Does Risk Seeking Drive Stock Prices? A Stochastic Dominance Analysis of Aggregate Investor Preferences and Beliefs

Thierry Post
Erasmus University Rotterdam

Haim Levy
The Hebrew University of Jerusalem

We use various stochastic dominance criteria that account for (local) risk seeking to analyze market portfolio efficiency relative to benchmark portfolios formed on market capitalization, book-to-market equity ratio and price momentum. Our results suggest that reverse S-shaped utility functions with risk aversion for losses and risk seeking for gains can explain stock returns. The results are also consistent with a reverse S-shaped pattern of subjective probability transformation. The low average yield on big caps, growth stocks, and past losers may reflect investors' twin desire for downside protection in bear markets and upside potential in bull markets.

The Traditional Mean-Variance Capital Asset Pricing Model (CAPM) by Sharpe (1964) and Lintner (1965) fares poorly in explaining observed cross-sectional average stock returns. Specifically, the value-weighted market portfolio of risky assets seems highly mean–variance inefficient relative to portfolios formed on factors like market capitalization, book-to-market equity (BE/ME) ratio (Fama and French, 1992), and momentum (Jagadeesh and Titman, 1993). Specifically, it seems possible to achieve a substantially higher mean and/or a substantially lower variance with portfolios with a higher weight of small caps, value stocks, and past winners.

One way to extend the mean–variance CAPM is by changing the maintained assumptions on the preferences and beliefs of investors. If we do not restrict the shape of the return distribution, then mean– variance CAPM is consistent with expected utility theory only if utility takes a quadratic form.[1] Extensions can be

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[1] Less restrictive assumptions are obtained if we do restrict the shape of the return distribution; see, for example, Berk (1997).

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obtained by using alternative classes of utility. For example, Kraus and Litzenberger (1976) and Harvey and Siddique (2000) assume that utility can be approximated using a third-order polynomial, and Dittmar (2002) uses a fourth-order polynomial. The higher-order polynomials fit stock return data better than the standard quadratic utility function. Still, the market portfolio remains inefficient and size, value, and momentum effects remain. Three potential problems of the extended models may help to explain these results:

- *Increasing marginal utility*. The extended models typically maintain the assumption that the utility of wealth is everywhere concave (i.e., marginal utility is diminishing) and hence investors are globally risk averse, that is, they never accept gambles with a negative expected outcome ("unfair" gambles). However, there is evidence that decision-makers are not globally risk averse, but rather they exhibit local risk-seeking behavior (that is, the utility function has convex segments) and hence they accept some unfair gambles. For example, Friedman and Savage (1948) and Markowitz (1952) argue that the willingness to purchase both insurance and lottery tickets implies that marginal utility is increasing over a range (Hartley and Farrell, 2002). In addition, psychologists led by Kahneman and Tversky (1979) find experimental evidence for local risk-seeking behavior. Finally, principal agent problems may explain why investors are risk seeking. Specifically, if the compensation scheme for investor agents is (locally) convex (e.g., because the reward for good performance exceeds the "penalty" for poor performance), then the agents may exhibit (local) risk seeking over the principals' wealth (Brennan, 1993).
- *Probability transformation*. Expected utility theory assumes that decision-makers use the true probability distribution. By contrast, experimental evidence suggests that decision-makers subjectively transform the true return distribution and use subjective decision weights that overweight or underweight the true probabilities. The most common pattern of probability transformation overweights small probabilities of large gains and losses, and underweights large and intermediate probabilities of small and intermediate gains and losses (Tversky and Kahneman, 1992). Decision-makers may accept unfair gambles because they estimate the expected outcome to be non-negative due to probability transformation. For investment choice, probability transformation may yield "abnormal" demand for investment assets that perform relatively well during extreme market scenarios that occur with a low frequency.
- *Specification error*. A difficulty in changing the preference and belief assumptions is the need to give parametric specifications of the functional form of the utility function, the probability distribution, and the transformation function. Unfortunately, economic theory gives

> minimal guidance for the functional specification, and there is a substantial risk of specification error. For example, a third-order polynomial utility function implies that investors care only about the first three central moments of the return distribution (mean, variance, and skewness). This approach is problematic if investors care about the higher-order central moments (for example, kurtosis) or about lower partial moments (for example, semivariance), which generally cannot be expressed in terms of the first three central moments. Another problem associated with low-order polynomials is the difficulty of imposing restrictions on the derivatives that apply globally. For example, we cannot impose non-satiation by restricting a quadratic polynomial to be monotone increasing and we cannot impose risk aversion by restricting a cubic polynomial to be globally concave (Levy, 1969).

To circumvent these problems, we may use criteria of Stochastic Dominance (SD; Levy, 1998). Attractively, SD criteria do not require a parametric specification, but rather they rely only on general preference and belief assumptions. SD is generally formulated in terms of expected utility theory and rational expectations. Still, the SD criteria are economically meaningful also for many non-expected utility theories (Starmer, 2000). Most notably, the criteria are invariant to some patterns of subjective probability transformation (Levy and Wiener, 1998) and the discussion in Section 1.3). Despite their theoretical appeal, applying SD criteria in practice is not a panacea. However, recently developed tests greatly reduce the computational burden and improve the statistical power relative to traditional SD tests (Post, 2003; Kuosmanen, 2004).

The SD literature involves a multitude of different criteria, associated with different sets of assumptions. The general First-order Stochastic Dominance (FSD) criterion assumes only non-satiation, that is, utility is monotone increasing. The Second-order Stochastic Dominance (SSD) criterion adds the assumption of global risk aversion, that is, utility is everywhere concave. Recently, a number of "intermediate" criteria have been developed on the basis of non-concave utility functions. Most notably, Levy (1998) developed Prospect Stochastic Dominance (PSD), which assumes an S-shaped utility function that is convex for losses and concave for gains. In addition, Levy and Levy (2002) developed Markowitz Stochastic Dominance (MSD), which assumes a reverse S-shaped utility function that is concave for losses and convex for gains.

In this article, we use the SSD, PSD, and MSD criteria to analyze investor behavior. To focus on the role of preferences and beliefs, we largely adhere to the remaining assumptions of the mean–variance CAPM: we use a single-period, portfolio-oriented model of a frictionless and competitive capital market. Within this model, we test whether the value-weighted market portfolio is efficient relative to benchmark portfolios formed on size, BE/ME, and momentum, under the SSD, PSD, and MSD criteria. For this purpose, we assume a simple data-generating process with a serially independent and identical distribution for the excess returns. Of course, there are good reasons to doubt our maintained assumptions and to believe that our results are affected by these assumptions in a non-trivial way. Still, we believe that our analysis can form the starting point for further research based on more general economic and statistical assumptions.

Our analysis raises two fundamental questions regarding efficiency of the market portfolio and, in addition, the methods for testing efficiency if investors are locally risk seeking:

- Again, we test efficiency of the market portfolio relative to various general classes of utility functions (concave, convex–concave and concave–convex). One problem is that under general utility functions, market portfolio efficiency is not necessarily a property of equilibrium: the efficient sets are not necessarily convex, and so even if individual investors hold efficient portfolios, the market, which is a weighted-average of all individual portfolios, will not necessarily be efficient. For example, Dybvig and Ross (1982) demonstrate that the SSD efficient set generally is not convex and hence the market portfolio may be inefficient even if individuals hold efficient portfolios. This problem arises also for the PSD and MSD efficient sets. This would seem to make a test of the efficiency of the market portfolio less interesting. However, given the large number of investors who appear to hold the market portfolio in the form of passive mutual funds and exchange traded funds that track broad value-weighted equity indexes, it is still interesting to ask what kind of utility functions could rationalize such behavior, in the face of attractive premiums offered by size, book-to-market, and momentum portfolios.
- We test efficiency by checking if the necessary first-order condition for portfolio optimization is satisfied for some admissible utility function. For concave utility functions, the first-order condition is necessary and sufficient for establishing a global optimum. Post (2003) developed a Linear Programming (LP) test for SSD efficiency that exploits this insight. However, for the generalized case that allows for local risk seeking, the first-order condition gives only a necessary condition. Specifically, portfolios that represent a local optimum will also satisfy the first-order condition. There exist various multivariate global optimization methods for locating the global optimum if the objective function is not concave (Horst and Pardalos, 1995). Unfortunately, these methods generally are computationally too demanding to be included in a SD test (where the objective function and the return distribution are not exactly known). For this reason, we use the first-order conditions only, and hence our PSD and MSD tests give necessary tests only. This

introduces the possibility of wrongly accepting PSD or MSD efficiency for an inefficient portfolio. However, the tests are still sufficient if we assume that investors can deviate only marginally from the market portfolio. This assumption seems valid if, for example, principal agent problems lead investment principals to impose investment constraints, so as to narrow the portfolio possibilities for investment agents.

The remainder of this article is structured as follows. Section 1 introduces a general SD efficiency criterion and it discusses the special cases used in our study. Section 2 discusses the issue of empirical testing. Specifically, we develop a general LP test for fitting SD efficiency criteria to empirical data, and we derive the asymptotic sampling distribution of the test results. This section effectively generalizes Post's (2003) treatment of SSD efficiency to our general SD efficiency criterion. We also discuss the use of bootstrapping to account for sampling error. Section 3 applies the methodology to test if the value-weighted market portfolio is SD efficient relative to benchmark portfolios based on size, BE/ME, and momentum. Finally, Section 4 gives concluding remarks and suggestions for further research. The Appendix gives the formal proofs for our theorems.

