Extreme Value Dependence in Financial Markets: Diagnostics, Models, and Financial Implications

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This article presents a general framework for identifying and modeling the joint-tail distribution based on multivariate extreme value theories. We argue that the multivariate approach is the most efficient and effective way to study extreme events such as systemic risk and crisis. We show, using returns on five major stock indices, that the use of traditional dependence measures could lead to inaccurate portfolio risk assessment. We explain how the framework proposed here could be exploited in a number of finance applications such as portfolio selection, risk management, Sharpe ratio targeting, hedging, option valuation, and credit risk analysis.

Recent studies in finance have highlighted the importance of rare events in asset pricing and portfolio choice. These rare events might be in the form of a large change in investment returns, a stock market crash, major defaults, or the collapse of risky asset prices. In many cases the most efficient and sometimes the only effective way of studying these rare events is through extreme value theories. This article is the first to explore the extremal dependence structure among risky asset returns and present a general framework for identifying and modeling joint-tail returns distributions.

The most important evidence that rare events matter comes from option prices: for example, the tendency of stock index options to overpredict volatility and jump risk, the implicit pricing kernel puzzle [Jackwerth (2000)], and the left-skewed risk-neutral density derived from put option

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prices, [see Bates (2001), Bakshi and Cao (2002), and Pan (2002)]. These phenomena are consistent with an equilibrium model of rare event premia when the rare events are unpredictable and cannot be hedged using tradable instruments [Collin-Dufresne and Hugonnier (2000) and Liu, Pan, and Wang (2002)]. Another manifestation of the impact of rare events is the effect of event risk on portfolio holding strategies. Liu, Longstaff, and Pan (2002) show that an investor facing event risk is less willing to take leverage or short positions and acts as if some portion of their wealth has become illiquid. Jansen, Koedijk, and de Vries (2000) show how univariate extreme value techniques could be used to construct a portfolio when an additional constraint is placed on downside risk. Basak and Shapiro (2001) show that value-at-risk (VaR)-constrained managers will hold a very different portfolio and will often choose a larger exposure to risky assets than a non-risk-regulated manager. Finally, there is an extensive literature devoted to developing asset pricing models that involve high moments (viz. skewness and kurtosis) of risky asset returns [see, e.g., Levy (1969), Samuelson (1970), Rubinstein (1973), Kraus and Litzenberger (1976), Scott and Horvath (1980), Friend and Westerfield (1980), Kane (1982), Sears and Wei (1985), Harvey and Siddique (2000), and Dittmar (2002)]. Rare events, by definition, appear in the tails of returns distributions and directly influence the magnitude of all moments. In credit risk analysis, default is often classified as a rare event. In this case, default is usually modeled as a binary variable with the probability of default controlled by a Poisson jump process.

This article describes a multivariate framework for studying rare events in finance. Such a framework is particularly useful when an asset return consists of more than one component, as in the case of a portfolio, or when it is driven by more than one pricing factor. The relationship among the constituents (or pricing factors), when they each are extreme, will give us a better understanding of the tail behavior of the asset of interest. In the extreme value literature, such a relationship is called the extremal dependence structure.

Basically dependence structures can be grouped into four types: independent, perfect dependent, asymptotic independent, and asymptotic dependent.[1] For positively related and asymptotically dependent/independent variables, large values of each variable will occur simultaneously more often (less often) than if these variables are independent (perfectly dependent). The distinction between the two asymptotic dependence structures occurs as both variables approach their respective upper limits. As one variable tends to its upper limit, the chance of the other variable being close to its upper limit goes to zero for asymptotically independent

[1] Note that if a random variable is rescaled such that it is in the interval [0,1], asymptotic refers to the situation when the variable is approaching 1, the upper limit.

random variables, but to a nonzero limit for asymptotically dependent variables. Consequently the extreme values in each variable can occur simultaneously only for asymptotically dependent variables, but will always arise at separate times for asymptotically independent variables. Very few finance articles make a clear distinction between these dependence structures. Most adopt multivariate extreme value models that assume asymptotic dependence. We show in this article, using returns on five major stock market indices, that international stock market returns tend to be asymptotically independent. Hence conventional multivariate models that are based on the assumption of asymptotic dependence will overestimate the probability of joint occurrence.

The conventional dependence measure, the Pearson correlation ρ, is constructed as an average of deviations from the mean. It makes no distinction between large and small realizations, and it does not distinguish between positive and negative returns. It assumes a linear relationship and a multivariate Gaussian distribution, which might lead to a significant underestimation of the risk from joint extreme events. Here we illustrate how two distribution-free dependence measures, χ and $\bar{\chi}$, may be used to identify the type of extremal dependence structure. We also show how parametric methods might be used to model the joint-tail distribution for calculating VaR as an example of practical application. We discuss how the correct identification of extremal dependence structure can be crucial in many other finance applications such as portfolio selection, investment hedging strategy, Sharpe ratio targeting, option pricing, and credit risk analysis.

Further analyses on returns on the five major stock markets reveal that, consistent with previous literature, left-tail dependence is much stronger than right-tail dependence. Moreover, tail dependence appears to increase in more recent time and especially among the European countries. With the use of both univariate and bivariate volatility filters, we find that extremal dependence among volatility-filtered residual returns is much weaker. The choice of volatility filter turns out to be unimportant. Heteroscedasticity is a major source of tail dependence, but it cannot explain away all the stock market co-crashes.

The remaining sections are organized as follows: Section 1 describes the two-tail dependence measures, χ and $\bar{\chi}$ that quantify the degree of asymptotic dependence and asymptotic independence, respectively. Section 2 describes how these two dependence measures may be estimated nonparametrically. Section 3 gives two illustrative parametric models that we use for asymptotically dependent and asymptotically independent variables. Section 4 contains the empirical analyses, which include a description of the data sources and a report of empirical findings. Section 5 demonstrates, with a VaR example, the impact of dependence structure on portfolio risk. Specifically, we quantify the impact of an invalid assumption of asymptotic dependence on portfolio risk assessment. Section 6 discusses the importance of the distinction between asymptotic dependence and asymptotic independence in other key finance applications. Section 7 concludes.

1. Dependence Measures for Multivariate Extreme

The joint distribution of a set of variables can be separated into their respective marginal distributions and the dependence structure among them. In the study of multivariate dependence structure, it is helpful to remove the influence of marginal aspects first by transforming the raw data to a common marginal distribution. After such a transformation, differences in distributions are purely due to dependence aspects.[2] So here we transform the bivariate returns (X, Y) to unit Fréchet marginals (S, T) as follows:

$$S = -1/\log F_X(X) \quad \text{and} \quad T = -1/\log F_Y(Y), \tag{1}$$

where F_X and F_Y are the respective marginal distribution functions for X and Y. The Fréchet transformation is used because of the widely documented fat-tail distributions for risk asset returns [Loretan and Phillips (1994)]. Consequently S and T have the distribution function $F(s) = e^{-1/s}$ for $s > 0$, so Pr $(S > s) = \Pr(T > s) = s^{-1} + O(s^{-2})$ as $s \to \infty$. The variables (S, T) possess the same dependence structure as (X, Y) since[3]

$$\begin{aligned} P(q) &= \Pr[F(T) > q | F(S) > q] \\ &= \Pr[Y > F_Y^{-1}(q) | Y > F_X^{-1}(q)]. \end{aligned} \tag{2}$$

We will focus our discussions on the dependence estimation in the bivariate context, though the ideas and techniques extend naturally to higher dimensions. The variables S and T are said to be *asymptotically independent* if $P(q)$ has a limit equal to zero as $q \to 1$. If the limit in Equation (2) is nonzero, S and T are described as *asymptotically dependent*. Values of $P(q)$ greater than, equal to, and less than $1 - q$ indicate positive dependence, independence, and negative dependence, respectively, at percentile q.

Putting this concept into context, Figures 1a and 1b present scatterplots of 1000 daily stock market returns in the United States versus Japan and those of Germany versus France for the period November 28, 1990, to November 12, 2001. The dependence for the German-French stock market returns is persistent for both positive and negative extreme returns,

[2] Embrechts, McNeil, and Strautman (1999) explain the importance of separating the marginal distributions and the dependence structure.

[3] In practice, the values of F_X and F_Y in Equation (1) are estimated separately from the empirical distribution functions of X and Y, possibly supplemented by estimates of the univariate tail model.

