Option Coskewness and Capital Asset Pricing

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This article shows how the market coskewness model of Rubinstein (1973) and Kraus and Litzenberger (1976) is altered when a nonredundant call option is optimally traded. Owing to the option's nonredundancy, the economy's stochastic discount factor (SDF) depends not only on the market return and the square of the market return but also on the option return, the square of the option return, and the product of the market and option returns. This leads to an asset pricing model in which the expected return on any risky asset depends explicitly on the asset's coskewness with option returns. The empirical results show that the option coskewness model outperforms several competing benchmark models. Furthermore, option coskewness captures some of the same risks as the Fama–French factors small minus big (SMB) and high minus low (HML). These results suggest that the factors that drive the pricing of nonredundant options are also important for pricing risky equities. (*JEL* G11, G12, D61)

Although the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) is undoubtedly one of the most popular models of modern financial economics, the accumulated empirical evidence suggests that the model is likely misspecified. Historically, one method of addressing the CAPM's shortcomings has been to investigate the implications of higher order moments, such as coskewness and cokurtosis, for capital asset pricing.[1] The usual route is to assume the existence of a representative agent, expand the agent's marginal utility in a Taylor series, and then drop all of the higher order terms that the econometrician believes are unimportant for explaining risky returns. This procedure produces a stochastic discount factor (SDF) that is a polynomial function of the market's return and leads to an asset pricing model in which moments other than covariance are economically important. For example, a quadratic SDF produces a model in which coskewness with the market return is priced, whereas a cubic SDF produces a model in which cokurtosis with the market return is

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[1] For market coskewness, see Rubinstein (1973), Kraus and Litzenberger (1976, 1983), Friend and Westerfield (1980), Scott and Horvath (1980), Barone Adesi (1985), Sears and Wei (1985), Lim (1989), Harvey and Siddique (2000a,b), Barone Adesi, Gagliardini, and Urga (2004), Post, Levy, and van Vliet (2004) and Poti (2005). For market cokurtosis, see Fang and Lai (1997) and Dittmar (2002). For other approaches, see Bawa and Lindenberg (1977), Nantell and Price (1979), Chung, Johnson, and Schill (2006), Chapman (1997), Bansal and Viswanathan (1993), and Bansal, Hsieh, and Viswanathan (1993).

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priced. The former model has recently been analyzed in Harvey and Siddique (2000a,b), whereas the latter model is analyzed in Dittmar (2002).

One drawback to these types of models is that they ignore the possibility that derivative securities might be important for explaining risky equity returns. Because derivatives are in zero net supply, the representative agent's equilibrium demand for derivative securities must be equal to zero. In turn, the agent's marginal rate of substitution does not depend on any derivative security, and thus derivatives play no role in capital asset pricing.

The assumption that derivative securities are unimportant for asset pricing is at odds with the recent empirical evidence. As shown by Bakshi, Cao, and Chen (2000), Buraschi and Jackwerth (2001), Coval and Shumway (2001), and Jones (2001), options on the S&P 500 index appear to be nonredundant assets. For example, the prices of call options sometimes increase even when the underlying index decreases. If S&P options are truly nonredundant, then they should have an effect on the functional form of the SDF (Detemple and Selden, 1991) and they should be relevant for explaining the cross-section of risky returns. These statements are made precise by analyzing a particular class of economies in which options are optimally traded. The resulting asset pricing model, which involves coskewness with option returns, is a generalization of the market coskewness model of Rubinstein (1973) and Kraus and Litzenberger (1976).

Rather than assuming the existence of a representative agent, this article assumes that there are two or more agents with heterogeneous preferences. If all of the agents' preferences fall within a certain class, call options on the market portfolio are optimally traded. Furthermore, the equilibrium allocation is Pareto efficient. Thus, any agent's marginal rate of substitution, which depends on the traded options, can be used as the economy's SDF. Traditional risk variables such as latent volatility do not directly enter the SDF but instead are subsumed by the traded option returns and the higher order powers of the traded option returns.

The most basic version of the model involves two agents who trade a single call option contract. The SDF for this economy, denoted by ξ, is equal to

$$\xi = b_0 + b_1 R_M + b_2 R_O + b_3 R_M^2 + b_4 R_O^2 + b_5 R_M R_O,$$

where R_M is the market return and R_O is the traded option return. Because the SDF involves R_M and R_M^2, covariance and coskewness with the market return remain important elements for asset pricing. However, because the SDF also involves R_O, R_O^2, and $R_M R_O$, covariance with the option return and other option-related moments, such as coskewness with the option return, are also important for asset pricing. Due to this latter fact, the asset pricing model that is associated with the above SDF is referred to as the *option coskewness model*. In the special case of a zero demand for the call option, the coefficients b_2, b_4, and b_5 are all equal to zero, and the option coskewness model collapses to the market coskewness model.

The above expression for ξ clearly illustrates how this article differs from the existing literature. For example, Vanden (2004) derived an asset pricing model that involves traded option returns by imposing nonnegative wealth constraints on a group of agents that have linear risk tolerance. Vanden (2004) tested his model by considering a SDF that is a linear function of the traded option's return ($b_3 = b_4 = b_5 = 0$ in the above expression for ξ). Along different lines, Glosten and Jagannathan (1994) and Agarwal and Naik (2004) used traded option returns in a performance measurement context to explain managed portfolio returns. Their option-based models are consistent with a SDF that is linear in the market portfolio and a collection of options on the market portfolio. In contrast, this article does not rely on wealth constraints to motivate options trading, nor does it arbitrarily assume that options are the relevant pricing factors. Instead, a new class of preferences with nonlinear risk tolerance is introduced, and options trading is shown to be an endogenous feature of the equilibrium.[2] Because the optimal risk sharing rules in this article are piecewise linear, it is apparent that Vanden's (2004) conditions for piecewise linear risk sharing rules are sufficient but not necessary. In addition, the empirical tests in this article employ a SDF that is nonlinear in the traded option's return ($b_3,\ b_4$, and b_5 in the above expression for ξ are not restricted to be zero). Thus, this article sheds some light on the ability of option coskewness to explain risky asset returns. None of the existing literature covers this case.

The remainder of the article is organized as follows. In Section 1, I present a theory of option coskewness and show how and why option returns enter the SDF. The results of Section 1 are summarized in Theorem 1. In Section 2, I provide examples of the theory and derive testable implications for asset pricing. Then, in Section 3, I describe the data and present the empirical results, which include an analysis of the option coskewness model, the option cokurtosis model, and a variety of competing benchmark models (e.g., the CAPM, the Fama–French three-factor model, and the market coskewness and market cokurtosis models). Lastly, in Section 4, I offer concluding remarks and discuss some extensions of the framework that might be useful for future research.

1. Theoretical Results

A theory of option coskewness is derived by examining a class of economies in which options trading is optimal. The agents are indexed by

[2] The equilibrium demand for options is driven by the agents' nonlinear risk tolerance functions. A similar rationale for options trading can be found in Leland (1980).

$i = 1, 2, \ldots, I$, and there are two dates, date 0 and date 1. The ith agent's utility function is $U_i(W_i)$, where W_i is the ith agent's date 1 wealth. Both U_i and U_i' are continuous functions of W_i, and the agents are risk averse, that is, U_i' is strictly decreasing. The agents have homogeneous beliefs about the economy's aggregate date 1 payoff which is denoted by the random variable S. The ith agent's date 0 wealth is exogenously fixed and is given by W_{0i}.

Pareto optimality is used as a normative criterion to determine how the agents should share S. From Wilson (1968), a necessary and sufficient condition for Pareto optimality is the existence of a function $\phi(S)$, such that

$$U_i'(\hat{W}_i) = \lambda_i \phi(S) \quad \text{for all } i \tag{1}$$

$$\sum_{j=1}^{I} \hat{W}_j = S, \tag{2}$$

where $\hat{W}_i$ is the ith agent's optimal wealth and $\{\lambda_j\}_{j=1,2,\ldots,I}$ are nonnegative weights. Equation (1) says that all agents have identical marginal rates of substitution, whereas Equation (2) is the resource constraint. Since $\hat{W}_i$ is generally a function of S, the above expressions can be rewritten with $\hat{W}_i = g_i(S)$, where g_i is the ith agent's risk-sharing rule. The weights $\{\lambda_j\}_{j=1,2,\ldots,I}$ can then be identified by using the agents' budget constraints, that is, the weights must solve

$$E[\phi(S) g_i(S)] = W_{0i} \quad \text{for all } i. \tag{3}$$

As discussed in Pratt (2000), analytically solving (1) and (2) for $\{g_j\}_{j=1,2,\ldots,I}$ is usually impossible.[3] In contrast, this article introduces a new class of utility functions that not only admits a solution to (1) and (2) but produces sharing rules $\{g_j\}_{j=1,2,\ldots,I}$ that are spanned by S, a riskless payoff, and a finite number of options on S. Thus, the SDF $\phi(S)$ depends on the options, which has strong implications for asset pricing.

The agents' utility functions are constructed in the following manner. First, fix a positive integer N_i and a collection of positive constants $\{\eta_n^i\}_{n=1,2,\ldots,N_i}$ that satisfy the restriction $\eta_1^i < \eta_2^i < \cdots < \eta_{N_i}^i$. Each η_n^i is a switching level, that is, a wealth level at which the parameters of the ith agent's utility function switch to a different set of values. For the ith agent, there are N_i distinct switching levels, and these switching levels create $N_i + 1$ different wealth regions. Next, define the function $h_i(W_i)$ as

[3] Exceptions to this statement include the linear risk-sharing rules in Wilson (1968) and Amershi and Stoeckenius (1983) and the nonlinear rules in Wang (1996), Dumas (1989), and Vanden (2004).

$$h_i(W_i) = \begin{cases} \theta_1^i & , \quad W_i \le \eta_1^i \\ \theta_2^i & , \quad \eta_1^i < W_i \le \eta_2^i \\ \vdots & \qquad \vdots \\ \theta_{N_i+1}^i & , \quad \eta_{N_i}^i < W_i \end{cases}, \tag{4}$$

where $\theta_1^i, \theta_2^i, \ldots, \theta_{N_i+1}^i$ are positive constants. Lastly, let the ith agent have the risk tolerance function

$$-\frac{U_i'(W_i)}{U_i''(W_i)} = aW_i + h_i(W_i), \tag{5}$$

where the parameter a does not depend on i. In the special case of $\theta_1^i = \theta_2^i = \cdots = \theta_{N_i+1}^i$, the class of preferences in (5) reduces to the linear risk tolerance class with cautiousness parameter a. Thus, the preferences analyzed here generalize those that are analyzed by Wilson (1968), Cass and Stiglitz (1970), and Rubinstein (1974).

To recover U_i, fix a given wealth region $\eta_n^i < W_i \le \eta_{n+1}^i$. Next, invert (5) to get the agent's risk aversion function and integrate twice to get U_i for the given region. Repeat this task for all $N_i + 1$ regions. Lastly, choose the constants of integration such that U_i and U_i' are both continuous functions of W_i at each of the switching levels $\eta_1^i, \eta_2^i, \ldots, \eta_{N_i}^i$. Upon carrying out these instructions, the utility function is

$$U_i(W_i) = \begin{cases} \frac{1}{a-1}\left(aW_i + \theta_1^i\right)^{1-\frac{1}{a}} k_1^i + c_1^i & , \quad W_i \le \eta_1^i \\ \frac{1}{a-1}\left(aW_i + \theta_2^i\right)^{1-\frac{1}{a}} k_2^i + c_2^i & , \quad \eta_1^i < W_i \le \eta_2^i \\ \vdots & \qquad \vdots \\ \frac{1}{a-1}\left(aW_i + \theta_{N_i+1}^i\right)^{1-\frac{1}{a}} k_{N_i+1}^i + c_{N_i+1}^i & , \quad \eta_{N_i}^i < W_i \end{cases}, \tag{6}$$

where $\{k_n^i\}_{n=1,2,\ldots,N_i+1}$ and $\{c_n^i\}_{n=1,2,\ldots,N_i+1}$ are the constants of integration. Because there are $2N_i + 2$ constants to be determined but only $2N_i$ continuity conditions, it is assumed without loss of generality that $k_1^i = 1$ and $c_1^i = 0$. The remaining constants for $n = 1, 2, \ldots, N_i$ are determined recursively via the expressions

$$k_{n+1}^i = k_n^i \left(a\eta_n^i + \theta_n^i\right)^{-\frac{1}{a}} \left(a\eta_n^i + \theta_{n+1}^i\right)^{\frac{1}{a}} \tag{7}$$

$$c_{n+1}^i = c_n^i + \frac{1}{a-1}\left[k_n^i \left(a\eta_n^i + \theta_n^i\right)^{1-\frac{1}{a}} - k_{n+1}^i \left(a\eta_n^i + \theta_{n+1}^i\right)^{1-\frac{1}{a}}\right] \tag{8}$$

The utility function in (6) is a piecewise power-type utility function. Although it is more general than the standard power utility function, it does satisfy the properties of a well-behaved utility function. By construction, it is a continuous function with a continuous first derivative. It is strictly increasing, and it is strictly concave whenever it is well defined (e.g., $aW_i + \theta_n^i > 0$ within the nth wealth region). For $a > -1$, the third derivative of (6) in each region is positive, indicating a preference for positive skewness. Furthermore, if $a \geq 0$ and $\theta_{n+1}^i \geq \theta_n^i$ for all n, the ith agent's risk tolerance [absolute risk aversion (ARA)] is nondecreasing [nonincreasing] in wealth.