1. Stochastic Dominance Efficiency Criteria

1.1 General model

We consider a single-period, portfolio-based model of a competitive capital market that satisfies the following assumptions:

Assumption 1. *The investment universe consists of N assets, one of which is a riskless asset. Throughout the text, we index the assets by* $\mathrm{I} \equiv \{i\}_{i=1}^{N}$. *The excess returns* $\boldsymbol{x} \in \Re^N$ *are treated as random variables with a continuous joint cumulative distribution function (CDF)* $G : \Re^N \rightarrow [0, 1]$.[2] *Investors may diversify between the assets, and we will use* $\boldsymbol{\lambda} \in \Re^N$ *for a vector of portfolio weights. The portfolio possibilities are represented by the simplex* $\Lambda \equiv \{\boldsymbol{\lambda} \in \Re_+^N : \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\lambda} = 1\}$.[3]

Assumption 2. *Investors select investment portfolios* $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}$ *to maximize the expected value of a once directionally differentiable, strictly increasing utility function* $u{:}\Re \rightarrow \Re$ *that is defined over portfolio return* $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}$.[4] *Formally, the*

[2] Throughout the text, we will use $\Re^N$ for an N-dimensional Euclidean space, and $\Re_-^N$ and $\Re_+^N$ denote the negative and positive orthants. To distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars. Further, all vectors are column vectors and we use $\boldsymbol{x}^{\mathrm{T}}$ for the transpose of $\boldsymbol{x}$. Finally, $\mathbf{0}$ and $\boldsymbol{e}$ denote a zero vector and a unity vector with dimensions conforming to the rules of matrix algebra.

[3] By using the simplex Λ, we exclude short selling. Short selling typically is difficult to implement in practice due to margin requirements and explicit or implicit restrictions on short selling for institutional investors (e.g., Sharpe (1991) and Wang (1998)). Still, we may generalize our analysis to include (bounded) short selling. In fact, the analysis applies to any portfolio set that takes the form of a polytope (roughly speaking, a non-empty and closed set that is defined by linear restrictions) if we replace the set I with the set of extreme points of the polytope.

[4] We use a directional derivative to allow for kinked utility functions, including piecewise-linear utility functions.

investment problem is $\max_{\boldsymbol{\lambda}\in\Lambda} \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})d\mathrm{G}(\boldsymbol{x})$. *We represent all admissible utility functions by* $U_0 \equiv \{u{:}u'(x) \geq 1\ \forall x \in \Re\}$, *with* $u'(x) \equiv \lim_{\delta\downarrow 0} \frac{u(x+\delta)-u(x)}{\delta}$ *for the directional derivative or "marginal utility" at* $x\in\Re$.[5]

We may characterize different SD efficiency criteria by different classes of utility functions, associated with different sets of (conditional) linear restrictions on marginal utility. Formally, we will denote different classes of utility functions by

$$U(\Psi) = \{u\in U_0\colon u'(x) \geq u'(y)\ \forall (x,y)\in\Psi\}. \tag{1}$$

Here, $\Psi \subseteq \Re^2$ represents the restrictions that are placed on marginal utility (special cases are given below). Using this notation, we may define the following general SD efficiency criterion:[6]

Definition 1. *Portfolio* $\boldsymbol{\tau} \in \Lambda$ *is* $U(\Psi)$*–SD efficient if and only if it is optimal for some* $u \in U(\Psi)$, *that is,*

$$\min_{u\in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda}\in\Lambda} \left\{ \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})dG(\boldsymbol{x}) - \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})dG(\boldsymbol{x}) \right\} \right\} = 0. \tag{2}$$

Alternatively, portfolio $\boldsymbol{\tau}\in\Lambda$ is $U(\Psi)$–SD inefficient if and only if

$$\min_{u\in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda}\in\Lambda} \left\{ \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})dG(\boldsymbol{x}) - \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})dG(\boldsymbol{x}) \right\} \right\} > 0. \tag{3}$$

In this article, we test if the value-weighted market portfolio of risky assets, say $\boldsymbol{\mu}\in\Lambda$, is $U(\Psi)$–SD efficient under Assumptions 1 and 2.

As discussed in the introductory section, a potential problem is that under the general SSD, MSD, and PSD utility functions, market portfolio efficiency is not necessarily a property of equilibrium: the efficient sets are not necessarily convex, and so even if individual investors hold efficient portfolios, the market, which is a weighted average of all individual portfolios, will not necessarily be efficient. This would seem to make a test of market portfolio efficiency less interesting.[7] Nevertheless, such a test remains of interest because of *revealed*

[5] U_0 restricts marginal utility to exceed unity. This restriction effectively standardizes the test statistic $\xi(\boldsymbol{\tau},\Psi)$ (see Section 2) and forces it away from zero if the evaluated portfolio is inefficient. Still, the restriction is harmless because SD rules are invariant to strictly positive affine transformation, that is, $u \in U(\Psi) \Rightarrow bu\in U(\Psi)\ \forall b > 0$.

[6] We focus on a definition in terms of utility functions, because we analyze the role of preference assumptions. SD may be defined equivalently in terms of distribution functions or their quantiles [e.g., Levy (1992, 1998)]. In fact, the LP dual of test statistic (15) is formulated in terms of quantiles; associated with each primal variable β_t, $t \in \Theta$ is a dual restriction on the running mean $\Sigma_{s=1}^{t} \boldsymbol{x}_s^{\mathrm{T}}\boldsymbol{\tau}/t$. Post (2003) gives more details for the SSD case.

[7] In asset pricing theories, efficiency of the market portfolio generally follows from underlying assumptions on investor preferences and beliefs. For example, following Rubinstein (1974), efficiency of the market follows from assuming that the preferences and beliefs of the different investors are sufficiently similar. In this case, we may use a *representative agent* whose preferences and beliefs are an aggregate of the preferences and beliefs of the actual investors. In this paper, we do not take this route, because detailed preference and belief assumptions are not consistent with the SD approach of using minimal assumptions.

preferences. Specifically, our motivation for testing market efficiency lies in the popularity of passive mutual funds and exchange traded funds that track broad value-weighted equity indexes.[8] In other words, (some) investors reveal a preference for market indexes, and our objective is to rationalize their choice and to analyze their preferences and beliefs. Of course, we could directly analyze the efficiency of actual funds. Still, for the sake of data availability and comparability, we focus on the Fama and French market portfolio (see Section 3), which is used in many comparable studies. Further, many actual funds, including total market index funds based on the very broad Wilshire 5000 index (for example, the Vanguard Total Stock Market Index Fund) are likely to be very highly correlated with the Fama and French market portfolio.

1.2 Special cases

In our empirical analysis, we will use three different SD criteria based on three different sets of restrictions on marginal utility (represented by Ψ). The traditional criterion of *Second-order Stochastic Dominance* (SSD) assumes risk aversion for the entire domain of returns:

$$\Psi_{SSD} \equiv \left\{(x,y) \in \Re^2 : x \leq y\right\}. \tag{4}$$

Models of decision-making and equilibrium under uncertainty traditionally use utility functions from $U(\Psi_{SSD})$. However, as discussed in the introductory section, there is compelling evidence that many decision-makers are risk seeking over a range.

The psychological experiments by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) suggest that preferences are best described by an S-shaped function that is convex for losses and concave for gains. In recent years, Prospect Theory has attracted much attention as a framework for understanding investor behavior and for explaining financial market anomalies (Benartzi and Thaler, 1995, Barberis and Huang, 2001, Barberis, Huang, and Santos, 2001). Prospect Stochastic Dominance (Levy, 1998) assumes an S-shaped utility function with risk seeking for losses and risk aversion for gains:[9]

$$\Psi_{PSD} \equiv \left\{(x,y) \in \Re^2_- : x \geq y\right\} \cup \left\{(x,y) \in \Re^2_+ : x \leq y\right\}. \tag{5}$$

Apart from risk seeking for losses and risk aversion for gains, "loss aversion" is another key element of Prospect Theory. Loss aversion refers to the

[8] If a riskless asset is available, then a sufficient condition for Assumption 3 is that some investor holds a combination of the market portfolio and the riskless asset. Let $\boldsymbol{\mu}' = \boldsymbol{\mu}\kappa + (1-\kappa)\boldsymbol{\delta}_F$ for some $\kappa \in \langle 0, 1]$, with $\boldsymbol{\delta}_F$ to denote a coordinate vector of zeros with a unity value for the riskless asset $F \in \mathrm{I}$. If $u'$is the optimal solution for $u \in U(\Psi)$, then $\boldsymbol{\mu}$ is optimal relative to $v(x) \equiv u(\kappa x) \in \mathrm{U}(\Psi)$ and hence efficient.

[9] Gains and losses are typically measured relative to a subjective reference point. For simplicity, we set the reference point at zero. The use of excess returns implies that the nominal reference point effectively equals the riskless rate. However, a follow-up analysis demonstrates that the empirical results are not significantly affected by lowering the nominal reference point to zero, or by raising it to the average return on the market portfolio. If the reference point is not known, it can be included as an additional model variable (at the cost of a possible loss of power).

phenomenon that decision-makers are distinctly more sensitive to losses than to gains. This phenomenon introduces a kink in the utility function at the reference point distinguishing gains from losses. We use directionally differentiable utility functions that may exhibit kinks (Assumption 2). Hence, all SD criteria used in this study are consistent with loss aversion.

