Options Trading and the CAPM

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This article studies equilibrium asset pricing when agents face nonnegative wealth constraints. In the presence of these constraints it is shown that options on the market portfolio are nonredundant securities and the economy's pricing kernel is a function of both the market portfolio and the nonredundant options. This implies that the options should be useful for explaining risky asset returns. To test the theory, a model is derived in which the expected excess return on any risky asset is linearly related (via a collection of betas) to the expected excess return on the market portfolio and to the expected excess returns on the nonredundant options. The empirical results indicate that the returns on traded index options are relevant for explaining the returns on risky asset portfolios.

There are many institutional restrictions on the amount of credit that an individual can obtain from the financial markets. For example, individuals face margin requirements when borrowing to purchase risky securities, they must qualify for and collateralize their home mortgages, and they face well-defined spending limits on their credit card agreements. Collectively these restrictions can be thought of as "bounded credit" and are designed to prevent individuals from reaching a state of nature in which they are unable to repay their accumulated debts. As noted by Dybvig and Huang (1988), bounded credit is a plausible economic assumption and is essentially equivalent to a nonnegative wealth constraint. This article examines the implications of nonnegative wealth constraints for asset pricing and shows that, in equilibrium, options on the market portfolio are nonredundant assets.

If some options in the economy are nonredundant, then, as Detemple and Selden (1991) have shown, there should be a general interaction between the returns on risky assets and the returns on the nonredundant options. In fact, the option returns should help to explain the cross section of risky asset returns. The theory presented here sheds some light on the

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exact nature of the pricing relationship when nonredundant options are present in the economy. In particular, a multibeta model is derived in which the expected excess return on any risky asset is linearly related (via a collection of betas) to the expected excess return on the market portfolio *and* to the expected excess returns on the nonredundant options. For example, let R_F denote the gross riskless return, let R_X denote the gross return on some risky asset, let R_C denote the gross return on the market portfolio, and let R_{Oj} denote the gross return on the jth traded (i.e., nonredundant) option. It is shown that

$$E(R_X) - R_F = \beta_{XC}[E(R_C) - R_F] + \sum_j \beta_{XOj}\left[E(R_{Oj}) - R_F\right],$$

where β_{XC} and the collection $\{\beta_{XOj}\}$ are appropriately defined betas, and $E(\cdot)$ denotes expectation. The assumptions used to generate the above expression are parsimonious since, in the absence of the wealth constraints, the options are redundant (i.e., they have no allocational role in equilibrium), the option betas vanish, and the above expression reduces to the capital asset pricing model (CAPM). Thus the above pricing relationship shows precisely how the CAPM of Sharpe (1964) and Lintner (1965) is altered when agents face nonnegative wealth constraints.[1]

As already mentioned, the particular friction that makes options nonredundant is the nonnegative wealth constraint that all agents face.[2] Thus one role of options is their ability to help agents avoid insolvency while still allowing agents to obtain a degree of leverage that may not otherwise be available through direct borrowing. A secondary role of options in this article is their ability to effectively complete the market. In analyzing the role of options for capital markets theory, Ross (1976) shows that standard call and put options are powerful instruments for completing a securities market. In addition, Breeden and Litzenberger (1978) show how a complete set of options (i.e., options whose strike prices coincide with every possible level of aggregate wealth) are sufficient to characterize the prices of Arrow-Debreu securities. In contrast to these two articles, the securities markets studied here are strongly incomplete given the traded

[1] The nonnegative wealth constraints considered here are less stringent than the borrowing constraints analyzed by Black (1972) and Ross (1977) or the margin requirements analyzed by Heath and Jarrow (1987). In particular, agents in the model are free to borrow (i.e., sell short the riskless asset), but they must arrive at the end of the economy with an ability to pay off their debts. In addition, the theory presented here is consistent with the recent empirical results of Bakshi, Cao, and Chen (2000), Buraschi and Jackwerth (2001), and Coval and Shumway (2001), who show that S&P index options appear to be nonredundant.

[2] For articles that motivate options trading along different lines, see Grossman (1988), John, Koticha, and Subrahmanyam (1992), Back (1993), Biais and Hillion (1994), Jarrow (1994), Brennan and Cao (1996), Kraus and Smith (1996), Leland (1996), Bowman and Faust (1997), Easley, O'Hara, and Srinivas (1998), Franke, Stapleton, and Subrahmanyam (1998), Demange and Laroque (1999), and Carr and Madan (2001).

options. However, the traded options do effectively complete the market and therefore lead to an efficient allocation among agents. In equilibrium, the agents in the economy agree on the value of all stochastic payoffs, whether they are traded or not. Hence the model derived here is potentially useful for valuing nontraded assets such as private placements, for estimating a company's cost of capital, and for making capital budgeting decisions.

The option-based model is easily contrasted with the existing asset pricing literature by focusing on the functional form of the stochastic discount factor (SDF). For example, one approach that has received considerable attention in the literature is to use a SDF that is a simple nonlinear function of the market return, such as a polynomial. This type of SDF can be found in the coskewness models of Rubinstein (1973), Kraus and Litzenberger (1976, 1983), Friend and Westerfield (1980), and Harvey and Siddique (2000) and the cokurtosis model of Dittmar (2002). An alternative approach is to assume that the SDF depends on a small number of factor portfolios that are based on firm characteristics, such as size and book-to-market. The model of Fama and French (1993, 1996) can be derived using this type of SDF. In contrast to these approaches, the nonnegative wealth constraints in the current model imply that the SDF depends on both the market return and the returns on a collection of traded options. Thus, this article explicitly shows how traded zero net supply derivatives can affect the SDF. In turn, this form of the SDF directly addresses the problem studied by Fama (1998), that is, the theory presented here suggests that, in addition to the market portfolio, the returns on the traded options are the relevant additional state variables for asset pricing.

The remainder of the article is organized as follows. Section 1 introduces the model, formally shows how the nonnegative wealth constraints lead to options trading in equilibrium, and derives the multibeta pricing model. Section 1 also examines the model's predictions concerning option writing (as a function of agent characteristics, such as wealth levels) and the equilibrium level of options trading volume. In addition, it is shown how the model can be extended to a continuous-time framework using Duffie and Huang's (1985) "continuous trading of few long-lived securities." Section 2 presents the empirical tests. The results indicate that option returns help to capture significant asymmetries in the relationship between risky portfolio returns and the market return. After controlling for the mismatch between the market proxy and the options' underlying asset, the results indicate that growth stocks on average have negative put option betas and positive call option betas. Value stocks have just the opposite risk profile. Small and big stocks both have positive call betas on average, but big stocks have negative put betas while small stocks have positive put betas. Lastly, Section 3 offers concluding remarks.

1. The Model

As in most models, the derivation of a positive theory of asset pricing requires certain simplifying assumptions. In this article the following assumptions are used: (i) all agents have linear risk tolerance with identical cautiousness; (ii) all agents have identical beliefs; (iii) all agents face nonnegative wealth constraints; and (iv) at least one agent has finite marginal utility at zero wealth. Conditions (i) and (ii) deserve no explanation because they are common in various parts of the literature. Conditions (iii) and (iv), on the other hand, drive the model.[3] The nonnegative wealth constraint in (iii) is the main focus of the analysis and condition (iv) simply guarantees that this constraint binds in some states of nature for at least one agent in the economy.

It is shown that conditions (i)–(iv) collectively are sufficient for the risk-sharing rules among the agents to be *piecewise linear* functions of the economy's aggregate wealth.[4] Once the sharing rules are derived, it is shown that the rules can be implemented in a securities market economy by trading options on the market portfolio.[5] The optimal allocation is efficient in the sense that all agents whose nonnegativity constraints do not bind in a given state of nature have identical marginal rates of substitution. This fact allows for a stochastic discount factor to be constructed such that the optimal allocation can be supported in a decentralized economy in which price-taking agents optimally choose positions in the market portfolio and in a collection of options on the market portfolio. The slopes of the various pieces of the sharing rules are used to construct an agent's optimal demand for options while the nondifferentiable points of the sharing rules represent the strike prices of the traded options.

Proceeding with the model, suppose that there are two dates, date 0 and date 1. Fix a number of agents I and suppose that the agents are indexed by $i = 1, 2, \dots, I$. Let the ith agent have the utility function $u_i(c_i)$, where $c_i \geq 0$ denotes the ith agent's date 1 consumption.[6] Because the economy lasts for only one period, date 1 consumption and date 1 wealth are identical. Since

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[3] Indeed, in the absence of (iii) and (iv), the model here reduces to the familiar setting analyzed by Cass and Stiglitz (1970) and Rubinstein (1974) in which borrowing and lending among the agents and trading the market portfolio are sufficient to obtain an optimal allocation. In addition, in the special case of quadratic utility, the CAPM is obtained.

[4] A risk-sharing rule is simply a function that describes how an agent's consumption is related to the aggregate consumption. The idea of describing an optimal allocation by a collection of risk-sharing rules goes back at least to Wilson (1968). Also see Appendix 2 of Rubinstein (1974), Amershi and Stoeckenius (1983), and Chapter 5 of Huang and Litzenberger (1988).

[5] Cox and Huang (1989), Grossman and Vila (1989), and Merton (1990) analyze the individual agent's optimal consumption and investment problem in the presence of nonnegativity constraints. In contrast, the focus here is on equilibria in which *all* agents simultaneously face nonnegativity constraints.

[6] As most would agree, negative consumption is economically infeasible. However, there is nothing particularly special about having zero as a lower bound on consumption. A positive subsistence level of consumption could instead be used as long as the aggregate consumption distribution is consistent with subsistence consumption.

it is assumed that all agents have linear risk tolerance with identical cautiousness and that at least one agent has finite marginal utility at zero, the utility functions take one of three forms.[7] Either all agents have utility functions of the form

$$u_i(c_i) = -\tau_i \exp\left(-\frac{c_i}{\tau_i}\right), \tag{1}$$

where $\tau_i > 0$ is a parameter, or all agents have utility functions of the form

$$u_i(c_i) = \frac{\gamma}{\gamma - 1}(c_i + \tau_i)^{1-1/\gamma}, \tag{2}$$

where $\gamma > 0$ and at least one agent has $\tau_i > 0$ but some agents may have $\tau_i \leq 0$, or all agents have utility functions of the form

$$u_i(c_i) = \tau_i c_i - \tfrac{1}{2}c_i^2, \tag{3}$$

where $\tau_i > 0$.