Table 1 presents two additional forms of the utility function in (6). The top panel of the table summarizes the piecewise exponential utility function. This utility function generalizes the standard exponential constant absolute risk aversion (CARA) utility function and is obtained by setting

Table 1
Piecewise exponential and piecewise logarithmic utility functions

Piecewise exponential: a = 0

$$U_i(W_i) = \begin{cases} -\theta_1^i \exp\left(-\frac{W_i}{\theta_1^i}\right) & , \quad W_i \leq \eta_1^i \\ -\theta_2^i k_2^i \exp\left(-\frac{W_i}{\theta_2^i}\right) + c_2^i & , \quad \eta_1^i < W_i \leq \eta_2^i \\ \vdots & \qquad \vdots \\ -\theta_{N_i}^i k_{N_i}^i \exp\left(-\frac{W_i}{\theta_{N_i}^i}\right) + c_{N_i}^i & , \quad \eta_{N_i-1}^i < W_i \leq \eta_{N_i}^i \\ -\theta_{N_i+1}^i k_{N_i+1}^i \exp\left(-\frac{W_i}{\theta_{N_i+1}^i}\right) + c_{N_i+1}^i & , \quad \eta_{N_i}^i < W_i \end{cases}$$

$$k_{n+1}^i = k_n^i \exp\left(\frac{\eta_n^i}{\theta_{n+1}^i} - \frac{\eta_n^i}{\theta_n^i}\right)$$

$$c_{n+1}^i = c_n^i + k_{n+1}^i \theta_{n+1}^i \exp\left(-\frac{\eta_n^i}{\theta_{n+1}^i}\right) - k_n^i \theta_n^i \exp\left(-\frac{\eta_n^i}{\theta_n^i}\right)$$

Piecewise logarithmic: a = 1

$$U_i(W_i) = \begin{cases} \log\left(W_i + \theta_1^i\right) & , \quad W_i \leq \eta_1^i \\ k_2^i \log\left(W_i + \theta_2^i\right) + c_2^i & , \quad \eta_1^i < W_i \leq \eta_2^i \\ \vdots & \qquad \vdots \\ k_{N_i}^i \log(W_i + \theta_{N_i}^i) + c_{N_i}^i & , \quad \eta_{N_i-1}^i < W_i \leq \eta_{N_i}^i \\ k_{N_i+1}^i \log\left(W_i + \theta_{N_i+1}^i\right) + c_{N_i+1}^i & , \quad \eta_{N_i}^i < W_i \end{cases}$$

$$k_{n+1}^i = k_n^i \left(\frac{\eta_n^i + \theta_{n+1}^i}{\eta_n^i + \theta_n^i}\right)$$

$$c_{n+1}^i = c_n^i + k_n^i \log\left(\eta_n^i + \theta_n^i\right) - k_{n+1}^i \log\left(\eta_n^i + \theta_{n+1}^i\right)$$

The table illustrates the piecewise exponential and piecewise logarithmic utility functions that are obtained by setting $a = 0$ and $a = 1$, respectively, in Equation (5) and integrating region by region. Without loss of generality, $k_1^i = 1$ and $c_1^i = 0$. The remaining constants of integration, which are given in the table, are chosen, such that $U_i(W_i)$ and $U_i{}'(W_i)$ are both continuous functions of W_i at each of the switching levels $\eta_1^i,\ \eta_2^i, \ldots, \eta_{N_i}^i$.

$a = 0$ in (5). The bottom panel of the table summarizes the piecewise logarithmic utility function. This function is obtained by setting $a = 1$ in (5). Of course, there are other possibilities of economic interest, for example, Equation (6) admits a piecewise cubic utility function $[a = -(1/2)]$ and a piecewise quadratic utility function $[a = -1]$.

Given that the utility functions have been fully characterized, it is now useful to return to the risk-sharing problem in Equations (1) and (2). The following theorem summarizes the solution to this risk-sharing problem for the class of economies in which every agent's utility function takes the form given in Equation (6).

Theorem 1. *Let all of the agents in the economy have identical beliefs. Furthermore, let all of the agents have utility functions of the form given in (6). If the parameter a is the same for every agent, then the Pareto optimal risk-sharing rules for the economy are piecewise linear functions of aggregate wealth.*

Proof. In the Appendix.

Note that the piecewise linear sharing rules are precisely the type of sharing rules that can be implemented in a securities market economy by trading options on the market portfolio. The strike prices of the traded options, which correspond to the "kink" points of the sharing rules, are endogenously determined and will generally depend on the weights $\{\lambda_j\}_{j=1,2,\ldots,I}$ as well as on the parameters of the agents' utility functions.[4] Furthermore, because every agent optimally holds options, the economy's SDF, and thus the prices of all assets, will depend on the traded options.

It is useful to point out that the above result is quite different from those found in the existing literature. For example, Ross (1976) showed how options can be used to complete an otherwise incomplete market, whereas Breeden and Litzenberger (1978) characterized an economy's Arrow–Debreu prices in terms of the prices of a complete set of options (i.e., a set of options whose strike prices correspond to every possible level of aggregate wealth). Thus, in both of these papers, the market structure is complete given the traded options. In contrast, the market structure in the above model may be incomplete given the traded options. For example, if S is continuously distributed, then the agents cannot possibly insure against all of the possible outcomes with only a single round of trading at date 0.

It is also useful to connect the above result with the spanning results of Carr and Madan (2001), Bakshi and Madan (2000), and Bakshi, Kapadia, and Madan (2003). These authors show how the optimal wealth

[4] In turn, because the weights $\{\lambda_j\}_{j=1,2,\ldots,I}$ are determined by Equation (3), the strike prices will generally depend on the initial wealth allocation $\{W_{0j}\}_{j=1,2,\ldots,I}$. Furthermore, for arbitrary weights, the aggregation property fails to hold (Brennan and Kraus, 1978), and thus the SDF will also depend on the initial wealth allocation.

of any agent with an increasing concave utility function can be spanned by the riskless asset, the market portfolio, and an infinite number of options on the market portfolio. In contrast, although the allocation in Theorem 1 generally involves only a finite number of options, it does rely on a specific class of utility functions. These results are connected because the risk tolerance in (5) can be used to closely approximate the risk tolerance of an arbitrary increasing concave utility function. This is easy to see by letting $a = 0$, in which case the risk tolerance in (5) is a step function of wealth. Because step functions can be used to approximate other functions, the risk tolerance that is associated with any increasing concave utility function can be approximated to any degree of precision by using (4) and (5) and letting N_i be large. In this case, because N_i directly influences the number of traded options in Theorem 1, a very large number of options will be required to span the agent's optimal allocation, just like in Carr and Madan (2001).

2. Examples and Implications

Several examples are presented to illustrate the implications of Theorem 1. In particular, it is shown how option coskewness and option cokurtosis enter the asset pricing relationship, it is shown how multiple options are potentially useful as explanatory variables, and it is shown how the option-based models preserve all of the economic intuition that is found in the simpler market-based models.

2.1 An example with one option

The first example involves two agents and shows how the standard cubic utility economy of Rubinstein (1973) can be generalized to accommodate a traded call option.[5] The first agent's utility function is obtained from (6) by letting $a = -(1/2)$ and letting $\theta_1^i = \theta_2^i = \cdots = \theta_{N_i+1}^i \equiv \rho > 0$. Thus, the first agent's utility is

$$U_1(W_1) = -\frac{2}{3}\left(\rho - \frac{1}{2}W_1\right)^3. \tag{9}$$

The second agent's utility function is obtained from (6) by letting $a = -(1/2)$ and $N_i = 1$, in which case there is a single switching level. Thus, the second agent's utility is

$$U_2(W_2) = \begin{cases} -\frac{2}{3}\left(\theta_1 - \frac{1}{2}W_2\right)^3 & , \quad W_2 \le \eta \\ -\frac{2}{3}\left(\theta_2 - \frac{1}{2}W_2\right)^3\left[\frac{\theta_1-(1/2)\eta}{\theta_2-(1/2)\eta}\right]^2 + \frac{2}{3}(\theta_2-\theta_1)[\theta_1-(1/2)\eta]^2 & , \quad W_2 > \eta \end{cases}, \tag{10}$$

[5] Also see Levy (1969) and Scott and Horvath (1980) for a discussion of the cubic utility function and its connection with skewness preference.

where the superscripts on η, θ_1, and θ_2 have been omitted for the ease of exposition. To ensure that the agent is risk averse over the appropriate range of wealth levels, the parameters are assumed to satisfy $\theta_2 > (\eta/2)$ and $\theta_1 > (\eta/2)$. Furthermore, to prevent either agent from becoming satiated in equilibrium, it is assumed that the aggregate wealth S is bounded above by $2(\rho + \theta_2)$.

If K, A, and B are given by

$$K = 2\rho + 2(\theta_1 B - \theta_2 A)(B - A)^{-1} \tag{11}$$

$$A = \sqrt{\lambda_1} + \sqrt{\lambda_2} \tag{12}$$

$$B = \sqrt{\lambda_1} + \sqrt{\lambda_2}\left(\theta_2 - \frac{1}{2}\eta\right)\left(\theta_1 - \frac{1}{2}\eta\right)^{-1}, \tag{13}$$

then the efficient allocation from solving (1) and (2) can be written as

$$\hat{W}_1 = \frac{\sqrt{\lambda_1}}{A} S + \left[\frac{2\rho A - 2\sqrt{\lambda_1}(\rho + \theta_1)}{A}\right] + \left(\frac{\sqrt{\lambda_1}}{B} - \frac{\sqrt{\lambda_1}}{A}\right)\max(0, S - K) \tag{14}$$

$$\hat{W}_2 = \frac{\sqrt{\lambda_2}}{A} S - \left[\frac{2\rho A - 2\sqrt{\lambda_1}(\rho + \theta_1)}{A}\right] - \left(\frac{\sqrt{\lambda_1}}{B} - \frac{\sqrt{\lambda_1}}{A}\right)\max(0, S - K) \tag{15}$$

Note that each agent's wealth includes a term that is proportional to S, a constant term, and a term that is proportional to $\max(0, S - K)$. Thus, if S is viewed as the payoff on the market portfolio, the efficient allocation can be implemented in a securities market economy by trading the market portfolio, a riskless bond, and a zero-net-supply call option on the market portfolio with strike price K.

A simple way to build some intuition for the allocation in (14) and (15) is to condition on W_2. If $W_2 \leq \eta$, then the second agent's risk tolerance is $\theta_1 - (1/2)W_2$ and is linear in wealth. Because the first agent has linear risk tolerance for all wealth levels, the results of Wilson (1968) suggest that the conditional sharing rules for $W_2 \leq \eta$ are linear in S. On the contrary, for the region $W_2 > \eta$, the second agent's risk tolerance is $\theta_2 - (1/2)W_2$ and is again linear in wealth. Thus, the conditional sharing rules for this region are also linear in S, but in general they have different slopes than the rules for the first region. Because the conditional sharing rules are linear within each region but have different slopes across regions, the unconditional sharing rules are piecewise linear, as shown in Theorem 1.

In addition, note that when $S = K$, the second agent's wealth is equal to η. This reveals that the option moves in-the-money (ITM) precisely as

$\hat{W}_2$ crosses η. Thus, the events $\{S \leq K\}$ and $\{\hat{W}_2 \leq \eta\}$ are equivalent, as are the events $\{S > K\}$ and $\{\hat{W}_2 > \eta\}$. As illustrated in (14) and (15), the agents invest in the stock market as if the second agent has a risk-tolerance level that is everywhere equal to $\theta_1 - (1/2)W_2$, but then they hedge with the call option just in case the second agent's risk tolerance actually turns out to be $\theta_2 - (1/2)W_2$. Of course, in the special case of $\theta_1 = \theta_2$, the second agent has linear risk tolerance, the agents' option demands are both equal to zero, and the above economy collapses to the standard cubic utility economy.

Given an initial wealth allocation, λ_1 and λ_2 (and thus K) are endogenously determined from (3). The relationship between K and the initial wealth allocation is worth exploring. Equation (11) can be rearranged to show that K depends on λ_1 and λ_2 only through the ratio λ_1/λ_2. Because the ratios λ_1/λ_2 and $W_{01}/(W_{01} + W_{02})$ move inversely with one another and K is decreasing in λ_1/λ_2, it can be concluded that K increases (decreases) as the second agent becomes relatively less (more) wealthy at date 0. Thus, an out-of-the-money (OTM) option will be traded if the second agent has sufficiently low relative initial wealth, whereas an ITM option will be traded if the second agent has sufficiently high relative initial wealth.

It is also worth noting that if λ_1/λ_2 is sufficiently high, a negative value for K is possible. In this case, assuming that the market portfolio exhibits limited liability (i.e., $S \geq 0$), the sharing rules in (14) and (15) are linear.[6] This can be explained by again noting that the events $\{S \leq K\}$ and $\{\hat{W}_2 \leq \eta\}$ are equivalent. Thus, a negative K implies that the event $\{\hat{W}_2 \leq \eta\}$ is impossible. Of course, a negative K only arises (for some parameter values) when the second agent's relative initial wealth is sufficiently high. In these cases, the high initial wealth level of the second agent leads to an allocation that satisfies $\hat{W}_2 > \eta$, that is, the second agent's optimal investment in the riskless asset guarantees that $\hat{W}_2 > \eta$. Thus, the piece of the agent's utility function for $W_2 \leq \eta$ becomes irrelevant, and efficient risk sharing is linear in S. To avoid this situation, it will always be assumed in the sequel that the event $\{S \leq K\}$ is possible, so that the option is useful for implementing the optimal allocation. The empirical results will then shed some light on the reasonableness of this assumption.