Other than considering utility functions that are concave over gains and convex over losses, we also, for completeness, consider utility functions that are convex over gains and concave over losses. Interestingly, Markowitz (1952) already suggested this type of utility function.[10] Markowitz Stochastic Dominance (Levy and Levy, 2002) assumes a reverse S-shaped utility function with risk aversion for losses and risk seeking for gains:

$$\Psi_{MSD} \equiv \left\{(x, y) \in \Re_-^2 : x \leq y\right\} \cup \left\{(x, y) \in \Re_+^2 : x \geq y\right\}. \tag{6}$$

1.3 Probability transformation

We have thus far formulated in terms of expected utility theory and rational expectations. As discussed in the introductory section, experimental evidence suggests that decision-makers subjectively transform the true probability distribution. Interestingly, SD rules can also be economically meaningful also for non-expected utility theories that allow for probability transformation. Such transformations are typically modeled by means of a strictly increasing transformation function f: $[0, 1] \to [0, 1]$, which is applied to the true cumulative probability distribution (Tversky and Kahneman, 1992). Applying such a transformation to the CDF of portfolio $\boldsymbol{\lambda} \in \Lambda$, that is, $H_{\boldsymbol{\lambda}}(x) \equiv \int_{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda} \leq x} dG(\boldsymbol{x})$, the original portfolio choice problem $\max_{\boldsymbol{\lambda} \in \Lambda} \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}) dG(\boldsymbol{x})$ becomes $\max_{\boldsymbol{\lambda} \in \Lambda} \int u(x) df(H_{\boldsymbol{\lambda}}(x))$. Probability transformation does not affect the SD efficiency classification if the transformation function has the same shape as the utility function. For example, Levy and Wiener (1998) demonstrate that the SSD criterion is not affected by transformations that are increasing and concave, and the PSD criterion is not affected by transformations that are increasing and convex over losses and increasing and concave over gains, that is, S-shaped transformations. By analogy, we can show that the MSD criterion is invariant to reverse S-shaped transformations that are increasing and concave over losses and increasing and convex over gains. To formalize this point, let Π_{MSD} denote the class of all strictly increasing transformation functions that are concave for losses and convex for gains, that is, $f'(p_1) \geq f'(p_2)$ $\forall (p_1, p_2) \in \Psi_{MSD}$. We can derive the following result:

[10] Markowitz (1952) actually proposed a utility function with two S-shaped segments. Such a utility function has a convex segment for 'large' losses and a convex segment for 'small and moderate' gains. Hence, MSD is consistent with Markowitz type utility only if all outcomes are 'small or moderate' gains or losses. This assumption seems reasonable for our application, because we compare well-diversified benchmark portfolios; for the full sample period (July 1963 to October 2001), the minimum monthly excess return for the market portfolio is −23.09 percent and the maximum excess return is 16.05 percent (see Table 1 and 2).

Theorem 1. *Portfolio $\boldsymbol{\tau} \in \Lambda$ is MSD efficient if*

$$\min_{u \in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda} \in \Lambda} \left\{ \int u(x) df(H_{\boldsymbol{\lambda}}(x)) - \int u(x) df(H_{\boldsymbol{\tau}}(x)) \right\} \right\} = 0, \tag{7}$$

for some $f \in \Pi_{MSD}$.
Alternatively, portfolio $\boldsymbol{\tau} \in \Lambda$ is MSD inefficient only if

$$\min_{u \in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda} \in \Lambda} \left\{ \int u(x) df(H_{\boldsymbol{\lambda}}(x)) - \int u(x) df(H_{\boldsymbol{\tau}}(x)) \right\} \right\} > 0, \tag{8}$$

for all $f \in \Pi_{MSD}$.

In case of probability transformation, we effectively test a joint hypothesis on preferences and beliefs. For example, rejection of PSD efficiency may reflect that investor preferences cannot be described by an S-shaped utility function. However, it may equally well reflect that investor beliefs cannot be described by an S-shaped transformation function. In this respect, PSD efficiency is *not* a test of Cumulative Prospect Theory (CPT; Tversky and Kahneman, 1992), because CPT uses a reverse S-shaped transformation function rather than an S-shaped transformation function. Similarly, acceptance of MSD efficiency may reflect that investor utility takes a reverse S-shape and/or that probability transformation is reverse S-shaped. In fact, MSD efficiency can be interpreted as evidence for CPT. Strictly speaking, MSD is not consistent with CPT, because MSD assumes a reverse S-shaped utility function rather than an S-shaped utility function. However, for the Tversky and Kahneman (1992) estimates, the curvature of the utility function is very weak compared to the curvature of the transformation function. Hence, a CPT decision-maker generally does not violate the MSD rule.

2. Empirical Testing

2.1 Empirical SD analysis

In practical applications, the CDF generally is not known. Rather, information typically is limited to a discrete set of time series observations, say $\mathbf{X} \equiv (\boldsymbol{x}_1 \dots \boldsymbol{x}_T)$ with $\boldsymbol{x}_t \equiv (x_1 \dots x_{Nt})^{\mathrm{T}}$, and indexed by $\Theta \equiv \{t\}_{t=1}^{T}$.

Assumption 4. *The observations are independent random draws from the CDF.*[11] *Since, by assumption, the timing of the draws is inconsequential, we*

[11] There are compelling theoretical and empirical arguments in favor of a time varying return distribution. Unfortunately, the search for a satisfactory specification of the return dynamics is still far from accomplished. In fact, Ghysels (1998) finds that ill-specified conditional asset pricing models in many cases yield greater pricing errors than unconditional models. For this reason, we use an unconditional model here. Still, further research could relax the Section 2.4 assumption. One possible approach is to use, for example, cluster analysis to construct a subset of observations that is sufficiently similar in terms of conditioning variables like the riskless rate, the term spread and the credit spread. The SD tests can then be applied to this subset rather than the full sample.

are free to label the observations by their ranking with respect to the evaluated portfolio, that is, $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\tau} < \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{\tau} < \cdots < \boldsymbol{x}_T^{\mathrm{T}}\boldsymbol{\tau}$.[12] *Using the observations, we can construct the following empirical distribution function (EDF):*

$$F(\boldsymbol{x}) \equiv \sum_{t \in \Theta} 1(\boldsymbol{x}_t \leq \boldsymbol{x})/T. \tag{9}$$

By using EDF $F(\boldsymbol{x})$ in place of CDF $G(\boldsymbol{x})$, we arrive at the following empirical definition of SD efficiency:

Definition 2. *Portfolio* $\boldsymbol{\tau} \in \Lambda$ *is empirically* $U(\Psi)$*–SD efficient if and only if:*

$$\min_{u \in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda} \in \Lambda} \left\{ \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}) dF(\boldsymbol{x}) - \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}) dF(\boldsymbol{x}) \right\} \right\} = 0 \Longleftrightarrow \tag{10}$$

$$\min_{u \in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda} \in \Lambda} \left\{ \sum_{t \in \Theta} (u(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\lambda}) - u(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau}))/T \right\} \right\} = 0. \tag{11}$$

Alternatively, portfolio $\boldsymbol{\tau} \in \Lambda$ *is empirically* $U(\Psi)$*–SD inefficient if and only if:*

$$\min_{u \in U(\Psi)} \left\{ \max_{\boldsymbol{\lambda} \in \Lambda} \left\{ \sum_{t \in \Theta} (u(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\lambda}) - u(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau}))/T \right\} \right\} > 0. \tag{12}$$

2.2 Linear programming test

To implement Definition 2, we will use a test statistic that generalizes Post's (2003) test statistic for SSD efficiency. The test statistic essentially checks if the necessary first-order condition for portfolio optimality holds for some utility function $u \in U(\Psi)$. Specifically, if the evaluated portfolio $\boldsymbol{\tau} \in \Lambda$ is optimal for some utility function $u \in U(\Psi)$, that is, $\boldsymbol{\tau} = \arg\max_{\boldsymbol{\lambda} \in \Lambda} \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}) dG(\boldsymbol{x})$, then all assets must lie on or below the tangent hyperplane, that is,

$$\int u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau} - x_i) dG(\boldsymbol{x}) \geq 0 \ \ \forall i \in \mathrm{I}.^{13} \tag{13}$$

This condition basically states that marginal deviations from the market portfolio lower expected utility. Similarly, the first-order condition for

[12] As we assume a continuous return distribution, ties do not occur. Still, the analysis can be extended in a straightforward way to cases where ties do occur, for example, due to a discrete return distribution or due to measurement problems or rounding (Post, 2003).

[13] The inequality should hold with strict equality for all assets that are included in the evaluated portfolio, that is $i \in \mathrm{I} : \tau_i > 0$. Hence, if all assets are included in the evaluated portfolio, that is, $\boldsymbol{\tau} > 0$, as is true for, for example, the value-weighted market portfolio, then inequality (13) reduces to $\int u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau} - x_i) dG(\boldsymbol{x}) = 0 \ \forall i \in \mathrm{I}$.

optimality relative to the EDF, that is, $\boldsymbol{\tau} = \arg\max_{\boldsymbol{\lambda}\in\Lambda} \sum_{t\in\Theta} u(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\lambda})/T$, is the following sample equivalent of (13):

$$\sum_{t\in\Theta} u'(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau})(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau} - x_{it})/T \geq 0 \;\; \forall i\in \mathrm{I}. \tag{14}$$

We will test for efficiency by using the following test statistic:

$$\xi(\boldsymbol{\tau},\Psi) \equiv \min_{\boldsymbol{\beta}\in B(\Psi),\theta} \left\{ \theta : \sum_{t\in\Theta} \beta_t(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau} - x_{it})/T + \theta \geq 0 \;\forall i\in \mathrm{I} \right\}, \tag{15}$$

with

$$B(\Psi) \equiv \left\{ \boldsymbol{\beta}\in [1,\infty\rangle^{\mathrm{T}} : \beta_t \geq \beta_s \;\forall t,\, s\in\Theta \colon (\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau},\boldsymbol{x}_s^{\mathrm{T}}\boldsymbol{\tau})\in \;\Psi \right\}. \tag{16}$$

The test statistic $\xi(\boldsymbol{\tau},\Psi)$ essentially checks if the first-order necessary condition (14) holds for some utility function $u\in U(\Psi)$. The vector $\boldsymbol{\beta} \equiv (\beta_1,...,\beta_{\mathrm{T}})^{\mathrm{T}}$ represents the gradient vector $\nabla u(\mathbf{X}^{\mathrm{T}}\boldsymbol{\tau}) \equiv (u'(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\tau}),...,u'(\boldsymbol{x}_T^{\mathrm{T}}\boldsymbol{\tau}))^{\mathrm{T}}$ for some $u\in U(\Psi)$; the set $B(\Psi)$ represents the restrictions that are placed on the gradient vector for a utility function $u\in U(\Psi)$. We can derive the following result:

Theorem 2. *Portfolio $\boldsymbol{\tau}\in\Lambda$ is empirically $U(\Psi)$–SD efficient only if $\xi(\boldsymbol{\tau},\Psi) = 0$. Alternatively, portfolio $\boldsymbol{\tau}\in\Lambda$ is empirically $U(\Psi)$–SD inefficient if $\xi(\boldsymbol{\tau},\Psi)>0$.*

The test statistic involves a linear objective function and a finite set of linear constraints. Hence, the test statistic can be solved using straightforward LP. The model always has a feasible solution (for example, $\boldsymbol{\beta} = \boldsymbol{e}$) and the solution is bounded from below by zero (the case of efficiency) and bounded from above by $\max_{i\in \mathrm{I}} \sum_{t\in\Theta} (x_{it} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau})/T$ (the case with $\boldsymbol{\beta} = \boldsymbol{e}$). For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized hardware and solver software is recommended for large-scale problems involving thousands of observations and/or assets.

