasible for Equation (20) and define $\theta = E[\tilde{\theta}]$. Suppose $\theta > 0$. Then, interpreting $\tilde{\theta}/\theta$ as a change of measure, the optimal a_t^d solves

$$\max_{a_t^d} \ E^{\tilde{\theta}}\left[b_t^d\left(a_t^d\right)\right] \qquad \text{s.t. } E^{\tilde{\theta}}\left[a_t^d\right] = a_t^I/\theta.$$

Since b_t^d is concave, this is solved with $a_t^d = a_t^I/\theta$. Hence, holding θ fixed, the optimal $\tilde{\theta}$ solves

$$\begin{aligned} &\max_{\tilde{\theta}} \ \theta\, b_t^d\left(a_t^I/\theta\right) - E\left[\hat{c}_t(\tilde{\theta})\right] \quad \text{s.t. } E\left[\tilde{\theta}\right] = \theta, \tilde{\theta} \geq 0 \\ &= \max_{\theta \geq 0} \theta\, b_t^d\left(a_t^I/\theta\right) - c_t(\theta), \end{aligned}$$

from Equation (1). ■

Appendix B: Model extensions

B1 Stochastic investment opportunities

Here we consider the case in which the firm's prospects are influenced by a verifiable Markov state variable X_t. The state variable X_t summarizes all of the available information in period t regarding the future characteristics of the firm. For example, it might contain information about the return to investment, such as the state of the business cycle and/or information on the parameters of next period's agency problem, such as the distribution of the firm's cash flows. And, of course, it contains information relevant to the distribution of X_τ, for $\tau > t$.

Suppose X_t is revealed just prior to the investment stage (any other timing can be handled similarly). Since X_t influences the future payoff possibilities, the continuation function is now conditional on X_t. That is, we let $b_t^I(a_t^I \mid X_t)$ represent the highest possible preinvestment continuation payoff for investors given a preinvestment continuation payoff for the agent of a_t^I, conditional on the realization of the state X_t.

What about the pre-information continuation function? That is, what is the payoff frontier just prior to the realization of X_t? We denote this continuation function by $b_t^{I-}(a_t^{I-} \mid X_{t-})$, where X_{t-} is the prior realization of the state variable. We are interested in the properties of this continuation function and the dynamics of the agent's continuation payoff at this stage.

The pre-information continuation function must solve the following:

$$\begin{aligned} b_t^{I-}\left(a_t^{I-} \mid X_{t-}\right) &= \max_{a_t^I(X_t)} E\left[b_t^I\left(a_t^I(X_t) \mid X_t\right) \mid X_{t-}\right] \\ &\text{subject to} \quad E\left[a_t^I(X_t) \mid X_{t-}\right] = a_t^{I-}. \end{aligned} \tag{21}$$

The solution to Equation (21) is characterized by

$$b_t^{I-\prime}\left(a_t^{I-} \mid X_{t-}\right) = b_t^{I\prime}\left(a_t^I(X_t) \mid X_t\right). \tag{22}$$

The following result demonstrates that we can take the post-information payoff, $a_t^I(X_t)$, to be increasing in the pre-information payoff a_t^{I-}. In addition, the pre-information continuation function is concave.

Proposition 9. *There exists a solution to Equation (21), $a_t^I(X_t)$, representing the post-information payoff of the agent, that is increasing in the pre-information payoff a_t^{I-}. Also, $b_t^{I-}(\cdot \mid X_{t-})$ is concave, with $b_t^{I-\prime} \geq -1$.*

As a result, all of the key properties of the continuation function and the contract dynamics are preserved through the information arrival stage, and our earlier results continue to hold.