2.2 An exact option coskewness model

It is well known that the standard cubic utility economy generates an exact market coskewness pricing model (Rubinstein, 1973), that is, there

[6] The linear risk-sharing rules that arise when $K < 0$ do not contradict the existing literature since they hold only for *specific* weights λ_1 and λ_2. In contrast, the linear risk-sharing results of Wilson (1968) and Amershi and Stoeckenius (1983) hold for *all* (i.e., arbitrary) weights. Similarly, as shown in Theorem 1, the sharing rules in this article for arbitrary weights are of the piecewise linear class, of which the linear class is a special case.

is no need to truncate the equilibrium marginal rate of substitution. Similarly, the economy in Section 2.1 generates an exact option coskewness model. To see this, recall that the SDF is given by $\phi(S) = \lambda_i^{-1} U_i'(\hat{W}_i)$, which can be evaluated by using either (9) and (14) or (10) and (15). Upon doing so, the SDF is

$$\phi(S) = [d_0 - d_1 S - d_2 \max(0, S - K)]^2 \tag{16}$$

$$d_0 = (\rho + \theta_1) A^{-1} \tag{17}$$

$$d_1 = (2A)^{-1} \tag{18}$$

$$d_2 = (A - B)(2AB)^{-1}. \tag{19}$$

Because S takes values in the interval $(0, 2\rho + 2\theta_2)$, it is straightforward to verify that $\phi > 0, \phi' < 0$, and $\phi'' > 0$ for both of the regions $S \leq K$ and $S > K$. These properties are a direct result of the fact that both agents have utility functions that exhibit nonsatiation ($U' > 0$), risk aversion ($U'' < 0$), and a preference for positive skewness ($U''' > 0$). Thus, the option coskewness model is consistent with the preference restrictions that are advocated by Post, Levy, and van Vliet (2004) and Poti (2005).

By expanding the square in (16) and moving from payoffs to returns, the SDF can also be written as

$$\xi = b_0 + b_1 R_M + b_2 R_O + b_3 R_M^2 + b_4 R_O^2 + b_5 R_M R_O, \tag{20}$$

where R_M is the market's return, R_O is the traded option's return, and $b_0,\ b_1, \ldots, b_5$ are quantities that are known at date 0. To see how (16) leads to (20), expand the square in (16) and consider the term $-2d_0 d_2 \max(0, S - K)$. Given the initial option price $P_O = E[\max(0, S - K)\,\phi(S)]$, the realized option return is $R_O = \max(0, S - K)/P_O$ and the quantity b_2 in (20) is $-2d_0 d_2 P_O$. Thus, in a multiperiod setting, the quantities $b_0,\ b_1, \ldots,\ b_5$ will be time varying, as in the market coskewness model (Harvey and Siddique, 2000a,b).

Letting R_j denote the gross return on the jth risky asset and assuming the existence of a riskless asset, both agents agree that

$$1 = E(\xi R_j) = R_f^{-1} E(R_j) + Cov(\xi, R_j), \tag{21}$$

where R_f to the gross riskless return. Because the jth asset is arbitrary, (21) can be individually applied to R_M and R_O. This is obviously not a problem, because R_M and R_O represent the returns on traded assets. On the contrary, R_M^2, R_O^2, and $R_M R_O$ may not be spanned by the traded securities, and thus applying (21) to these quantities involves an additional assumption. Namely, it is assumed that if a market for these factors were opened, then their prices would coincide with the agents' marginal valuations. Due to the Pareto efficiency of the allocation, these valuations coincide. Furthermore, at these prices, both agents will have a zero demand for the unspanned factors, and the equilibrium prices of the other assets in the economy will not be affected. Applying (21) to each of the five factors gives a system of equations for the quantities $b_1,\ b_2, \ldots,\ b_5$. Solving for these quantities and substituting back into (21) then produces a multi-beta pricing model that involves option coskewness.

The above discussion implies that

$$E(R_j) - R_f = \omega_j^\top \Sigma^{-1} \mu, \tag{22}$$

where μ is the excess return column vector defined as

$$\mu = \begin{bmatrix} E(R_M) - R_f \\ E(R_O) - R_f \\ E\left(R_M^2\right) - R_f \\ E\left(R_O^2\right) - R_f \\ E(R_M R_O) - R_f \end{bmatrix}, \tag{23}$$

Σ is the covariance matrix for the vector $(R_M,\ R_O,\ R_M^2,\ R_O^2,\ R_M R_O)$, and $\omega_j^\top$ is a 1×5 row vector of covariances given by

$$\left[Cov(R_j, R_M) Cov(R_j, R_O) Cov(R_j, R_M^2) Cov(R_j, R_O^2) Cov(R_j, R_M R_O)\right]. \tag{24}$$

Equation (22) is the main testable implication of the theory. Interpreting $\beta_j^\top \equiv \omega_j^\top \Sigma^{-1}$ as a row vector of betas, (22) is a multi-beta pricing model that involves market coskewness *and* option coskewness. To see this, note that coskewness is a function of both the factor covariance and the squared-factor covariance. In particular, the option coskewness and the market coskewness for R_j satisfy

$$Cov(R_j, R_O^2) = Cos(R_j, R_O, R_O) + 2E(R_O) Cov(R_j, R_O) \tag{25}$$

$$Cov(R_j, R_M^2) = Cos(R_j, R_M, R_M) + 2E(R_M) Cov(R_j, R_M), \tag{26}$$

where $Cos(\cdot, \cdot, \cdot)$ is the coskewness operator (Rubinstein, 1973).

2.3 Expected returns and coskewness

The relationship between coskewness and expected returns is examined by analyzing (17)–(20). Since d_0 and d_1 are both positive, it is immediate that $b_1 < 0$. As in the CAPM, this implies that a risky asset's expected return is increasing in its covariance with the market return. Likewise, it is easy to show that $b_3 > 0$, because it is proportional to d_1^2. This fact implies that a risky asset's expected return is decreasing in its covariance with the squared-market return. All else equal, investors are willing to pay a higher price and accept a lower expected return for assets that are positively coskewed with the market. Both of these features are present in Rubinstein (1973) and Kraus and Litzenberger (1976). Thus, the option coskewness model preserves the implications for expected returns that are present in the market coskewness model.

Similarly, it is straightforward to show that $b_4 > 0$, because it is proportional to d_2^2. This implies that a risky asset's expected return is decreasing in its covariance with the squared-option return. All else equal, investors are willing to pay a higher price and accept a lower expected return for assets that are positively coskewed with the option return. Lastly, it is apparent that b_2 is proportional to $-d_0 d_2$, whereas b_5 is proportional to $d_1 d_2$. Because d_2 can be positive or negative depending on the relationship between the constants A and B, it is not possible to sign b_2 and b_5. The inability to sign b_2 is particularly interesting, because it leaves open the possibility that expected risky asset returns are *decreasing* in the asset's covariance with the call option return. This would be true even if the endogenous value for K resulted in the option being ITM and therefore highly correlated with the market.

The intuition behind this last result stems directly from the fact that the prices in the economy are partially determined by the nonredundant option. Along these lines, note that (16) can be interpreted as the marginal rate of substitution of a fictitious pricing agent whose portfolio includes d_1 units of the market portfolio and d_2 units of the call option. If $A > B$, the pricing agent holds a long call position and thus assigns a lower value to risky assets that positively covary with the option return. In this case, $b_2 < 0$ and $b_5 > 0$. On the contrary, if $A < B$, the pricing agent holds a short call position. In this case, $b_2 > 0$, and the agent assigns a *higher* value to risky assets that positively covary with the option return.

Because the expectation of the SDF is the price of a riskless bond, the above discussion has implications for the one-period interest rate. Since $b_1 < 0$ and $b_3 > 0$, the interest rate is increasing in $E(R_M)$ and is decreasing in $E(R_M^2)$. This is intuitive since, all else equal, the market is a more favorable investment relative to the riskless bond as $E(R_M)$ increases and $E(R_M^2)$ decreases.[7] Likewise, because $b_4 > 0$, the interest rate is decreasing in $E(R_O^2)$. All else equal, the call option is a more favorable investment relative to the riskless bond as $E(R_O^2)$ decreases. However, because b_2 and b_5 cannot be signed, the interest rate could be increasing or decreasing in $E(R_O)$ and $E(R_M R_O)$. To see the possible effects here, assume $\theta_2 > \theta_1$, so that the second agent's utility function in (10) displays a decrease in ARA at the switching level η. This is a plausible assumption

[7] Similar properties of the interest rate are present in the standard power utility models of Grossman and Shiller (1981) and Hansen and Singleton (1983).

given the experimental work of Levy (1994). In this case, $d_2 < 0$ and $b_2 > 0$, and the interest rate is *decreasing* in $E(R_O)$. The pricing agent holds a short option position when $d_2 < 0$, and thus an increase in $E(R_O)$ implies that the riskless bond is the more favorable investment.

2.4 Reduction to the market coskewness model

In the case of $\theta_1 = \theta_2$, the function in (10) reduces to a standard cubic utility function. In this case, no options are traded, $d_2 = 0$ in (19), and $b_2 = b_4 = b_5 = 0$ in (20). Thus, the pricing model in (22) reduces to the market coskewness model

$$E(R_j) - R_f = \Lambda_1\left[E(R_M) - R_f\right] + \Lambda_2\left[E(R_M^2) - R_f\right], \tag{27}$$

where

$$\Lambda_1 = \frac{Cov(R_M, R_j)\,Var(R_M^2) - Cov(R_M^2, R_j)\,Cov(R_M, R_M^2)}{Var(R_M)\,Var(R_M^2) - \left[Cov(R_M, R_M^2)\right]^2} \tag{28}$$

$$\Lambda_2 = \frac{Cov(R_M^2, R_j)\,Var(R_M) - Cov(R_M, R_j)\,Cov(R_M, R_M^2)}{Var(R_M)\,Var(R_M^2) - \left[Cov(R_M, R_M^2)\right]^2} \tag{29}$$

Except for notational differences,[8] (27)–(29) are identical to the model given in Equation (8) of Harvey and Siddique (2000a). Likewise, Expression (27) can be rearranged to solve for the loadings on $Cov(R_M, R_j)$ and $Cov(R_M^2, R_j)$ to produce an expression that is identical to Equation (7a) of Harvey and Siddique (2000a) and Equation (6) of Harvey and Siddique (2000b).

2.5 An approximate option coskewness model

As shown in Section 2.2, setting $a = -(1/2)$ produces an exact option coskewness model. For other values of a, an approximate option coskewness model can be obtained by truncating a Taylor series expansion of the equilibrium marginal rate of substitution.[9] This is illustrated by examining a two-agent economy in which the agents have exponential-type utility functions. The first agent's utility function is

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[8] The main difference is that $Cov(R_M, R_M^2)$ shows up in (28) and (29), whereas Harvey and Siddique (2000a,b) used $Skew(R_M)$ in their equations. This arises because the literature is inconsistent with regards to the definition of coskewness. Huang and Litzenberger (1988, p. 155) refered to $Cov(R_j, R_M^2)$ as the jth asset's coskewness with the market. Using this definition, $Cov(R_M, R_M^2)$ gives the market skewness, which is consistent with Harvey and Siddique's (2000a,b) notation. However, $Cov(R_M, R_M^2)$ is not equal to $E\{[R_M - E(R_M)]^3\}$, the traditional definition of skewness. Thus, (28) and (29) uses Rubinstein's (1973) definition of coskewness, that is, the jth asset's coskewness with the market is $E\{[R_j - E(R_j)][R_M - E(R_M)]^2\}$, which is clearly related to $Cov(R_j, R_M^2)$.

[9] Truncating the higher order terms of the Taylor series expansion in this fashion is consistent with Kraus and Litzenberger (1976), Harvey and Siddique (2000a,b), and Dittmar (2002).

$$U_1(W_1) = -\rho \exp\left(-\frac{W_1}{\rho}\right), \tag{30}$$

where $\rho > 0$ is a constant. Note that (30) is obtained from the top panel of Table 1 by letting $\theta_1^i = \theta_2^i = \cdots = \theta_{N_i+1}^i \equiv \rho$. The second agent's utility function is

$$U_2(W_2) = \begin{cases} -\theta_1 \exp\left(-\frac{W_2}{\theta_1}\right), & W_2 \leq \eta \\ -\theta_2 \exp\left[-\frac{W_2}{\theta_2} + \left(\frac{\eta}{\theta_2} - \frac{\eta}{\theta_1}\right)\right] + (\theta_2 - \theta_1)\exp\left(-\frac{\eta}{\theta_1}\right), & W_2 > \eta, \end{cases} \tag{31}$$

which is obtained from the top panel of Table 1 by letting $N_i = 1$. In this case, there is a single switching level given by η, and the superscripts on η, $\theta_1 > 0$, and $\theta_2 > 0$ have been omitted. In addition, unlike the cubic utility economy, no restriction is placed on the support of the distribution of S. If K is given by

$$K = \eta\theta_1^{-1}(\theta_1 + \rho) + \rho[\ln(\lambda_2) - \ln(\lambda_1)], \tag{32}$$

then the efficient allocation from solving (1) and (2) can be written as

$$\hat{W}_1 = \frac{\rho S}{\rho + \theta_1} - \left(\frac{\rho\theta_1}{\rho + \theta_1}\right)\ln\left(\frac{\lambda_1}{\lambda_2}\right) + \left(\frac{\rho}{\rho + \theta_2} - \frac{\rho}{\rho + \theta_1}\right)\max(0, S - K) \tag{33}$$

$$\hat{W}_2 = \frac{\theta_1 S}{\rho + \theta_1} + \left(\frac{\rho\theta_1}{\rho + \theta_1}\right)\ln\left(\frac{\lambda_1}{\lambda_2}\right) - \left(\frac{\rho}{\rho + \theta_2} - \frac{\rho}{\rho + \theta_1}\right)\max(0, S - K). \tag{34}$$

Furthermore, either agent's marginal rate of substitution can be evaluated at the optimal allocation to get the SDF

$$\phi(S) = h_1 \exp[-h_2 S - h_3 \max(0, S - K)] \tag{35}$$

$$h_1 = (\lambda_1)^{\frac{-\rho}{\rho+\theta_1}}(\lambda_2)^{\frac{-\theta_1}{\rho+\theta_1}} \tag{36}$$

$$h_2 = (\rho + \theta_1)^{-1} \tag{37}$$

$$h_3 = (\theta_1 - \theta_2)[(\rho + \theta_2)(\rho + \theta_1)]^{-1}. \tag{38}$$

Equation (35) can be expanded in a Taylor series to get

$$\phi(S) = h_1 e^{-h_2 K} \left\{ \begin{array}{c} 1 - [h_2(S-K) + h_3 \max(0, S-K)] \\ + \frac{1}{2}[h_2(S-K) + h_3 \max(0, S-K)]^2 + \cdots \end{array} \right\}. \tag{39}$$

Ignoring terms that are higher order than quadratic and moving from payoffs to returns, the approximate SDF, $\hat{\xi}$, can be written as

$$\hat{\xi} = b_0 + b_1 R_M + b_2 R_O + b_3 R_M^2 + b_4 R_O^2 + b_5 R_M R_O. \tag{40}$$

Using $\hat{\xi}$ in (21) then produces a multi-beta pricing model that is identical to (22)–(24).