The polyhedron $B(\Psi)$ involves constraints on pairs of observations $s, t\in\Theta$. This suggests that the number of constraints increases quadratically with the number of observations. However, many of the constraints are redundant, and the effective number of constraints increases linearly with the number of observations. For example, using $z \in \Theta$ for the first observation in the domain of gains, that is, $\boldsymbol{x}_{z-1}^{\mathrm{T}}\boldsymbol{\tau} < 0 \leq \boldsymbol{x}_z^{\mathrm{T}}\boldsymbol{\tau}$, it is easily verified that:

$$B(\Psi_{SSD}) = \{\boldsymbol{\beta} \in \Re^{\mathrm{T}}:\beta_1 \geq \cdots \geq \beta_T \geq 1\}; \tag{17}$$

$$B(\Psi_{PSD}) = \{\boldsymbol{\beta} \in \Re^{\mathrm{T}}:1 \leq \beta_1 \leq \cdots \leq \beta_{z-1};\ \beta_z \geq \cdots \beta_T \geq 1\}; \tag{18}$$

$$B(\Psi_{MSD}) = \{\boldsymbol{\beta} \in \Re^{\mathrm{T}}:\beta_1 \geq \cdots \geq \beta_{z-1} \geq 1 \leq \beta_z \leq \cdots \beta_T\}. \tag{19}$$

Our Theorem 2 extends Post's (2003) Theorem 2 for SSD efficiency. SSD assumes that the utility function is concave. In this case, the first-order condition is necessary and sufficient for establishing a global optimum for the portfolio problem. Roughly speaking, for concave utility functions, linear extrapolation will overstate the expected utility level of portfolios that lie "far away" from the market portfolio. Hence, if marginal deviations from the market portfolio reduce expected utility for SSD investors, then "radical" changes will make them even worse off. However, for the generalized case that allows for local risk seeking, the first-order condition gives only a necessary condition. Specifically, linear extrapolation may understate the expected utility level of "remote" portfolios if utility is non-concave. Hence, the PSD and MSD efficiency tests may wrongly classify inefficient portfolios as efficient. Specifically, portfolios that represent a local optimum will also satisfy the first-order condition.

There exist various multivariate global optimization methods for locating the global optimum if the (known) objective function is not concave (see, for example, Horst and Pardalos (1995)). Unfortunately, these methods generally are computationally more complex than the single and simple linear programming problem required for computing $\xi(\boldsymbol{\tau}, \Psi)$, and the computational burden becomes prohibitive if the problem dimensionality is high (that is, many assets are included). This is especially true if we use bootstrapping to infer the sampling properties of the test results (see Section 2.3). In addition, we do not see a simple approach for the case where the objective function is not known (as is true for the SD rules) and where we have to determine if a given solution is the global maximum for some objective function in a class of functions. For this reason, we use the first-order conditions only.

This approach still yields a sufficient condition if we assume that investors have to select portfolios "in the neighborhood" of the market portfolio. After all, the first-order condition implies that marginal changes reduce expected utility. This assumption seems plausible if, for example, investment agents face a small subset of the portfolio possibility set, due to investment constraints that are imposed by their principals. However, from the first-order condition alone, we cannot logically conclude that the market is efficient relative to the full portfolio possibilities set. In this respect, future research should be directed at generalizing the current SD test by including additional necessary restrictions for global optimality.

2.3 Statistical inference

We have thus far discussed SD efficiency relative to the EDF $F(x)$ rather than the CDF $G(x)$. Generally, the EDF is very sensitive to sampling variation and the test results are likely to be affected by sampling error in a non-trivial way. The applied researcher must therefore have knowledge of the sampling distribution in order to make inferences about the true efficiency classification. In our analysis, we will use an asymptotic hypothesis testing procedure as well as the bootstrap methodology.

2.3.1 Asymptotic hypothesis testing procedure.[14] The asymptotic test procedure is based on two simplifications: (1) the use of a restrictive null hypothesis and (2) the use of the limiting least favorable distribution. Our null (H_0) is that the assets have an equal and finite mean, that is, $E[\boldsymbol{x}] = \mu\boldsymbol{e}$, $\mu < \infty$, and a covariance matrix $E\,[(\boldsymbol{x} - \mu)(\boldsymbol{x} - \mu)^{\mathrm{T}}] = \boldsymbol{\Omega}$ with finite elements $\boldsymbol{\omega}_{ij} < \infty, i, j \in \mathrm{I}$. This null is restrictive, because it gives a sufficient but not necessary condition for efficiency. In fact, under the null, all portfolios $\boldsymbol{\lambda} \in \Lambda$ are efficient, because they optimize expected utility for $u(x) = x$, that is, for risk neutral investors.

The shape of the distribution of $\xi(\boldsymbol{\tau}, \Psi)$ under the null generally depends on the shape of $G(\boldsymbol{x})$. Our approach will be to focus on the least favorable distribution, that is, the distribution that maximizes the size or relative frequency of Type I error (rejecting the null when it is true). This approach stems from the desire to be protected against Type I error. For each $G(\boldsymbol{x})$, the size is always smaller than the size for the least favorable distribution. Naturally, this approach comes at the cost of a high frequency of Type II error (accepting the null when it is not true) or a low power (1 – the relative frequency of Type II error).

Using $\mathbf{C} \equiv (\mathrm{I} - \boldsymbol{e\tau}^{\mathrm{T}})$, we can summarize the asymptotic sampling distribution of our test statistic as follows:[15]

[14] This section extends Post's (2003) Theorem 3 for SSD, and it exploits the result that $\xi\,(\boldsymbol{\tau}, \Psi)$ is bounded from above by $\omega(\boldsymbol{\tau}) \equiv \max_{i \in \mathrm{I}} \left\{\Sigma_{t \in \Theta}(x_{it} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau})/T\right\}$(see the proof in the Appendix). This argument applies not only for Ψ_{SSD}, but also more generally for all Ψ.

[15] We use $\mathbf{I}$ for an identity matrix with dimensions conforming to the rules of matrix algebra.

Theorem 3. *For the asymptotic least favorable distribution, the p-value* $\Pr[\xi(\boldsymbol{\tau},\Psi)>y|H_0], y \geq 0$, *equals the integral* $\left(1-\int_{z\leq ye} d\Phi_{\boldsymbol{\Sigma}}(z)\right)$, *with* $\Phi_{\boldsymbol{\Sigma}}(\mathbf{z})$ *for a N-dimensional multivariate normal distribution function with zero means and covariance matrix* $\boldsymbol{\Sigma} \equiv (\mathbf{C}\boldsymbol{\Omega}\mathbf{C}^{\mathrm{T}})/T$.

The theorem shows the crucial role of the length of the time series (T) and the length of the cross-section (N); the p-values decrease as the time series grows, and increase as the cross-section grows. We may test efficiency by comparing the *p*-value for the observed value of $\xi(\boldsymbol{\tau},\Psi)$ with a predefined level of significance; we may reject efficiency if the *p*-value is smaller than or equal to the significance level.

Computing the *p*-value requires the unknown covariance terms ω_{ij}. We may estimate these parameters in a distribution-free and consistent manner using the sample equivalents:

$$\hat{\omega}_{ij} \equiv \sum_{t\in\Theta}(x_{it} - \sum_{t\in\Theta} x_{it}/T)(x_{jt} - \sum_{t\in\Theta} x_{jt}/T)/T. \tag{20}$$

We stress that the asymptotic *p*-values relies on the least favorable distribution. For this reason, the *p*-values may involve low power in small samples. An alternative approach is to approximate the sampling distribution by means of bootstrapping. Bootstrapping can deal directly with the true distribution, and hence it can yield more powerful results. In addition, the bootstrap can deal with the null of SD efficiency rather than the null of equal means.

2.3.2 A bootstrap approach to testing market efficiency. Bootstrapping, first introduced by Efron (1979) and Efron and Gong (1983), is a well-established tool to analyze the sensitivity of empirical estimators to sampling variation in situations where the sampling distribution is difficult or impossible to obtain analytically. The technique is based on the idea of repeatedly simulating the CDF, usually through resampling, and applying the original estimator to each simulated sample or pseudo-sample so that the resulting estimators mimic the sampling distribution of the original estimator. Key to the success of bootstrapping is the selection of an appropriate approximation for the CDF. If the approximation is statistically consistent, then the bootstrap distribution gives a statistically consistent estimator for the original sampling distribution. Under the assumption that the return distribution is serially two-dimensional (see Section 2.1), the EDF $F(x)$ is a consistent estimator for the CDF $G(x)$. This suggests that bootstrap pseudo-samples can be obtained simply by random sampling with replacement from the EDF. We generate $B = 1000$ pseudo-samples $\hat{\mathbf{X}}_b, b = 1, ..., B,$ in this way and compute the test statistic $\xi(\boldsymbol{\tau},\Psi)$ for each pseudo-sample. Using $\overline{\xi}(\boldsymbol{\tau},\Psi)$ for the average value of $\xi_b(\boldsymbol{\tau},\Psi)$ over all pseudo-samples, we can estimate the bias of the original estimator $\xi_b(\boldsymbol{\tau},\Psi)$ as $bias = \overline{\xi}(\boldsymbol{\tau},\Psi) - \xi(\boldsymbol{\tau},\Psi)$. To correct for bias, we then compute the bootstrap bias-corrected estimators $\xi^*_b(\boldsymbol{\tau},\Psi) \equiv \xi_b(\boldsymbol{\tau},\Psi) - 2bias = \xi_b(\boldsymbol{\tau},\Psi) - 2\overline{\xi}(\boldsymbol{\tau},\Psi) + 2\xi(\boldsymbol{\tau},\Psi)$.[16] Finally, we compute the bootstrap *p*-value as the relative frequency of pseudo-samples in which the evaluated portfolio is classified as efficient, that is, $\xi^*_b(\boldsymbol{\tau},\Psi) = 0$. We reject efficiency if and only if the *p*-value is smaller than the significance level of five percent.