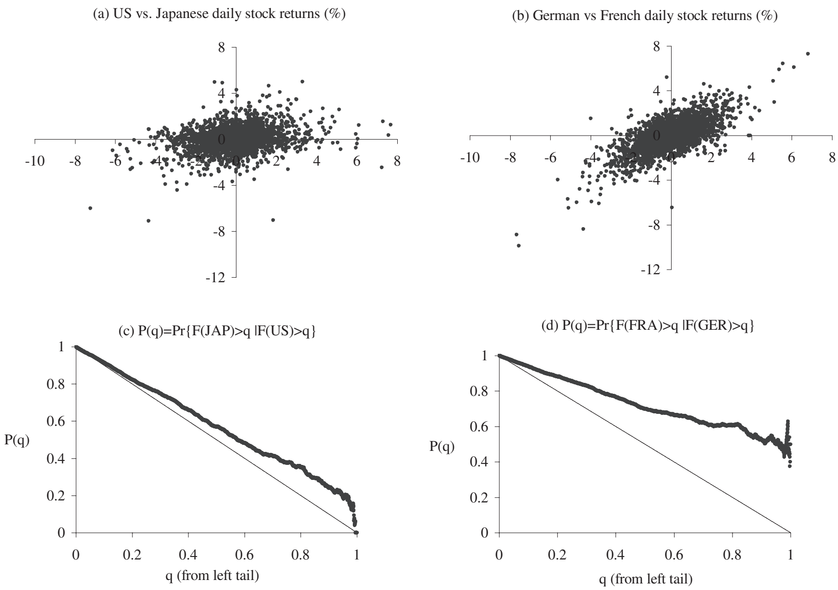

Figure 1
Scatter plots of daily stock market returns (%) and uniform marginal probabilities for the left tail of the stock return distributions for the period from November 28, 1990 to November 12, 2001.

which is indicative of the variables being asymptotically dependent. In contrast, the extremal dependence between U.S. and Japanese returns is much weaker, although the largest values in the left tail for one variable coincide with moderately large values of the same sign for the other variable, suggesting that the variables are positively dependent but *asymptotically independent*. Figures 1c and 1d present empirical estimates of $P(q)$ for the left tails of the two stock return pairs. In Figures 1c and 1d, $P(q)$ is greater than $1 - q$, suggesting some positive dependence for both pairs. But for the U.S. and Japanese pair, $P(q)$ drops steadily toward zero as $q \to 1$ in Figure 1c, indicating *asymptotic independence*. On the other hand, $P(q)$ approaches 0.4 as $q \to 1$ for the German and French pair in Figure 1d indicating *asymptotic dependence*. It follows that, systemic risk is more prominent for a portfolio of European stocks as compared with one that crosses the Pacific ocean.[4]

In Sections 1.1 and 1.2, we present measures of the degree of asymptotic dependence and asymptotic independence, respectively, that exploit the intuitive findings above. Though it will be seen that these measures have some similarities with the coskewness measure, they are not exactly the same as coskewness. Positive coskewness describes the case where large

[4] Such a detailed level of information is not possible to derive from Pearson correlation coefficients, which is 0.267 for the United States versus Japan and 0.695 for Germany versus France.

positive realizations of one series coincide with large absolute realizations of another series. The extremal measures here can be used to measure dependency of the same or different signs and they provide two further advantages: they are invariant to the marginal distributions of the random variables and they give the most relevant summary of the dependence between variables for extreme values as they quantify the characteristics of $P(q)$ as $q \to 1$. Harvey and Siddique (2000) find that systematic coskewness (or coskewness with market factor) is priced. If tail events are systematic as well, one might expect the extremal dependence between the asset returns and the market factor returns to also command a risk premium. Our extremal dependence measures provide the flexibility for testing such a pricing relationship separately for positive, negative, or unsigned systematic tail dependence.

1.1 Asymptotic dependence

As variables S and T are now on a common scale, events of the form $\{S > s\}$ and $\{T > s\}$, for large values of s, correspond to equally extreme events for each variable. The first nonparametric measure of dependence, χ, is based on $P(q)$ in Equation (2) with

$$\begin{aligned}\chi &= \lim_{q \to 1} P(q) \\ &= \lim_{s \to \infty} \Pr(T > s \mid S > s) \\ &= \lim_{s \to \infty} \frac{\Pr(T > s, S > s)}{\Pr(S > s)}\end{aligned}$$

and $0 \leq \chi \leq 1$.

Following from the previous discussion, S and T are asymptotically dependent if $\chi > 0$, and they are perfectly dependent if $\chi = 1$. In the last few years several multivariate extreme value applications have appeared in the finance literature, all assuming asymptotic dependence.[5] If $\chi = 0$, S and T are asymptotically independent. Note that all bivariate normal variables with a Pearson correlation of $\rho \neq 1$ are asymptotic independent [Sibuya (1960)] with $\chi = 0$. Yet these variables are not independent if $\rho \neq 0$. In practice, traditional multivariate extreme value methods[6] immediately assume $\chi > 0$ when independence of the variables is rejected. This approach leads to traditional methods over estimating probabilities of

[5] Longin and Solnik (2001) explored the use of multivariate extreme value methods for stock market returns, which Longin (2000) used to demonstrate how VaR of a position can be derived. Stariça (2000) found a high level of dependence between the extreme movements of most of the currencies in the European Union. Marsh and Wagner (2000) found extremal dependence between stock returns and trading volume among equity markets. Hartmann, Straetmans, and de Vries (2000) found co-crashes between stock and bond markets and some evidence of extreme returns cross-border linkages. Hauksson et al. (2001) studied extreme risk aggregation in a portfolio of two currencies.

[6] See, for example, de Haan (1985), Resnick (1987), de Haan and de Ronde (1998), Coles and Tawn (1991), Longin and Solnik's (2001) Equation (4), and the logistic multivariate tail modeled here in Equation (13).

extreme events if the true value of χ is zero, as in the case of a bivariate normal where $0 < \rho < 1$. The degree of such bias will depend on the rate at which $\Pr(T > s \mid S > s) \to 0$ as $s \to \infty$.

1.2 Asymptotic independence

A complementary measure $\overline{\chi}$, developed by Ledford and Tawn (1996), can be used to measure extremal dependence for variables that are asymptotically independent, that is, where $\chi = 0$. Coles, Heffernan, and Tawn (1999) defined $\overline{\chi}$ as

$$\overline{\chi} = \lim_{s \to \infty} \frac{2 \log \Pr(S > s)}{\log \Pr(S > s, T > s)} - 1, \tag{3}$$

where $-1 < \overline{\chi} \leq 1$, and $\overline{\chi}$ is a measure of the rate at which $\Pr(T > s \mid S > s)$ approaches zero. For perfect dependence, $\Pr(S > s, T > s) = \Pr(S > s)$ and $\overline{\chi} = 1$. For independence, $\Pr(S > s, T > s) = [\Pr(S > s)]^2$, so $\log \Pr(S > s, T > s) = 2 \log \Pr(S > s)$ and $\overline{\chi} = 0$. Hence values of $\overline{\chi} > 0, \overline{\chi} = 0$, and $\overline{\chi} < 0$ correspond, respectively, to when S and T are positively associated, independent, and negatively associated in the extremes. For the bivariate normal dependence structure, $\overline{\chi} = \rho$. For other examples, see Heffernan (2000).

The pair of dependence measures $(\chi, \overline{\chi})$ together provide all the necessary information for characterizing the form and the degree of extremal dependence. It is important to test if $\overline{\chi} = 1$ first before drawing conclusions about asymptotic dependence based on estimates of χ. For asymptotically dependent variables, $\overline{\chi} = 1$ with the degree of dependence given by $\chi > 0$. For asymptotically independent variables, $\chi = 0$ with the degree of dependence given by $\overline{\chi}$.

2. Nonparametric Estimation for χ and $\overline{\chi}$

2.1 Hill estimator

The estimation of χ and $\overline{\chi}$ is based on fitting a univariate model for exceedances of a high threshold. A number of diagnostic techniques exist for threshold selection. Here we adopt a bootstrap method from Danielsson and de Vries (1997) that produces an optimal threshold level above which inference conclusions should be insensitive to increases in the threshold level.