2.6 Higher order truncations and the option cokurtosis model

Expression (40) is derived by ignoring the terms in (39) that are higher order than quadratic. Of course, there are other economically interesting truncations, such as ignoring the terms that are higher order than cubic. In this case, $\hat{\xi}$ is

$$\begin{aligned} \hat{\xi} = b_0 &+ b_1 R_M + b_2 R_O + b_3 R_M^2 + b_4 R_O^2 + b_5 R_M R_O \\ &+ b_6 R_M^3 + b_7 R_O^3 + b_8 R_M^2 R_O + b_9 R_M R_O^2, \end{aligned} \tag{41}$$

and there are nine factors that drive stock returns instead of five. Because R_M^3 and R_O^3 show up in the SDF, cokurtosis with the market return and cokurtosis with the option return are both relevant for asset pricing. This generalizes the market cokurtosis model that is analyzed by Fang and Lai (1997) and Dittmar (2002).

2.7 Option portfolios as factors

The economies in Sections 2.1 and 2.5 involve only a single option, because there is only a single wealth switching level (given by η). More generally, if the second agent has more than one switching level, or if there are multiple agents with one or more switching levels, then multiple options are required to implement the efficient allocation. In this case, the SDF will depend on a portfolio of options, and R_O in (40) must be interpreted as an option portfolio return.

To illustrate, consider an economy with three agents. The first agent has the utility function given in (30), whereas the second agent has the utility function given in (31). The third agent has a piecewise exponential utility function that is given by

$$U_3(W_3) = \begin{cases} -\nu_1 \exp\left(-\frac{W_3}{\nu_1}\right) & , \quad W_3 \leq \zeta \\ -\nu_2 \exp\left(-\frac{W_3}{\nu_2} + \frac{\zeta}{\nu_2} - \frac{\zeta}{\nu_1}\right) + (\nu_2 - \nu_1)\exp\left(-\frac{\zeta}{\nu_2}\right) & , \quad W_3 > \zeta. \end{cases} \tag{42}$$

Note that the agent's risk tolerance above and below the switching level ζ is equal to ν_2 and ν_1, respectively. If it is assumed that[10]

$$\lambda_2^{-1} \exp\left(-\frac{\eta}{\theta_1}\right) > \lambda_3^{-1} \exp\left(-\frac{\zeta}{\nu_1}\right), \tag{43}$$

then the optimal allocation can be implemented by opening a market for borrowing and lending, by trading the market portfolio with payoff S, and by trading *two* call options on the market portfolio with strike prices K_1 and K_2. These strike prices are

$$K_1 = \frac{\eta}{\theta_1}(\rho + \nu_1 + \theta_1) + \rho \log\left(\frac{\lambda_2}{\lambda_1}\right) + \nu_1 \log\left(\frac{\lambda_2}{\lambda_3}\right) \tag{44}$$

$$K_2 = K_1 + (\rho + \nu_1 + \theta_2)\left[\log\left(\frac{\lambda_3}{\lambda_2}\right) - \frac{\eta}{\theta_1} + \frac{\zeta}{\nu_1}\right]. \tag{45}$$

Given (43), it can be verified that $K_2 > K_1$. The SDF for this economy is

$$\phi(S) = w_1 \exp[-w_2 S - w_3 \max(0, S - K_1) - w_4 \max(0, S - K_2)], \tag{46}$$

where the constants w_1 through w_4 are

$$w_1 = (\lambda_1)^{\frac{-\rho}{\rho+\nu_1+\theta_1}} (\lambda_2)^{\frac{-\theta_1}{\rho+\nu_1+\theta_1}} (\lambda_3)^{\frac{-\nu_1}{\rho+\nu_1+\theta_1}} \tag{47}$$

$$w_2 = (\rho + \nu_1 + \theta_1)^{-1} \tag{48}$$

$$w_3 = (\theta_1 - \theta_2)[(\rho + \nu_1 + \theta_1)(\rho + \nu_1 + \theta_2)]^{-1} \tag{49}$$

$$w_4 = (\nu_1 - \nu_2)[(\rho + \nu_1 + \theta_2)(\rho + \nu_2 + \theta_2)]^{-1}. \tag{50}$$

Equation (46) can be interpreted as the marginal rate of substitution of a fictitious pricing agent, whose portfolio includes w_2 units of the market, w_3 units of the first call option, and w_4 units of the second call option. If the sign of $\theta_1 - \theta_2$ and $\nu_1 - \nu_2$ are the same, then the pricing agent is on the same side of the market (i.e., long or short) for both options. However, if $\theta_1 - \theta_2$ and $\nu_1 - \nu_2$ have different signs, the pricing agent holds a position in a bull-type spread ($\theta_1 - \theta_2 > 0,\ \nu_1 - \nu_2 < 0$) or a bear-type spread ($\theta_1 - \theta_2 < 0,\ \nu_1 - \nu_2 > 0$). Thus, the theory presented here

[10] As S increases, (43) implies that W_2 exceeds η before W_3 exceeds ζ. Thus, given K_1 and $K_2 > K_1$ in (44) and (45), the events $\{S > K_1\}$ and $\{\hat{W}_2 > \eta\}$ are equivalent, as are the events $\{S > K_2\}$ and $\{\hat{W}_3 > \zeta\}$. If the initial allocation of wealth is such that the inequality in (43) is reversed, then the optimal sharing rules are still piecewise linear, but the "kink" points (i.e., the options' strike prices) are different.

justifies the use of option portfolios, such as spreads, in the empirical analysis that follows.

3. Empirical Results

Although the multi-beta model in (22) is stated in terms of expected returns, it is possible to use realized returns in a time-series test of the theory. In particular, suppose that the return on the jth risky asset portfolio is

$$R_j = E(R_j) + \beta_j^\top \pi + e_j, \tag{51}$$

where π is the column vector of return innovations given by

$$\pi = \begin{bmatrix} R_M - E(R_M) \\ R_O - E(R_O) \\ R_M^2 - E(R_M^2) \\ R_O^2 - E(R_O^2) \\ R_M R_O - E(R_M R_O) \end{bmatrix}, \tag{52}$$

e_j is a mean zero random variable, and $\beta_j^\top \equiv \omega_j^\top \Sigma^{-1}$. Using (51) to substitute for $E(R_j)$ in (22) and adding an intercept term α_j produces the regression equation

$$R_j - R_f = \alpha_j + \beta_j^\top (\pi + \mu) + e_j, \tag{53}$$

where μ is from (23) and $(\pi + \mu)$ is a vector of excess factor returns. Note that the betas from estimating (53) are the sample counterparts of the theoretical betas. Thus, if the excess factor returns are important for explaining risky returns, then the estimated betas should be statistically significant. Furthermore, if the model is correctly specified, then the estimated intercepts should not be statistically significant.

3.1 The data

The data consist of monthly value-weighted returns from January 1993 through December 2000. The CRSP portfolio of NYSE/AMEX/NASDAQ equities is used as the market proxy, whereas the CRSP Treasury-bill return is used to calculate excess returns. Two sets of dependent variables are used—the 25 Fama–French portfolios and 30 industry portfolios.[11] The Fama–French portfolios are described in Fama and French (1992, 1993, 1996) and are formed by sorting the NYSE/AMEX/NASDAQ equities into groups based on size and book to market.

[11] The data for the dependent variables, as well as the data for the three risk factors SMB, HML, and UMD, are obtained from Ken French's Website.

Because size is used as a sorting variable, these portfolios are useful for detecting model misspecification (Berk, 1995). The 30 industry portfolios are formed by sorting the NYSE/AMEX/NASDAQ equities into groups based on standard industrial classification (SIC) codes. These portfolios are useful for examining cross-sectional industry differences in option coskewness. They also provide a robustness check, because there are factors other than size and book to market that appear to explain industry returns (Moskowitz and Grinblatt, 1999; Hou and Robinson, 2003).

Because there are no traded options on the market proxy, data for S&P 500 index options (SPX) are used. The reason for using the SPX contract is threefold. First, as shown by Bakshi, Cao, and Chen (2000) and Buraschi and Jackwerth (2001), SPX options appear to be nonredundant. It thus seems natural to use the SPX contract when testing the option coskewness model. Second, the SPX contract is actively traded. The average daily trading volume for SPX calls and puts during the sample period is 39,027 and 54,758 contracts, respectively.[12] Lastly, the SPX contract is European. This is important because an unconditional version of the model is tested and the joint distribution of the factors is more likely to be stationary with options that do not allow for early exercise. This is also why the sample period begins in 1993, which corresponds to the point in time that the Chicago Board Options Exchange (CBOE) began listing near-term monthly expirations in addition to the standard quarterly expirations.[13] This allows for the option returns to be constructed in a manner that precisely controls for expiration (i.e., volatility term structure) effects.

The daily SPX data are used to construct four time series of monthly option returns with different levels of moneyness. To form the option returns for month n, all options that expire in month $n+1$ are identified. This produces a collection of near expiration options for month n from which the appropriate moneyness levels are selected. A time series of near-the-money (NTM) call option returns is constructed by calculating the value-weighted return for a portfolio that contains the nearest in-the-money and the nearest out-of-the-money call options. Likewise, ITM and OTM call returns are constructed from the option contracts whose strike prices are approximately 4% below and 4% above the prevailing index level, respectively. Options with moneyness levels that lie outside the $\pm 4\%$ range are avoided, because they tend to be more thinly traded.

[12] As noted by Harvey and Whaley (1991), the S&P 100 index option contract (OEX) is the most actively traded option on a broad market index. Although the OEX average daily trading volume is almost twice that of the SPX contract during the sample period, the average open interest for the SPX is three times higher than that of the OEX. The OEX is also an American exercise style option, which complicates the empirical analysis.

[13] The three near-term expiration months for SPX were first listed by the CBOE in November 1992. Adding the last two months of 1992 to the sample period does not affect the paper's results.

Lastly, a long–short (LS) portfolio is constructed that is long the OTM option and short the ITM option. This is a time series of bear spread returns, and its use in testing the option coskewness model helps to capture the effects discussed in Section 2.7.

Three additional time series of monthly option returns are constructed to implement a collection of competing benchmark models. This helps to assess the option coskewness model's relative performance and provides a way of determining whether latent volatility, as opposed to the option coskewness factors, is useful for explaining equity returns. The three additional time series include

- a zero-beta straddle (STR) portfolio that is long the NTM call options and long an otherwise similar portfolio of NTM put options such that the sample covariance with the market is zero. The STR portfolio allows for the model of Coval and Shumway (2001, Table 8) to be implemented as one of the benchmark models;
- a delta-hedged gains (DHG) portfolio that is long the NTM call options and short the S&P 500 index. The hedge ratio (i.e., the delta) for DHG is calculated using the method in Bakshi and Kapadia (2003, Section 3). As these authors discuss, DHG are related to the volatility risk premium;
- a volatility contract (VOL) that is constructed using the method of Bakshi, Kapadia, and Madan (2003). The VOL contract, which is designed to mimic the squared S&P 500 return, is rebalanced monthly using all available out-of-the-money SPX calls and puts.

3.2 Factor sample statistics

Sample statistics for the explanatory factors are given in Table 2. As the top panel illustrates, the NTM, ITM, and OTM portfolios have means, standard deviations (SDs), and skewness levels that are decreasing in moneyness, whereas their market correlations are increasing in moneyness. The correlation between the squared-market and squared-option factors is also increasing in moneyness, but it is relatively small compared with the market correlations. For example, OTM's market correlation is 0.70, but the correlation between $R_M^2 - R_f$ and $R_O^2 - R_f$ for OTM is only 0.12. For NTM and ITM, the latter correlation is 0.26 and 0.44, respectively. Thus, the squared-option factor, whose loading represents option coskewness, is not merely a substitute for the squared-market factor that is commonly used to capture market coskewness (Harvey and Siddique, 2000a,b; Barone Adesi, Gagliardini, and Urga, 2004).