3. Aggregate Investor Preferences and Beliefs

3.1 Data description

We analyze investor preferences and beliefs by testing if different SD criteria can rationalize the market portfolio. For this purpose, we need a proxy for the market portfolio and proxies for the individual assets in the investment universe. We will use the Fama and French market portfolio, which is the value-weighted average of all common stocks listed on NYSE, AMEX, and NASDAQ, and covered by CRSP. Further, we use the one-month US Treasury bill as the riskless asset. Finally, for the individual risky assets, we use two sets of benchmark portfolios:

- The 25 Fama and French benchmark portfolios constructed as the intersections of five quintile portfolios formed on size and five quintile portfolios formed on BE/ME. We use data on monthly returns (month-end to month-end) from July 1963 to October 2001 (460 months) obtained from the data library on the homepage of Kenneth French.[17,18] Also, we analyze the stability of the results by applying the test to two non-overlapping subsamples of equal length (230 months): (1) July 1963 to August 1982 and (2) September 1982 to October 2001.
- The set of 27 benchmark portfolios described in Carhart et al. (1996) and used in Carhart (1997). The portfolios are formed using a three-way classification on the basis of size, BE/ME, and momentum. These portfolios are formed by dividing all stocks into

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[16] In this way, the bias-corrected estimators $\xi^*_b(\boldsymbol{\tau},\Psi)$ are centered at $\xi(\boldsymbol{\tau},\Psi) - bias = 2\xi(\boldsymbol{\tau},\Psi) - \overline{\xi}(\boldsymbol{\tau},\Psi)$, which is the bias-corrected value for the original estimator $\xi(\boldsymbol{\tau},\Psi)$. The raw bootstrap distribution of the SSD test statistic generally involves positive bias, and not correcting for bias lowers the bootstrap p-values and hence increases statistical size and power.

[17] The data set starts in 1963 because the COMPUSTAT data used to construct the benchmark portfolios are biased towards big historically successful firms for the earlier years (Fama and French (1992)).

[18] Monthly returns are frequently used in empirical asset pricing. In our analysis, one consideration for using monthly data is the power of SD tests in small samples. Specifically, the tests are not sufficiently powerful to deal with a small data set of low-frequency returns (for example, yearly returns), especially if the number of benchmark assets is large.

thirds on the basis of B/M. Each of the three first-level portfolios is then divided into three portfolios on the basis of size. Finally, each of the nine second-level portfolios is divided into “past losers”, “middle”, and “past winners”. We use data on monthly returns (month-end to month-end) from July 1963 to December 1994 (378 months), as well as the 189-month subsamples from July 1963 to March 1979 and from April 1979 to December 1994.

These two sets of benchmark portfolios provide a challenge because many studies indicate that the cross-sectional pattern of returns across size, BE/ME, and momentum portfolios cannot be explained by the traditional approach based on concave utility functions. Tables 1 and 2 show some descriptive statistics for the excess returns of the benchmark portfolios for the full samples.

To give some feeling for the data, Figures 1*a* and 1*b* shows mean–variance diagrams for the two full samples, including the individual benchmark portfolios (the clear dots), the market portfolio (M) and the mean–variance efficient frontier (OA).[19] Clearly, the market portfolio is highly inefficient in terms of mean–variance analysis in both cases. Roughly speaking, there exist portfolios with the same standard deviation as the market portfolio but twice the average excess return, and there exist portfolios with the same average excess return as the market portfolio but only half the standard deviation. This result violates the central prediction of the mean–variance CAPM: mean–variance efficiency of the market portfolio.

The maintained utility assumptions of mean–variance CAPM are a possible explanation for this violation; variance does not fully capture the risk profile of assets unless investor utility is quadratic (if we impose no prior structure on the return distribution). This provides the motivation for testing if the market portfolio is efficient in terms of SD criteria.

3.2 SD test results

We employ the three SD efficiency criteria discussed in Section 2: SSD, PSD, and MSD. For each criterion, we compute the value of the test statistic $\xi(\boldsymbol{\mu}, \Psi)$ and the associated asymptotic least favorable *p*-value. We reject efficiency if the *p*-value is smaller than or equal to the significance level of five percent. As discussed in Section 2, the use of the least favorable distribution may involve low power. For this reason, we also compute *p*-values using the bootstrap procedure outlined in Section 2.3. Table 3 shows the test results for the two sets of benchmark portfolios.

[19] The frontier passes through the origin because we use returns in excess of the T-bill rate. Hence, the T-bill has a zero mean and a zero standard deviation. Also, the frontier is shown for the case without short selling; the portfolio possibilities are described by all convex combinations of the individual assets (Section 1).

Table 1
Descriptive statistics Fama and French portfolios

		Mean	SD	Skewness	Kurtosis	Minimum	Maximum
Market portfolio		0.462	4.461	−0.498	2.176	−23.09	16.05
Benchmark portfolios							
BE/ME	Size						
Growth	Small	0.235	8.246	0.003	2.466	−34.32	38.82
2	Small	0.733	7.064	0.037	3.338	−31.30	36.67
3	Small	0.815	6.140	−0.087	3.233	−29.17	27.86
4	Small	0.998	5.724	−0.149	3.636	−29.47	27.43
Value	Small	1.075	5.943	−0.098	4.128	−29.45	31.83
Growth	2	0.359	7.494	−0.316	1.673	−33.21	29.62
2	2	0.622	6.120	−0.472	2.778	−32.50	26.25
3	2	0.842	5.410	−0.488	3.739	−28.10	26.78
4	2	0.913	5.154	−0.385	3.979	−27.09	26.70
Value	2	0.966	5.660	−0.243	4.245	−29.83	29.43
Growth	3	0.380	6.908	−0.342	1.365	−29.96	23.39
2	3	0.665	5.511	−0.624	3.051	−29.65	23.35
3	3	0.673	4.962	−0.621	2.984	−25.56	21.14
4	3	0.818	4.710	−0.327	3.029	−23.20	22.61
Value	3	0.953	5.297	−0.423	4.384	−27.79	27.30
Growth	4	0.506	6.146	−0.182	1.683	−26.02	25.01
2	4	0.433	5.218	−0.571	3.317	−29.62	20.46
3	4	0.651	4.855	−0.423	3.348	−26.13	22.87
4	4	0.820	4.622	0.035	2.394	−18.30	24.57
Value	4	0.880	5.347	−0.198	2.697	−25.36	26.20
Growth	Big	0.442	4.841	−0.177	1.676	−22.15	21.70
2	Big	0.441	4.588	−0.334	1.897	−23.21	16.05
3	Big	0.493	4.344	−0.243	2.639	−22.47	18.22
4	Big	0.603	4.289	0.056	1.574	−15.35	18.85
Value	Big	0.583	4.645	−0.146	0.944	−19.28	15.39

Monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 months) for the value-weighted Fama and French market portfolio and 25 value-weighted benchmark portfolios based on market capitalization (size) and book-to-market equity ratio (BE/ME). Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. All data are obtained from the data library on the homepage of Kenneth French.

The results vary across the two different benchmark sets, the three different sample periods, and the two different methods for computing *p*-values. However, the relative goodness of the three criteria is remarkably robust; PSD performs worst and MSD performs best.

For the full sample and the first subsample, the market portfolio is significantly SSD inefficient relative to both sets of benchmark portfolios. For the second subsample, the SSD results are mixed, but the bootstrap *p*-value for the Carhart sample again suggests SSD inefficiency. On the basis of these findings, we reject the SSD criterion.[20] Several authors have suggested that concave higher-order polynomial utility functions substantially better explain the cross-sectional variation of stock returns than quadratic utility functions do. Our results suggest that *no* concave utility function can rationalize the market portfolio. Under our maintained

[20] These results are in line with Post's (2003) finding that the Fama and French market portfolio is SSD inefficient.

Table 2
Descriptive statistics Carhart portfolios

			Mean	SD	Skewness	Kurtosis	Minimum	Maximum
Market portfolio			0.387	4.399	−0.394	2.600	−23.09	16.05
Benchmark portfolios								
BE/ME	Size	Momentum						
Growth	Small	Loser	−0.279	6.644	0.003	2.784	−31.41	30.45
Neutral	Small	Loser	0.306	5.975	0.474	5.363	−24.81	37.51
Value	Small	Loser	0.552	6.440	0.974	7.774	−28.26	45.34
Growth	Small	Middle	0.349	6.070	−0.461	3.194	−31.33	27.54
Neutral	Small	Middle	0.631	4.957	−0.201	5.038	−27.05	27.02
Value	Small	Middle	1.062	5.583	0.199	6.219	−29.09	33.99
Growth	Small	Winner	0.933	6.755	−0.704	2.492	−33.52	19.45
Neutral	Small	Winner	1.157	6.088	−0.730	3.569	−32.60	23.17
Value	Small	Winner	1.341	6.246	−0.289	4.564	−32.00	30.20
Growth	Medium	Loser	−0.089	5.952	0.088	2.757	−26.49	27.54
Neutral	Medium	Loser	0.400	5.396	0.501	3.468	−19.08	31.20
Value	Medium	Loser	0.582	6.078	0.469	4.678	−25.79	38.07
Growth	Medium	Middle	0.143	5.341	−0.461	2.207	−26.81	18.96
Neutral	Medium	Middle	0.513	4.498	−0.344	4.720	−25.68	23.05
Value	Medium	Middle	0.909	5.303	−0.016	6.042	−28.30	31.03
Growth	Medium	Winner	0.895	6.033	−0.550	2.261	−29.90	20.73
Neutral	Medium	Winner	0.772	5.325	−0.896	3.683	−30.54	17.52
Value	Medium	Winner	1.287	5.881	−0.806	5.234	−33.86	25.76
Growth	Big	Loser	0.105	5.218	0.178	2.208	−20.40	25.40
Neutral	Big	Loser	0.438	4.801	0.477	2.287	−19.66	21.74
Value	Big	Loser	0.600	5.569	0.718	4.414	−16.99	35.34
Growth	Big	Middle	0.250	4.507	−0.161	1.885	−20.73	17.86
Neutral	Big	Middle	0.367	4.211	0.158	2.307	−15.57	21.03
Value	Big	Middle	0.589	4.752	−0.053	2.717	−23.47	20.41
Growth	Big	Winner	0.611	5.348	−0.298	2.035	−23.68	21.21
Neutral	Big	Winner	0.595	4.856	−0.347	2.565	−24.00	19.95
Value	Big	Winner	0.923	5.352	−0.254	2.846	−24.78	22.84