The agents share a date 1 quantity of aggregate consumption denoted by the continuously distributed random variable C, where the distribution of C is exogenously fixed, that is, the agents do not choose this distribution.[8] For the case in which all agents have constant absolute risk aversion in Equation (1), it is assumed that $C > 0$. For the power utility case, it is assumed that

$$C > -\sum_{i=1}^{I} \tau_i 1_{\{\tau_i < 0\}} \geq 0, \tag{4}$$

where $1_{\{.\}}$ equals one when the expression in brackets is true and equals zero otherwise. This lower bound on C is chosen simply to be consistent with the specification of preferences in Equation (2). For the quadratic utility case in Equation (3), it is assumed that $0 < C < \tau_1 + \tau_2 + \cdots + \tau_I$ so that no agent is ever satiated.[9]

[7] An agent has linear risk tolerance with cautiousness parameter γ if the agent's utility function satisfies the differential equation $-u'(c_i)/u''(c_i) = \tau_i + \gamma c_i$, for some constants τ_i and γ. Up to an increasing linear transformation, there are three solutions to this equation: (i) negative exponential utility, (ii) power form utility, and (iii) logarithmic utility. Equation (1) corresponds to type (i) and Equation (2) covers a subset of type (ii) and type (iii). Equation (3), a quadratic utility, is a special case of type (ii) and is treated separately here since quadratic utility generates strong implications for capital asset pricing.

[8] The conditions under which agents unanimously agree on the distribution of C and the allocation is optimal are rather limited [Wilson (1968, Theorem 11)]. See Gandhi, Hausmann, and Saunders (1985) for a model in which Pareto optimality is abandoned in favor of deriving sharing rules which ensure unanimity of project rankings.

[9] The main analysis here does not alleviate the fact that quadratic utility economies may admit arbitrage opportunities if the aggregate payoff is unbounded. Hence it is assumed that C is bounded for these economies. For a discussion of arbitrage in quadratic utility economies, see Dybvig and Ingersoll (1982) and Jarrow and Madan (1997).

To derive the sharing rules, a collection of weights $\{\lambda_i\}_{i=1}^{I}$ is exogenously fixed that depend on the initial distribution of resources among the agents. In a market context, λ_i is the ith agent's shadow price of initial wealth (i.e., the Lagrange multiplier that ensures that the agent's budget constraint is satisfied at the optimal allocation). Thus $\lambda_i^{-1}u_i'(c_i)$ is interpreted to be a marginal rate of substitution. If the allocation is efficient, then for any two agents i and j whose date 1 nonnegativity constraints *do not* bind, it must be the case that

$$\frac{u_i'(c_i)}{\lambda_i} = \frac{u_j'(c_j)}{\lambda_j}. \tag{5}$$

Let the common value of the expression in Equation (5) equal e^t, where t is a random variable to be determined in equilibrium. If the ith agent's constraint *does* bind, $c_i = 0$. Hence the allocation assigned to the ith agent is simply

$$c_i = \max\left[0, u_i'^{-1}(\lambda_i e^t)\right]. \tag{6}$$

Aggregating Equation (6) gives

$$C = \sum_{i=1}^{I} c_i = \sum_{i=1}^{I} \max\left[0, u_i'^{-1}(\lambda_i e^t)\right]. \tag{7}$$

If the real-valued function $\Phi(x)$ is now defined as

$$\Phi(x) \equiv \sum_{i=1}^{I} \max\left[0, u_i'^{-1}(\lambda_i e^x)\right], \tag{8}$$

then $t = \Phi^{-1}(C)$ and the ith agent's optimal sharing rule is

$$c_i = \max\left[0, u_i'^{-1}\left(\lambda_i e^{\Phi^{-1}(C)}\right)\right]. \tag{9}$$

In the above expression, Φ^{-1} is the piecewise inverse mapping of the function Φ on the interval where Φ is strictly decreasing. In addition, the price of consumption per unit of probability (i.e., the SDF) for this economy is $e^{\Phi^{-1}(C)}$. Given the strong assumptions on preferences, an analytical solution for Φ^{-1} exists for each of the utility functions given above. Thus a closed-form solution for the sharing rules can be obtained in these cases. This result is formally summarized in the following proposition.

Proposition 1. *Suppose that all agents have homogeneous beliefs, have linear risk tolerance with identical cautiousness, and face nonnegativity constraints $c_i \geq 0$. Furthermore, suppose that at least one agent has finite marginal utility at zero wealth. Then the optimal sharing rules in Equation* (9) *are piecewise linear functions of C.*

Proof. See the appendix.

The intuition underlying the proposition is as follows. First, consider the set of states (i.e., the levels of C) for which all agents consume. Since the agents have linear risk tolerance with identical cautiousness, sharing rules are linear *within* this set of states and the slopes of the agents' sharing rules are determined by the characteristics of all of the agents in the economy. As aggregate consumption decreases to a lower level, one or more agents will have binding wealth constraints and will not consume. For this new set of states, risk-sharing rules are still linear, but now the slopes are determined by the characteristics of the agents who *do* consume. Extending the argument, additional agents will face binding constraints as aggregate consumption decreases even further, and this changes the slopes of the sharing rules for those agents who do consume. This leads to piecewise linear sharing rules.

In addition, in the context of a securities market, the stochastic discount factor $e^{\Phi^{-1}(C)}$ can be interpreted as the marginal rate of substitution of an agent (with a smooth utility function) who holds a portfolio containing a long position in the market plus positions in options on the market. To see this, note that since all agents have homogeneous beliefs, associated with each agent is a threshold aggregate wealth level such that the agent has a binding nonnegativity constraint below the threshold and has strictly positive wealth above the threshold. These threshold levels are used to rank the agents. Since the resource constraint in Equation (7) must be satisfied in equilibrium, there is always one agent whose nonnegativity constraint never binds. This agent is called the *pricing agent* for the economy.[10] Since the pricing agent optimally holds a portfolio that includes options, this has strong implications for capital asset pricing.

The implications for asset pricing are investigated by recasting the model in terms of a securities market economy. This requires two steps. The first step is to consider a decentralized complete markets economy in which each agent takes the stochastic discount factor $e^{\Phi^{-1}(C)}$ as given and solves an expected utility maximization problem by choosing a nonnegative date 1 consumption pattern, subject to a budget constraint. The optimal consumption pattern chosen by each agent in this case is equal to the allocation described in Proposition 1. The second step is to identify C as the gross payoff (i.e., liquidating dividend) on the market portfolio and to introduce a collection of securities that the agents are allowed to

[10] Using the terminology of Dybvig and Ross (1986), there is a clientele effect in quantities (since not every agent is marginal on all assets), but not in prices (since assets are priced linearly using the pricing agent's marginal rate of substitution).

trade at date 0. Specifically, the market portfolio and a collection of zero net supply call options on the market portfolio (with appropriately specified strike prices) are taken as the traded securities. Since C is continuously distributed and since there is only one trading period, the securities market described here is strongly incomplete. However, the traded options effectively complete the market because each agent's optimal consumption pattern from the complete markets economy can be obtained in the incomplete securities market economy by purchasing a portfolio of securities at date 0.

Since the agents are as well off in the incomplete securities market economy as they are in the complete markets economy, financial innovation cannot improve the agents' welfare. Specifically, any additional limited liability asset that is introduced into the incomplete securities market economy does not alter the optimal portfolio of any agent. To achieve this, the price of the additional asset in equilibrium is determined by using the pricing agent's marginal rate of substitution. At this price, all agents in the economy view the asset as being fairly valued given their optimal portfolios. Thus no agent trades the additional asset and the asset is priced as if it is redundant.

For the first step of the implementation, let $W_{0i} > 0$ denote the ith agent's date 0 wealth. The ith agent's optimization problem is

$$\max_{c_i} E[u_i(c_i)], \tag{10}$$

subject to the constraints

$$c_i \geq 0 \tag{11}$$

$$E\left[c_i e^{\Phi^{-1}(C)}\right] \leq W_{0i}. \tag{12}$$

The solution to this programming problem is

$$c_i = \max\left[0, u_i'^{-1}\left(\lambda_i e^{\Phi^{-1}(C)}\right)\right], \tag{13}$$

where λ_i solves

$$E\left[e^{\Phi^{-1}(C)} \max\left[0, u_i'^{-1}\left(\lambda_i e^{\Phi^{-1}(C)}\right)\right]\right] = W_{0i}. \tag{14}$$

Since the above programming problem is valid for any agent i, Equation (14) for $i \in \{1, 2, \ldots, I\}$ gives a system of equations that must be solved for the weights $\{\lambda_i\}_{i=1}^{I}$. Thus solving for the equilibrium is equivalent to solving this system of equations.

For the second step of the implementation, relabel the agents so that the pricing agent for the economy is labeled agent 1 and introduce $I-1$ traded call options on the market portfolio where the jth call option has strike price $\Phi[\log(\lambda_j^{-1} u_j'(0))]$ for $j \in \{2, 3, \ldots, I\}$. The price of the market portfolio at date 0, denoted by P_C, is

$$P_C = E\left[e^{\Phi^{-1}(C)} C\right], \tag{15}$$

while the price of the jth traded call option, denoted by P_{Oj}, is

$$P_{Oj} = E\left[e^{\Phi^{-1}(C)} \max\left[0, C - \Phi\left(\log\left(\lambda_j^{-1} u_j'(0)\right)\right)\right]\right], \tag{16}$$

where $j = 2, 3, \ldots, I$. Given that the market portfolio and the call options are traded securities, agents construct portfolios in order to achieve their optimal date 1 piecewise linear payoff patterns given by Equation (13). At the prices given by Equations (15) and (16), each agent's optimal portfolio is budget feasible by construction. Suppose now that a limited liability asset with payoff X is introduced into the economy. The introduction of this asset does not alter any agent's optimal portfolio, since at the price $P_X = E[e^{\Phi^{-1}(C)} X]$ the asset is viewed by all agents as being fairly valued.[11] Thus, in equilibrium, only the market portfolio and the call options on the market portfolio are traded.

Given this structure of asset prices, the risk-neutral pricing measure can be easily identified. If agents' subjective probability beliefs are described by a continuous density $p(C)$, then the risk-neutral density $q(C)$ is

$$q(C) \equiv \mathcal{M}(C)p(C)\left[\int \mathcal{M}(C)p(C)dC\right]^{-1}. \tag{17}$$

Since the stochastic discount factor $\mathcal{M}(C) \equiv e^{\Phi^{-1}(C)}$ is not a smooth function of C, even if agents' subjective probability beliefs are described by a smooth density $p(C)$, the risk-neutral probability density $q(C)$ is not smooth. On the other hand, there may exist a nonsmooth subjective density that makes the risk-neutral density smooth, for example, suppose that $f(C)$ is a smooth density and suppose $p(C) = [f(C)/\mathcal{M}(C)]$ $[\int f(C)/\mathcal{M}(C)dC]^{-1}$. Then by construction $q(C) = f(C)$. Thus when agents face nonnegativity constraints, it is not generally possible to have both the subjective density and the risk-neutral density as smooth functions of the aggregate payoff, C. As an example, suppose that $f(C)$ is a lognormal density and suppose that, for a given $\mathcal{M}(C)$, the subjective density $p(C) = [f(C)/\mathcal{M}(C)][\int f(C)/\mathcal{M}(C)dC]^{-1}$ is well-defined. Then

[11] To see that the agents agree that P_X is the fair value of the asset, it is necessary to explicitly account for the role of the constraints in Equation (11). If $p(C)$ denotes the common subjective probability density for C, then the multiplier for the ith agent's nonnegativity constraint is zero if $c_i > 0$ and is equal to $p(C)\lambda_i[e^{\Phi^{-1}(C)} - u_i'(0)\lambda_i^{-1}] > 0$ if $c_i = 0$. From the ith agent's first-order condition

$$P_X = E[u_i'(c_i)\lambda_i^{-1} X + 1_{\{c_i=0\}}(e^{\Phi^{-1}(C)} - u_i'(0)\lambda_i^{-1})X],$$

where $1_{\{\cdot\}}$ is the indicator function for the event in curly brackets. Hence all agents agree that $P_X = E[e^{\Phi^{-1}(C)} X]$.