For a fixed moneyness level, Table 2 also summarizes considerable variation across the five factors' sample moments. Focusing on NTM,

Table 2
Factor sample statistics

Option portfolio	Factor	Mean	Standard deviation	Skewness	Excess kurtosis	Correlation with $R_M - R_f$	Correlation with $R_O - R_f$	Correlation with $R_M^2 - R_f$	Correlation with $R_O^2 - R_f$
Panel A									
NTM	$R_M - R_f$	0.0093	0.041	−1.012	2.266	1.00			
	$R_O - R_f$	0.0947	0.802	0.369	−0.971	0.81	1.00		
	$R_M^2 - R_f$	−0.0020	0.003	5.174	37.530	−0.18	0.08	1.00	
	$R_O^2 - R_f$	0.6486	0.721	2.059	5.302	0.18	0.54	0.26	1.00
	$R_M R_O - R_f$	0.0237	0.032	1.531	2.685	0.03	0.44	0.71	0.80
ITM	$R_M - R_f$	0.0093	0.041	−1.012	2.266	1.00			
	$R_O - R_f$	0.0827	0.543	−0.114	−0.838	0.86	1.00		
	$R_M^2 - R_f$	−0.0020	0.003	5.174	37.530	−0.18	−0.01	1.00	
	$R_O^2 - R_f$	0.2979	0.319	1.868	4.819	−0.02	0.19	0.44	1.00
	$R_M R_O - R_f$	0.0163	0.024	2.281	8.037	−0.12	0.15	0.81	0.83
OTM	$R_M - R_f$	0.0093	0.041	−1.012	2.266	1.00			
	$R_O - R_f$	0.1406	1.191	1.130	0.556	0.70	1.00		
	$R_M^2 - R_f$	−0.0020	0.003	5.174	37.530	−0.18	0.12	1.00	
	$R_O^2 - R_f$	1.4346	2.481	3.205	10.772	0.28	0.77	0.12	1.00
	$R_M R_O - R_f$	0.0320	0.050	1.768	3.254	0.18	0.71	0.54	0.81
LS	$R_M - R_f$	0.0093	0.041	−1.012	2.266	1.00			
	$R_O - R_f$	0.0540	0.788	1.555	2.410	0.46	1.00		
	$R_M^2 - R_f$	−0.0020	0.003	5.174	37.530	−0.18	0.19	1.00	
	$R_O^2 - R_f$	0.6206	1.346	3.960	17.686	0.28	0.77	0.02	1.00
	$R_M R_O - R_f$	0.0119	0.032	2.301	5.916	0.37	0.89	0.22	0.81

Option portfolio	Factor	Mean	Standard deviation	Skewness	Excess kurtosis	Correlation with $R_M - R_f$	Correlation with STR	Correlation with DHG	Correlation with VOL
Panel B									
STR	$R_O - R_f$	−0.0989	0.293	0.827	0.412	0.00	1.00		
DHG	$R_O - R_f$	0.0046	0.021	−0.942	3.288	−0.01	−0.89	1.00	
VOL	$R_O - R_f$	−0.2513	0.476	−0.902	8.733	−0.07	0.76	−0.64	1.00

NTM, near-the-money; ITM, 4% in-the-money; OTM, 4% out-of-the-money. Sample statistics for the five factors that comprise the option coskewness model are summarized in the top panel for four different option portfolios. The option portfolios that are used in the benchmark models are summarized in the bottom panel. The market factors are calculated from the monthly value-weighted return of the NYSE/AMEX/NASDAQ portfolio, whereas the option factors are calculated from the monthly value-weighted return of near expiration S&P 500 index options (SPX). The monthly average Treasury-bill return is 38 bp. Long–short LS is a bear spread that is long the OTM option and short the ITM option, STR is the zero-beta straddle portfolio of Coval and Shumway (2001), DHG is the delta-hedged gains portfolio of Bakshi and Kapadia (2003), and VOL is the volatility contract of Bakshi, Kapadia, and Madan (2003).

$R_O - R_f$ exhibits positive skewness and negative excess kurtosis, which is just the opposite of $R_M - R_f$. Although $R_M^2 - R_f$ has a small mean and SD, this factor by far has the largest skewness and excess kurtosis. In addition, the relationship between $R_O - R_f$ and $R_O^2 - R_f$ is rather interesting. The mean of $R_O^2 - R_f$ is higher than that of $R_O - R_f$, but the SD of $R_O^2 - R_f$ is the smaller of the two. A similar relationship holds for $R_M - R_f$ and $R_M R_O - R_f$. In a standard mean-variance world, these results would appear to be counterfactual. In this article, however, the higher order moments play an important role, and the standard mean-variance thinking must be refined.

Table 2 also summarizes the sample statistics for STR, DHG, and VOL. Although STR has a zero market correlation by construction, DHG and VOL also have low market correlations. This is consistent with the notion that all three portfolios are driven by a nonmarket factor, such as latent volatility. However, given the differences in the other sample moments, it is apparent that these portfolios are not perfect substitutes.[14]

3.3 Fama–French and industry sample statistics

Table 3 summarizes the sample statistics for the Fama–French portfolios and the industry portfolios. As the top panel illustrates, big stocks in the sample tend to have lower SDs, less skewness, less excess kurtosis, and lower CAPM betas relative to smaller stocks, whereas growth stocks tend to have higher SDs, more skewness, and higher CAPM betas relative to value stocks. The five right-hand columns show the univariate regression results for the option coskewness model factors. Although the beta on the squared-market return is significant for most of the portfolios, the significance level tends to favor small stocks. Furthermore, the beta is increasing in size. Both of these features can be found in Harvey and Siddique (2000a). For the last two columns, the beta on the squared-option return is significant for only bigger stocks, whereas the beta on the market-option product of returns tends to be significant for only smaller stocks. The magnitudes of these betas also vary as a function of portfolio size. This suggests that the higher order option-related factors, in addition to squared-market factor, might be useful for explaining some of the cross-sectional size effects that are so prevalent in equity returns.

The univariate betas in Table 3 can be combined with sample statistics in Table 2 to calculate each portfolio's market and option coskewness

[14] For example, although STR and VOL both have negative sample means, the magnitudes are quite different. The negative means arise because both portfolios are constructed using put options. Put options are useful for portfolio insurance and investors pay for this insurance with an expected holding period loss. The different magnitudes, however, arise because the put options that are used in STR and VOL have different moneyness levels. Thus, STR and VOL should have different sensitivities to a nonmarket factor such as latent volatility.

Table 3
Sample statistics for the Fama–French portfolios and the industry portfolios

Size	BTM	Mean	Standard deviation	Skewness	Excess kurtosis	β on $R_M - R_f$	β on $R_O - R_f$	β on $R_M^2 - R_f$	β on $R_O^2 - R_f$	β on $R_M R_O - R_f$
Panel A										
S	L	0.0006	0.092	0.603	3.976	1.48***	0.05***	−7.78**	−0.02	−0.87*
S	2	0.0103	0.079	0.985	6.247	1.14***	0.03**	−6.13**	−0.02	−0.75*
S	3	0.0106	0.057	0.469	4.602	0.93***	0.03***	−5.53**	−0.01	−0.57*
S	4	0.0124	0.049	0.050	4.575	0.81***	0.03***	−5.13**	−0.00	−0.44
S	H	0.0120	0.046	−0.779	2.920	0.76***	0.03***	−5.48***	−0.01	−0.44*
2	L	0.0051	0.079	0.010	2.074	1.43***	0.05***	−6.68**	−0.02	−0.69*
2	2	0.0068	0.053	−0.419	2.223	1.01***	0.04***	−4.93**	−0.00	−0.40
2	3	0.0088	0.042	−0.764	3.009	0.78***	0.03***	−4.46**	−0.00	−0.33
2	4	0.0103	0.044	−1.023	3.617	0.76***	0.03***	−5.08***	−0.00	−0.38
2	H	0.0079	0.042	−0.546	0.958	0.76***	0.03***	−4.28***	−0.00	−0.34
3	L	0.0051	0.075	−0.224	1.445	1.42***	0.05***	−6.54**	−0.02	−0.66*
3	2	0.0082	0.050	−0.765	3.387	0.99***	0.04***	−4.90*	0.01	−0.24
3	3	0.0096	0.043	−0.862	4.477	0.80***	0.04***	−4.69**	0.01	−0.18
3	4	0.0085	0.040	−0.543	3.500	0.65***	0.03***	−3.47*	0.01	−0.09
3	H	0.0119	0.036	−0.949	3.879	0.62***	0.03***	−3.55*	0.01	−0.10
4	L	0.0096	0.066	0.172	2.420	1.36***	0.06***	−4.86*	0.00	−0.36
4	2	0.0101	0.044	−0.900	4.716	0.90***	0.05***	−4.34*	0.02*	−0.12
4	3	0.0110	0.043	−0.565	4.888	0.80***	0.04***	−4.00*	0.02*	−0.03
4	4	0.0115	0.036	−0.465	1.941	0.65***	0.03***	−2.31	0.02**	0.00
4	H	0.0088	0.040	0.072	0.891	0.60***	0.04***	−1.13	0.02**	0.16
B	L	0.0106	0.046	−0.465	0.164	1.04***	0.06***	−2.95	0.02**	−0.00
B	2	0.0107	0.043	−0.724	2.324	0.88***	0.05***	−4.10*	0.02**	−0.01
B	3	0.0106	0.044	−0.791	3.031	0.83***	0.05***	−4.18*	0.02**	0.04
B	4	0.0103	0.043	−0.479	1.583	0.65***	0.04***	−2.90	0.02**	0.12
B	H	0.0100	0.038	−0.508	−0.275	0.44***	0.03***	−1.32	0.01**	0.09

Table 3
(continued)

Industry	Mean	Standard deviation	Skewness	Excess kurtosis	β on $R_M - R_f$	β on $R_O - R_f$	β on $R_M^2 - R_f$	β on $R_O^2 - R_f$	β on $R_M R_O - R_f$
Panel B									
Food products	0.0075	0.051	−0.296	1.330	0.53***	0.04***	−3.44*	0.01	−0.05
Beer, liquor	0.0105	0.045	0.006	−0.233	0.24	0.02**	−0.69	0.01	0.09
Tobacco products	0.0084	0.080	−0.154	1.273	0.45**	0.04***	−0.43	0.02**	0.23
Entertainment	0.0078	0.063	−0.456	1.159	1.14***	0.06***	−5.57**	0.03*	−0.07
Printing, publishing	0.0089	0.041	−0.240	0.196	0.75***	0.04***	−2.02	0.01*	0.02
Consumer goods	0.0142	0.045	−0.205	0.621	0.83***	0.05***	−1.38	0.01**	0.14
Clothing, apparel	−0.0006	0.061	−0.020	3.153	0.70***	0.05***	−4.02	0.03*	0.16
Healthcare, medical	0.0135	0.048	−0.423	0.240	0.64***	0.04***	−2.53	0.01	−0.03
Chemicals	0.0069	0.047	0.448	2.587	0.68***	0.04***	−2.55	0.02**	0.07
Textiles	−0.0074	0.050	−0.512	0.299	0.63***	0.03***	−4.12**	0.00	−0.18
Construction, materials	0.0050	0.047	−0.662	2.001	0.85***	0.05***	−3.56	0.02*	−0.00
Steel works	0.0057	0.061	0.440	3.343	1.08***	0.05***	−4.16*	0.01	−0.20
Fabricated products	0.0074	0.054	−0.582	1.887	1.01***	0.05***	−4.45*	0.02**	−0.04
Electrical equipment	0.0157	0.082	1.074	4.758	1.49***	0.06***	−3.91	0.01	−0.29
Autos, trucks	0.0072	0.058	−0.705	1.287	0.84***	0.05***	−5.25**	0.01	−0.09
Aircraft, ships, rail	0.0144	0.054	−0.675	1.989	0.72***	0.04***	−3.20	0.02**	0.09
Precious metals	−0.0031	0.066	−0.143	0.655	0.83***	0.04***	−4.19*	−0.00	−0.28
Coal	0.0041	0.071	0.433	2.038	0.64***	0.04***	−3.22*	0.00	−0.15
Petroleum, gas	0.0095	0.048	0.414	0.653	0.56***	0.03***	−2.52	0.02*	0.06
Utilities	0.0073	0.041	0.197	−0.131	0.16*	0.02***	1.69*	0.01*	0.29*
Telecommunications	0.0080	0.059	−0.612	1.818	1.17***	0.06***	−5.86**	−0.00	−0.44
Business services	0.0135	0.076	−0.218	1.367	1.53***	0.07***	−4.99*	0.00	−0.33
Business equipment	0.0189	0.076	−0.423	0.596	1.46***	0.07***	−4.34	0.02	−0.17
Office supplies	0.0049	0.052	0.345	2.453	0.75***	0.04***	−2.81	0.01	−0.01
Transportation	0.0056	0.049	−0.400	1.736	0.83***	0.05***	−3.63	0.02*	−0.01
Wholesale products	0.0075	0.043	−0.683	1.995	0.81***	0.04***	−4.09*	0.01	−0.13
Retail products	0.0081	0.052	−0.106	0.499	0.83***	0.05***	−1.48	0.02	0.15
Restaurants, hotels	0.0042	0.049	−0.533	1.725	0.74***	0.05***	−3.17	0.02	0.03
Financial	0.0125	0.050	−0.590	3.077	0.87***	0.05***	−4.70*	0.03**	0.04
Other	−0.0009	0.054	−0.618	1.643	0.72***	0.04***	−3.23	0.01	−0.08

Sample statistics are shown for the monthly excess returns of the Fama–French portfolios and the industry portfolios. The betas are from univariate regressions of each portfolio's excess return onto each explanatory factor. The market return is from the value-weighted NYSE/AMEX/NASDAQ portfolio, whereas the option return is from the value-weighted NTM call option portfolio. The monthly average Treasury-bill return is 38 bp. The table uses Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (three lags). ***, **, and * indicates statistical significance at the 1% level, at the 5% level, and at the 10% level, respectively.

level [see Equations (25) and (26)]. Upon doing this, it is revealed that the variation in the market (option) coskewness level across size and book to market is very similar to that given by the squared-market (squared-option) beta. For example, the correlation between the market (option) coskewness level and the squared-market (squared-option) beta is 0.96 (0.99). Thus, the univariate betas in Table 3 contain most of the relevant information for assessing the cross-sectional market and option coskewness levels of the Fama–French portfolios.