Monthly excess returns (month-end to month-end) from July 1963 to December 1994 (378 months) for the value-weighted Fama and French market portfolio and 27 value-weighted benchmark portfolios based on market capitalization (size), book-to-market equity (BE/ME) and momentum. Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. Data on the 27 portfolios are courtesy of Mark Carhart. The remaining data on the market portfolio and the Treasury bill are obtained from the homepage of Kenneth French.

assumptions, this implies that investors who hold the market portfolio (for example, index funds or exchange traded funds) are not globally risk averse and utility is not everywhere concave, and we have to account for (local) risk-seeking behavior.

The PSD criterion is also rejected for both sets of benchmark portfolios. In fact, the evidence against PSD is even more convincing than the evidence against SSD; the test statistic takes higher values and the p-values are smaller. This implies that no S-shaped utility function can rationalize the market portfolio (if we ignore probability transformation).

For both benchmark sets and all three sample periods, the MSD criterion performs best. In fact, we cannot reject MSD efficiency of the market portfolio, with the exception of the bootstrap results for the

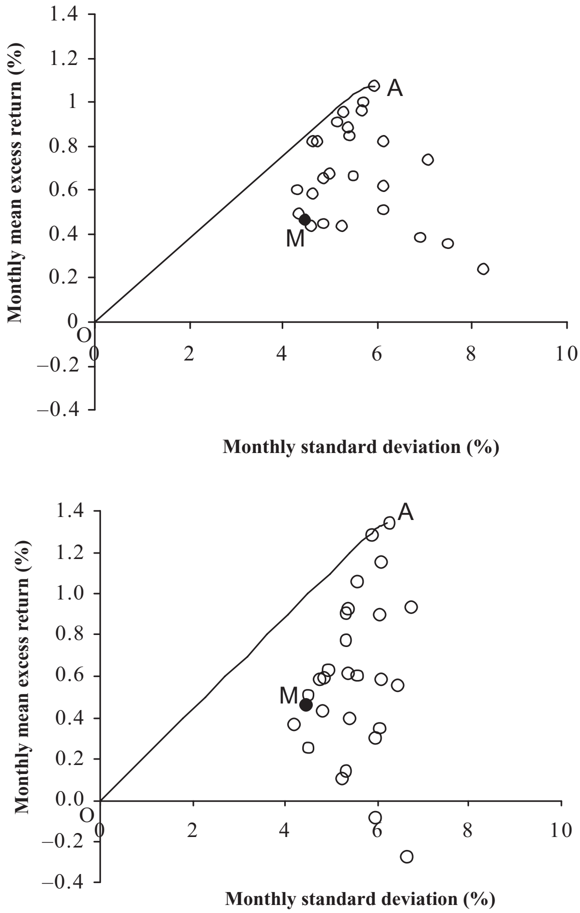

Figure 1
Mean–variance diagram
(*a*) The figure shows mean–variance diagram for the historical excess returns of the Fama and French market portfolio (M) and the 25 Fama and French benchmark portfolios (the clear dots) for the full sample period from July 1963 to October 2001. OA represents the efficient frontier (with the one-month US Treasury bill as the riskless asset and with short sales not allowed). (*b*) Mean–variance diagram for the historical excess returns of the Fama and French market portfolio (M) and the 27 Carhart benchmark portfolios (the clear dots) for the full sample period from July 1963 to December 1994. OA represents the efficient frontier (with the one-month US Treasury bill as the riskless asset and with short sales not allowed).

second subsample of the Carhart data set. This suggests that reverse S-shaped utility functions best capture investor preferences, and that risk aversion over losses and risk seeking over gains helps to explain the cross-sectional pattern of stock returns. If investors are risk averse for losses and risk seeking for gains, then they will pay a premium for stocks that have low downside risk in bear markets and high upside potential in bull market. Hence, stocks with low downside risk in bear markets and

Table 3
Test results

	Full sample	Subsample 1	Subsample 2
25 Fama and French portfolios			
Ψ_{SSD}	0.434	0.585	0.299
	(0.031)	**(0.034)**	(0.490)
	(0.000)	**(0.000)**	(0.157)
Ψ_{PSD}	0.432	0.905	0.356
	(0.031)	**(0.000)**	(0.324)
	(0.000)	**(0.000)**	**(0.009)**
Ψ_{MSD}	0.207	0.226	0.192
	(0.501)	(0.748)	(0.865)
	(0.077)	(0.587)	(0.052)
27 Carhart portfolios			
Ψ_{SSD}	0.469	0.780	0.442
	(0.013)	**(0.004)**	(0.169)
	(0.002)	**(0.000)**	**(0.010)**
Ψ_{PSD}	0.861	0.930	0.737
	(0.000)	**(0.000)**	**(0.004)**
	(0.000)	**(0.000)**	**(0.000)**
Ψ_{MSD}	0.291	0.346	0.384
	(0.275)	(0.547)	(0.300)
	(0.104)	(0.439)	**(0.020)**

We test whether the Fama and French market portfolio is $U(\Psi)$–*SD* efficient relative to all portfolios formed from the US Treasury bill and a set of risky benchmark portfolios (25 Fama and French portfolios or 27 Carhart portfolios). Each cell shows the observed value for the test statistic $\xi(\boldsymbol{\tau}, \Psi)$. the asymptotic least favorable *p*-value (first value within brackets) and the bootstrap *p*-value (second value within brackets). The bootstrap *p*-value is computed using 1000 pseudo-samples and after correcting for possible bias in the test statistic. We use a bold font for *p*-values that are smaller than or equal to five percent, that is, the level of significance used in this study.

high upside potential in bull markets provide low expected returns. As will be discussed in Section 3.3 and 3.4 below, this pattern is also consistent with the skewness preference of the three-factor CAPM and the probability transformation of the Cumulative Prospect Theory.

For illustration, Figures 2*a* and 2*b* shows example utility functions that rationalize the market portfolio relative to the two sets of benchmark portfolios for the full sample periods. These particular utility functions are piecewise-linear functions that are formed from the optimal solution $\boldsymbol{\beta}^* \in B(\Psi_{MSD})$ as

$$p(x|\boldsymbol{\beta}^*) = \begin{cases} \alpha_1 + \beta_1^* x & x \leq \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_t + \beta_t^* x & \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\mu} < x \leq \boldsymbol{x}_{t+1}^{\mathrm{T}}\boldsymbol{\mu} < 0 \\ \alpha_{z-1} + \beta_{z-1}^* x & \boldsymbol{x}_{z-1}^{\mathrm{T}}\boldsymbol{\mu} \leq x < 0 \\ \alpha_z + \beta_z^* x & 0 \leq x < \boldsymbol{x}_z^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_t + \beta_t^* x & 0 < \boldsymbol{x}_{t-1}^{\mathrm{T}}\boldsymbol{\mu} < x < \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_T + \beta_T^* x & x \geq \boldsymbol{x}_T^{\mathrm{T}}\boldsymbol{\mu} \end{cases}, \tag{21}$$

with $\boldsymbol{\alpha} \equiv (\alpha_1, ..., \alpha_{\mathrm{T}})^{\mathrm{T}}$ such that the linear line segments are connected, that is, $\alpha_t + \beta_t^* x = \alpha_{t+1} + \beta_{t+1}^* x$ for all $t \in \Theta \backslash T$, and $\alpha_z = 0$. We stress that

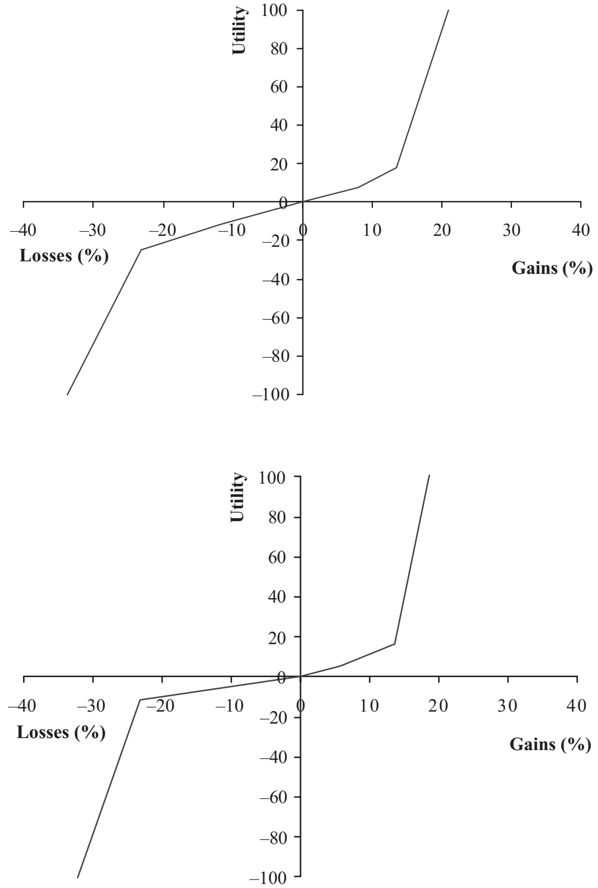

Figure 2
Optimal piecewise linear MSD utility function
(*a*) This utility function rationalizes the Fama and French market portfolio relative to the US Treasury bill and the 25 Fama and French benchmark portfolios for the full sample period from July 1963 to October 2001. (*b*) This utility function rationalizes the Fama and French market portfolio relative to the US Treasury bill and the 27 Carhart benchmark portfolios for the full sample period from July 1963 to December 1994. Both utility function are constructed from the optimal solution $\boldsymbol{\beta}^* \in B(\Psi_{MSD})$ to (15) as