The tail of a univariate heavy-tailed variable Z above a high threshold u satisfies

$$\Pr(Z > z) = \mathcal{L}(z) z^{-1/\xi} \quad \text{for } z > u, \tag{4}$$

where ξ is a shape parameter, also called the tail index, and $\mathcal{L}(z)$ is a slowly varying function of z, that is

$$\lim_{u\to\infty}\frac{\mathcal{L}(uz)}{\mathcal{L}(u)} = 1 \quad \text{for all } z > 0. \tag{5}$$

Treating the slowly varying function as a constant for all $z > u$ and under the assumption of independent observations, the maximum-likelihood estimators for ξ, known as Hill's estimator [Hill (1975)], and $\mathcal{L}(z)$ are

$$\hat{\xi} = \frac{1}{n_u}\sum_{j=1}^{n_u}\log\left(\frac{z_{(j)}}{u}\right), \tag{6}$$

$$\widehat{\mathcal{L}(z)} = \frac{n_u}{n}u^{1/\hat{\xi}}, \tag{7}$$

where $z_{(1)},\ldots,z_{(n_u)}$ are the n_u observations of variable Z that exceed u. There have been many studies into the properties of Hill's estimator, with the problem of bias found when the slowly varying function exhibits nonconstancy above u due to higher-order terms in $\mathcal{L}(z)$. However, there is no easy way to overcome this bias in practice, so we attempt to be cautious in drawing conclusions using this estimator.

The above discussion applies to *i.i.d.* variables. Unfortunately, temporal dependence adds complications to the threshold exceedance method. One possible solution is to ignore the dependence and apply the methods as if the data were independent. This leads to unbiased estimators, but with standard errors that are too small.[7] In the empirical study in Section 4, we assess the possible bias of the standard error estimate that is due to volatility clustering and autocorrelation of returns.

2.2 The estimation of $\overline{\chi}$

Ledford and Tawn (1996, 1998) established that under weak conditions

$$\Pr(S > s, T > s) \sim \mathcal{L}(s)s^{-1/\eta} \quad \text{as } s\to\infty, \tag{8}$$

where $0 < \eta \le 1$ is a constant and $\mathcal{L}(s)$ is a slowly varying function. From this representation it follows that

$$\overline{\chi} = 2\eta - 1 \tag{9}$$

and that if $\overline{\chi} = 1$, corresponding to $\eta = 1$, then $\chi = \lim_{s\to\infty}\mathcal{L}(s)$. Thus the estimation of η and $\lim_{s\to\infty}\mathcal{L}(s)$ provide the basis for estimating

[7] Two other solutions are to (i) decluster the exceedances of the threshold to produce approximately independent data [see Davison and Smith (1990)]; and (ii) use robust methods for standard error evaluation [see Coles and Walshaw (1994)] and Drees (2000)]. Many finance applications of extreme value techniques published to date have directly assumed temporal independence.

χ and $\bar{\chi}$.[8] We make the modeling assumption that Equation (8) is an equality for $s > u$.

Inference follows using univariate extreme value techniques by identifying that if $Z = \min(S, T)$ then

$$\begin{aligned}\Pr(Z > z) &= \Pr\{\min(S, T) > z\} \\ &= \Pr(S > z, T > z) \\ &= \mathcal{L}(z) z^{-1/\eta} \quad \text{for } z > u,\end{aligned} \tag{10}$$

for some high threshold u. From Equation (10) and the univariate form in Equation (4), it can be seen that η is the tail index of the univariate variable Z, and so can be estimated using the Hill estimator from Equation (6), truncated to the interval $(0, 1]$, and $\mathcal{L}(z)$ can be estimated by Equation (7). This requires us to make the same assumption of a constant slowly varying function as was used in obtaining Equation (7). The following development is based on the assumption of independent observations on Z. From this formulation we obtain our estimator for $\bar{\chi}$ to be

$$\hat{\bar{\chi}} = \frac{2}{n_u}\left(\sum_{j=1}^{n_u} \log\left(\frac{z_{(j)}}{u}\right)\right) - 1$$
$$\operatorname{var}(\hat{\bar{\chi}}) = (\hat{\bar{\chi}} + 1)^2 / n_u,$$

with the notations as in Equations (6) and (7).[9] Results in Draisma (2000) suggest this estimator performs well (in terms of mean squared error) and does not suffer from biases found in other applications of Hill's estimator.

2.3 The estimation of χ

If $\hat{\bar{\chi}}$ is significantly less than one then we infer the variables to be asymptotically independent and take $\chi = 0$. Only if there is no significant evidence to reject $\bar{\chi} = 1$ do we estimate χ, which we do under the assumption that $\bar{\chi} = \eta = 1$. Using the maximum-likelihood estimator given by Equation (7) under the constraint $\hat{\bar{\chi}} = 1$, our estimator of χ is

$$\hat{\chi} = \frac{u n_u}{n},$$
$$\operatorname{var}(\hat{\chi}) = \frac{u^2 n_u (n - n_u)}{n^3}.$$

[8] There is the possibility that $\eta = 1$ and $\mathcal{L}(s) \to 0 \to 0$ as $s \to \infty$, leading to asymptotic independence. This boundary case cannot be identified from the data, as the slowly varying function cannot be identified other than as a constant. Misspecification of the dependence structure in this situation is unlikely to be important. Thus we focus on inference for η and $\lim_{s\to\infty} \mathcal{L}(s)$, *treating the slowly varying function as constant over some threshold u, that is,* $\mathcal{L}(s) = d$ *for* $s > u$.

[9] Alternative estimators of η, or equivalently $\bar{\chi}$, have been proposed [see Peng (1999) and Draisma et al. (2001)].

3. Parametric Joint-Tail Models

The two extremal dependence measures χ and $\overline{\chi}$, described in Section 1 and estimated nonparametrically in Section 2, help us to identify the type of tail dependence. This section shows how the joint tails of asymptotically independent and asymptotically dependent variables may be modeled parametrically. Such a framework is required for making specific inferences about the joint occurrence of extreme values. We present only parametric estimates of the joint tail as parametric models exist for both asymptotically dependent and asymptotically independent cases. In contrast, nonparametric methods [e.g. Draisma (2000)] are currently restricted to asymptotic dependence cases, so despite their flexibility in that situation, they are overly restrictive for our purposes.

We need the parametric models to have the same tail dependence characteristics as that identified by the nonparametric summaries. Here we are interested in applying the joint-tail model for the bivariate variable (X, Y) over regions of the form

$$R = \{(-\infty, u_X) \times (-\infty, u_Y)\}^c, \tag{11}$$

where u_X and u_Y are high marginal thresholds. We separate the marginal distributions and the dependence structure, and describe them in turn.

3.1 Marginal structure

As we require a model over the joint-tail region, R, it is not sufficient to model the two marginal distributions above the thresholds u_X and u_Y, respectively. Conditional on being above the threshold, a generalized Pareto distribution (GPD) is used. Below the threshold we have much more data and no natural parametric model. So the empirical distribution function, $\tilde{F}(x)$ is used. Consequently our model for $F_X(x)$ (and similarly for $F_Y(y)$) is

$$F_X(x) = \begin{cases} \tilde{F}_X(x) & \text{if } x < u_X \\ 1 - \{1 - \tilde{F}_X(u_X)\}\{1 + \xi_X(x - u_X)/\sigma_X\}^{-1/\xi_X} & \text{if } x \geq u_X, \end{cases} \tag{12}$$

where σ_X and ξ_X are the GPD scale and shape parameters, respectively. After estimation of the marginal parameters we transform, via Equation (1), the marginal variables (X, Y) to unit Fréchet form (S, T) to model the dependence structure.

3.2 Dependence structure

There are many parametric dependence models that have identical values for the pair $(\chi, \overline{\chi})$, see [Heffernan (2000)]. As our aim is to emphasize the difference between asymptotically dependent and asymptotically independent forms of dependence structure, here we consider only a single parametric model for each class. Further justification for this approach comes from experience gained from the studies by Ledford and Tawn (1996) and Dupuis and Tawn (2001) that if the model has an appropriate $\bar{\chi}$ value, the precise form of the dependence structure is relatively unimportant.

We present both dependence models over the whole sample space. However, as we are only interested in the extreme joint tail, these models are fitted to data over a pair of high thresholds. Specifically we fit the model parameters by matching their associated $(\chi, \bar{\chi})$ values with our nonparametric estimates $(\hat{\chi}, \hat{\bar{\chi}})$. This form of statistical inference is unique in this study and has not appeared in the literature before.