B2 Renegotiation-proofness and short-term securities

Note from Figure 4 that for $a < a_t^r$, the payoff pair $(a, b_t^I(a))$ is Pareto-inferior to other feasible outcomes. If renegotiation is possible, then they will both agree to some superior outcome. Thus, allowing the parties to renegotiate restricts the payoffs that may occur to points on the frontier with $a \geq a_t^r$. According to the renegotiation principle (see Hart and Tirole, 1988), the optimal renegotiation-proof contract (i.e., the optimal contract once we allow for renegotiation) is characterized by history contingent payoffs that remain on the efficient frontier of the payoff possibility set. We summarize this discussion with the following:

Proposition 10. *An optimal renegotiation-proof contract entails a preinvestment continuation payoff for the agent $a \geq a_t^r$, where*

$$a_t^r \equiv \inf\left\{a \mid b_t^{I\prime}(a) \leq 0\right\}.$$

Outside the agency stage, this is the only requirement imposed by renegotiation-proofness.

In particular, from Proposition 10, in the agency stage the feasibility constraint (AF) should be replaced by,

$$\text{(AF-R)} \quad a_t^I(\cdot) \geq a_t^r.$$

Also, depending on the agency problem, renegotiation may restrict the lowest feasible payoff for b_t^y. Outside of the agency stage there are no changes because the contract is on the Pareto frontier. Replacing (AF) with (AF-R), the underlying dynamics of the optimal contract are unaltered, and our main conclusions continue to apply.

With a renegotiation-proof contract, long-term securities are not needed to implement the optimal long-term contract. Short-term securities, refinanced over time, will also suffice. For instance, suppose the optimal contract calls for agent and investor continuation payoffs (at the end of period t) of a_t^e and $b_t^e(a_t^e)$, respectively. These payoffs can be implemented with short-term securities, perhaps short-term debt, as follows. Short-term (one-period) debt with a required payoff of $b_t^e(a_t^e)$ can be refinanced with new short-term debt with a value of $b_t^e(a_t^e)$. The original debtholder receives his payoff of $b_t^e(a_t^e)$ in cash, and the new debtholder receives a continuation payoff of $b_t^e(a_t^e)$.

B3 Infinite horizon

All of our analysis can be extended to the infinite horizon case, as we briefly describe here. Consider an infinite horizon setting that we arbitrarily truncate to a T-period horizon by forcing liquidation at the end of period T. Let $\hat{b}$ be the optimal continuation function for this truncated model. Then in the final period, $\hat{b}_T^I(a_T^I) = -c_T(0) - a_T^I$, and we can solve for earlier periods as before. Now let b be the optimal continuation function with an infinite horizon. Then since the contract could call for liquidation in period T,

$$-c_T(0) - a_T^I \leq b_T^I(a_T^I) \leq V_T^{I,fb} - a_T^I$$

where $V_T^{I,fb} = \theta_T^{fb} V_T^{fb} - c_T(\theta_T^{fb})$ is the first-best value at the investment stage in the infinite horizon case. Therefore,

$$\hat{b}_0^y(a) \leq b_0^y(a) \leq \hat{b}_0^y(a) + e^{-rT}\left(V_T^{I,fb} + c_T(0)\right),$$

and the optimal continuation function for the infinite horizon model coincides with the limit of the truncated models as long as

$$e^{-rT}\left(V_T^{I,fb} + c_T(0)\right) \to 0. \tag{23}$$

Condition (23) is easily satisfied, given the natural assumptions of a positive discount rate and boundedness of the first-best value of the firm relative to its liquidation value. In this case we can solve for the optimal continuation function for the infinite horizon case by considering the limit of finite horizon models.

Appendix C: Proofs

Proof of Proposition 2. First, note that if $\theta_t > a_t^I/a_t^0$, then $a_t^I/\theta < a_t^0$ and investors receive $-\infty$. Thus, $\theta_t \leq a_t^I/a_t^0$. The first-order condition of Equation (8) implies that if $\theta_t > 0$, then

$$b_t^d\left(a_t^I/\theta_t\theta_t\right) - \frac{a_t^I}{\theta_t} b_t^{d\prime}\left(a_t^I/\theta_t\right) = c_t'(\theta_t) \tag{24}$$