The bottom panel of Table 3 summarizes the sample statistics for the industry portfolios. The CAPM and option betas are significant for most portfolios, whereas the squared-market and squared-option betas are significant for about one half of the portfolios. As in Harvey and Siddique (2000a), the only portfolio with a positive squared-market beta is the utility portfolio. This is also the only portfolio for which the beta on the market-option product of returns is significant. Using (25) and (26), the calculated correlation between the market (option) coskewness level and the squared-market (squared-option) beta is 0.93 (0.94). Thus, as in the case of the Fama–French portfolios, the univariate industry betas contain most of the relevant information for assessing the cross-sectional industry differences in market and option coskewness levels.

3.4 Size, book to market, and the option-related betas

Table 4 summarizes the multivariate regression results for the NTM option coskewness model. As the table illustrates, the market beta remains significant for most portfolios even after controlling for the other factors. Although the other betas are significant for only about one half of the portfolios, the portfolios for which they are significant include the extreme size portfolios (i.e., the smallest and the biggest stocks) and the nonbig growth stocks. This suggests that the option coskewness factors are useful for capturing some of the size and book-to-market effects that are present in stock returns.

To see the economic impact of including the extra factors, the marginal return that is associated with a one SD move in each excess factor return is analyzed. For illustration, consider portfolio S-L (small growth stocks), which has the largest magnitude beta for each factor. If the excess option return increases by one SD, then the excess return on S-L, whose option beta is −0.06, decreases by 4.8%. Likewise, a one SD increase in the squared-market excess return produces a 3.7% increase, a one SD increase in the squared-option excess return produces a 3.6% increase, and a one SD increase in the market-option product of returns produces a 5.5% decrease. Thus, the magnitudes are similar across all of the extra factors. Because the betas in Table 4 vary systematically with size and book to

Table 4
Option coskewness model regressions for the Fama–French portfolios

Size	BTM	α	β on $R_M - R_f$	β on $R_O - R_f$	β on $R^2{}_M - R_f$	β on $R^2{}_O - R_f$	β on $R_M R_O - R_f$
Panel A							
S	L	0.0145	2.50***	−0.06**	12.41**	0.05**	−1.72**
S	2	0.0203	2.08***	−0.06**	9.63*	0.04**	−1.31*
S	3	0.0137	1.51***	−0.04**	5.34	0.03*	−0.88*
S	4	0.0077	1.32***	−0.03**	3.41	0.03**	−0.68
S	H	0.0090	1.05***	−0.02*	1.97	0.02**	−0.56
2	L	0.0094	2.20***	−0.05**	7.96**	0.03*	−1.00*
2	2	0.0011	1.50***	−0.03***	3.47	0.02*	−0.50
2	3	−0.0002	1.06***	−0.02***	0.48	0.01	−0.22
2	4	−0.0006	0.88***	−0.01	−1.49	0.01	−0.15
2	H	0.0085	0.88***	−0.01	2.64	0.02**	−0.62**
3	L	0.0096	2.09***	−0.04***	7.73***	0.03**	−0.99**
3	2	−0.0108	1.37***	−0.03***	−0.93	0.01	0.11
3	3	−0.0107*	0.91***	−0.01	−2.50	0.01	0.03
3	4	−0.0125*	0.60***	0.00	−5.06**	−0.01	0.52
3	H	−0.0047	0.54***	0.00	−3.46**	−0.00	0.21
4	L	0.0088	1.96***	−0.04**	7.26**	0.03**	−0.78*
4	2	−0.0083	0.94***	−0.01	−2.15	0.00	0.12
4	3	−0.0136*	0.70***	0.00	−4.83**	−0.00	0.41
4	4	−0.0058	0.64***	−0.00	−2.72	−0.00	0.32
4	H	−0.0043	0.35**	0.01	−2.09	−0.00	0.28
B	L	−0.0006	0.91***	0.01*	−0.59	−0.01	0.18
B	2	−0.0141**	0.57***	0.02**	−6.79***	−0.02***	0.76***
B	3	−0.0206**	0.63***	0.01	−7.97***	−0.02**	0.94***
B	4	−0.0151*	0.18	0.03**	−7.80***	−0.02**	0.82**
B	H	0.0020	−0.09	0.03***	−4.03	−0.02	0.41

Without SMB and HML	β on $R_M - R_f$	β on $R_O - R_f$	β on $R^2{}_M - R_f$	β on $R^2{}_O - R_f$	β on $R_M R_O - R_f$
Panel B					
Number of significant betas at 5%	23	13	10	10	6
χ^2 test (beta vector = 0)	4038.65 (<0.001)	96.41 (<0.001)	128.15 (<0.001)	122.70 (<0.001)	105.51 (<0.001)
With SMB and HML					
Panel C					
Number of significant betas at 5%	25	5	5	2	6
χ^2 test (beta vector = 0)	2428.26 (<0.001)	42.64 (0.015)	130.34 (<0.001)	109.85 (<0.001)	85.13 (<0.001)

The top panel summarizes the multivariate regression results for the option coskewness model using the Fama–French portfolios as the dependent variables. The option return is from the valued-weighted portfolio of near-the-money (NTM) call options, whereas the market return is the value-weighted portfolio of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{j,t} - R_{f,t} = \alpha_j + \beta_j^\top (\pi + \mu)_t + e_{j,t}$, which is Equation (53). The bottom panel evaluates the significance of the betas when SMB and HML are included as regressors. The χ^2 test is a joint test of the 25 betas for each factor. The test statistic is $\hat{\beta}^\top \hat{\Sigma}_{\beta}^{-1} \hat{\beta}$ and has 25 degrees of freedom. The p values are in parentheses. All of the tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (three lags). ***, **, and * indicates statistical significance at the 1% level, at the 5% level, and at the 10% level, respectively.

market, the economic impact of the factors also varies systematically with these variables.[15]

Due to measurement error in the betas, the economic impact can also be analyzed by holding the factor returns constant and letting the betas increase by one SD. For portfolio S-L, a one SD increase in the option beta (the squared-market beta, the squared-option beta, and the market-option beta) produces a change in S-L's excess return equal 0.2% (−0.9, 1.5, and 1.6%, respectively). In this case, the higher order option-related factors have a larger economic impact.

The bottom panel of Table 4 investigates the significance of the betas when the Fama–French factors small minus big (SMB) and high minus low (HML) are included as regressors. With SMB and HML, there is at least a 50% reduction in the number of significant betas for the option return and the two squared returns. However, there is no decrease in the number of significant betas for the market-option product of returns, and the hypothesis of a zero-beta vector for each factor is rejected both with and without SMB and HML. Thus, SMB and HML do not drive out the joint significance of the extra factor betas.

Although size is typically interpreted as a risk factor (Berk, 1995), the fact that SMB does not drive out the joint significance of the extra betas suggests that SMB imperfectly captures coskewness risk. Returning to Tables 2 and 3 and Equations (25) and (26), the option coskewness levels between small and big stocks differ by about a factor of three. Furthermore, as in Harvey and Siddique (2000a) and Barone Adesi, Gagliardini, and Urga (2004), market coskewness is negative for small stocks and is increasing in size. Thus, *both* market coskewness and option coskewness are correlated with size, suggesting that size may be proxying for omitted market and option coskewness. The bottom panel of Table 4, however, indicates that this proxy is imperfect.

3.5 Industries and the option-related betas

Table 5 summarizes the multivariate regression results for the NTM option coskewness model using the industry portfolios as the dependent variables. Unlike Table 4, the option-related betas in Table 5 are significant for far fewer portfolios, and the intercepts are significant for more portfolios. Although this is not a formal test of the model, which is the topic of the next section, it does suggest that the five-factor option coskewness model might be misspecified. There might also be time variation in the true factor loadings. As discussed in Fama and French (1997), industry loadings in the three-factor Fama–French model exhibit strong

[15] The systematic variation in the betas is easy to see by calculating the average beta for each size and book-to-market category. For example, small stocks on average have a large market beta, a large beta on the two-squared returns, and a small beta on the other two factors. Big stocks, on average, have the opposite characteristics.

Table 5
Option coskewness model regressions for the industry portfolios

Industry	α	β on $R_M - R_f$	β on $R_O - R_f$	β on $R_M^2 - R_f$	β on $R_O^2 - R_f$	β on $R_M R_O - R_f$
Panel A						
Food products	−0.0154*	−0.02	0.03**	−9.40***	−0.03**	0.89*
Beer, liquor	−0.0052	−0.06	0.02	−5.99*	−0.02	0.79
Tobacco products	−0.0066	−0.27	0.04*	−5.71	−0.01	0.43
Entertainment	−0.0117	1.11***	−0.00	−1.67	0.00	0.11
Printing, publishing	−0.0022	0.68***	0.01	−1.55	−0.01	0.39
Consumer goods	−0.0012	0.53***	0.02**	−4.42**	−0.03***	0.89***
Clothing, apparel	−0.0467***	0.48*	0.01	−11.72***	−0.02	1.31***
Healthcare, medical	−0.0021	0.36	0.02	−4.38	−0.01	0.40
Chemicals	−0.0026	0.28	0.02***	−1.84	−0.00	0.06
Textiles	−0.0249***	0.47**	0.01	−5.72**	−0.02	0.55
Construction, materials	−0.0143*	0.55***	0.02*	−4.68**	−0.01	0.52*
Steel works	−0.0046	1.25***	−0.01	0.18	−0.00	0.08
Fabricated products	−0.0107	1.01***	−0.00	−0.94	0.01	0.01
Electrical equipment	0.0181	2.34***	−0.05**	10.58*	0.04*	−0.91
Autos, trucks	−0.0212	0.59**	0.01	−8.50*	−0.03	0.96
Aircraft, ships, rail	−0.0199*	0.57*	0.01	−8.80**	−0.03*	1.16**
Precious metals	−0.0027	0.72**	0.01	0.60	−0.00	−0.18
Coal	−0.0032	0.40	0.01	−2.59	−0.01	0.10
Petroleum, gas	−0.0105	0.39	0.01	−4.63	−0.01	0.45
Utilities	−0.0027	−0.13	0.02*	−3.70	−0.02	0.71
Telecommunications	0.0133*	1.18***	0.00	3.54	0.01	−0.59
Business services	0.0141	1.91***	−0.02*	5.76**	0.01	−0.42
Business equipment	0.0216**	1.85***	−0.02	8.98***	0.04**	−1.00**
Office supplies	−0.0096	0.53**	0.01	−2.90	−0.01	0.32
Transportation	−0.0097	0.46**	0.02**	−3.70*	−0.01	0.28
Wholesale products	−0.0148*	0.85***	−0.01	−4.42*	−0.01	0.51
Retail products	−0.0085	0.66***	0.01	−3.38	−0.02*	0.79**
Restaurants, hotels	−0.0089	0.13	0.04***	−5.44*	−0.02*	0.55
Financial	−0.0241***	0.56***	0.01	−9.88***	−0.02**	1.04***
Other	−0.0195**	0.39	0.02*	−5.98**	−0.02	0.58

Without SMB and HML	β on $R_M - R_f$	β on $R_O - R_f$	β on $R_M^2 - R_f$	β on $R_O^2 - R_f$	β on $R_M R_O - R_f$
Panel B					
Number of significant betas at 5%	18	6	10	4	6
χ^2 test (beta vector = 0)	3705.63	155.96	202.03	78.28	138.10
	(< 0.001)	< (0.001)	(< 0.001)	(< 0.001)	(< 0.001)
With SMB and HML					
Panel C					
Number of significant betas at 5%	24	2	5	2	2
χ^2 test (beta vector = 0)	3534.87	125.11	112.59	57.31	139.74
	(< 0.001)	(< 0.001)	(< 0.001)	(0.002)	(< 0.001)

The multivariate regression results are shown for the option coskewness model using the industry portfolios as the dependent variables. The option return is from the value-weighted portfolio of near-the-money (NTM) call options, whereas the market return is the value-weighted portfolio of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{j,t} - R_{f,t} = \alpha_j + \beta_j^\top (\pi + \mu)_t + e_{j,t}$, which is Equation (53). The bottom panel evaluates the significance of the betas when SMB and HML are included as regressors. The χ^2 test statistic is $\hat{\beta}^\top \hat{\Sigma}_\beta^{-1} \hat{\beta}$, which has 30 degrees of freedom and is a joint test of the 30 betas for each factor. The p values are in parentheses. All of the tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (three lags). ***, **, and * statistically significant at the 1% level, at the 5% level, and at the 10% level, respectively.

variation through time. Likewise, the option-related betas in Table 5 might not be very precise estimates of each industry's true loadings on the option-related factors. Although the option-related betas in Table 5 are not always individually significant, the bottom panel shows that they are jointly significant. Adding SMB and HML as regressors tends to decrease the number of individual betas that are significant at the 5% level, but the addition of these factors does not drive out the joint significance. As in Table 4, this suggests that SMB and HML are imperfect proxies for capturing coskewness risk.

To see the economic impact of the extra factors, consider the electrical equipment portfolio, which has some of the largest betas by magnitude. For a one SD increase in the option excess return (the squared-market excess return, the squared-option excess return, and the market-option excess return), the portfolio excess return changes by −4.0% (3.2, 2.9, and −2.9%, respectively). Thus, like the Fama–French portfolios, the magnitudes are similar across the extra factors. If instead the factor returns are held constant and the option beta (the squared-market beta, the squared-option beta, and the market-option beta) for the electrical equipment portfolio is increased by one SD, then the return impact is equal to 0.15% (−0.77, 1.17, and 1.32%, respectively). In this case, the higher order option-related factors have a slightly larger economic impact.