$q(C)$ is lognormal and the options are priced in equilibrium according to a Black and Scholes (1973) type of model. In addition, since the options are nonredundant, the options are actually traded at date 0 at these prices.[12]

If preferences are specialized to quadratic utility, then a remarkable feature of the securities market economy presented above is that covariance with the market portfolio is *not* the appropriate quantity for determining the risk premium on any asset. Instead, covariance with a portfolio containing options is the appropriate quantity. In particular, risky assets are priced in equilibrium by a (quadratic utility) pricing agent that holds the market portfolio plus a short position in all of the traded call options. This implies that the risk premium on any asset is proportional to the covariance between the asset's payoff and the payoff on the pricing agent's optimal portfolio. To see this, consider a risky asset with a date 1 gross payoff denoted by X. In equilibrium, the price of the risky asset is given by

$$P_X = E\left[e^{\Phi^{-1}(C)}X\right] = \frac{E(X)}{R_F} + \operatorname{cov}\left(e^{\Phi^{-1}(C)}, X\right), \tag{18}$$

where $(R_F)^{-1}$ is the date 0 price of a riskless discount bond and $e^{\Phi^{-1}(C)}$ is the stochastic discount factor for the quadratic utility economy. From the proof of Proposition 1, $e^{\Phi^{-1}(C)} = \lambda_1^{-1}(\tau_1 - c_1)$, where c_1 is the payoff on the pricing agent's portfolio, that is,

$$c_1 = C + \lambda_1 \sum_{j=2}^{I} \left\{ \left(\bar{\lambda}_j\right)^{-1} - \left(\bar{\lambda}_{j-1}\right)^{-1} \right\} \max\left[0, C - K_j\right] \tag{19}$$

and $\bar{\lambda}_j \equiv \lambda_1 + \lambda_2 + \cdots + \lambda_j$. Since Equation (18) holds for any risky asset or any portfolio of assets, it must hold for the pricing agent's portfolio. Thus the excess return on the risky asset can be written as

$$E(R_X) - R_F = \frac{\operatorname{cov}(R_X, R_{c_1})}{\operatorname{Var}(R_{c_1})}[E(R_{c_1}) - R_F], \tag{20}$$

where R_X is the asset's gross return and R_{c1} is the gross return on the pricing agent's portfolio. The above expression is a CAPM-like relation in which the covariance with the return on the pricing agent's portfolio, which includes options, determines the excess return on the risky asset.

In addition, if Equation (18) is applied to each of the traded assets and if Equation (19) is used, then the excess return on any risky asset can be written in terms of the excess return on the market portfolio and the excess returns on the traded options. Let R_C denote the gross return on the market portfolio and let R_{Oj} denote the gross return on the option with strike price

[12] See Rubinstein (1976) for a derivation of the Black and Scholes (1973) model in a single-period framework.

K_j, where $j \in \{2, 3, \ldots, I\}$. Also define the excess return vector μ as

$$\mu \equiv \begin{bmatrix} E(R_C) - R_F \\ E(R_{O2}) - R_F \\ \vdots \\ E(R_{OI}) - R_F \end{bmatrix}, \tag{21}$$

define the covariance vector ρ as

$$\rho \equiv \begin{bmatrix} \mathrm{cov}(R_X, R_C) \\ \mathrm{cov}(R_X, R_{O2}) \\ \vdots \\ \mathrm{cov}(R_X, R_{OI}) \end{bmatrix}, \tag{22}$$

and let the variance-covariance matrix Σ be defined as

$$\Sigma \equiv \begin{bmatrix} \mathrm{Var}(R_C) & \mathrm{cov}(R_C, R_{O2}) & \cdots & \mathrm{cov}(R_C, R_{OI}) \\ \mathrm{cov}(R_{O2}, R_C) & \mathrm{Var}(R_{O2}) & \cdots & \mathrm{cov}(R_{O2}, R_{OI}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(R_{OI}, R_C) & \mathrm{cov}(R_{OI}, R_{O2}) & \cdots & \mathrm{Var}(R_{OI}) \end{bmatrix}. \tag{23}$$

Then the expected excess return on the risky asset is

$$E(R_X) - R_F = \rho^\top \Sigma^{-1} \mu, \tag{24}$$

where $\rho^\top$ is the transpose of ρ. Equation (24) shows explicitly that the excess return on any risky asset is linearly related to the excess return on the market and the excess returns on a collection of traded options. In particular, by defining a row vector of betas as $\rho^\top \Sigma^{-1}$, Equation (24) is the equation for a security market hyperplane. This equation is the main empirically testable result of the theory. It suggests that a collection of betas (one for the market and one for each of the traded options) exactly explains the cross-sectional variation in expected excess returns. The fact that all of the traded options show up in Equation (24) is consistent with the empirical findings of Buraschi and Jackwerth (2001), who show that options of different moneyness levels are needed for spanning purposes.

1.1 Results on options trading

This section investigates which agents buy and which agents write options in terms of both wealth levels and risk tolerances. Also included are results on which options in a particular economy have the highest trading volume. Before proceeding, a definition is needed:

Definition. *Consider two agents i and j. Agent i is a net writer of options relative to agent j if, for every option contract traded by agent j, either both agents write the same number of contracts or agent i writes strictly more contracts than agent j.*

For example, suppose that two contracts are traded with strike prices 50 and 55. Furthermore, suppose that both agents are short one contract with strike 55, but agent i holds a short position in the contract with strike 50, while agent j holds a long position in the contract with strike 50. Then agent i is a net writer of options relative to agent j. Note that the definition leaves open the possibility that agent i trades in more contracts than agent j. For example, agent i might be long the contract with strike 45, while agent j takes no position in this contract. Essentially the definition says that agent i is a net supplier of the contracts demanded by agent j. Using this definition, the following proposition highlights the main results of this section.

Proposition 2. *The following statements are true*:

- (a) *In the case of the CARA utility given by Equation* (1):
  - (i) *Fix two agents i and j and suppose* $\tau_i = \tau_j$. *If* $W_{0i} > W_{0j}$, *then agent i is a net writer of options relative to agent j. Furthermore, the call option with strike price* $\Phi(\log(\lambda_i^{-1}))$ *has higher trading volume than the call option with strike price* $\Phi(\log(\lambda_j^{-1}))$, *that is, the option with the lower strike has the higher trading volume*;
  - (ii) *Fix two agents i and j and let* $W_{0i} = W_{0j}$. *If agent i is a net writer of options relative to agent j, then* $\tau_j > \tau_i$;
- (b) *In the case of the power utility given by Equation* (2):
  - (i) *Fix two agents i and j and suppose* $\tau_i = \tau_j > 0$. *If* $W_{0i} > W_{0j}$, *then agent i is a net writer of options relative to agent j. Furthermore, the call option with strike price* $\Phi(\log(\tau_i^{-1/\gamma}\lambda_i^{-1}))$ *has higher trading volume than the call option with strike price* $\Phi(\log(\tau_j^{-1/\gamma}\lambda_j^{-1}))$, *that is, the option with the lower strike has the higher trading volume*;
  - (ii) *Fix two agents i and j and let* $W_{0i} = W_{0j}$. *If agent i is a net writer of options relative to agent j, then* $\tau_j > \tau_i$;
- (c) *In the case of the quadratic utility given by Equation* (3):
  - (i) *Fix two agents i and j and suppose* $W_{0i} > W_{0j}$. *If* $\lambda_i = \lambda_j$, *then agent i is a net writer of options relative to agent j. Furthermore, the call option with strike price* $\Phi(\log(\tau_i\lambda_i^{-1}))$ *has a higher trading*

volume than the call option with strike price $\Phi(\log(\tau_j \lambda_j^{-1}))$, *that is, the option with the lower strike has the higher trading volume*;

(*ii*) *Fix two agents i and j and let* $W_{0i} = W_{0j}$. *If agent i is a net writer of options relative to agent j, then* $\tau_j > \tau_i$.

Proof. See the appendix.

The above proposition can be interpreted as follows. Results (*a*)(*i*), (*b*)(*i*), and (*c*)(*i*) present conditions under which wealthier agents are net suppliers of options to less wealthy agents. These results are consistent with the intuition that, all else being equal, wealthier agents can afford to absorb larger losses and therefore write options, while less wealthy agents prefer the limited liability nature of long option positions. These results also show that call option trading volume is decreasing in the strike price, which indicates that in-the-money call options have higher trading volume than out-of-the-money call options. While this fact is at odds with empirical observation, put-call parity can be used to rewrite the sharing rules in terms of put options instead of call options. In this case the proposition indicates that out-of-the-money puts tend to have higher trading volume than in-the-money puts, and this result is observed in many financial markets (essentially investors buy out-of-the-money puts as a form of inexpensive portfolio insurance). On balance, the trading volume results presented here are partially consistent with observed facts.

For results (*a*)(*ii*), (*b*)(*ii*), and (*c*)(*ii*), wealth levels are held constant and the relationship between options writing and preferences is investigated. In particular, if agent *i* is a net supplier of options to agent *j*, then, for a fixed wealth level, agent *j* is more risk tolerant than agent *i*. While agent *i* is a net writer of options, in equilibrium agent *i* covers these short option positions by taking a long position in the market or by taking a long position in call options with lower strike prices.

All of the above results are general in the sense that they do not depend on a particular distribution for the aggregate wealth, *C*. However, while the proposition outlines several results that appear to be intuitive, it does not cover every possible case. For example, it is possible that if a wealthy agent is sufficiently risk tolerant relative to a less wealthy agent, the equilibrium allocation will involve the less wealthy agent writing an option and the more wealthy agent buying this option. Hence the model is very rich in its predictions.

1.2 A dynamic model

Before proceeding to the empirical results, it is demonstrated how the model can be extended to a continuous-time setting. Since the allocation in the economy is efficient, the main tool for accomplishing this extension is Duffie and Huang's (1985) "continuous trading of few long-lived securities." In particular, if the market portfolio and a collection of options on the market portfolio are traded, each agent can achieve his optimal end-of-period wealth by adopting a buy-and-hold portfolio strategy.