(where we choose some super-gradient of b_t and some sub-gradient of c_t at any kinks). Because $b_t^d(a_t^d) \leq V_t^{fb} - a_t^d$ and $b_t^{d\prime} \geq -1$, Equation (24) implies $c_t'(\theta_t) \leq V_t^{fb}$ and thus

$\theta_t \le \theta_t^{fb}$. Next note that because b_t^d is concave, the function $f(\alpha) \equiv b_t^d(\alpha) - \alpha b_t^{d\prime}(\alpha)$ is increasing in $\alpha \ge 0$. Also, the convexity of c_t implies that c_t' is increasing. Since $a_t^d = a_t^I/\theta$, we can rewrite Equation (24) as

$$f\left(a_t^d\right) = c_t'\left(a_t^I/a_t^d\right) = c_t'(\theta_t),$$

so that if a_t^I increases, both a_t^d and θ_t must weakly increase as well. In fact, if a_t^d is constant θ_t strictly increases. If a_t^d is increasing, then if b_t^d is strictly concave, f is strictly increasing. Thus, $c_t'(\theta_t)$ must strictly increase, and therefore so must θ_t unless c_t is "kinked" at θ_t.

Note that by the envelope theorem,

$$b_t^{I\prime}(a_t^I) = b_t^{d\prime}(a_t^d). \tag{25}$$

Thus, $b_t^{I\prime}(a_t^I) \ge -1$ follows from Proposition 1. Also, from the concavity of b_t^d and the monotonicity of a_t^d, Equation (25) implies that $b_t^{I\prime}$ is weakly decreasing in a_t^I so that b_t^I is also concave.

Finally, suppose $a_t^I > 0$ and $\theta = 0$ is optimal. Then

$$c_t'(\theta) \ge \lim_{\theta \to 0} b_t^d\left(a_t^I/\theta\right) - \frac{a_t^I}{\theta} b_t^{d\prime}\left(a_t^I/\theta\right) = \lim_{a \to \infty} b_t^d(a) - ab_t^{d\prime}(a) = \lim_{a \to \infty} f(a),$$

and this equation holds for all $a_t^I > 0$. Hence $\theta_t = 0$ is optimal for all a_t^I. ■

Proof of Proposition 3. The optimality condition (24) for Equation (8), together with $b_t^{d\prime} \ge -1$, implies that for all a_t^I,

$$c_t'(\theta_t) \le b_t^d\left(a_t^d\right) + a_t^d \le b_t^d\left(a_t^1\right) + a_t^1.$$

Now suppose cash dividends are paid, $d_t > 0$. Then $a_t^d > a_t^1$. If this is the case, $b_t^{d\prime}(a_t^d) = -1$, and from Equation (24),

$$c_t'(\theta_t) = b_t^d\left(a_t^1\right) + a_t^1.$$

Thus, $\theta_t = \bar{\theta}_t$ unless c_t is linear on the interval $[\theta_t, \bar{\theta}_t]$, in which case any scale in the interval is optimal. But in that case our selection rule of choosing the smallest optimal scale implies $\theta_t = \bar{\theta}_t$. ■

Proof of Theorem 1. Immediate from Proposition 2 and Proposition 3. ■

Proof of Proposition 4. Equation (4) computes b^d from b^e, Equation (8) computes b^I from b^d, Equation (11) computes b^y from b^I, and Equation (12) computes b^e (in the prior period) from b^y. Finally, Equation (1) relates expected growth to actual growth. ■

Proof of Theorem 2. From Figure 5, starting from a_s^I, the continuation payoff of the agent evolves according to transformations that are monotone conditional on any state of the world regarding randomizations. Thus, $\tilde{\theta}_t$ is FOSD increasing in a_s^I. Finally, from Equation (1), the conditional distribution of θ_s is increasing $\tilde{\theta}_s$ in an MLRP sense. Since a_s^I is monotone in θ_s, this proves the result. ■

Proof of Proposition 5. From (PK-1), the mean of any feasible policy $\hat{a}_t^I$ is fixed. From (IC-1),

$$\hat{a}_t^I(0) \le \hat{a}_t^I(Y_t) - \lambda Y_t$$

so that, by taking expectations $\hat{a}_t^I(0) \le a_t^y - \lambda\mu_t$. Thus, if there exists a feasible policy, the proposed policy a_t^I satisfies (AF-1) and is feasible.