3.2.1 An asymptotically dependent model: logistic. As an illustrative model for asymptotic dependence, we use a bivariate extreme value distribution with logistic dependence model. This dependence structure is the most widely used in parametric approaches to bivariate extreme values [e.g., Longin and Solnik (2001)]. For unit Fréchet margins, the logistic model has the joint distribution

$$\Pr(S \le s,\ T \le t) = \exp\{-(s^{-1/\gamma} + t^{-1/\gamma})^{\gamma}\} \tag{13}$$

with $0 < \gamma \le 1$, where decreasing γ increases dependence. For this model $(\chi, \bar{\chi}) = (2 - 2^{\gamma}, 1)$. When $\gamma = 1$, the variables are exactly independent and $\chi = 0$. When $\gamma < 1$, the variables are asymptotically dependent and $\chi > 0$. Hence it is clear that the logistic model has implicitly excluded the case of asymptotic independence. We estimate γ by $\hat{\gamma} = \log(2 - \hat{\chi})/\log 2$.

3.2.2 An asymptotically independent model: Gaussian. As an illustrative model for asymptotic independence we use the Gaussian dependence structure. The Gaussian joint-tail model was proposed by Bortot, Coles, and Tawn (2000). For unit Fréchet margins, the Gaussian model has the joint distribution

$$\Pr(S \le s,\ T \le t) = \Phi_2(\Phi^{-1}\{\exp(-1/s)\},\ \Phi^{-1}\{\exp(-1/t)\};\ \rho_{ext}), \tag{14}$$

where Φ_2 is the bivariate normal distribution with mean vector $(0, 0)$ and a variance-covariance matrix where the diagonal elements are equal to one and the covariance equal to the dependence parameter ρ_{ext}, with $-1 \le \rho_{ext} \le 1$. For this model $(\chi, \bar{\chi}) = (0, \rho_{ext})$ provided $\rho_{ext} < 1$. We estimate ρ_{ext} by $\hat{\rho}_{ext} = \hat{\bar{\chi}}$. Only if the dependence structure over the whole distribution is Gaussian will ρ_{ext} be equal to the Pearson correlation coefficient.

Bortot, Coles, and Tawn (2000) suggest a method of assessing the fit of the Gaussian joint-tail model based on the Mahalanobis distance of points in the joint-tail region. Specifically, if $A_1, \ldots, A_{N_R}$ denote the two-dimensional Gaussian variables (with standard Gaussian margins and correlation matrix Σ) that fall in the joint-tail region R, then the test statistic

$$MD_R = \frac{1}{N_R}\sum_{i=1}^{N_R} A_i \hat{\Sigma}^{-1} A_i', \tag{15}$$

where $\hat{\Sigma}$ is the estimate of Σ, so has off-diagonal entry $\hat{\bar{\chi}}$. Unlike the usual Mahalanobis distance, MD_R does not follow a standard distribution, so we simulate the critical values of the test statistic.

4. Empirical Analyses

Our data consist of closing stock index levels of the S&P 500 from the United States, FTSE 100 from the United Kingdom, DAX 30 from Germany, CAC 40 from France, and Nikkei 225 from Japan. Our sample period spans from December 26, 1968 to November 12, 2001, giving rise to 8577 daily return observations for each series. Three of the indices (viz. S&P 500, FTSE, and CAC) were created by grafting two returns series from the same country. For example, the U.K. returns are represented by the FT All Shares returns before January 1, 1980, and FTSE returns after that date. Daily index returns are generated by taking first differences of the logarithmic indices. Although some of the returns series do not include the dividend distribution, this is not a problem for our analysis, as dividends do not generate extreme movements.

It has been widely documented elsewhere that the U.S. market has, by far, the greatest influence on all the other stock markets [see Martens and Poon (2001), for example]. The U.S. market is also the latest to close on any particular day among the five stock markets in our sample. This means that any extreme movements in the U.S. stock market are likely to impact on the other stock markets on the following day. Hence, in the following analyses, we use the previous day's U.S. returns whenever the returns pair involves S&P 500 returns. Applying returns synchronization procedures described in Martens and Poon (2001) to our data did not qualitatively change our empirical findings and conclusions.

4.1 Descriptive statistics and univariate tail behavior

Table 1 presents some summary statistics for the five stock index returns. Panel A presents the mean, variance, skewness, and excess kurtosis, evaluated using generalized method of moments to ensure robustness to heteroscedasticity.[10] The average mean returns for all five returns is 0.0216% (or 5.62% per annum) excluding dividends. All series have a significant negative skewness and a kurtosis that is significantly greater than three.

[10] We achieve this using the method described in Richardson and Smith (1993).

Table 1
Summary statistics and tail indices for stock market returns over the period December 27, 1968, to November 12, 2001

	U.S.	U.K.	Germany	France	Japan
	Panel A: Summary statistics				
Mean	0.023	0.027	0.021	0.029	0.008
Standard deviation	0.997	1.058	1.146	1.129	1.168
Skewness	−1.82	−0.31	−0.60	−0.54	−0.27
Excess kurtosis	43.54	9.27	8.79	8.39	11.94
	Panel B: Tail index for daily stock index returns				
Left tail	0.32	0.28	0.29	0.30	0.31
	(0.019)	(0.023)	(0.024)	(0.025)	(0.026)
Right tail	0.28	0.31	0.30	0.26	0.34
	(0.023)	(0.030)	(0.029)	(0.019)	(0.028)
	Panel C: Tail index for daily return residuals from the asymmetric GARCH(1, 1) model				
Left tail	0.27	0.25	0.25	0.24	0.26
	(0.023)	(0.017)	(0.016)	(0.017)	(0.021)
Right tail	0.19	0.21	0.21	0.20	0.29
	(0.018)	(0.014)	(0.017)	(0.018)	(0.019)

The stock market indices are S&P 500 (for the U.S.), FTSE 100 (for the U.K.), DAX 30 (for Germany), CAC 40 (for France), and Nikkei 225 (for Japan).

There are 8577 observations in each returns series and the returns are defined as log differences of stock index.

Mean, standard deviation, skewness and kurtosis are computed based on the generalized method of moments using a weighting matrix correcting for possible first order autocorrelation. Hence, they are robust against heteroskedasticity and serial correlation. Standard errors are in parentheses.

The tail index was estimated using the Hill estimator and the threshold selection is based on the bootstrap method of Danielsson and de Vries (1997).

To disentangle variations in volatility, known to generate excess kurtosis, from extreme returns and to remove clustering of extreme values caused by volatility persistence, we also consider a filtered version of stock returns. The filter we used is an asymmetric version of the GARCH(1, 1) model [see Zakoïan (1994)] as shown in the appendix. The second panel of Table 1 presents the Hill index for both left and right tails of the unfiltered and filtered univariate data series. There are approximately 2% of the 8577 returns observations falling into the tail region for each variable when the method of Danielsson and de Vries (1997) is used for threshold selection. All the tail indices reported in Table 1 are significantly greater than zero. The GARCH filter reduces the tail index by 16.3% on average for the left tail and by 26.8% on average for the right tail, indicating that heteroscedasticity is a contributing factor to extreme price movements.

4.2 Estimates of $\bar{\chi}$

Estimates of $\bar{\chi}$, for selected pairs of raw and filtered returns, are reported in Table 2 for three nonoverlapping subperiods.[11] Two types of GARCH

[11] Note that $-1 < \bar{\chi} \leq 1$ from Equation (3), but our estimator based on the Hill's estimator in Equation (6) is not constrained to ensure that the upper bound is satisfied. Table 2 presents the unconstrained estimates of $\bar{\chi}$.