To illustrate, an example with two CARA utility agents is examined. The first agent's risk tolerance is τ_1, while the second agent's risk tolerance is τ_2. Time is continuous and is indexed by $t \in [0, T]$. At date T the market portfolio pays a liquidating dividend after which the agents consume and the economy ends. From the previous results of the article, assuming $\lambda_2 > \lambda_1$, the optimal allocation at date T for these two agents is

$$c_{1T} = C_T - \left(\frac{\tau_2}{\tau_1 + \tau_2}\right) \max[C_T - K, 0] \tag{25}$$

$$c_{2T} = \left(\frac{\tau_2}{\tau_1 + \tau_2}\right) \max[C_T - K, 0], \tag{26}$$

where $K \equiv \tau_1[\log(\lambda_2) - \log(\lambda_1)] > 0$ is the strike price of the traded option. Using Equations (25) and (26), agent 1's optimal wealth is attainable by buying the market at date 0, shorting $\tau_2(\tau_1 + \tau_2)^{-1}$ units of the call option at date 0, and holding these positions until date T. Similarly agent 2's optimal wealth is attainable by buying $\tau_2(\tau_1 + \tau_2)^{-1}$ units of the call option at date 0 and holding this position until date T.

For simplicity, uncertainty in the economy is represented by a single stochastic state variable, S_t, whose evolution is governed by the process

$$dS_t = \sigma dz_t, \tag{27}$$

where z_t is a standard Brownian motion and σ is a constant. The market's liquidating dividend is assumed to be equal to the square of the state variable at date T, that is,

$$C_T \equiv S_T^2. \tag{28}$$

The stochastic discount factor for the economy is

$$\begin{aligned}\mathcal{M}(C_T) =& \frac{1}{\lambda_1} e^{-C_T/\tau_1} 1_{\{C_T < K\}} \\ &+ \left(\frac{1}{\lambda_1}\right)^{\tau_1/(\tau_1+\tau_2)} \left(\frac{1}{\lambda_2}\right)^{\tau_2/(\tau_1+\tau_2)} e^{-C_T/(\tau_1+\tau_2)} 1_{\{C_T \geq K\}},\end{aligned} \tag{29}$$

and hence the pricing kernel is defined as

$$\eta_t = \frac{E_t[\mathcal{M}(C_T)]}{E_0[\mathcal{M}(C_T)]}. \tag{30}$$

The price of a discount bond is thus normalized to one (i.e., the interest rate is zero) for all $t \in [0, T]$ and the value of the market portfolio at date t, denoted C_t, is

$$C_t = E_t\left[\frac{\eta_T}{\eta_t} C_T\right] = E_t\left[\frac{\eta_T}{\eta_t} S_T^2\right]. \tag{31}$$

Similarly the value of the traded call option at date t, denoted F_t, is

$$F_t = E_t\left[\frac{\eta_T}{\eta_t}\max[C_T - K, 0]\right] = E_t\left[\frac{\eta_T}{\eta_t}\max\left[S_T^2 - K, 0\right]\right]. \qquad (32)$$

Since S_T given S_t is a normally distributed random variable, closed-form solutions for η_t, C_t, and F_t can easily be calculated.

Figure 1 shows several graphs for the dynamic economy using the parameter values $\tau_1 = 10$, $\tau_2 = 8$, $\lambda_1 = 0.15$, $\lambda_2 = 1.0$, and $\sigma = 0.25$. These values imply that $K = 18.97$. The top left panel of the figure shows the value of the market portfolio and the value of the traded call option as a function of the underlying state variable. The upper curve in this panel is the market portfolio value, while the lower curve is the call option value. Using only positive values of the underlying state variable, the bottom left panel shows the call option value as a function of the market portfolio value. Next, the top right panel shows the implied volatility skew as a function of option moneyness. To determine the volatility skew, call options with various strike prices and 0.5 years until expiration are valued using the above model. The implied volatility for each call option is then calculated by equating the call option value to the Black-Scholes formula using a zero interest rate, a time to expiration of 0.5 years, and an underlying asset value of 18.97. As the top right panel shows, the particular

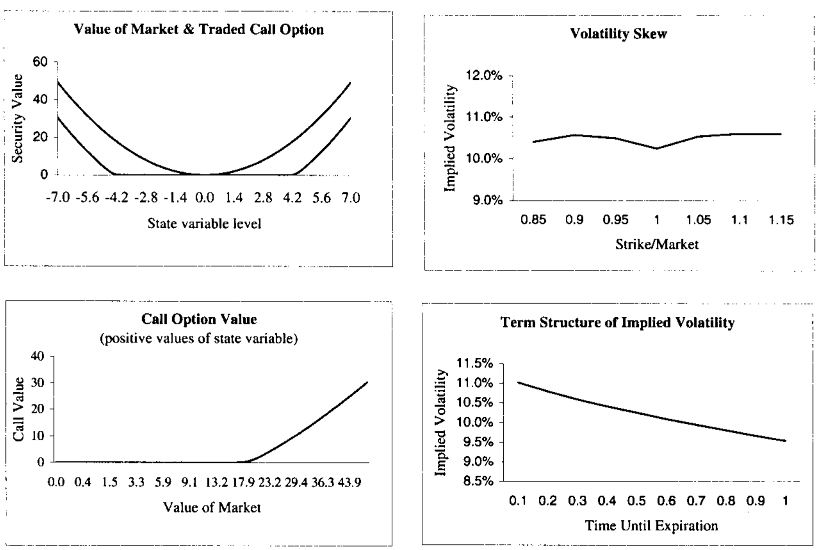

Figure 1.
Market values and implied volatility for the dynamic economy
The top left panel shows the value of the market portfolio (the upper curve) and the value of the traded call option (the lower curve) as a function of the underlying state variable, S_t. The bottom left panel shows the call option value as a function of the market portfolio value. The two panels on the right show the volatility skew and the term structure of implied volatility.

parameterization of the model analyzed here does not produce a skew of the type that is typically observed in the options markets. Other model parameterizations may do a better job. Lastly, the bottom right panel shows the term structure of implied volatility using at-the-money options with varying expiration dates.

In summary, the example illustrates that the model can easily be extended to a continuous-time setting. Equilibrium option prices will depend on the number of different types of agents in the economy (which affects the functional form of the pricing kernel) as well as on how the market's liquidating dividend relates to the Brownian uncertainty that is driving the economy (e.g., there may be multiple state variables that affect C_T). Only in very special cases should we expect to obtain the Black and Scholes (1973) model as the equilibrium option pricing model.

2. Empirical Results

A time-series regression analysis is used to empirically evaluate the performance of the option-based pricing model. Recall from Equation (24) that the theoretical pricing relationship can be written as

$$E(R_j) - R_F = \beta_j \mu, \tag{33}$$

where R_j is the gross return on the jth risky asset portfolio, β_j is a row vector of betas for the jth portfolio, and μ is the vector of expected excess returns from Equation (21). Although Equation (33) is stated in terms of expected returns, it is possible to use realized returns to test the theory. To see this, suppose that the return on the jth risky asset portfolio is

$$R_j = E(R_j) + \beta_j \pi + e_j, \tag{34}$$

where π is a column vector of return innovations (i.e., realized returns minus expected returns) for the market portfolio and the traded options and e_j is a random variable with zero mean. Using Equation (33) to substitute for $E(R_j)$ in Equation (34) and adding an intercept term produces the regression equation

$$R_j - R_F = \alpha_j + \beta_j[\pi + \mu] + e_j, \tag{35}$$

where the elements of $\pi + \mu$ are the realized excess returns on the traded securities. Since the betas from estimating Equation (35) in a time-series regression are the sample counterparts of the theoretical betas, two empirical tests are immediate. First, if the option returns and the market return are important variables for explaining risky portfolio returns, the estimated option and market betas should be statistically different than zero. Second, the estimated intercept term should not be statistically different than zero.

The data used in the analysis covers the period from January 1993 through December 2000 and consists of the following components: (i) total returns for the Center for Research in Security Prices (CRSP) index of NYSE, AMEX, and NASDAQ stocks; (ii) CRSP Treasury-bill total returns; (iii) total returns for the 25 Fama and French portfolios that sort the NYSE, AMEX, and NASDAQ stocks by size and book-to-market; and (iv) S&P 500 index option prices. Since microstructure effects tend to distort daily returns, the empirical analysis relies solely on monthly (equally weighted) holding period returns. Component (i) is used as a proxy for the market portfolio's return, while component (ii) is used to calculate excess returns. As for component (iii), the excess returns on the 25 Fama and French portfolios are the dependent variables for the regression analysis, and a detailed description of the construction procedure for these portfolios can be found in Fama and French (1993). Lastly, since there are no actively traded options in the NYSE/AMEX/NASDAQ portfolio, the S&P 500 index options of component (iv) are used as a proxy for options on the market portfolio.[13] This allows the results presented here to be compared with those of Buraschi and Jackwerth (2001) and Coval and Shumway (2001), who also use S&P 500 options.

Component (iv) consists of daily S&P 500 index option settlement prices for both calls and puts. The daily data are used to construct a time series of monthly call option returns and a time series of monthly put option returns. To ensure that stale prices and microstructure effects do not influence the results, only near-the-money options and options that are near expiration are used in the analysis.[14] Specifically, to form the call option return for month n, all call options that expire in month $n+1$ are first identified. This produces a collection of near-expiration options for month n from which the nearest in-the-money and the nearest out-of-the-money call options are selected. To determine the moneyness of each option, the prevailing S&P 500 level from the beginning of month n is used. Finally, the monthly holding period returns for the two nearest-the-money call options are equally weighted to produce the call return for month n that is used in the regression analysis. The time series of put returns is determined in a similar manner.

Table 1 presents summary statistics for the market portfolio and the two option portfolios. As shown in the top panel, the average excess call return is positive while the average excess put return is negative. For

[13] The mismatch between the market proxy and the options' underlying asset is investigated once the initial results are presented.

[14] As reported by Buraschi and Jackwerth (2001), most of the trading activity in S&P 500 options is concentrated in the nearest contract (0 to 30 days to expiration) and the second-nearest contract (30 to 60 days to expiration). In addition, according to contractual specifications obtained from the Chicago board options exchange (CBOE), the S&P 500 option contract is European and hence early exercise is not an issue. The exact expiration date is the Saturday following the third Friday of the expiration month.

Table 1
Summary statistics

	Market	Puts	Calls
Panel A. Sample moments			
Mean	0.0072	−0.3107	0.0975
Std. dev.	0.0475	0.7758	0.8156
Skewness	−1.0016	1.7988	0.3809
Panel B. Correlation matrix			
Market	1.0	−0.6509	0.5179
Puts	−0.6509	1.0	−0.7475
Calls	0.5179	−0.7475	1.0

Panel A shows the sample mean, standard deviation, and skewness for the market portfolio, the put options, and the call options. Panel B shows the sample correlation matrix for the three portfolios.

comparison, the average excess return on the S&P 500 during the sample period is about 1.17% per month. These results are consistent with the fact that call options are riskier than their underlying asset while put options, due to their negative correlation with the market, are useful instruments for portfolio insurance. The bottom panel shows the sample correlation matrix for the three portfolios. Although the sample period differs from those of Coval and Shumway (2001) and Buraschi and Jackwerth (2001), the results of Table 1 are largely consistent with their sample statistics. Coval and Shumway (2001) report average weekly returns for near-the-money calls and puts of 2.0% and −7.71%, respectively. This implies a monthly gain of about 8.2% for the calls and a monthly loss of about 27.5% for the puts. Buraschi and Jackwerth (2001) convert all puts into calls using put-call parity and report a mean return for the at-the-money contract of 3 basis points (bp) per trading day and a standard deviation of 1,832 bp. Assuming 23 trading days per month, these numbers imply a monthly mean and standard deviation of about 6.9% and 87.8%, respectively.