$$p(x|\boldsymbol{\beta}^*) = \begin{cases} \alpha_1 + \boldsymbol{\beta}_1^* x & x \leq \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_t + \beta_t^* x & \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\mu} < x \leq \boldsymbol{x}_{t+1}^{\mathrm{T}}\boldsymbol{\mu} < 0 \\ \alpha_{z-1} + \beta_{z-1}^* x & \boldsymbol{x}_{z-1}^{\mathrm{T}}\boldsymbol{\mu} \leq x < 0 \\ \alpha_z + \beta_z^* x & 0 \leq x < \boldsymbol{x}_z^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_t + \beta_t^* x & 0 < \boldsymbol{x}_{t-1}^{\mathrm{T}}\boldsymbol{\mu} < x < \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\mu} \\ \alpha_T + \beta_T^* x & x \geq \boldsymbol{x}_T^{\mathrm{T}}\boldsymbol{\mu} \end{cases},$$

with $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_{\mathrm{T}})^{\mathrm{T}}$ such that the linear line segments are connected, that is, $\alpha_t + \beta_t^* = \alpha_{t+1} + \beta_{t+1}^* x$ for all $t \in \Theta \backslash T$, and with $\alpha_z = 0$.

these utility functions are not unique, as we can construct alternative utility functions with the same gradient vector at $\mathbf{X}^{\mathrm{T}}\boldsymbol{\mu}$. Also, the utility functions are likely to be very sensitive to estimation error and we cannot claim any confidence in their statistical estimation. Still, the utility functions do suggest that the market portfolio is optimal, and hence the benchmark assets are correctly priced, for investors that are very sensitive to large losses, say returns of less than minus ten percent per month, and to large gains, say returns of more than ten percent per month.

To illustrate the return pattern found with the SD tests, it is insightful to analyze the returns of two extreme portfolios in two extreme months. For our samples, the worst month, that is, the month with the lowest market return, is October 1987, with an excess market return of –23.09 percent, and the best month is October 1974, with an excess market return of 16.05 percent. Interestingly, the "inexpensive" or high average return portfolio of small value winner stocks (with an average monthly excess return of 1.341) substantially underperforms the market with an excess return of –32.00 percent in October 1987 and 8.63 percent in October 1974. By contrast, the "low yield" big-growth-loser portfolio (with average 0.105) involves an excess return of –20.40 percent in October 1987 and of 25.40 percent in October 1974, hence substantially outperforming the market. These observations are consistent with the idea that investors are willing to pay a premium for stocks that offer systematic downside protection and/or upside potential. Of course, these observations involve only two portfolios and two months. However, our SD test results suggest that this pattern applies also for the full sample of all portfolios and all months.

3.3 Three-moment CAPM

Further support for reverse S-shaped utility can be found from the empirical results on the three-moment CAPM by Kraus and Litzenberger (1976), and used by Friend and Westerfield (1980), and Harvey and Siddique (2000), among others. This model substitutes a cubic utility function for the standard quadratic utility function used in mean–variance CAPM. While the mean–variance CAPM predicts an exact linear relationship between mean and market covariance, the three-moment model predicts an exact linear relationship between mean, covariance, and coskewness with the market portfolio.

Interestingly, coskewness seems to explain a substantial part of the cross-sectional variation of mean returns not explained by beta.[21] Specifically, there is evidence for a substantial *coskewness discount*, that is, securities that increase market skewness systematically earn negative

[21] Nevertheless, a significant part also remains unexplained and the market portfolio remains inefficient relative to portfolios formed on size, B/M, and momentum. As the market portfolio appears MSD efficient in our study, this suggests that higher-order central moments (such as kurtosis) and lower partial moments (such as semivariance) play an important role in explaining the variation of mean return not explained by beta and coskewness.

mean–variance CAPM excess returns. A first glance, the coskewness discount is also present in our samples; for example, the "high yield" portfolio of small value winner stocks has a negatively skewed return distribution, while the "low yield" big-growth-loser portfolio has a positively skewed return distribution (see Table 2).

Empirical studies of the three-moment model typically analyze the relationship between mean, covariance, and coskewness, and they do not report the underlying utility function. Still, there are compelling arguments to expect that the underlying utility function takes a reverse S-shaped form. Specifically, while the studies generally assume concavity in their theory section, they do not *a priori* restrict the risk premiums such that the underlying utility function is concave in their empirical section. Rather, the empirical studies typically require a non-negative premium for covariance and a non-positive premium (or a non-negative discount) for coskewness. It is easily verified that these restrictions are necessary but not sufficient for concavity.[22] In fact, using US stock data, Post, Levy, and Vliet (2003) show that the sizeable coskewness discount implies a reverse S-shaped utility function and that the discount disappears if concavity is imposed.

3.4 Cumulative prospect theory

As discussed in Section 1, MSD efficiency can be interpreted as evidence for Cumulative Prospect Theory (CPT). Specifically, MSD is consistent with the two key elements of CPT: (1) loss aversion and (2) a reverse S-shaped transformation function. MSD is not consistent with the third element, an S-shaped utility function. However, according to the Tversky and Kahneman (1992) estimates, the curvature of the utility function is very weak compared to the curvature of the transformation function. Hence, we may expect that CPT decision-makers generally do not violate the MSD rule. Like MSD investors, CPT investors will have an "abnormal" demand for assets that offer systematic downside protection (due to loss aversion and the overweighting of small probabilities of large losses) or systematic upside potential (due to loss aversion and the overweighting of small probabilities of large gains).

[22] Consider the (standardized) cubic utility function $u(x) = x + ax^2 + bx^3$. Starting from the Euler equation $E[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})x_i] = 0$, it is straightforward to derive the following relationship between expected return $E[x_i]$, market covariance $\mathrm{cov}_i \equiv E[x_i\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}] - \mathrm{E}[x_i]E[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}]$ and market coskewness $\mathrm{cos}_i \equiv E[x_i(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})^2] - E[x_i]E[(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})^2]$:

$$E[x_i] = -(2a/E[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})])(\mathrm{cov}_i - (3b/E[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})])\mathrm{cos}_i.$$

Assuming non-satiation or $u'(x) \geq 0$ for all x, the sign of a is the opposite of the sign of the covariance premium $-(2a/E[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})])$ and the sign of b is the opposite of the sign of the coskewness premium $-(3b/E[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})])$. Hence, a non-negative covariance premium and a non-positive coskewness premium is equivalent to $a \leq 0$ and $b \geq 0$. These conditions are not sufficient to guarantee concavity or $u''(x) = 2a + 6bx \leq 0 \ \forall x \leq 0$. Specifically, for every $a \leq 0$ and $b \geq 0$, $u(x)$ is convex for all $x >$ *-a/3b*.

4. Conclusions

We cannot reject MSD efficiency of the market portfolio relative to benchmark portfolios formed on size, BE/ME, and momentum. Hence, the market portfolio seems efficient relative to reverse S-shaped utility functions. By contrast, the alternative criteria of SSD and PSD are convincingly rejected and utility functions with global risk aversion or risk seeking for losses and risk aversion over gains cannot rationalize the market portfolio. If investors are risk averse for losses and risk seeking for gains, then they are willing to pay a premium for stocks that give downside protection in bear markets and upside potential in bull markets. Our results suggest that this explanation is consistent with the historical cross-sectional pattern of stock returns. Roughly speaking, stock returns suggest that investors are driven by the twin desires for security and potential, and that investment portfolios are designed to avoid poverty and to give a chance at riches. In this respect, our findings are consistent with the predictions of several behavioral finance models like the behavioral portfolio theory (BPT) by Shefrin and Statman (2000).

We jointly analyze preferences and beliefs, and our results may reflect increasing marginal utility of wealth for gains and/or the subjective overweighing of the probability of large gains. Both phenomena effectively yield risk-seeking behavior (that is, taking unfair gambles). The low average return of stocks that perform relatively well in bull market scenarios may reflect a high marginal utility of wealth for these scenarios, but it may equally well reflect the overweighting of the frequency with which these scenarios occur. In this respect, our findings do not contradict Cumulative Prospect Theory (CPT). We find evidence against S-shaped utility functions with risk seeking for losses. Still, the results do not exclude the CPT combination of an S-shaped utility function and a reverse S-shaped transformation function. In fact, if probability transformation is sufficiently strong, then an investor with arbitrary preferences and a reverse S-shaped transformation function behaves as if he or she has rational expectations and a reverse S-shaped utility function.

Our analysis focuses on the necessary first-order condition for the portfolio-optimization problem. Unfortunately, for reverse S-shaped utility functions, the first-order condition is not sufficient for establishing a global optimum. Hence, the MSD efficiency classification for the market portfolio may mean that the market portfolio is a local optimum rather than the global optimum for portfolio selection. The first-order condition implies that all marginal deviations from the market portfolio make MSD investors worse off. However, "radical" deviations from the market portfolio may still make MSD investors better off. For example, it is possible that MSD investors would invest all their wealth in small value winner stocks only (an extreme point of the portfolio possibilities set). After all, this strategy involves a better mean-to-variance ratio and less negative skewness than the market portfolio (see, for example, the descriptive statistics in Table 2).