Table 2
Measures of extreme tail independency, $\bar{\chi}$, for selected daily stock market return pairs over three subperiods

	Unfiltered					Residuals from AGARCH					Residuals from ADC				
		Left tail		Right tail			Left tail		Right tail			Left tail		Right tail	
	ρ	$\bar{\chi}$	s.e.	$\bar{\chi}$	s.e.	ρ	$\bar{\chi}$	s.e.	$\bar{\chi}$	s.e.	ρ	$\bar{\chi}$	s.e.	$\bar{\chi}$	s.e.
Subperiod 1: December 27, 1968–December 12, 1979 (2859 observations)															
US-UK	0.218	0.520	0.126	0.625	0.185	0.214	0.396	0.150	0.221	0.093	0.215	0.416	0.154	0.204	0.093
US-GER	0.216	0.792*	0.191	0.374	0.093	0.191	0.111	0.153	0.234	0.083	0.191	0.416	0.147	0.216	0.082
US-FRE	0.274	0.466	0.107	0.594	0.206	0.240	0.425	0.099	0.178	0.088	0.237	0.393	0.097	0.160	0.097
US-JAP	0.117	0.350	0.104	0.372	0.092	0.112	−0.111	0.117	0.140	0.165	0.115	−0.201	0.118	0.102	0.147
UK-GER	0.105	0.342	0.095	0.319	0.089	0.086	0.207	0.096	−0.090	0.130	0.086	0.182	0.083	0.017	0.136
UK-FRE	0.141	0.477	0.117	0.324	0.094	0.131	0.239	0.083	0.119	0.114	0.133	0.256	0.084	0.059	0.112
GER-FRE	0.166	0.434	0.106	0.164	0.132	0.172	0.324	0.093	−0.091	0.116	0.172	0.334	0.091	−0.225	0.110
Subperiod 2: December 13, 1979–November 27, 1990 (2859 observations)															
US-UK	0.345	0.794*	0.259	1.000*	0.263	0.303	0.438	0.126	0.541	0.121	0.303	0.440	0.106	0.545	0.135
US-GER	0.310	0.604	0.112	0.430	0.100	0.355	0.511	0.105	0.181	0.081	0.358	0.547	0.104	0.183	0.109
US-FRE	0.243	0.483	0.104	0.354	0.176	0.324	0.506	0.104	0.318	0.090	0.335	0.551	0.125	0.316	0.088
US-JAP	0.373	0.648	0.120	0.861*	0.224	0.295	0.460	0.098	0.382	0.097	0.292	0.318	0.157	0.365	0.099
UK-GER	0.358	0.641	0.111	0.908*	0.246	0.283	0.511	0.094	0.256	0.094	0.281	0.404	0.120	0.264	0.092
UK-FRE	0.317	0.730	0.130	0.451	0.103	0.271	0.487	0.103	0.378	0.097	0.273	0.477	0.100	0.349	0.093
GER-FRE	0.409	1.381*	0.330	0.734*	0.154	0.286	0.533	0.105	0.291	0.103	0.289	0.547	0.104	0.343	0.094
Subperiod 3: November 28, 1990–November 12, 2001 (2859 observations)															
US-UK	0.298	0.737*	0.152	0.420	0.096	0.286	0.553	0.107	0.330	0.117	0.291	0.534	0.105	0.293	0.082
US-GER	0.323	0.525	0.103	0.376	0.099	0.343	0.492	0.101	0.206	0.135	0.348	0.516	0.111	0.291	0.088
US-FRE	0.277	0.420	0.120	0.278	0.115	0.271	0.394	0.102	0.232	0.096	0.278	0.350	0.093	0.282	0.092
US-JAP	0.267	0.549	0.104	0.496	0.115	0.265	0.422	0.095	0.312	0.117	0.265	0.440	0.098	0.324	0.115

UK-GER	0.625	1.043*	0.139	0.495	0.207	0.547	0.724	0.117	0.365	0.092	0.550	0.779*	0.120	0.407	0.096
UK-FRE	0.713	0.906*	0.129	0.805*	0.131	0.680	0.849*	0.125	0.597	0.111	0.679	0.866*	0.126	0.633	0.111
GER-FRE	0.695	0.949*	0.144	0.940*	0.153	0.646	0.702	0.115	0.549	0.106	0.648	0.748*	0.152	0.522	0.104

The stock market indices are S&P 500 (for the U.S.), FTSE 100 (for the U.K.), DAX 30 (for Germany), CAC 40 (for France), and Nikkei 225 (for Japan).

ρ is the Pearson correlation coefficient calculated using all observations in the subperiod.

$\overline{\chi}$ is computed based on tail index estimation on Frechet transformed margins of daily coexceedances of stock market returns pair.

*Indicates cases where $\overline{\chi}$ is not significantly different from one.

The univariate filter used in the residuals from AGARCH is an asymmetric version of univariate GARCH,

$$R_t = \omega + \sqrt{h_t}\varepsilon_t,$$
$$h_t = \alpha_0 + \alpha^+ \varepsilon_{t-1}^2 D_{\varepsilon_{t-1} \geq 0} + \alpha^- \varepsilon_{t-1}^2 D_{\varepsilon_{t-1} < 0} + \beta h_{t-1},$$

where R_t is the stock return at time t, α_0, α^+, α^-, and β are parameters, and D_E is the indicator function that event E occurs.

The bivariate GARCH filter used in the residuals from ADC is the Kroner and Ng's asymmetric dynamic covariance (ADC) model,

$$\begin{aligned} R_{it} &= \omega_i + \sqrt{h_{it}}\varepsilon_{it}, && \text{for } i = 1, 2 \\ h_{ijt} &= \theta_{ijt} && \text{for } i = j = 1, 2 \\ h_{12t} &= \rho_{12}\sqrt{\theta_{11t}}\sqrt{\theta_{22t}} + \phi_{12}\theta_{12t} && \\ \theta_{ijt} &= \alpha_{ij} + B_{ij}h_{ij-1} + a_i'\varepsilon_{t-1}'\varepsilon_{t-1}'a_j + g_i'N_{t-1}N_{t-1}'g_j && \text{for all } i, j \end{aligned}$$

where $a_i = [a_{1i}, \alpha_{2i}]'$, $g_i = [g_{1i}, g_{2i}]'$, $N_i = [\min(0, \varepsilon_{1t}), \min(0, \varepsilon_{2t})]'$.

filter were used in the bivariate case here. One is the univariate asymmetric GARCH used in Section 4.1. The second is a bivariate version of the asymmetric dynamic covariance (ADC) model [Kroner and Ng (1998)], as shown in the appendix. The bivariate ADC model was fitted to each returns pair for all subperiods.

Table 2 also presents the Pearson correlation coefficients for raw and filtered returns. All the correlation coefficients are significantly positive and appear to have increased through time, especially for the European countries, possibly due to the effect of economic policy integration as they moved toward a European union. The correlation for the filtered series is slightly weaker, but is still statistically significant.

All but one of the $\overline{\chi}$'s estimated for the unfiltered returns are significantly greater than zero and most of these $\overline{\chi}$ estimates are much larger than the corresponding Pearson correlation coefficients. This reinforces our previous conjecture that the Pearson correlation measure is a poor measure for tail dependence. Moreover, many $\overline{\chi}$ estimates are significantly less than one, which means asymptotic dependence is rejected in these cases.

For all series, the $\overline{\chi}$ estimates are larger for unfiltered than those for filtered returns, indicating that volatility is a major contributing factor to the between-series extremal dependence. Qualitatively there is little difference if a univariate or a bivariate GARCH filter is used. The fact that many of the $\overline{\chi}$'s estimated from the filtered returns are statistically significantly different from zero suggests that GARCH filters cannot completely remove the tail dependency. The volatility scaling has removed some but not all of the extremal dependence.[12] Toward the end of the sample period, there is a strong dominance among unfiltered returns of the left-tail dependence over the right tails.[13] The finding of an increase in dependence through time is often attributed to markets integration and is consistent with findings reported in Speidall and Sappenfield (1992). Here we have been able to show it persists into the extreme events, even after volatility filtering.

4.3 Estimates of $\overline{\chi}$

Estimates of χ for pairs where asymptotic dependence cannot be rejected (i.e., $\overline{\chi} = 1$) are shown in Table 3 with standard errors shown in parentheses.

[12] The univariate and bivariate GARCH models used here model the returns level and conditional variance with quasi-maximum likelihood, assuming that the errors are Gaussian. Future research could test the performance of GARCH models with high moments and non-Gaussian errors. Univariate versions of such GARCH models have appeared in Harvey and Siddique (1999) and Rockinger and Jondeau (2002). The implementation of the bivariate version would require further research.

[13] The dominance of left-tail dependence is consistent with findings in Longin and Solnik (2001) and Martens and Poon (2001).

Table 3
Measures of extreme tail dependence, χ, in subperiods for which market return pairs exhibit asymptotic dependence

		Left tail	Right tail
Subperiod 1	US-GER	0.240 (0.025)	NA
Subperiod 2	US-UK	0.186 (0.027)	0.211 (0.028)
	US-JAP	NA	0.211 (0.025)
	UK-GER	NA	0.202 (0.026)
	GER-FRA	0.286 (0.040)	0.281 (0.024)
Subperiod 3	US-UK	0.290 (0.025)	NA
	UK-GER	0.459 (0.030)	NA
	UK-FRA	0.532 (0.035)	0.470 (0.033)
	GER-FRA	0.517 (0.037)	0.429 (0.033)

Recall that the χ estimate gives the probability of joint occurrence of the most extreme values. For example, the estimates in Table 3 indicate that in the third subperiod, there is about a 50% chance that United Kingdom will experience the largest gain (fall) in the stock market when the largest stock market positive (negative) return is recorded in France. So χ is a true measure for systemic risk in international stock markets. Since a nonzero χ estimate corresponds to only 13 of the 84 pairs of unfiltered returns, we may conclude that the assumption of asymptotic dependence is inappropriate in many cases. Multivariate extreme value theory (EVT) that assumes asymptotic dependency will overestimate the systemic risk of portfolios that consist of asset returns that are asymptotically independent. We will quantify, in Section 5, the impact of such a misspecification for a portfolio of two assets.