Table 2 shows the results from estimating Equation (35) using only the market portfolio as an explanatory variable (i.e., the CAPM is tested). This creates a benchmark case against which the other regressions can be evaluated in order to illustrate the marginal contribution of the option returns. While the market beta in Table 2 is statistically significant for all of the portfolios, the intercept term is also significant for 16 of the 25 portfolios. To explore this further, the bottom panel shows the joint test results of the regression intercepts where the tests are conducted for all 25 portfolios, the 5 small stock portfolios, and the remaining 20 nonsmall portfolios. The test statistic of Gibbons, Ross, and Shanken (1989) is shown as $\text{GRS}(N, T-N-L)$. This statistic has an F-distribution in small samples with N and $T-N-L$ degrees of freedom, where T is the number of time series observations, N is the number of portfolios, and L is the number of explanatory variables. Since the GRS statistic assumes

Table 2
Benchmark CAPM regressions

Size	B/M	α_j	$\beta_{j,\text{mkt}}$	R_j^2	μ_j	σ_j
S	L	−0.0139	1.7096	0.830	−0.0016	0.0890
		(−3.68)***	(21.57)***			
S	2	−0.0025	1.2989	0.903	0.0068	0.0649
		(−1.22)	(29.69)***			
S	3	0.0039	1.0594	0.912	0.0116	0.0527
		(2.44)**	(31.41)***			
S	4	0.0050	0.9091	0.922	0.0116	0.0449
		(3.88)***	(33.45)***			
S	H	0.0075	0.9699	0.852	0.0145	0.0499
		(3.79)***	(23.40)***			
2	L	−0.0095	1.4617	0.842	0.0011	0.0756
		(−3.06)***	(22.51)***			
2	2	−0.0014	0.9946	0.812	0.0058	0.0524
		(−0.59)	(20.26)***			
2	3	0.0041	0.6960	0.654	0.0091	0.0408
		(1.67)*	(13.45)***			
2	4	0.0044	0.6662	0.569	0.0092	0.0417
		(1.56)	(11.26)***			
2	H	0.0021	0.6894	0.565	0.0070	0.0433
		(0.70)	(11.16)***			
3	L	−0.0061	1.3021	0.732	0.0033	0.0721
		(−1.85)*	(16.16)***			
3	2	0.0018	0.8102	0.591	0.0077	0.0498
		(0.55)	(11.77)***			
3	3	0.0057	0.6064	0.439	0.0101	0.0432
		(1.72)*	(8.69)***			
3	4	0.0053	0.4857	0.287	0.0088	0.0425
		(1.42)	(6.26)***			
3	H	0.0086	0.4512	0.331	0.0118	0.0368
		(2.76)***	(6.93)***			
4	L	0.0003	1.1279	0.722	0.0084	0.0629
		(0.08)	(15.74)***			
4	2	0.0057	0.5793	0.338	0.0098	0.0468
		(1.44)	(7.04)***			
4	3	0.0073	0.5287	0.299	0.0111	0.0453
		(1.87)*	(6.46)***			
4	4	0.0081	0.4272	0.264	0.0112	0.0389
		(2.36)**	(5.93)***			
4	H	0.0061	0.3472	0.159	0.0086	0.0402
		(1.95)*	(4.36)***			
B	L	0.0037	0.7247	0.507	0.0089	0.0481
		(1.05)	(9.93)***			
B	2	0.0059	0.4543	0.231	0.0092	0.0441
		(1.94)*	(5.44)***			
B	3	0.0085	0.4195	0.174	0.0115	0.0467
		(1.94)*	(4.58)***			
B	4	0.0087	0.2723	0.091	0.0106	0.0408
		(2.16)**	(3.24)***			
B	H	0.0089	0.1620	0.028	0.0101	0.0391
		(2.24)**	(1.94)*			

All 25:	GRS (25,70) = 4.27 (8.4×10^{-7})	$\chi^2(25) = 119.6$ (2.6×10^{-14})
5 small:	GRS (5,90) = 8.67 (9.7×10^{-7})	$\chi^2(5) = 44.6$ (1.7×10^{-8})
20 nonsmall:	GRS (20,75) = 1.82 (0.0328)	χ^2 (20) = 39.1 (0.0064)

The table shows the regression results for the 25 Fama and French portfolios. Included is the adjusted R^2, the mean excess return, and the standard deviation. The *t*-statistics are in parentheses, ***, **, * denote significance at the 1%, 5%, and 10% levels, respectively; S = small, B = big, L = low, H = high. The bottom panel shows the GRS *F*-test (homoscedasticity) and the chi-square test (heteroscedasticity), with *p*-values in parentheses.

homoscedasticity of the residuals, a chi-square test is also performed to accommodate heteroscedasticity.[15] This is shown in the table as $\chi^2(N)$, where N denotes the degrees of freedom. Collectively the tests indicate that the CAPM is misspecified, but the model fits bigger stocks better than it does smaller stocks.

As predicted by the theory, the returns on traded options should help to explain the risky portfolio returns. In particular, Equation (33) is valid regardless of whether only put options, only call options, or both puts and calls are traded. Thus, to test the theory, additional regressions are performed that include the put returns and the call returns as explanatory variables.[16] Table 3 shows the results from including the put return along with the market return as an explanatory variable. The put beta is statistically significant for 24 of the 25 portfolios, suggesting that there is an asymmetric relationship between a risky portfolio's return and the return on the market portfolio. In addition, the adjusted R^2 values are higher than their counterparts from Table 2 for all but one portfolio and the intercept term is significant for only 7 of the 25 portfolios. The joint tests, however, suggest that the inclusion of the put return offers only a very slight improvement over the CAPM. This improvement is clearly driven by the model's ability to explain the nonsmall stock portfolios. The model can be rejected when small stocks are included, and it appears that there is a missing factor that affects small stocks and smaller growth stocks. The next set of regressions examines this further.

Table 4 shows the results from adding the call return as an explanatory variable. As the table shows, there is substantial variation in the call beta with portfolio size. Small stock portfolios have negative call betas while big stock portfolios have positive call betas. The call beta, however, is significant for only 11 portfolios and the significance is concentrated among the 10 largest portfolios. For these portfolios, the adjusted R^2 values increase slightly relative to those reported in Table 3, while for some of the smaller size portfolios the adjusted R^2 values decrease slightly. In addition, relative to Table 3, the inclusion of the call return eliminates the significance of the intercept for only the bigger growth stocks and appears to play very little role in explaining the returns on small stocks and smaller growth stocks. The joint tests in

[15] For the chi-square test, the covariance matrix of the regression intercepts is estimated assuming serially independent residuals, but allowing for conditional heteroscedasticity. For details of the procedure, see MacKinlay and Richardson (1991) and Cochrane (2001, chap. 12). In addition, following MacKinlay and Richardson (1991), the chi-square statistic is adjusted by the factor $(T - N - L)/T$. This scaling factor has no effect on the asymptotic properties of the statistic and improves the small sample performance of the test (i.e., without the adjustment the test rejects the null too frequently).

[16] In a recent article on hedge funds, Lo (2001) uses the positive and negative parts of the S&P 500 return as separate explanatory variables and demonstrates that the beta asymmetry (i.e., the difference in the betas for up and down markets) can be quite pronounced for certain hedge fund trading strategies. The option returns should also be useful for capturing this type of asymmetry.

Table 3
Market and put option regressions

Size	B/M	α_j	$\beta_{j,\text{mkt}}$	$\beta_{j,\text{put}}$	R_j^2
S	L	−0.0080	1.9807	0.0255	0.857
		(−2.14)**	(20.71)***	(4.35)***	
S	2	0.0016	1.4859	0.0176	0.928
		(0.81)	(29.90)***	(5.78)***	
S	3	0.0070	1.1995	0.0132	0.933
		(4.64)***	(31.00)***	(5.56)***	
S	4	0.0073	1.0105	0.0095	0.937
		(5.80)***	(31.42)***	(4.84)***	
S	H	0.0112	1.1358	0.0156	0.885
		(5.95)***	(23.61)***	(5.30)***	
2	L	−0.0095	1.4640	0.0002	0.840
		(−2.82)***	(17.02)***	(0.04)	
2	2	−0.0032	0.9130	−0.0077	0.817
		(−1.28)	(14.33)***	(−1.97)*	
2	3	0.0016	0.5805	−0.0109	0.676
		(0.62)	(8.79)***	(−2.69)***	
2	4	0.0016	0.5368	−0.0122	0.595
		(0.53)	(7.10)***	(−2.63)***	
2	H	−0.0019	0.5113	−0.0168	0.614
		(−0.62)	(6.66)***	(−3.57)***	
3	L	−0.0099	1.1277	−0.0164	0.748
		(−2.47)**	(10.95)***	(−2.60)**	
3	2	−0.0024	0.6167	−0.0182	0.634
		(−0.73)	(7.19)***	(−3.46)***	
3	3	0.0008	0.3812	−0.0212	0.519
		(0.24)	(4.47)***	(−4.06)***	
3	4	−0.0005	0.2239	−0.0246	0.398
		(−0.14)	(2.39)**	(−4.29)***	
3	H	0.0029	0.1918	−0.0244	0.479
		(0.97)	(2.54)**	(−5.27)***	
4	L	−0.0051	0.8847	−0.0229	0.766
		(−1.51)	(10.21)***	(−4.31)***	
4	2	−0.0018	0.2413	−0.0318	0.495
		(−0.48)	(2.55)**	(−5.48)***	
4	3	0.0001	0.1999	−0.0309	0.458
		(0.02)	(2.11)**	(−5.32)***	
4	4	0.0019	0.1438	−0.0267	0.423
		(0.58)	(1.71)*	(−5.18)***	
4	H	−0.0013	0.0094	−0.0318	0.371
		(−0.38)	(0.10)	(−5.71)***	
B	L	−0.0051	0.3269	−0.0374	0.716
		(−1.78)*	(4.48)***	(−8.37)***	
B	2	−0.0036	0.0192	−0.0409	0.528
		(−1.08)	(0.22)	(−7.75)***	
B	3	−0.0011	−0.0181	−0.0411	0.440
		(−0.29)	(−0.18)	(−6.76)***	
B	4	0.0003	−0.1082	−0.0358	0.354
		(0.08)	(−1.16)	(−6.27)***	
B	H	0.0003	−0.2313	−0.0369	0.335
		(0.08)	(−2.54)**	(−6.65)***	
All 25:	GRS (25,69) = 3.89 (4.1×10^{-6})		$\chi^2(25) = 115.2$ (1.5×10^{-13})		
5 small:	GRS (5,89) = 10.4 (7.1×10^{-8})		χ^2 (5) = 51.6 (6.5×10^{-10})		
20 nonsmall:	GRS (20,74) = 1.46 (0.1213)		$\chi^2(20) = 31.8$ (0.0446)		

The table shows the regression results for the 25 Fama and French portfolios when both the market and the puts are used as explanatory variables. The *t*-statistics are in parentheses, ***, **, * denote significance at the 1%, 5%, and 10% levels, respectively; S = small, B = big, L = low, H = high. The bottom panel shows the results of the GRS *F*-test (homoscedasticity) and the chi-square test (heteroscedasticity), with *p*-values in parentheses.