Unfortunately, our tests imply nothing about the expected utility of such "remote" portfolios. First, the tests identify the gradient vector with the smallest possible violation of the first-order condition for MSD utility functions. In general, there are an infinite number of MSD utility functions that are consistent with the optimal gradient vector, and hence we cannot recover unique expected utility levels. For example, the piecewise-linear utility functions shown in Figures 2*a* and 2*b* are merely examples of MSD utility functions that do not significantly violate the first-order conditions. Second, even if the market portfolio is not the global optimum for all utility functions that are consistent with the optimal gradient vectors, then we still have to check all possible gradient vectors that do not significantly violate the first-order conditions. In brief, there is no way to distinguish between a local optimum and a global optimum on the basis of the first-order conditions alone. Hence, future research should be directed at generalizing the current SD tests by including additional necessary restrictions for global optimality.

Still, the first-order condition is sufficient for investors who can deviate only marginally from the market portfolio. Hence, our results are relevant if, for example, investment principals impose investment constraints that restrict their agents to select a portfolio in the "proximity" of the market portfolio. In fact, as discussed in the introductory section, principal agent problems may also explain why investors are risk seeking for gains; compensation schemes may lead agents to exhibit risk seeking over their principals' wealth.

Of course, our results may be biased by our maintained assumptions: we used a single-period, portfolio-oriented model of a competitive capital market without frictions (apart from short selling restrictions). In addition, we assumed a simple data-generating process with a serially independent and identical distribution for the excess returns. There are good reasons to doubt these maintained assumptions. Further research could focus on relaxing our economic assumptions (e.g., by considering the multiperiod consumption-investment problem, imperfect competition, and market imperfections like transaction costs and taxes) and on relaxing our statistical assumptions (e.g., by using econometric time series estimation techniques to estimate a conditional CDF). In addition, any explanation of asset returns should be capable of explaining both the returns on stocks and the returns on fixed income securities, or the so-called equity premium puzzle (Mehra and Prescott, 1985). Hence, further research could focus on including data on bond indexes, for example, along the lines of Fama and French (1993). Still, at the very least, our results suggest that reverse S-shaped utility functions are capable of capturing the cross-sectional pattern of stock returns even under very simple economic and statistical assumptions.

Finally, we hope that our results provide a stimulus for further research based on Markowitz type utility functions (and non-concave utility functions in general). Also, we hope that this study contributes to the further proliferation of the SD methodology. Since large, statistically representative samples often are available in financial economics, the "nonparametric" SD approach is a useful complement to existing parametric approaches.

Appendix

Proof of Theorem 1. We start by repeating the necessary first-order condition for the case without probability transformation, that is, $\max_{\boldsymbol{\lambda} \in \Lambda} \int u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})dG(\boldsymbol{x}), u \in U(\Psi_{MSD})$:

$$\int u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau} - x_i)dG(\boldsymbol{x}) \geq 0 \ \forall i \in \mathrm{I}. \tag{A1}$$

Note that this inequality applies for all extreme points of Λ, that is, for all coordinate vectors $\boldsymbol{\delta}_i$ of zeros with a unity value for $i \in \mathrm{I}$.
Assume that $\boldsymbol{\tau}$ is optimal for some $v \in U(\Psi_{MSD})$ and $f \in \Pi_{MSD}$, that is,

$$\boldsymbol{\tau} = \arg\max_{\boldsymbol{\lambda} \in \Lambda} \int v(x)df(H_{\boldsymbol{\lambda}}(x)) = \int v(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})f'(H_{\boldsymbol{\lambda}}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}))dG(x). \tag{A2}$$

We cannot directly apply the first-order condition (A.1) in this situation, because the transformed probability $f'(H_{\boldsymbol{\lambda}}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}))$ depends on the portfolio weights $\boldsymbol{\lambda}$. We therefore take another route. Consider the polytope $K(\delta) \equiv \{(1-\delta)\boldsymbol{\tau} + \delta\boldsymbol{\lambda}: \boldsymbol{\lambda} \in \Lambda\}, \delta \in [0,1]$. Since $G(x)$ is smooth, we can always find $\delta^* \in [0,1]$ such that $H_{\boldsymbol{\lambda}}(x) = H_{\boldsymbol{\tau}}(x)$ for all $\boldsymbol{\lambda} \in K(\delta^*)$. Since $\boldsymbol{\tau} \in \mathrm{K}(\delta^*) \subseteq \Lambda$, (A.2) implies

$$\boldsymbol{\tau} = \arg\max_{\boldsymbol{\lambda} \in \mathrm{K}(\delta^*)} \int v(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})f'(H_{\boldsymbol{\tau}}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}))dG(\boldsymbol{x}). \tag{A3}$$

Here, $f'(H_{\boldsymbol{\tau}}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}))$ does not depend on the portfolio weights $\boldsymbol{\lambda}$ and we may directly apply the first-order condition. The extreme points of $K(\delta^*)$ are given by $(1-\delta)\boldsymbol{\tau} + \delta\boldsymbol{\delta}_i, i \in \mathrm{I}$, and hence the first-order condition is:

$$\int v'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})f'(H_{\boldsymbol{\tau}}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau}))(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau} - x_i)dG(\boldsymbol{x}) \geq 0 \ \forall i \in \mathrm{I}. \tag{A4}$$

This inequality also gives the first-order condition for the case without probability transformation, that is, (A.1), if we set $u(x)$ such that $u'(x) = v'(x)f'(H_{\boldsymbol{\tau}}(x))$. The relevant utility function can be obtained by integrating $v'(x)f'(H_{\boldsymbol{\tau}}(x))$ by parts, that is,

$$u(x) = v(x)f'(H_{\boldsymbol{\tau}}(x)) + \int v(x)f''(H_{\boldsymbol{\tau}}(x))d\mathrm{H}_{\boldsymbol{\tau}}(x). \tag{A5}$$

To complete the proof, we need to show that this utility function has a reverse S-shape, that is, $u(x) \in U(\Psi_{MSD})$. This follows from the assumptions that $v \in U(\Psi_{MSD})$ and $f \in \Pi_{MSD}$. Specifically, these assumptions imply $v'(x) \geq v'(y)$ and $f'(H_{\boldsymbol{\tau}}(x)) \geq f'(H_{\boldsymbol{\tau}}(y))$ and hence

$v'(x)f'(H_{\tau}(x)) \geq v'(y)f'(H_{\tau}(y))$ or $u'(x) \geq u'(y)$ for all $(x,y) \in \Re_{-}^{2} : x \leq y$ and $(x,y) \in \Re_{+}^{2} : x \geq y$.

Proof of Theorem 2. If $\boldsymbol{\tau}$ is empirically $U(\Psi)-$SD efficient, then there exists some utility function $u \in U(\Psi)$ for which the first-order optimality condition (14) holds. By construction, the gradient vector $\nabla u(\mathbf{X}^{\mathrm{T}}\boldsymbol{\tau}) \equiv (u'(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\tau}) \cdots u'(\boldsymbol{x}_T^{\mathrm{T}}\boldsymbol{\tau}))^{\mathrm{T}}$, $u \in U(\Psi)$, is a feasible solution, that is, $\nabla u(\mathbf{X}^{\mathrm{T}}\boldsymbol{\tau}) \in B(\Psi)$. The first-order optimality condition (14) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; $\boldsymbol{\tau}$ is $U(\Psi)-$SD efficient only if $\xi(\boldsymbol{\tau}, \Psi) = 0$.

Proof of Theorem 3. Since the unity vector is a feasible solution to the primal problem, that is, $\boldsymbol{e} \in B(\Psi)$, we know that $\xi(\boldsymbol{\tau}, \Psi) \leq \omega(\boldsymbol{\tau}) \equiv \max_{i \in \mathrm{I}} \left\{ \sum_{t \in \Theta} (x_{it} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau})/T \right\}$. Using known results, we can derive the exact asymptotic sampling distribution of $\omega(\boldsymbol{\tau})$. Under the null $H_0 : E[\boldsymbol{x}] = \mu\boldsymbol{e}$, the vectors $\mathbf{C}\boldsymbol{x}_t = ((x_{1t} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau}) \cdots (\boldsymbol{x}_{Nt} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau}))^{\mathrm{T}}$, $t \in \Theta$, are serially IID random vectors with zero mean and covariance matrix $\mathbf{C\Omega C}^{\mathrm{T}}$. Hence, the Lindeberg–Levy central limit theorem implies that the vector $\mathbf{C}\mathbf{X}\boldsymbol{e}/T = (\sum_{t \in \Theta} (x_{1t} - \boldsymbol{x}_t^{\mathrm{T}})/\mathbf{T} \cdots \sum_{t \in \Theta} (x_{Nt} - \boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\tau})/T)^{\mathrm{T}}$ obeys an asymptotically joint normal distribution with zero mean and covariance matrix $\boldsymbol{\Sigma} \equiv (\mathbf{C\Omega C}^{\mathrm{T}})/T$. Hence, $\omega(\boldsymbol{\tau})$ asymptotically behaves as the largest order statistic of N random variables with a multivariate normal distribution, and $\Pr[\omega(\boldsymbol{\tau}) > y|H_0] = 1 - \Pr[\mathbf{C}\mathbf{X}\boldsymbol{e}/T \leq y\boldsymbol{e}]$ asymptotically equals the multivariate normal integral $1 - \int_{z \leq y\boldsymbol{e}} d\Phi_{\boldsymbol{\Sigma}}(z)$. Since $\xi(\boldsymbol{\tau}, \Psi) \leq \omega(\boldsymbol{\tau})$, $\Pr[\xi(\boldsymbol{\tau}, \Psi) > y|H_0]$ is bounded from above by $\Pr[\omega(\boldsymbol{\tau}) > y|H_0]$ for all return distributions $G(\boldsymbol{x})$. Moreover, it is easily verified that there exist $G(\boldsymbol{x})$ for which $\xi(\boldsymbol{\tau}, \Psi)$ approximates $\omega(\boldsymbol{\tau})$, and therefore the asymptotic distribution of $\omega(\boldsymbol{\tau})$ also represents the asymptotic least favorable distribution for $\xi(\boldsymbol{\tau}, \Psi)$.

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