Now define $g(y) = \hat{a}_t^I(y) - a_t^I(y) = \hat{a}_t^I(y) - \lambda y - a_t^y + \lambda\mu_t$. Then $E[g(Y_t)] = 0$, and since $\hat{a}_t^I$ satisfies (IC-1), $g(y)$ is weakly increasing. Therefore, because b_t^I is concave,

$$\begin{aligned} E\left[b_t^I\left(\hat{a}_t^I(Y_t)\right)\right] &= E\left[b_t^I\left(a_t^I(Y_t) + g(Y_t)\right)\right] \\ &\le E\left[b_t^I\left(a_t^I(Y_t)\right) + b_t^{I\prime}\left(a_t^I(Y_t)\right)g(Y_t)\right] \\ &= E\left[b_t^I\left(a_t^I(Y_t)\right)\right] + cov\left(b_t^{I\prime}\left(a_t^I(Y_t)\right), g(Y_t)\right) \\ &\le E\left[b_t^I\left(a_t^I(Y_t)\right)\right]. \end{aligned}$$

Finally, concavity of b_t^y follows since $b_t^I(a_t^y + \lambda(Y_t - \mu_t))$ is concave in a_t^y and expectation is a positive linear transformation. Indeed, we have $b_t^{y\prime}(a_t^y) = E[b_t^{I\prime}(a_t^y + \lambda(Y_t - \mu_t))]$, implying that b_t^y is renegotiation-proof ($b_t^{y\prime} \le 0$) if b_t^I is renegotiation-proof. ■

Proof of Proposition 6. Suppose the agent is compensated through the dividends on a share λ of the firm's equity. The firm pays dividends using cash flows Y_t in excess of existing debt D_t. This is consistent with the agent's payments from the optimal contract (see Figure 5 and Proposition 5), if and only if

$$\begin{aligned} \lambda(Y_t - D_t)^+ &= \theta_t\left(a_t^d - a_t^1\right)^+ = \left(a_t^I - \theta_t a_t^1\right)^+ \\ &= \left[(a_t^y + \lambda(Y_t - \mu_t)) - \bar{\theta}_t a_t^1\right]^+ \\ &= \lambda\left[Y_t - \left(\left(\bar{\theta}_t a_t^1 - a_t^y\right)/\lambda + \mu_t\right)\right]^+ \end{aligned}$$

where we use the fact, from Proposition 3, that growth is maximal if the agent receives a cash dividend. Therefore,

$$D_t = \left(\bar{\theta}_t a_t^1 - a_t^y\right)/\lambda + \mu_t. \tag{26}$$

Using Equation (26), we have

$$\lambda(D_t - Y_t) = \left(\bar{\theta}_t a_t^1 - a_t^y\right) + \lambda(\mu_t - Y_t) = \bar{\theta}_t a_t^1 - a_t^I = \bar{\theta}_t a_t^1 - \theta_t a_t^d, \tag{27}$$

and since $Y_t < D_t$ is equivalent to $a_t^d < a_t^1$, this implies

$$\lambda(D_t - Y_t)^+ = \bar{\theta}_t a_t^1 - \theta_t a_t^d = \bar{\theta}_t a_t^1 - \theta_t a_t^e = \bar{\theta}_t a_t^1 - \theta_t \delta_t a_{t+1}^y$$

or

$$a_{t+1}^y = \theta_t^{-1}\delta_t^{-1}\left[\bar{\theta}_t a_t^1 - \lambda(D_t - Y_t)^+\right]. \tag{28}$$