Table 4
Market, put, and call option regressions

Size	B/M	α_j	$\beta_{j,\text{mkt}}$	$\beta_{j,\text{put}}$	$\beta_{j,\text{call}}$	R_j^2
S	L	−0.0093 (−2.44)**	1.9896 (20.89)***	0.0184 (2.45)**	−0.0095 (−1.49)	0.859
S	2	0.0013 (0.65)	1.4876 (29.77)***	0.0162 (4.10)***	−0.0019 (−0.57)	0.927
S	3	0.0061 (4.02)***	1.2059 (32.20)***	0.0080 (2.71)***	−0.0069 (−2.77)***	0.938
S	4	0.0071 (5.45)***	1.0119 (31.33)***	0.0084 (3.28)***	−0.0016 (−0.73)	0.936
S	H	0.0106 (5.51)***	1.1394 (23.69)***	0.0127 (3.36)***	−0.0038 (−1.20)	0.886
2	L	−0.0088 (−2.55)**	1.4598 (16.90)***	0.0036 (0.53)	0.0045 (0.78)	0.839
2	2	−0.0023 (−0.89)	0.9070 (14.31)***	−0.0029 (−0.57)	0.0064 (1.52)	0.820
2	3	0.0021 (0.81)	0.5769 (8.71)***	−0.0079 (−1.53)	0.0039 (0.87)	0.675
2	4	0.0021 (0.68)	0.5334 (7.03)***	−0.0094 (−1.57)	0.0037 (0.73)	0.593
2	H	−0.0012 (−0.38)	0.5069 (6.59)***	−0.0132 (−2.18)**	0.0047 (0.92)	0.613
3	L	−0.0088 (−2.14)**	1.1204 (10.87)***	−0.0106 (−1.30)	0.0078 (1.13)	0.749
3	2	−0.0012 (−0.35)	0.6085 (7.13)***	−0.0116 (−1.72)*	0.0088 (1.55)	0.640
3	3	0.0018 (0.54)	0.3744 (4.40)***	−0.0157 (−2.34)**	0.0073 (1.28)	0.522
3	4	0.0006 (0.16)	0.2165 (2.31)**	−0.0187 (−2.53)**	0.0079 (1.27)	0.402
3	H	0.0036 (1.19)	0.1870 (2.47)**	−0.0205 (−3.43)***	0.0052 (1.03)	0.480
4	L	−0.0031 (−0.90)	0.8712 (10.33)***	−0.0121 (−1.83)*	0.0143 (2.55)**	0.779
4	2	0.0007 (0.21)	0.2245 (2.46)**	−0.0183 (−2.55)**	0.0179 (2.96)***	0.533
4	3	0.0024 (0.68)	0.1840 (2.00)**	−0.0181 (−2.51)**	0.0171 (2.80)***	0.495
4	4	0.0039 (1.21)	0.1302 (1.59)	−0.0158 (−2.45)**	0.0145 (2.68)***	0.459
4	H	0.0013 (0.40)	−0.0086 (−0.09)	−0.0173 (−2.54)**	0.0193 (3.36)***	0.434
B	L	−0.0019 (−0.71)	0.3053 (4.76)***	−0.0202 (−3.99)***	0.0230 (5.40)***	0.782
B	2	−0.0003 (−0.09)	−0.0029 (−0.04)	−0.0232 (−3.74)***	0.0236 (4.52)***	0.609
B	3	0.0019 (0.52)	−0.0383 (−0.41)	−0.0249 (−3.36)***	0.0216 (3.45)***	0.499
B	4	0.0031 (0.89)	−0.1269 (−1.43)	−0.0207 (−2.97)***	0.0201 (3.41)***	0.420
B	H	0.0026 (0.75)	−0.2468 (−2.81)***	−0.0246 (−3.55)***	0.0166 (2.85)***	0.382
All 25:		GRS (25,68) = 3.75 (7.9×10^{-6})		$\chi^2 = 111.7$ (6.1×10^{-13})		
5 small:		GRS (5,88) = 9.32 (3.7×10^{-7})		$\chi^2 = 49.3$ (1.9×10^{-9})		
20 nonsmall:		GRS (20,73) = 1.24 (0.2453)		$\chi^2 = 30.2$ (0.0663)		

The table shows the regression results for the 25 Fama and French portfolios when the market portfolio, the puts, and the calls are used as explanatory variables. The *t*-statistics are in parentheses, ***, **, * denote significance at the 1%, 5%, and 10% levels, respectively; S = small, B = big, L = low, H = high. The bottom panel shows the GRS *F*-test (heteroscedasticity) and the chi-square test (heteroscedasticity), with *p*-values in parentheses.

the bottom panel of the table provide a similar set of conclusions. For all 25 portfolios, the model is only slightly better than the CAPM and can clearly be rejected. However, for the nonsmall portfolios, the inclusion of the call return offers a significant improvement over the results of Table 3. For the GRS test, the p-value increases from 0.1213 in Table 3 to 0.2453 in Table 4. For the chi-square test, the p-value increases from 0.0446 in Table 3 to 0.0663 in Table 4.

The inability of the model to explain small stock returns might be directly related to the underlying economic assumptions. Specifically, in the presence of the nonnegative wealth constraints, the traded options help the agents to avoid insolvency. Since there is no financial distress at the individual agent level, the economy's stochastic discount factor and the resulting asset pricing model also do not include the effects of financial distress. As shown by Chan and Chen (1991), small firms tend to be marginal firms that are under financial distress. Thus, it might be expected that the option-based model will not perform well for these firms. On the other hand, Fama and French (1995) and Chen and Zhang (1998) show that firms with high book-to-market ratios (i.e., value stocks) typically have depressed earnings and are also under distress. If the omission of financial distress in the option-based model is the main reason for the poor overall fit, it should be expected that the model will also perform poorly for value stocks. An examination of Tables 3 and 4, however, indicates that this is not the case. The model performs better for value stocks and worse for growth stocks. Thus the omission of financial distress may not fully explain the model's shortcomings.

For the final set of regressions shown in Table 5, the S&P return is added as an explanatory variable to control for the mismatch between the market proxy and the options' underlying asset. This is important since the significance of the option betas in Tables 3 and 4 might be solely due to this mismatch and not due to the nonredundancy of the options. As Table 5 shows, the S&P beta is significant for most portfolios and there is substantial variation in the S&P beta with portfolio size. Small stock portfolios have positive market betas and negative S&P betas while big stock portfolios, on average, have just the opposite. In addition, the adjusted R^2 values are higher than their counterparts from Tables 3 and 4 for every portfolio. While the inclusion of the S&P return alters the variation of the option betas across portfolios relative to those found in Table 4, the S&P return does not drive out the significance of the option betas. The hypothesis of a zero option beta vector is easily rejected at the 1% level. This is also true for the option betas in Tables 3 and 4. As the bottom panel of Table 5 shows, the S&P return has very little impact on the overall performance of the model. The model can be rejected for all 25 portfolios and does about the same as the model in Table 4 for the nonsmall stock portfolios.

Table 5
Market, put, call, and S&P regressions

Size	B/M	α_j	$\beta_{j,\text{mkt}}$	$\beta_{j,\text{S\&P}}$	$\beta_{j,\text{put}}$	$\beta_{j,\text{call}}$	R_j^2
S	L	−0.0085	2.0998	−1.1864	−0.0061	0.0144	0.939
		(−3.31)***	(32.37)***	(−10.60)***	(−1.09)	(2.99)***	
S	2	0.0016	1.5357	−0.5133	0.0055	0.0084	0.956
		(1.02)	(38.21)***	(−7.40)***	(1.61)	(2.83)***	
S	3	0.0061	1.2203	−0.1543	0.0048	−0.0038	0.943
		(4.19)***	(33.02)***	(−2.42)**	(1.52)	(−1.38)	
S	4	0.0071	1.0195	−0.0808	0.0067	0.0001	0.940
		(5.52)***	(31.34)***	(−1.44)	(2.40)**	(0.02)	
S	H	0.0108	1.1673	−0.2972	0.0066	0.0022	0.904
		(6.01)***	(25.66)***	(−3.79)***	(1.69)*	(0.64)	
2	L	−0.0087	1.4779	−0.2068	−0.0006	0.0086	0.848
		(−2.53)**	(16.98)***	(−1.38)	(−0.08)	(1.33)	
2	2	−0.0027	0.8538	0.5638	0.0088	−0.0049	0.875
		(−1.25)	(15.61)***	(5.98)***	(1.88)*	(−1.22)	
2	3	0.0016	0.5108	0.7062	0.0066	−0.0104	0.813
		(0.80)	(9.82)***	(7.87)***	(1.48)	(−2.69)***	
2	4	0.0015	0.4553	0.8346	0.0078	−0.0131	0.776
		(0.64)	(7.80)***	(8.29)***	(1.57)	(−3.03)***	
2	H	−0.0017	0.4393	0.7307	0.0018	−0.0099	0.745
		(−0.66)	(6.80)***	(6.56)***	(0.33)	(−2.08)**	
3	L	−0.0088	1.1168	0.0270	−0.0099	0.0071	0.757
		(−2.13)**	(10.64)***	(0.15)	(−1.11)	(0.92)	
3	2	−0.0019	0.5072	1.0799	0.0107	−0.0129	0.850
		(−0.88)	(8.90)***	(10.99)***	(2.20)**	(−3.07)***	
3	3	0.0010	0.2674	1.1461	0.0079	−0.0158	0.836
		(0.50)	(5.19)***	(12.89)***	(1.80)*	(−4.13)***	
3	4	−0.0003	0.0986	1.2620	0.0074	−0.0175	0.795
		(−0.12)	(1.74)*	(12.88)***	(1.52)	(−4.15)***	
3	H	0.0030	0.1054	0.8756	−0.0024	−0.0124	0.735
		(1.37)	(1.88)*	(9.08)***	(−0.50)	(−2.99)***	
4	L	−0.0033	0.8414	0.3162	−0.0056	0.0079	0.797
		(−0.99)	(10.06)***	(2.19)**	(−0.78)	(1.27)	
4	2	−0.0002	0.1012	1.3177	0.0089	−0.0086	0.884
		(−0.09)	(2.15)**	(16.26)***	(2.21)**	(−2.48)**	
4	3	0.0015	0.0599	1.3299	0.0093	−0.0097	0.876
		(0.83)	(1.27)	(16.39)***	(2.32)**	(−2.79)***	
4	4	0.0032	0.0292	1.0815	0.0066	−0.0073	0.804
		(1.60)	(0.57)	(12.32)***	(1.51)	(−1.92)*	
4	H	0.0007	−0.1088	1.0798	0.0049	−0.0025	0.756
		(0.28)	(−1.85)*	(10.68)***	(0.98)	(−0.56)	
B	L	−0.0020	0.2695	0.3841	−0.0122	0.0153	0.816
		(−0.84)	(4.43)***	(3.66)***	(−2.34)**	(3.37)***	
B	2	−0.0011	−0.1138	1.1891	0.0013	−0.0004	0.930
		(−0.79)	(−3.29)***	(19.99)***	(0.45)	(−0.14)	
B	3	0.0010	−0.1662	1.3717	0.0033	−0.0061	0.881
		(0.54)	(−3.50)***	(16.77)***	(0.83)	(−1.72)*	
B	4	0.0023	−0.2383	1.1988	0.0039	−0.0041	0.804
		(1.11)	(−4.48)***	(13.06)***	(0.88)	(−1.02)	
B	H	0.0021	−0.3296	0.9023	−0.0059	−0.0015	0.625
		(0.74)	(−4.66)***	(7.40)***	(−0.99)	(−0.29)	