3.6 Model specification tests

The option coskewness model is formally evaluated by performing a collection of joint tests on the regression intercepts from Tables 4 and 5. The first test is the GRS F-test of Gibbons, Ross, and Shanken (1989). Although this test assumes homoskedasticity of the regression residuals, it has the advantage that the small sample distribution for the test statistic is known. The second test is a large sample χ^2 test that accommodates heteroskedasticity and autocorrelation of the residuals, as advocated by MacKinlay and Richardson (1991). The covariance matrix for this test is estimated using the technique of Newey and West (1987). The third test involves the HJ distance of Hansen and Jagannathan (1997), which is the least squares distance between a given candidate SDF and the set of SDFs that prices all assets correctly.[16]

As a basis for comparison, several competing benchmark models are also tested. The benchmark models include the CAPM, the market coskewness and market cokurtosis models, the Fama–French three-factor model, and a triad of models that use the variables STR, DHG, and VOL. Several of the benchmark models are nested within the option coskewness

[16] As shown by Jagannathan and Wang (1996), T times the squared HJ distance is asymptotically distributed as a weighted sum of independent χ^2 random variables, each with one degree of freedom. Jagannathan and Wang (1996) showed how to empirically estimate the weights as well as the p values for this statistic.

model. For example, the CAPM is obtained by using the SDF in (20) and imposing the restrictions $b_2 = b_3 = b_4 = b_5 = 0$, whereas the market coskewness model is obtained from (20) by setting $b_2 = b_4 = b_5 = 0$. Likewise, the STR, DHG, and VOL models are each obtained from (20) by setting $b_3 = b_4 = b_5 = 0$ and substituting the STR, DHG, and VOL portfolio returns, respectively, for R_O. On the contrary, the Fama–French model is not nested within (20) nor is the market cokurtosis model since it includes R_M^3 in the SDF.

Table 6 summarizes the joint test results. For the 25 Fama–French portfolios, the GRS test rejects the benchmark models at traditional significance levels but does not reject the NTM option coskewness model ($p = 0.126$). Although the χ^2 test rejects all of the models, the option coskewness model's HJ distance is lower than all but one of the competing models. For the industry portfolios, the GRS test rejects several of the benchmark models at the 5% level while not rejecting the NTM option coskewness model ($p = 0.351$). The χ^2 test again rejects all of the models, but the NTM model's HJ distance measure is more than 10% lower than any of the benchmark models.[17]

Adding SMB and HML to the option coskewness model has the effect of increasing both the GRS and the χ^2 statistics, thus leading to a stronger rejection of the model. This effect is particularly acute for the GRS test of the Fama–French portfolios, because the NTM model is not rejected in the absence of SMB and HML. Because the GRS and χ^2 statistics are scaled by the residual covariance matrix, these effects might be because of lower standard errors rather than an increase in the estimated intercepts (Chapman, 1997). The squared HJ distance does not suffer from this problem, because it relies on the same weighting matrix across all models. As summarized in Table 6, adding SMB and HML to the NTM option coskewness model produces a squared HJ distance that is more than 13% lower than that of the NTM model alone. Thus, although size is an imperfect proxy for omitted coskewness (Tables 4 and 5), coskewness does not drive out the ability of size to explain the data.

The bottom panel of Table 6 shows the average alpha, the average absolute alpha, and the average root mean-square regression error (RMSE). The one feature of the Fama–French model that stands out is the fact that its RMSE is about 50% lower than that of any other model for the Fama–French portfolios and is about 10% lower than that of any other model for the industry portfolios. Thus, the

[17] As discussed in Ahn and Gadarowski (2004) and Kan and Zhou (2004), the asymptotic HJ distance test tends to over-reject, especially when the sample size is small relative to the number of dependent variables, as in this article. However, as Kan and Zhou (2004) also showed, the probability of rejection is decreasing in the number of factors. Thus, the HJ p value for the option coskewness model might be overstated relative to those of the benchmark models.

Table 6
Joint test results for the regression intercepts

Model	Factors	25 Fama–French portfolios			30 industry portfolios		
		GRS F-test	χ^2 test	Squared HJ distance	GRS F-test	χ^2 test	Squared HJ distance
Panel A							
CAPM	1	2.76	94.49	54.32	1.60	64.24	48.30
		(0.001)	(< 0.001)	(0.016)	(0.058)	(< 0.001)	(0.021)
Market coskewness	2	2.49	123.64	54.17	2.27	124.20	45.70
		(0.002)	(< 0.001)	(0.004)	(0.003)	(< 0.001)	(0.023)
Market cokurtosis	3	2.38	147.85	50.29	1.35	59.93	43.15
		(0.003)	(< 0.001)	(0.009)	(0.156)	(0.001)	(0.020)
Fama–French	3	17.08	960.73	51.48	2.02	57.64	40.50
		(< 0.001)	(< 0.001)	(0.005)	(0.010)	(0.002)	(0.005)
Straddle (STR)	2	2.80	96.36	53.27	1.78	57.18	48.19
		(< 0.001)	(< 0.001)	(0.004)	(0.027)	(0.002)	(0.022)
Delta-hedged gains (DHG)	2	2.87	93.15	53.85	1.76	59.15	48.26
		(< 0.001)	(< 0.001)	(0.006)	(0.029)	(0.001)	(0.021)
Volatility contract (VOL)	2	2.57	126.82	53.45	1.71	70.73	48.30
		(0.001)	(< 0.001)	(0.004)	(0.036)	(< 0.001)	(0.016)
Option coskewness NTM	5	1.43	168.22	50.67	1.12	178.59	34.89
		(0.126)	(< 0.001)	(< 0.001)	(0.351)	(< 0.001)	(0.092)
Option coskewness NTM + SMB and HML	7	2.65	215.76	43.68	1.13	264.38	27.07
		(0.001)	(< 0.001)	(0.021)	(0.339)	(< 0.001)	(0.116)

Table 6
(continued)

Model	Average alpha	Average absolute alpha	Average RMSE	Average alpha	Average absolute alpha	Average RMSE
Panel B						
CAPM	0.0009	0.0038	0.035	−0.0001	0.0042	0.043
Market coskewness	−0.0014	0.0034	0.034	−0.0007	0.0054	0.043
Market cokurtosis	−0.0051	0.0123	0.034	−0.0084	0.0230	0.042
Fama–French	0.0027	0.0030	0.018	−0.0013	0.0045	0.037
Straddle (STR)	−0.0002	0.0045	0.034	0.0003	0.0045	0.043
Delta-hedged gains (DHG)	0.0000	0.0042	0.034	0.0001	0.0043	0.043
Volatility contract (VOL)	−0.0010	0.0045	0.034	0.0003	0.0050	0.043
Option coskewness NTM	−0.0007	0.0091	0.032	−0.0079	0.0124	0.041
Option coskewness NTM + SMB and HML	−0.0033	0.0054	0.017	−0.0061	0.0093	0.036

The top panel presents the joint test results of the regression intercepts. The intercepts are obtained by estimating the regression $R_{j,t} - R_{f,t} = \alpha_j + \beta_j^\top f_t + e_{j,t}$ where the factors f_t depend on the model. The dependent variables are the 25 Fama–French portfolios (first three columns) and the 30 industry portfolios (last three columns). The option return is from the value-weighted portfolio of near-the-money (NTM) call options, whereas the market return is the value-weighted portfolio of NYSE/AMEX/NASDAQ equities. The p values for all tests are in parentheses. The Gibbons, Ross, and Shanken (1989) F-statistic has $(N, T - N - k)$ degrees of freedom where N, T, and k are the number of portfolios, observations, and factors, respectively. The χ^2 test statistic is $\hat{\alpha}^\top \hat{\Sigma}_\alpha^{-1} \hat{\alpha}$, has N degrees of freedom, and is robust to heteroskedasticity and autocorrelation (three lags). The reported value for the squared Hansen-Jagannathan distance is multiplied by T and has a p value that is calculated using the method of Jagannathan and Wang (1996). The bottom panel shows the average alpha, the average absolute alpha, and the average root mean-square regression error (RMSE).

Fama–French model explains a much larger proportion of the cross-sectional variation in risky asset returns. Adding SMB and HML to the NTM option coskewness model only slightly improves the RMSE relative to the Fama–French model. Without SMB and HML, the option coskewness model is only about 5–6% better than the remaining benchmark models when the RMSE is used as a ranking criterion.

3.7 Alternative option factors

If the option coskewness model is instead implemented with the OTM or ITM option portfolio rather than the NTM portfolio, then several striking features are revealed. First, for both sets of dependent variables, the p values of the GRS statistic are decreasing in the moneyness of the option returns. This suggests that OTM option returns are a better explanatory variable than ITM option returns.[18] Second, although the χ^2 test rejects the OTM and ITM versions of the model, the values of the χ^2 statistic again indicate that the models can be ranked according to moneyness. A similar phenomenon shows up in the squared HJ distance—the lowest HJ distance among all of the models, including the benchmark models, belongs to the OTM option coskewness model. The squared HJ distance times the sample size T for the OTM model is 49.46 for the Fama–French portfolios and 29.79 for the industry portfolios. The latter number implies a HJ distance of 0.56, which is comparable with the lowest HJ distance that is reported by Dittmar (2002).

The above moneyness phenomenon is corroborated by the fact that the average absolute alpha of the option coskewness model is increasing in the moneyness of the option returns. For the Fama–French portfolios, the average absolute alpha for the ITM (OTM) option coskewness model is 117 bp (57 bp), whereas for the industry portfolios the average absolute alpha for ITM (OTM) is 180 bp (88 bp). The NTM average absolute alpha lies strictly between these values for each set of dependent variables. In addition, the LS option coskewness model has an average absolute alpha of 26 bp for the Fama–French portfolios and 60 bp for the industry portfolios. These are lower than those of the OTM model, suggesting that option spreads might be useful as explanatory variables.

A comparison of the average sample returns and the predicted model returns for the Fama–French portfolios reveals that the CAPM, the market coskewness model, and the STR, DHG, and VOL models tend to fare the worst. These models exhibit a negative relationship between actual and predicted returns that in each case is statistically significant at the 5% level. On the contrary, the market cokurtosis model and the four

[18] If the OTM, NTM, and ITM option returns are individually regressed onto the market portfolio, OTM has the lowest adjusted R^2. A similar result holds if a squared-market factor and/or a cubed-market factor are added as regressors.

option coskewness models (NTM, OTM, ITM, and LS) all have positive-fitted slopes that are statistically significant at the 5% level. While the Fama–French model also has a positive-fitted slope, it is not statistically different than zero at the 10% level. Of all the models, the LS option coskewness model offers the largest fitted slope (equal to 0.34) and the smallest intercept (equal to 60 bp). Of course, a perfectly fitted model would have a slope of 1.0 and a zero intercept. In the case of the 30 industry portfolios, six of the models have positive-fitted slopes—the CAPM, the Fama–French model, the OTM option coskewness model, and the STR, DHG, and VOL models. However, only the CAPM, the STR model, and the DHG model have slopes that are significant at the 5% level. All of the remaining models have fitted slopes that are insignificant at the 10% level.

3.8 Option coskewness and common risk factors

Although the joint beta tests in Tables 4 and 5 indicate that size and book to market are not perfect proxies for coskewness, the HJ distances in Table 6 suggest that the inclusion of SMB and HML improves the overall fit. The relationship between the Fama–French factors and coskewness is formally investigated by regressing SMB and HML onto the five factors of the option coskewness model. As summarized in the top left portion of Table 7, SMB and HML have very different loadings. SMB has a negative beta on both the option return and the market-option product of returns and has a positive beta on the other factors. All five betas are significant, whereas the intercept is insignificant. HML has essentially the opposite profile with four significant betas and an insignificant, but economically large, intercept. In terms of the fit, the adjusted R^2 in the absence of the three option-related factors is 0.03 for SMB and 0.23 for HML. Adding the three option-related factors increases these values to 0.29 and 0.37, respectively. This suggests that at least some of the risks that are captured by SMB and HML are the risks that are normally associated with options.

As a robustness test, the momentum portfolio UMD is regressed onto the five option coskewness factors. This is important since, as Carhart (1997) had shown, the momentum effect of Jegadeesh and Titman (1993) helped to explain risky returns even after controlling for size and book to market. As Table 7 illustrates, the intercept for up minus down (UMD) is large (326 bp) and highly significant, which suggests that option coskewness does not explain momentum. Because momentum is largely an industry-related phenomenon (Moskowitz and Grinblatt, 1999), this might explain why the results in Table 5 are not better. Furthermore, although Amin, Coval, and Seyhun (2004) had shown that option prices are influenced by past stock returns, it appears that this influence, as captured by the option coskewness model, is unable to explain UMD.