4.4 Discussions of model assumptions

The autocorrelation for all returns series are low except in subperiod 1, probably due to index level rescaling and financial markets being less liquid in the earlier period. We used the extremal index, the standard measure of clustering of extreme values in univariate stationary time series [see Smith and Weissman (1994)], to estimate the average size of clusters of extreme values. Small average cluster sizes were found, typically 1.2 observations for the raw series and 1.1 for the filtered and squared filtered series. Consequently the effect of ignoring temporal dependence on the standard errors used in this article is likely to be minimal.

Omitting cases where asymptotic dependence cannot be rejected, we assessed the fit of the Gaussian joint-tail model using the test statistics MD_R in Equation (15). The test statistic casts some doubt over the quality of fit over R in Equation (11), as generally the test statistic is marginally significant at the 95% level. Hence we examined the fit in the key region $(u_X, \infty) \times (u_Y, \infty)$ using the same form of test statistic and found the Gaussian joint-tail model to be a good fit except for the case of United States–United Kingdom in subperiod 3.

4.5 Implications for finance practice

The empirical findings here have a direct impact on practices in the finance industry. For example, the fact that the U.K.–U.S. and U.K.–German markets are asymptotically dependent, while the U.S.–German market is not, means tail diversification and a reduction of portfolio extreme risk can better be achieved by holding U.S. and German stocks. Hence a high χ value indicates a greater exposure to systemic risk in time of financial crisis. To reduce such a systemic risk exposure, one should identify investments that have $\chi = 0$ or $\overline{\chi} < 1$. The smaller the value of $\overline{\chi}$, the greater is the diversification of tail event risk. However, the scope for tail diversification may be decreasing, as our findings indicate that $\overline{\chi}$ is increasing through time and the number of cases where $\chi \neq 0$ has also increased through time. Nevertheless, the measures χ and $\overline{\chi}$ are quick to compute. They can be used as screening tools in the search for suitable portfolio combinations that are less vulnerable to systematic risk.

5. Example on Portfolio Risk Assessment and VaR

In this section we demonstrate the importance of tail dependence in portfolio risk assessment. In particular, we show the consequence of assuming asymptotic dependence when asset returns are asymptotically independent.

To analyze the impact of stock market crashes, we multiply stock returns (X, Y) by -1. For a portfolio $P_a = aX + (1-a)Y$, we are concerned about the distribution of large losses, that is

$$\Pr(P_a > c_a) \tag{16}$$

for large c_a. Given the scarcity of observations in the region $P_a > c_a$, Equation (16) needs to be estimated using a statistical model.

There are two approaches to this problem. The first approach involves constructing the univariate time series of the portfolio returns and analyzing its extreme values using the univariate methods of Section 3.1, with a different analysis being required for each value of a. The second approach estimates the joint distribution of (X, Y) from which $\Pr(P_a > c_a)$ is derived for any portfolio weight a. Though the univariate model is simpler to apply, there are two clear advantages of using the joint distribution approach adopted here. First, by modeling the joint distribution, the changes in features of this distribution over time can be easily built into the extrapolations, examples being the changes in dependence structure between stock returns and the volatility structure of the individual stock return series. Second, for large portfolio losses, the separate losses for the constituent assets in the portfolio can be studied, examples being $E(X \mid P_a > c_a)$ and $E(Y \mid P_a > c_a)$.

5.1 Simulating the joint-tail distribution

To be able to use the parametric models in Section 3, we need $au_X + (1-a)u_Y < c_a$ for each portfolio weight, a, of interest. This condition may be restrictive for small losses, but is fine if c_a is a high value. Evaluation of Equation (16) using the estimated parametric models is not possible in closed form, so Monte Carlo methods need to be employed. We simulate directly from the joint distribution of (X, Y) for each of the parametric models by first simulating from the dependence structure model with a specified marginal distribution and then transforming to the marginal model Equation (12). The simulation of the Gaussian dependence is standard, and we use the scheme of Shi (1995) for generating from the logistic model of Equation (13). The final stage is to evaluate the portfolio for the simulated pairs, noting that for different portfolio weights the same simulated (X, Y) pairs are used, and from this we obtain the proportion with $P_a > c_a$. Importance sampling can reduce the amount of required simulations. We simulated 300,000 pairs of (X, Y) under each parametric specification.

5.2 Portfolio risk and implications for practice

The simulations in the previous subsection produced two pieces of information. The first, analogous to VaR, is the estimated quantile $c_a(\alpha)$, obtained by setting Equation (16) equal to α, with the estimates shown in Figure 2. The second is $E(P_a \mid P_a > c_a(\alpha))$, also known as the expected shortfall [see Artzner et al. (1997, 1999)]. The expected shortfalls for various α's are presented in Figure 3. The left columns of Figures 2 and 3 show risk measures of a portfolio of U.S. and Japanese stocks, whereas the right columns of Figures 2 and 3 show similar information for a portfolio of German and French stocks. To aid comparison, we also repeated these calculations assuming a Gaussian dependence structure with Pearson correlation estimated using all returns data (after transformation to Gaussian margins to reduce the influence of extreme values).

In Figures 2a and 3a, the bivariate Gaussian tail model provides the best description of the dependence structure between large losses in U.S. and Japanese stock markets, since they are asymptotically independent with $\overline{\chi} < 1$. In comparison, a bivariate Gaussian model based on the Pearson correlation coefficient calculated using all returns observations and a bivariate logistic model underestimates[14] and overestimates[15], respectively, the portfolio risk measured in VaR and the expected shortfall. In general, the difference in risk estimates increases as we move toward the limit, that

[14] As one moves out into the tail, the amount of underestimation ranges between 29 and 111 basis points in the case of one-day VaR, and 40 to 184 basis point in the case of the one-day expected shortfall.

[15] As one moves out into the tail, the amount of overestimation ranges between 12 and 87 basis points in the case of one-day VaR, and 23 to 203 basis point in the case of the one-day expected shortfall.