All 25:	GRS $(25,67) = 3.72$ (9.5×10^{-6})	$\chi^2(25) = 120.5$ (1.8×10^{-14})
5 small:	GRS $(5,87) = 9.19$ (4.7×10^{-7})	$\chi^2(5) = 46.3$ (7.8×10^{-9})
20 nonsmall:	GRS $(20,72) = 1.24$ (0.2438)	$\chi^2(20) = 30.1$ (0.0678)

The table shows the results for the 25 portfolios when the market, the S&P 500, the puts, and the calls are used as explanatory variables. Included is the adjusted R^2. The t-statistics are in parentheses; ***, **, * denote significance at the 1%, 5%, and 10% levels, respectively; S = small, B = big, L = low, H = high. The bottom panel shows the GRS F-test (homoscedasticity) and the chi-square test (heteroscedasticity), with p-values in parentheses.

In summary, the mixed empirical results offer limited support for the theory presented in this article. Due to the significance of the option betas in Tables 3–5, it appears that the option returns are important variables for explaining risky portfolio returns. In addition, if small stocks are excluded, the model offers a statistically significant improvement over the CAPM. In terms of overall model performance, however, the inclusion of the option returns offers only a very slight improvement over the CAPM. As discussed earlier, one interpretation of these results is that the theory is missing a risk factor (such as financial distress) that affects small stocks. This is consistent with Berk (1995), who shows that firms with lower market values have higher risk. However, it might be possible that the theory is correct but the particular version of the model tested is misspecified (e.g., the previous tests rely on only near-the-money options). In other words, a possible explanation that is consistent with the theory is that *additional* options are required to fully capture the economic effects of the nonnegative wealth constraints. Since near-the-money options are already included in the model, this suggests that it might be beneficial to add away-from-the-money options as additional explanatory variables. This interpretation is also consistent with the results of Buraschi and Jackwerth (2001), who show that away-from-the-money options are potentially driven by different factors than those that affect at-the-money options. It is possible that the returns on these additional options are important for explaining the returns on small stocks and smaller growth stocks.

3. Conclusion

This article analyzes asset pricing with nonnegative wealth constraints. In the presence of these constraints it is shown that options on the market portfolio are nonredundant securities and the economy's pricing kernel depends on both the market's return and the options' returns. This leads to a pricing model in which the expected excess return on any risky asset is linearly related to the expected excess return on the market portfolio and to the expected excess returns on the nonredundant options. The empirical results indicate that the inclusion of the option returns can improve the CAPM and this improvement is significant for nonsmall stocks. Consistent with the theory, this suggests that nonnegative wealth constraints are likely important for asset pricing.

Appendix

Proof of Proposition 1. Using Equation (8), the function Φ when all agents have preferences in the form of Equation (1) is $\Phi(x) \equiv \sum_{i=1}^{I} \max(0, -\tau_i x - \tau_i \log(\lambda_i))$, where $x \in (-\infty, \infty)$. Similarly when preferences are of the form of Equation (2), the function Φ takes the form $\Phi(x) \equiv \sum_{i=1}^{I} \max\left(0, \lambda_i^{-\gamma} e^{-\gamma x} - \tau_i\right)$ and when preferences are given by Equation (3), Φ is $\Phi(x) \equiv \sum_{i=1}^{I} \max(0, \tau_i - \lambda_i e^{x})$. In each case it is straightforward to show that Φ is continuous, nonincreasing, and differentiable at all but a finite number of points. By restricting the domain of Φ to the interval over which Φ is strictly decreasing, the inverse mapping Φ^{-1} can easily be calculated. In addition, for the collection of agents whose marginal utility at zero is well defined and is finite, the following assumption is imposed:[17]

Assumption 1. *The agents can be strictly ranked by the quantity* $u_i'(0)\lambda_i^{-1}$.

To understand the assumption, suppose that there are two agents, say agent i and agent j, such that $\lambda_i^{-1}u_i'(0) = \lambda_j^{-1}u_j'(0)$. Using Equation (9), the marginal rates of substitution for these two agents are

$$\lambda_i^{-1}u_i'(c_i) = \lambda_i^{-1}u_i'\left(\max\left[0, u_i'^{-1}\left(\lambda_i e^{\Phi^{-1}(C)}\right)\right]\right) \tag{36}$$

$$\lambda_j^{-1}u_j'(c_j) = \lambda_j^{-1}u_j'\left(\max\left[0, u_j'^{-1}\left(\lambda_j e^{\Phi^{-1}(C)}\right)\right]\right). \tag{37}$$

When $C \leq \Phi(\log[u_i'(0)/\lambda_i])$, the expressions in Equations (36) and (37) are equal to $\lambda_i^{-1}u_i'(0)$, and when $C > \Phi(\log[u_i'(0)/\lambda_i])$, the expressions in Equations (36) and (37) are equal to $\exp[\Phi^{-1}(C)]$. Thus Equations (36) and (37) are identical for all states of nature. Since the allocation between these two agents is Pareto optimal, these two agents can be replaced by a single (representative) agent. This process can be repeated until Assumption 1 holds. Assumption 1 is therefore very weak and is simply a sufficient condition for there to be J different types of agents in the economy whose marginal utility at zero is well defined and is finite, where the jth agent's type is defined by the value of $u_j'(0)\lambda_j^{-1}$. For the J types, the agents will always be labeled such that

$$\frac{u_1'(0)}{\lambda_1} > \frac{u_2'(0)}{\lambda_2} > \cdots > \frac{u_J'(0)}{\lambda_J}. \tag{38}$$

For preferences of the form of Equation (1) or Equation (3) it is typically the case that $J = I$ (i.e., the number of types equals the actual number of agents). However, for preferences of the form of Equation (2), $J < I$ is possible since some agents may have $\tau_i < 0$.

The three cases corresponding to the utility functions given by Equations (1)–(3) are now evaluated. In the case where utility functions are given by Equation (1), define the constants $K_2 \equiv \Phi(\log(\lambda_2^{-1})), K_3 \equiv \Phi(\log(\lambda_3^{-1})),\ldots, K_I \equiv \Phi(\log(\lambda_I^{-1}))$, where Φ is given above. The inverse mapping Φ^{-1} can be written as

$$\Phi^{-1}(C) = \begin{cases} \dfrac{-\tau_1 \log\lambda_1 - C}{\bar{\tau}_1}, & 0 < C \leq K_2 \\ \dfrac{-\tau_1 \log\lambda_1 - \tau_2 \log\lambda_2 - C}{\bar{\tau}_2}, & K_2 < C \leq K_3 \\ \vdots & \vdots \\ \dfrac{-\tau_1 \log\lambda_1 - \tau_2 \log\lambda_2 - \cdots - \tau_{I-1}\log\lambda_{I-1} - C}{\bar{\tau}_{I-1}}, & K_{I-1} < C \leq K_I \\ \dfrac{-\tau_1 \log\lambda_1 - \tau_2 \log\lambda_2 - \cdots - \tau_I \log\lambda_I - C}{\bar{\tau}_I}, & K_I < C, \end{cases} \tag{39}$$

where $\bar{\tau}_k \equiv \sum_{j=1}^k \tau_j$. Thus the optimal allocation from Equation (9) is

$$c_1 = C + \sum_{j=2}^{I}\left(\frac{\tau_1}{\bar{\tau}_j} - \frac{\tau_1}{\bar{\tau}_{j-1}}\right)\max\left[0, C - K_j\right] \tag{40}$$

[17] Marginal utility at zero is always finite for utility functions of the form of Equation (1) or Equation (3). However, in the case of the power utility given by Equation (2), marginal utility at zero is well defined and finite only for $\tau_i > 0$.

$$c_k = \frac{\tau_k}{\bar{\tau}_k}\max[0, C - K_k] + \sum_{j=k+1}^{I}\left(\frac{\tau_k}{\bar{\tau}_j} - \frac{\tau_k}{\bar{\tau}_{j-1}}\right)\max\left[0, C - K_j\right] \tag{41}$$

$$c_I = \frac{\tau_I}{\bar{\tau}_I}\max[0, C - K_I], \tag{42}$$

where Equation (41) holds for $k \in \{2, 3, \ldots, I-1\}$. Thus the sharing rules are piecewise linear functions of C for every agent in the economy.

Now consider the case in which every agent has a power utility function of the form of Equation (2). The group of agents is partitioned into two subgroups based on the parameter τ_i. Let $J \leq I$ and suppose that if $i \in \{1, 2, \ldots, J\}$, then the ith agent has $\tau_i > 0$. For these agents, the nonnegativity constraint may bind in some states of nature. If $i \in \{J+1, J+2, \ldots, I\}$, then the ith agent has $\tau_i \leq 0$. For these agents, the nonnegativity constraint never binds. For notational purposes, let $\hat{\tau} \equiv \sum_{j=J+1}^{I}\tau_j$, let $\hat{\lambda} \equiv \sum_{j=J+1}^{I}\lambda_j^{-\gamma}$, and let $\bar{\lambda}_k \equiv \sum_{j=1}^{k}\lambda_j^{-\gamma}$ for $k = 1, 2, \ldots, J$. If $J = I$, then the second set of agents is empty (and empty summations are defined to be zero). Next define the constants $K_1 \equiv \Phi(\log(\tau_1^{-1/\gamma}\lambda_1^{-1})), K_2 \equiv \Phi(\log(\tau_2^{-1/\gamma}\lambda_2^{-1})), \ldots, K_J \equiv \Phi(\log(\tau_J^{-1/\gamma}\lambda_J^{-1}))$, where Φ is given above. The inverse mapping Φ^{-1} can be written as