Table 7
Option coskewness and option cokurtosis model regressions for SMB, HML, and UMD

	Option coskewness model			Option cokurtosis model		
Coefficient	SMB	HML	UMD	SMB	HML	UMD
Panel A						
α	0.0039	−0.0174	0.0326***	−0.0104	0.0211	−0.0032
β on $R_M - R_f$	0.91***	−1.29***	0.71	1.29***	−1.20*	0.42
β on $R_O - R_f$	−0.05***	0.05**	−0.03	−0.04*	0.01	0.01
β on $R_M^2 - R_f$	4.97*	−8.10*	9.17**	−4.98	−8.99	11.52
β on $R_O^2 - R_f$	0.03*	−0.04*	0.03*	0.06*	−0.11***	0.09*
β on $R_M R_O - R_f$	−0.73*	0.93	−1.19**	−0.23	1.97	−2.25
β on $R_M^3 - R_f$				33.58	−7.21	4.26
β on $R_O^3 - R_f$				−0.11***	0.12***	−0.11***
β on $R_M^2 R_O - R_f$				−27.74**	17.10	−16.54
β on $R_M R_O^2 - R_f$				2.73***	−2.49**	2.78*
Adjusted R^2						
Panel B						
All factors	0.29	0.37	0.17	0.47	0.51	0.29
No option factors	0.03	0.23	0.04	0.04	0.22	0.03
Joint beta tests						
Panel C						
F-test (option-related betas = 0)	12.94	7.99	5.77	13.57	10.01	6.89
	(< 0.001)	(< 0.001)	(< 0.001)	(< 0.001)	(< 0.001)	(< 0.001)
χ^2 test (option-related betas = 0)	8.92	3.88	4.54	124.18	37.07	135.11
	(0.030)	(0.275)	(0.208)	(< 0.001)	(< 0.001)	(< 0.001)
Joint intercept tests						
Panel D						
GRS F-test (alphas = 0)		2.65			0.59	
		(0.054)			(0.623)	
χ^2 test (alphas = 0)		7.50			2.52	
		(0.058)			(0.472)	

The top panel summarizes the regression results for the option coskewness and option cokurtosis models when SMB, HML, and UMD are used as dependent variables. The option return is from the value-weighted portfolio of near-the-money (NTM) call options, whereas the market return is from the NYSE/AMEX/NASDAQ value-weighted portfolio. The adjusted R^2 are shown for the unrestricted models and for a set of restricted models that have no option-related factors. The joint beta tests are performed for the three option-related betas (six option-related betas) of the option coskewness (option cokurtosis) model. The beta F-test for the option coskewness model has (3, 90) degrees of freedom (d.o.f.), whereas the χ^2 test, which uses the statistic is $\hat{\beta}^\top \hat{\Sigma}_\beta^{-1} \hat{\beta}$, has 3 d.o.f. The F-test and χ^2 test for the option cokurtosis model have (6, 86) and 6 d.o.f., respectively. The bottom panel summarizes the joint test results of the regression intercepts. The Gibbons, Ross, and Shanken (1989) F-statistic has $(N, T - N - k)$ d.o.f., where $N = 3$ is the number of portfolios, T is the number of time-series observations, and k is the number of factors. The χ^2 test statistic is $\hat{\alpha}^\top \hat{\Sigma}_\alpha^{-1} \hat{\alpha}$, which has $N = 3$ d.o.f. The p values are shown in parentheses. The regressions and all of the χ^2 tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (three lags). ***, significant at the 1% level; ** at the 5% level; * at the 10% level.

The top right columns of Table 7 summarize the results from regressing SMB, HML, and UMD onto the nine factors of the option cokurtosis model. Unlike the option coskewness results, none of the regression intercepts are statistically significant. Furthermore, it appears that both coskewness and cokurtosis with option returns, as captured by the betas on the squared and cubed option returns, respectively, are important for all three portfolios. In fact, *only* the higher order option-related factors are significant for stock market momentum. Furthermore, the increase in the adjusted R^2 when the six option-related factors are included is much larger than in the case of the option coskewness model.

The bottom two panels of Table 7 summarize the joint test results for the betas and the regression intercepts. For the beta tests, the three (six) option-related betas of the option coskewness (option cokurtosis) model for each regression are jointly tested for their statistical significance. In the case of the option cokurtosis model, all of the joint beta tests strongly reject the null of a zero option-related beta vector. In the case of the option coskewness model, the F-test strongly rejects the null of a zero-beta vector for each regression, but the χ^2 test produces a rejection at the 5% level only for the SMB regression ($p = 0.03$). This corroborates the results of Table 4 and indicates that the option coskewness factors are useful for explaining some of the size effect in returns. Lastly, the joint intercept tests indicate that neither the option coskewness model nor the option cokurtosis model are rejected at the 5% level.

3.9 Option coskewness versus option cokurtosis

Table 7 raises the possibility that the option cokurtosis model might offer a better overall fit to the data. This is investigated by returning to the Fama–French and industry portfolios and performing a Wald test on the estimated coefficients of the SDF. Although not reported in Table 7, the coefficients in (41) were estimated using a two-stage generalized method of moments (GMM) procedure. This allows for a "horse race" to be run between the option coskewness model (the null) and the option cokurtosis model. Using (41), the null hypothesis is $b_6 = b_7 = b_8 = b_9 = 0$, the test statistic is $\hat{b}^\top \hat{\Sigma}_b^{-1} \hat{b}$, which is asymptotically distributed χ^2 with four degrees of freedom, and $\hat{\Sigma}_b$ is a robust estimate of the covariance matrix (Newey and West, 1987). For the Fama–French portfolios, the LS version of the model produces a p value of 0.004, the NTM and ITM models both have p values that are slightly less than 0.045, and the OTM model has a p value of 0.573. Thus, moneyness again plays a key role in the analysis. The OTM option coskewness model is not rejected, but the ITM and NTM models are rejected in favor of the option cokurtosis model, which uses higher order factors. For the industry portfolios, the null is not rejected for any of the option portfolios. The smallest p value is 0.122, which occurs when OTM is used. Thus, although the option cokurtosis model in Table 7 is somewhat better at jointly explaining SMB, HML, and UMD, the horse race tends to favor the OTM option coskewness model.

The GMM estimation also provides some evidence on the appropriate signs of b_2 and b_5. The theory predicts that these two coefficients should have opposite signs. Furthermore, the sign of b_2 depends upon whether the pricing agent is long or short the traded option (see Section 2.3). For the eight option coskewness models that were estimated (OTM, ITM, NTM, and LS, each paired with the Fama–French portfolios and the industry portfolios), all produced $\hat{b}_2 > 0$ and $\hat{b}_5 < 0$. Although not all of the estimates were statistically significant, the opposite signs are consistent with the theory. Furthermore, the positive sign for $\hat{b}_2$ suggests that the pricing agent is short the traded call option, which is consistent with the second agent having a decrease in ARA at the switching level η. It is also consistent with the Fama–French portfolios in Table 3, for which there is a negative relationship between the average portfolio returns and the option beta and a positive relationship between the average portfolio returns and the beta for the market-option product of returns.

All eight GMM estimations also produced a negative value for $\hat{b}_1$, which is consistent with the theory. Combining this fact with $\hat{b}_2 > 0$ and $\hat{b}_5 < 0$, the GMM estimations are consistent with $\hat{d}_0 > 0, \hat{d}_1 > 0$, and $\hat{d}_2 < 0$ in Equations (17)–(19). Furthermore, because the estimated magnitude of $\hat{b}_1$ is larger than that of $\hat{b}_2$, it can be deduced that $\hat{d}_1 + \hat{d}_2 > 0$. Assuming as in Post, Levy, and van Vliet (2004) that the nonsatiation condition holds, it can therefore be concluded that the SDF in (16) satisfies $\phi' < 0$ and $\phi'' > 0$ for both regions, that is, the estimated SDF is consistent with risk-aversion and skewness preference. Thus, the estimated option coskewness model appears to satisfy the restrictions advocated by Post, Levy, and van Vliet (2004) and Poti (2005), at least for the sample used here.

4. Summary and Concluding Remarks

The market coskewness model of Rubinstein (1973) and Kraus and Litzenberger (1976) is generalized to the case in which a nonredundant option is traded. The equilibrium SDF depends not only on the market return and the square of the market return but also on the option return, the square of the option return, and the market-option product of returns. This implies that the expected return on any risky asset depends not only on its covariance and coskewness with the market return but also on its covariance and coskewness with the option return. Thus, the theory provides one way of integrating the derivatives pricing literature, which relies so heavily on risk factors such as stochastic volatility, with the equities pricing literature, which has instead focused on competing risk variables such as size and book to market.

Future research in this area could address at least four issues. First, although the empirical analysis contains some conditioning information due to the periodic rebalancing of the Fama–French portfolios, the above analysis is not a formal conditional test of the option coskewness model. The theory predicts that the coefficients of the SDF (and thus the betas) will vary through time, and it remains an open issue to test the model along these lines. Second, the choice of the moneyness level for the nonredundant option portfolio needs more attention. In the above analysis, a constant moneyness level is used within each nonredundant option portfolio. However, the theory indicates that the true moneyness level will vary through time due to the fluctuating wealth levels of the agents. A tractable way to address this issue is perhaps to rely on trading volume or open interest when choosing which options to include in the nonredundant option portfolio. Third, this article provides empirical evidence only on the option coskewness and option cokurtosis models. It remains an open issue to investigate what role, if any, the higher order option-related factors might have for explaining risky asset returns. This type of investigation would be in the spirit of Chung, Johnson, and Schill (2006) who recently performed a similar analysis for the higher order factors of the market model. Lastly, there appears to be some interest in explaining option returns themselves by using the higher order factors of the market model (O'Brien and Shackleton, 2005) and the Fama–French model (Santa-Clara and Saretto, 2004). Indeed, a correctly specified SDF should price *all* assets correctly, including options. Thus, to the extent that nonredundant options exist in the economy, the option coskewness and option cokurtosis models should be useful for extending this type of analysis. In summary, the theory presented here offers a justifiable reason for using options as explanatory variables.

Appendix

Proof of Theorem 1. Recall that if the ith agent's wealth W_i is in the nth wealth region, then W_i satisfies $\eta^i_{n-1} < W_i \leq \eta^i_n$. Of course, for a given level of S, different agents may have wealth levels that fall into different wealth regions (e.g., the first agent's wealth might be in his third region, the second agent's wealth might be in her fifth region). Thus, the notation is expanded to index the region by i. Specifically, the ith agent's wealth region is denoted by $n(i)$, and the parameters of the agent's utility function for this region are denoted by $\theta^i_{n(i)}$, $k^i_{n(i)}$, and $c^i_{n(i)}$. This allows the proof to be carried out in a general way for the set of wealth regions $\{n(j)\}_{j=1,2,\ldots,I}$ that corresponds to a given level of S.

Proceeding with the proof, the ith agent's wealth from (1) is $\hat{W}_i = U_i'^{-1}[\lambda_i\phi(S)]$, where $U_i'^{-1}$ is the inverse mapping of the agent's marginal utility function. Substituting this expression for $\hat{W}_i$ into (2) produces

$$\sum_{j=1}^{I} U_i'^{-1}[\lambda_j\phi(S)] = S. \tag{A1}$$

If the function $f(x)$ is defined as $f(x) \equiv \sum_{j=1}^{I} U_j'^{-1}(\lambda_j x)$, then $\phi(S) = f^{-1}(S)$, and the ith agent's sharing rule is

$$g_i(S) = U_i'^{-1}[\lambda_i f^{-1}(S)]. \tag{A2}$$

It will be shown that g_i is a piecewise linear function of S for all i. Using the piece of the ith agent's utility function that corresponds to the region $n(i)$, the ith agent's marginal utility is

$$U_i'(W_i) = k_{n(i)}^i \left[aW_i + \theta_{n(i)}^i\right]^{-\frac{1}{a}}$$

and thus the inverse marginal utility for this region is

$$U_i^{'-1}(z) = \frac{1}{a}\left(\frac{z}{k_{n(i)}^i}\right)^{-a} - \frac{\theta_{n(i)}^i}{a}. \tag{A3}$$

The function $f(x)$ is therefore

$$f(x) = \frac{x^{-a}}{a}\left[\sum_{j=1}^{I}\left(\frac{\lambda_j}{k_{n(j)}^j}\right)^{-a}\right] - \frac{1}{a}\sum_{j=1}^{I}\theta_{n(j)}^j \tag{A4}$$

and thus

$$\phi(S) = f^{-1}(S) = \left[\frac{aS + \sum_{j=1}^{I}\theta_{n(j)}^j}{\sum_{j=1}^{I}\left(\frac{\lambda_j}{k_{n(j)}^j}\right)^{-a}}\right]^{-\frac{1}{a}}. \tag{A5}$$

From (55) and (56), the sharing rule for the ith agent in region $n(i)$ is

$$g_i(S) = \frac{\left(\frac{\lambda_i}{k_{n(i)}^i}\right)^{-a} S + \frac{1}{a}\left[\left(\frac{\lambda_i}{k_{n(i)}^i}\right)^{-a}\sum_{j=1}^{I}\theta_{n(j)}^j - \theta_{n(i)}^i\sum_{j=1}^{I}\left(\frac{\lambda_j}{k_{n(j)}^j}\right)^{-a}\right]}{\sum_{j=1}^{I}\left(\frac{\lambda_j}{k_{n(j)}^j}\right)^{-a}}, \tag{A6}$$

which is linear in S for the collection $\{n(j)\}_{j=1,2,\ldots,I}$. Likewise, for the piecewise exponential utility function in Table 1,

$$\phi(S) = f^{-1}(S) = \exp\left\{\frac{-S + \sum_{j=1}^{I}\left[\theta_{n(j)}^j \log\left(\frac{k_{n(j)}^j}{\lambda_j}\right)\right]}{\sum_{j=1}^{I}\theta_{n(j)}^j}\right\} \tag{A7}$$

and

$$g_i(S) = \left(\frac{\theta_{n(i)}^i}{\sum_{j=1}^{I}\theta_{n(j)}^j}\right)\left\{S - \sum_{j=1}^{I}\left[\theta_{n(j)}^j \log\left(\frac{k_{n(j)}^j}{\lambda_j}\right)\right]\right\} + \theta_{n(i)}^i \log\left(\frac{k_{n(i)}^i}{\lambda_i}\right). \tag{A8}$$

For the piecewise logarithmic utility function in Table 1, $\phi(S)$ and $g_i(S)$ are obtained by setting $a = 1$ in expressions (A5) and (A6), respectively. Since U_i' is continuous for all i, the functions f and f^{-1} are also continuous. In turn, recalling that $\phi(S) = f^{-1}(S)$, each $g_i(S)$ is continuous. Thus, the sharing rules are continuous functions of S and are linear for a given collection of wealth regions. However, because the slopes of the sharing rules vary across wealth regions, each $g_i(S)$ is a piecewise linear function of S. This completes the proof.

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