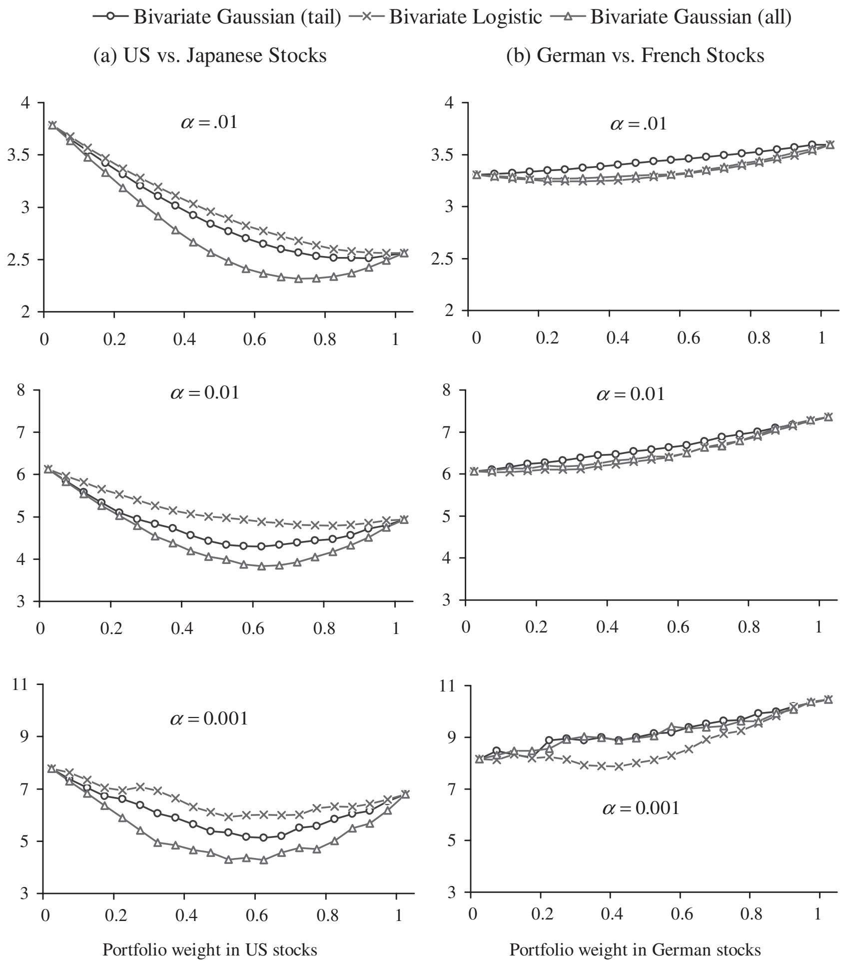

Figure 2
Portfolio one-day value at risk at different tail size based on returns data from December 27, 1968, to November 12, 2001
Portfolio risk is estimated based on 300,000 simulations assuming that the tail dependent structure is (i) bivariate Gaussian (tail) with $\hat{\rho} = \hat{\bar{\chi}}$ and (ii) bivariate logistic with $\hat{\gamma} = \log(2 - \hat{\chi})/\log 2$. In the case of bivariate Gaussian (all), ρ is calculated using all returns data after transformation to Gaussian margins. α is the tail size in terms of probability of returns falling in the tail region. The y-axis is the one-day VaR in daily percentage returns. The x-axis is the portfolio weight. In the left column it is the proportion of investment in U.S. (vis-à-vis Japanese) stocks. In the right column it is the proportion of investment in German (vis-à-vis French) stocks.

is as $c_a(\alpha)$ becomes larger, and is the largest when the portfolio is about equally split between the two stock markets. Hence the failure to recognize asymptotic independence leads to an overestimation of portfolio risk and an understatement of the benefit of tail diversification.

For the case of German and French stock portfolios, the two stock markets are highly related and are asymptotically dependent. This

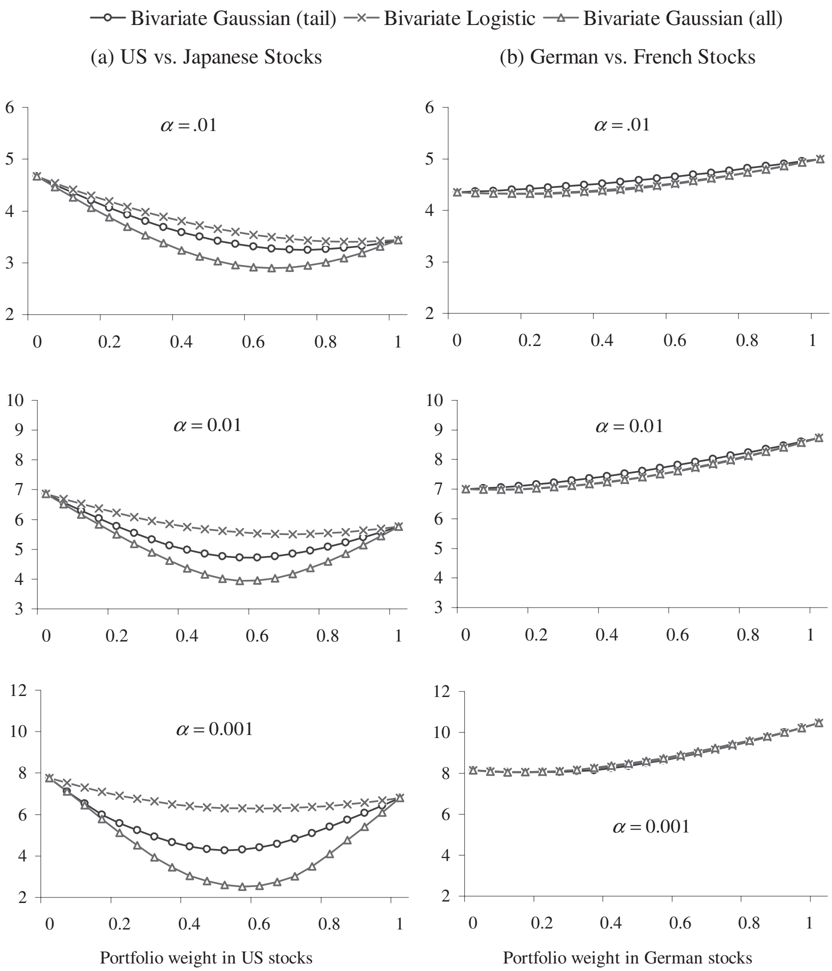

Figure 3
Portfolio one-day expected shortfall at different tail sizes based on returns data from December 27, 1968, to November 12, 2001
Portfolio risk is estimated based on 300,000 simulations assuming that the tail dependent structure is (i) bivariate Gaussian with $\hat{\rho} = \hat{\bar{\chi}}$ and (ii) bivariate logistic with $\hat{\rho} = \log(2 - \hat{\chi})/\log 2$. In the case of bivariate Gaussian (A), ρ is calculated using all returns data after transformation to Gaussian margins.

combination does not provide any tail-level diversification. The VaR and expected shortfall estimates obtained from different models are very similar except for the VaR estimates for the most extreme $c_a(\alpha)$. The difference in expected shortfall estimates produced from different models is negligible in most cases. The implication is that if asset returns are asymptotic dependent, the risk estimation is not sensitive to the wrong assumption for the tail dependence and the methods used to estimate large event risk.

5.3 Relation to other works

The above examples of portfolio risk assessment are mainly illustrative. A full-scale empirical study is warranted in order to explore comprehensively how stock markets of different regions are related to each other under extreme conditions. At the time of writing, there are two other studies that also deal with portfolio VaR. The first, by Glasserman, Heidelberger and Shahabuddin (2000), approaches the problem by using a multivariate t-distribution fitted to all observations. This might overcome some of the weaknesses of the Gaussian model, but it is still insufficient to capture all the characteristics of the tails, as all variables are asymptotically dependent under this model. Longin (2000) uses multivariate extreme value theory that assumes asymptotic dependence and a simplistic aggregation rule $(VaR_P)^2 = \sum_i \sum_j \rho_{ij} \omega_i \omega_j VaR_i VaR_j$. This aggregation formula will misspecify portfolio VaR even for the case of asymptotic dependent variables.[16] The approach we adopt here, for illustrative purposes, is a static one. A time dynamic extension could follow along the line of research in Diebold, Schuermann, and Stroughair (1999) and McNeil and Frey (2000) for the univariate case.

6. Impact of Asymptotic Independence on Other Applications

In the following subsections we describe areas where extremal dependence structure plays a crucial role; in these areas the diagnostic tools and the modeling framework outlined in this article will be useful.

6.1 The impact on portfolio choice

The concept of portfolio diversification was established in the seminal work by Markowitz (1952) on mean-variance efficiency. The empirical findings reported in Section 4 are directly relevant for portfolio diversification. Issues related to tail dependence and implications for portfolio tail diversification and risk management have already been discussed in Sections 4.5 and 5.2. It is important as well to differentiate between asymptotic independence and asymptotic dependence in the studies of systemic risk and contagion, as in Das and Uppal (2001), Bekaert, Harvey and Ng (2002), and Forbes and Rigobon (2002).

For example, Das and Uppal (2001) find systemic risk affects the allocation of wealth between the riskless and risky assets but has little effect on the composition of a portfolio of only-risky assets. But the sensitivity of these findings to the degree of asymptotic independence remains to be assessed. Factor models for the tails of asset returns could

[16] Take the case of the German-French stock portfolio, for example, where the tail distributions are asymptotic dependent, the simple rule underestimates portfolio one-day VaR by 33 to 113 basis points across different values of $c_d(\alpha)$.

potentially be very useful for distinguishing between systemic risk and contagion as the cause for extremal dependence. Note that the factors returns structure could be asymptotically independent or asymptotically dependent depending on the heaviness of the tails of the factors and the conditional distribution of the asset returns given the factors. The methods proposed here allow this structure to be studied through the unconditional dependence structure. However, the extremal properties of the factor models are probably best explored using conditional approaches, as in Heffernan and Tawn (2003).

6.2 Sharpening the Sharpe ratio[17]

The Sharpe ratio, defined as returns in excess of the risk-free rate divided by the standard deviation, is widely used to measure investment performance. A higher Sharpe ratio is generally considered as good, since it usually means higher returns or lower volatility. The Sharpe ratio has its weakness, however, as it erroneously penalizes high volatility that is purely driven by occasional high returns.