Then, using Equation (26) to compute D_{t+1} and Equation (28) to replace a_{t+1}^y,

$$
\begin{aligned}
\theta_t D_{t+1} &= \theta_t\left[\left(\bar{\theta}_{t+1}a^1_{t+1} - a^y_{t+1}\right)/\lambda + \mu_{t+1}\right] \\
&= \theta_t\left[\left(\bar{\theta}_{t+1}a^1_{t+1} - \theta_t^{-1}\delta_t^{-1}\left[\bar{\theta}_t a^1_t - \lambda(D_t - Y_t)^+\right]\right)/\lambda + \mu_{t+1}\right] \\
&= \delta_t^{-1}\left[(D_t - Y_t)^+ + \theta_t\delta_t\left(\bar{\theta}_{t+1}a^1_{t+1}/\lambda + \mu_{t+1}\right) - \bar{\theta}_t a^1_t/\lambda\right] \\
&= \delta_t^{-1}\left[(D_t - Y_t)^+ + m_t\theta_t - n_t\right]
\end{aligned}
$$

verifying Equation (14). Note also that from Equation (27), $a^I_t = \bar{\theta}_t a^1_t - \lambda(D_t - Y_t)$, so that growth is decreasing in $(D_t – Y_t)$. ■

Proof of Proposition 7. The concavity of b^I_t implies that a^I_t should be constant unless the IC constraint binds, which gives the result. ■

Proof of Proposition 8. Note that the probability of a high cash flow given mean μ is given by

$$\pi_t(\mu) = \frac{\mu - y^L_t}{\Delta Y_t}.$$

The agent chooses effort to maximize:

$$\pi_t(\mu)a^I_t\left(y^H_t\right) + (1 - \pi_t(\mu))a^I_t\left(y^L_t\right) - k_t(\mu) = a^I_t\left(y^L_t\right) + \pi_t(\mu)\Delta a_t - k_t(\mu)$$

which implies Equation (17). (As in the proof of Proposition 5, the concavity of b^I_t implies that if k_t has multiple sub-gradients at μ, the smallest is used.) Equation (18) then follows from the promise-keeping constraint. For monotonicity of the agent's payoff with respect to a^y_t, note that

$$\frac{\partial}{\partial a^y_t}a^I_t(y) = 1 + (y - \mu_t)\frac{\partial}{\partial a^y_t}k'_t(\mu_t)$$

from which the inequalities follow. The remaining results then follow from Theorem 1and Theorem 2. ■

Proof of Proposition 9. Concavity follows from the concavity of $b^I_t(\cdot \mid X_t)$ since if $\tilde{a}_j$ is a solution for a_j, then $\Sigma_j\lambda_j\tilde{a}_j$ is feasible for $\Sigma_j\lambda_j a_j$. From Equation (22), if $b^{I-}_t(a \mid X_{t-})$ is strictly decreasing at a, then by the concavity of $b^I_t(\cdot \mid X_t)$, a^I_t must be strictly increasing in a^{I-}_t. Finally, if $b^{I-}_t{}'(a \mid X_{t-})$ is not strictly decreasing at a, then there are multiple solutions to Equation (22). Let $\tilde{a}_0$ be the component-wise minimal solution and $\tilde{a}_1$ be the maximal solution. Since $b^I_t(\cdot \mid X_t)$ is concave, $\lambda\,\tilde{a}_1 + (1 - \lambda)\tilde{a}_0$ also solves Equation (22) and represents an increasing selection. ■

Proof of Proposition 10. Much of the proof follows immediately from the discussion in the text and the references therein. The only remaining issue is whether renegotiation-proofness prior to the choice of firm scale implies renegotiation-proofness after the scale decision is announced. This follows, however, since from Equation (25),

$$b^I_t{}'(a^I_t) = b^d_t{}'(a^d_t),$$

and so $a^d_t = a^r_t/\theta_t(a^r_t)$ maximizes b^d_t. Finally, if $b^d_t(a^d_t)$ is efficient, it must be derived from efficient points in β_t. ■

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