$$\Phi^{-1}(C) = \begin{cases} \frac{1}{\gamma}\log\left[\frac{\hat{\lambda}}{C + \hat{\tau}}\right], & 0 < -\hat{\tau} < C \leq K_1 \\ \frac{1}{\gamma}\log\left[\frac{\bar{\lambda}_1 + \hat{\lambda}}{C + \bar{\tau}_1 + \hat{\tau}}\right], & K_1 < C \leq K_2 \\ \vdots & \vdots \\ \frac{1}{\gamma}\log\left[\frac{\bar{\lambda}_{J-1} + \hat{\lambda}}{C + \bar{\tau}_{J-1} + \hat{\tau}}\right], & K_{J-1} < C \leq K_J \\ \frac{1}{\gamma}\log\left[\frac{\bar{\lambda}_J + \hat{\lambda}}{C + \bar{\tau}_J + \hat{\tau}}\right], & K_J < C \end{cases} \tag{43}$$

and therefore the allocation in Equation (9) is

$$\begin{aligned} c_i = & -\tau_i + \lambda_i^{-\gamma}\left(\hat{\lambda}\right)^{-1}(C + \hat{\tau}) \\ & + \lambda_i^{-\gamma}\left\{\left(\bar{\lambda}_1 + \hat{\lambda}\right)^{-1} - \left(\hat{\lambda}\right)^{-1}\right\}\max[0, C - K_1] \\ & + \lambda_i^{-\gamma}\sum_{j=2}^{J}\left\{\left(\bar{\lambda}_j + \hat{\lambda}\right)^{-1} - \left(\bar{\lambda}_{j-1} + \hat{\lambda}\right)^{-1}\right\}\max\left[0, C - K_j\right] \end{aligned} \tag{44}$$

for $i \in \{J+1, J+2, \ldots, I\}$ and

$$\begin{aligned} c_i = & \lambda_i^{-\gamma}\left(\bar{\lambda}_i + \hat{\lambda}\right)^{-1}\max[0, C - K_i] \\ & + \lambda_i^{-\gamma}\sum_{j=i+1}^{J}\left\{\left(\bar{\lambda}_j + \hat{\lambda}\right)^{-1} - \left(\bar{\lambda}_{j-1} + \hat{\lambda}\right)^{-1}\right\}\max\left[0, C - K_j\right] \end{aligned} \tag{45}$$

for $i \in \{1, 2, \ldots, J-1\}$ and

$$c_J = \lambda_J^{-\gamma}\left(\bar{\lambda}_J + \hat{\lambda}\right)^{-1}\max[0, C - K_J]. \tag{46}$$

Thus, from Equations (44)–(46), the sharing rules are piecewise linear functions of C for every agent in the economy. For the case of generalized logarithmic utility functions, the constrained optimal sharing rules are given by Equations (44)–(46) by setting $\gamma = 1$. In the case of quadratic utility functions, define the constants $K_2 \equiv \Phi(\log(\tau_2/\lambda_2)), K_3 \equiv \Phi(\log(\tau_3/\lambda_3)), \ldots, K_I \equiv \Phi(\log(\tau_I/\lambda_I))$, where Φ is given above. The inverse mapping Φ^{-1} can be written as

$$\Phi^{-1}(C) = \begin{cases} \log(\tau_1 - C) - \log(\lambda_1), & 0 < C \leq K_2 \\ \log(\bar{\tau}_2 - C) - \log(\bar{\lambda}_2), & K_2 < C \leq K_3 \\ \vdots & \vdots \\ \log(\bar{\tau}_{I-1} - C) - \log(\bar{\lambda}_{I-1}), & K_{I-2} < C \leq K_{I-1} \\ \log(\bar{\tau}_I - C) - \log(\bar{\lambda}_I), & K_{I-1} < C < \bar{\tau}_I, \end{cases} \tag{47}$$

and therefore the allocation in Equation (9) is

$$c_1 = C + \lambda_1 \sum_{j=2}^{I} \left\{ \left(\bar{\lambda}_j\right)^{-1} - \left(\bar{\lambda}_{j-1}\right)^{-1} \right\} \max\left[0, C - K_j\right] \tag{48}$$

$$c_i = \lambda_i \left(\bar{\lambda}_i\right)^{-1} \max[0, C - K_i] + \lambda_i \sum_{j=i+1}^{I} \left\{ \left(\bar{\lambda}_j\right)^{-1} - \left(\bar{\lambda}_{j-1}\right)^{-1} \right\} \max\left[0, C - K_j\right] \tag{49}$$

$$c_I = \lambda_I \left(\bar{\lambda}_I\right)^{-1} \max[0, C - K_I], \tag{50}$$

where Equation (49) holds for $i \in \{2, \ldots, I-1\}$. Thus, from Equations (48)–(50), the sharing rules are piecewise linear functions of C for every agent in the economy. Since in this case $0 < C < \bar{\tau}_I$, the sharing rules imply that $0 \leq c_i < \tau_i$ so that each agent has positive marginal utility at the optimal allocation, that is, agents never reach a state of satiation. This completes the proof. ■

Proof of Proposition 2.

(*a*)(*i*): Let $\tau_i = \tau_j$ and suppose $W_{0i} > W_{0j}$. These conditions imply that $\lambda_i^{-1} > \lambda_j^{-1}$ and $\Phi(\log(\lambda_i^{-1})) < \Phi(\log(\lambda_j^{-1}))$, that is, agent i consumes for a larger set of states than does agent j. The sharing rules in Equations (40)–(42) reveal that, for any traded option contract, either agent i writes options and agent j takes no position or a long position, or both agents take identical short positions. Thus agent i is a net writer of options relative to agent j. Since the trading volume of the option with strike price $\Phi(\log(\lambda_i^{-1}))$ is $\tau_i(\bar{\tau}_i)^{-1}$, while the trading volume of the option with strike price $\Phi(\log(\lambda_j^{-1}))$ is $\tau_j(\bar{\tau}_j)^{-1}$, the option with the lower strike price has the higher trading volume.

(*a*)(*ii*): Since agent i is a net writer of options relative to agent j, it must be the case that $\Phi(\log(\lambda_i^{-1})) < \Phi(\log(\lambda_j^{-1}))$. In addition, since $W_{0i} = W_{0j}$,

$$\int_{\Phi\left(\log\left(\lambda_i^{-1}\right)\right)}^{\infty} p(C) e^{\Phi^{-1}(C)} \left[-\tau_i \log\left(\lambda_i e^{\Phi^{-1}(C)}\right)\right] dC \tag{51}$$

is equal to

$$\left(\frac{\tau_j}{\tau_i}\right) \int_{\Phi\left(\log\left(\lambda_j^{-1}\right)\right)}^{\infty} p(C) e^{\Phi^{-1}(C)} \left[-\tau_i \log\left(\lambda_j e^{\Phi^{-1}(C)}\right)\right] dC, \tag{52}$$

where $p(C)$ is a probability density. Since the value of the first integral exceeds the value of the integral in the second expression, $\tau_j > \tau_i$.

(*b*)(*i*): Let $\tau_i = \tau_j > 0$ and suppose $W_{0i} > W_{0j}$. These conditions imply that $\lambda_i^{-1} > \lambda_j^{-1}$ and $\Phi(\log(\tau_i^{-1/\gamma}\lambda_i^{-1})) > \Phi(\log(\tau_j^{-1/\gamma}\lambda_j^{-1}))$, that is, agent i consumes for a larger set of states than does agent j. The sharing rules of Equations (44)–(46) reveal that, for any traded option contract, either agent i writes options and agent j takes no position or a long position, or both agents take identical short positions. Thus agent i is a net writer of options relative to agent j. Since the trading volume of the option with strike price $\Phi(\log(\tau_i^{-1/\gamma}\lambda_i^{-1}))$ is $\lambda_i^{-\gamma}(\bar{\lambda}_i+\hat{\lambda})^{-1}$, while the trading volume of the option with strike price $\Phi(\log(\tau_j^{-1/\gamma}\lambda_j^{-1}))$ is $\lambda_j^{-\gamma}(\bar{\lambda}_j+\hat{\lambda})^{-1}$, the option with the lower strike price has the higher trading volume.

(*b*)(*ii*): Since agent i is a net writer of options relative to agent j, it must be the case that $\Phi(\log(\tau_i^{-1/\gamma}\lambda_i^{-1})) < \Phi(\log(\tau_j^{-1/\gamma}\lambda_j^{-1}))$. In addition, since $W_{0i}=W_{0j}$,

$$\int_{\Phi\left(\log\left(\tau_i^{-1/\gamma}\lambda_i^{-1}\right)\right)}^{\infty} p(C)e^{\Phi^{-1}(C)}\left[\frac{\left(\lambda_i e^{\Phi^{-1}(C)}\right)^{-\gamma}}{\tau_i}-1\right]dC \tag{53}$$

is equal to

$$\left(\frac{\tau_j}{\tau_i}\right)\int_{\Phi\left(\log\left(\tau_j^{-1/\gamma}\lambda_j^{-1}\right)\right)}^{\infty} p(C)e^{\Phi^{-1}(C)}\left[\frac{\left(\lambda_j e^{\Phi^{-1}(C)}\right)^{-\gamma}}{\tau_j}-1\right]dC. \tag{54}$$

Since the value of the first integral exceeds the value of the integral in the second expression, $\tau_j > \tau_i$.

(*c*)(*i*): $W_{0i} > W_{0j}$ implies that $\Phi(\log(\tau_i\lambda_i^{-1})) < \Phi(\log(\tau_j\lambda_j^{-1}))$, that is, agent i consumes for a larger set of states than does agent j. Since $\lambda_i=\lambda_j$, the sharing rules of Equations (48)–(50) reveal that, for any traded option contract, either agent i writes options and agent j takes no position or a long position, or both agents take identical short positions. Thus, agent i is a net writer of options relative to agent j. Also, the trading volume of the option with strike price $\Phi(\log(\tau_i\lambda_i^{-1}))$ is $\lambda_i(\bar{\lambda}_i)^{-1}$, while the trading volume of the option with strike price $\Phi(\log(\tau_j\lambda_j^{-1}))$ is $\lambda_j(\bar{\lambda}_j)^{-1}$, that is, the option with the lower strike price has the higher trading volume.

(*c*)(*ii*): Since agent i is a net writer of options relative to agent j, it must be the case that $\Phi(\log(\tau_i\lambda_i^{-1})) < \Phi(\log(\tau_j\lambda_j^{-1}))$. In addition, since $W_{0i}=W_{0j}$,

$$\int_{\Phi\left(\log\left(\tau_i\lambda_i^{-1}\right)\right)}^{\bar{\tau}_I} p(C)e^{\Phi^{-1}(C)}\left[1-\frac{\lambda_i}{\tau_i}e^{\Phi^{-1}(C)}\right]dC \tag{55}$$

is equal to

$$\left(\frac{\tau_j}{\tau_i}\right)\int_{\Phi\left(\log\left(\tau_j\lambda_j^{-1}\right)\right)}^{\bar{\tau}_I} p(C)e^{\Phi^{-1}(C)}\left[1-\frac{\lambda_j}{\tau_j}e^{\Phi^{-1}(C)}\right]dC. \tag{56}$$

Since the value of the first integral exceeds the value of the integral in the second expression, $\tau_j > \tau_i$. This completes the proof. ■

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