To avoid being penalized for exceptionally good returns, fund managers might truncate the right tail of the returns distribution by selling an out-of-the-money call option written at a high strike price. When the stock price rises, all the gains above the option exercise price will be passed on to the option holder if the option is exercised. Although this has the effect of reducing portfolio returns, it will certainly bring down the portfolio volatility and the variability of the fund manager's performance. The premium derived from selling the call option also enhances portfolio performance from the start. A more cautious manager might use the call premium to purchase out-of-the-money puts so that the left tail is similarly truncated and the downside risk protected.

The use of derivatives to massage the Sharpe ratio has several disadvantages. First, the timing when the option is exercised (usually at option maturity) needs not coincide with the evaluation of the portfolio performance. Second, the fund manager might be reluctant to use put options to protect downside risk, as this has the immediate effect of reducing portfolio returns. Finally, such a derivative package has to be reinitialized each time the options expire.

An alternative strategy is to combine assets that are asymptotically independent. A portfolio of assets that are asymptotically independent will have thinner tails than those with assets that are asymptotically dependent. A small $\overline{\chi}$ value for the left (right) tail reduces the need to use put (call) options in Sharpe ratio management. As such, $\overline{\chi}$ will be very

[17] This subsection heading is inspired by a article called "Sharpening Sharpe Ratios" written by Goetzmann et al. (2002).

useful as a screening tool for identifying the most suitable portfolio constituents.

6.3 Impact on hedging strategies

For hedging applications, the hedging instrument should have a high Pearson correlation and be asymptotically dependent with the underlying asset. Our findings here suggest a hedging strategy that is based on modeling time-varying correlation alone [e.g., Muthuswamy et al. (2001)] may be inadequate.

The Multivariate GARCH model imposes some multivariate distributional assumption before measuring the conditional second moments. Hence a hedge ratio that is based on conditional correlation generated from multivariate GARCH models [e.g., Lee (1999)] will be distorted by the marginal distribution assumption imposed. Two findings here weaken the usefulness of GARCH conditional correlation as a hedge ratio: (i) the finding here and in Danielsson and de Vries (1997) of remaining tail dependence after the returns were scaled by conditional volatility; and (ii) the magnitude of conditional correlation is still much smaller than that generated using our distribution-free multivariate EVT technique. The lower partial moment hedge ratio proposed by Lien and Tse (1998) appropriately focuses on the tails, but it is restricted by the assumption of bivariate Gaussianity.

6.4 Valuation of complex options

Tail dependence structure critically affects the valuation of many exotic options such as the Asian option, quanto, options written on related assets [e.g., the power reverse dual in Peterson and Stapleton (2002)], and swaptions just to name a few. Yet the distinction between asymptotic dependence and asymptotic independence has never been made in these applications, whether it is concerned with intertemporal or cross-sectional dependence. In the case of swaptions, it will be the dependence structure of the movements of interest rates of different maturities that is the focus of attention.

The framework described in Section 5 can be used to price these options given the market price of (large event) risk. Alternatively, if option market prices are already available, the market price of (large event) risk can be extracted under the equivalent martingale measure. As the tail dependence structure also differs between left and right tails, this has to be taken into account when valuing deep out-of-the-money puts and calls.

6.5 Credit risk analysis

Recently a handful of studies [e.g., Lucas (1995), Erturk (2000), Zhou (2001) and Cathcart and El-Jahel (2002)] consider the relationship between defaults by different borrowers, in which case the tail dependence structure of the type outlined in this study plays a crucial role in the assessment of credit risk and the valuation of credit risk derivatives. Again, the literature does not make clear the distinction between asymptotic dependence and asymptotic independence. The studies listed above assume Gaussian distributed variables, which will underestimate the probability of joint default since default events could potentially be asymptotically dependent, especially when they are triggered by macroeconomic conditions.

Andreasen's (2001) credit explosives, on the other hand, assume all defaults are asymptotic dependent. A recent empirical study by Das et al. (2002) finds the correlation of default probabilities exhibits regime shifts and that highest-quality issuers show higher default correlations than medium-grade firms. We know that correlation is not a reliable dependence measure as far as tail events are concerned. The two-tail dependence measures described here are better suited for such a study. Li (2000) suggests the use of copula functions in modeling default correlation. This concept is closely related to the extremal dependence structure outlined here. As shown in Section 5, a Gaussian copula corresponds to the case of asymptotic independence, while a logistic copula corresponds to the case of asymptotic dependence.

7. Conclusion

In this article, we describe a multivariate framework whereby two types of extreme value dependence structures may be identified and modeled. These new techniques, which are based on multivariate extreme value theories, enable us to document, for the first time, the widespread asymptotic independence among stock market returns; a phenomenon that has been overlooked in the finance literature. The omission of the case of asymptotic independence could lead to estimation errors of portfolio risk and hence suboptimal portfolio choice, which we demonstrated here using real stock market returns data.

Empirical findings reported here include a confirmation that extreme value cross-sectional dependence is much stronger in bear markets than in bull markets, and that some of this dependency is due to correlated conditional volatilities. However, volatility clustering alone cannot explain all the tail dependence between stock market returns. In general, the dependence between volatilities has increased over time to produce asymptotically dependent stock markets within Europe and strong, but still asymptotically independent stock markets between Europe (United Kingdom, Germany, and France), North America (United States), and Asia (Japan).

Although some of the multivariate results could arguably be obtained using univariate methods, we argue in this article that the multivariate approach is the most efficient and effective way to study extreme events such as systemic risk and crisis. We explain how the dependence measures and modeling framework discussed here may be exploited in a number of important finance applications such as portfolio selection and tail-risk reduction, Sharpe ratio targeting, hedging, option valuation, and credit risk analysis.

Future research could study the time dynamic and aggregation properties of extreme values, the effect of investment time horizons on tail dependence, and the dependence structure between exogenous and leading indicators for financial crises. The tail dependence measures, χ and $\overline{\chi}$, and the framework outlined in this study could serve as a useful starting point.

Appendix: GARCH filters

The univariate GARCH filter used in Section 4.1 is an asymmetric version of the GARCH(1, 1) model:

$$\begin{aligned} R_t &= \omega + \sqrt{h_t}\varepsilon_t \\ h_t &= \alpha_0 + \alpha^+\varepsilon_{t-1}^2 D_{\varepsilon_{t-1}\geq 0} + \alpha^-\varepsilon_{t-1}^2 D_{\varepsilon_{t-1}<0} + \beta h_{t-1}, \end{aligned}$$

where α_0, α^+, α^-, and β are parameters, and D_E is the indicator function that event E occurs. This GARCH specification first appeared in Zakoïan (1994).

We use the bivariate GARCH filter of Kroner and Ng (1998) in Section 4.2. This filter, known as the asymmetric dynamic covariance (ADC) model, specifies that the joint returns $R_t = [R_{1t}, R_{2t}]'$ are distributed according to a bivariate normal density with mean $\omega_t = [\omega_1, \omega_2]'$ and conditional variance covariance matrix $H_t = [h_{ijt}]$, $i, j = 1, 2$:

$$\begin{aligned} R_{it} &= \omega_i + \sqrt{h_{it}}\varepsilon_{it} && \text{for } i = 1, 2 \\ h_{ijt} &= \theta_{ijt} && \text{for } i = j = 1, 2 \\ h_{12t} &= \rho_{12}\sqrt{\theta_{11t}}\sqrt{\theta_{22t}} + \phi_{12}\theta_{12t} \\ \theta_{ijt} &= \alpha_{ij} + b_{ij}h_{ijt-1} + a_i'\varepsilon_{t-1}\varepsilon_{t-1}'a_j + g_i'\zeta_{t-1}\zeta_{t-1}'g_j && \text{for all } i, j, \end{aligned}$$

where $[\varepsilon_t = \varepsilon_{1t}, \varepsilon_{2t}]'$, $a_i = [a_{1i}, a_{2i}]'$, $g_{\mathrm{i}} = [g_{1i}, g_{2i}]'$ and $\zeta_t = [\min(0, \varepsilon_{1t}), \min(0, \varepsilon_{2t})]'$. This is the most general form of bivariate GARCH which encompasses the constant correlation model of Bollerslev (1990), the BEKK model of Engle and Kroner (1995), a constrained version of the VECH model of Bollerslev, Engle and Wooldridge (1988), and the factor ARCH (FARCH) model of Engle, Ng, and Rothschild (1990).

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