

Valuation and Return Dynamics of New Ventures

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A dynamic model of a multistage investment project that captures many features of research and development (R&D) ventures and start-up companies is developed. An important feature these problems share is that firms learn about the potential profitability of the project throughout its life, but that technical uncertainty about the R&D effort is only resolved through additional investment. Consequently the risks associated with the ultimate cash flows have a systematic component even while the purely technical risks are idiosyncratic. Our model captures these different sources of risk and allows us to study their interaction in determining the value and risk premium of the venture.

The current valuation of ventures that are expected to produce positive cash flows only in the future is an important problem for financial analysts. Within established organizations, managers are under increasing pressure to quantitatively justify expenditures on research and development (R&D). Financial intermediaries, such as venture capitalists, routinely allocate capital resources on the basis of implicit or explicit valuations of firms with no assets, or few assets, other than the potential to generate cash flows in the future. Such firms are a growing presence on organized equity markets as well, and were at the center of the high-tech boom and bust of the late 1990s.

New ventures are subject to several, qualitatively different sources of risk. There is the “technical” uncertainty associated with the success of the

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venture itself, or the time and cost required to bring it to fruition. Examples are the clinical trials of a new drug, marketing strategy of an Internet firm, or exploratory drilling for oil. There is the exogenous risk associated with the actions of a competitor or changes in the environment before or soon after product introduction. For example, a new drug may be rendered unnecessary by a superior treatment option. A software product may fail because of technological advances in hardware. Finally, there are the more traditional risks associated with demand for the product and production costs.

How do these types of risk jointly determine the value of a new venture? How much systematic versus unsystematic risk is borne by investors who invest in R&D, and how does this influence the risk premium they require? In this article we develop a model that allows for all three types of uncertainty and is still sufficiently tractable to provide qualitative and quantitative guidance regarding these questions.

Textbooks in corporation finance, in response to the tendency of practitioners and students to confuse systematic and unsystematic risks, often point to the technical risks of R&D as examples of diversifiable risks that do not require compensation. Brealey and Myers (1996), for instance, list the risk of a dry hole in oil exploration and denial of approval for a drug that cures baldness by the U.S. Food and Drug Administration (FDA) as examples of “bad outcomes,” that “appear to reflect unique (i.e., diversifiable) risks which would not affect the expected rate of return demanded by investors” (p. 220, their parentheses). Brealey, Myers, and Marcus (1995), again considering the example of oil exploration, argue “...for the company as a whole, it’s the *average* success rate that matters. Geological risks (is there oil or not?) should average out. The risk of a worldwide drilling program is much less than the apparent risk of any single wildcat well” (p. 237, their emphasis and parentheses). They go on to say that the investors in an oil company’s stock will “...naturally — and realistically — assume that your successes and failures in drilling oil wells will average out with the thousands of independent bets made by the companies in their portfolio” (p. 237).

These arguments might be construed to imply that R&D projects would not require a particularly high expected return. In this article we explain why this conclusion is naive. In our model we value simultaneously a multistage R&D project and the stream of cash flows it will deliver if successful. We show that the required risk premium for the R&D is *higher* than it would be were the R&D complete and the venture a traditional, cash-producing, project.

Reasoning based simply on the idiosyncratic nature of purely technical risk ignores an important feature of actual R&D projects. Decisions to continue with R&D are made conditioning on the resolution of systematic as well as unsystematic uncertainty. For example, while “geological risks”

are arguably diversifiable, the decision to continue drilling at any particular site is made with the current price of oil in mind. This imparts to the project the characteristics of a compound option on systematic uncertainty. Options, because of the implicit leverage they impart, have higher systematic risk than the underlying asset.

We are able to obtain a number of interesting results on the interaction of these different sources of risk in determining the value and expected return on the venture. Perhaps most interesting is the role the resolution of idiosyncratic uncertainty plays in determining the current risk premium. As technical (idiosyncratic) uncertainty is resolved, the probability of successful and timely completion changes. This changes the properties of the option to suspend investment in the project, which is exercised with the potential future cash flows in mind. Thus, although there is no risk premium earned on idiosyncratic risk per se, the resolution of idiosyncratic uncertainty can dramatically alter the risk premium earned on the venture as a whole.

Analytically simple upper and lower bounds on the risk premium of the R&D, and thus on the cost of capital of “pure growth” firms, are derived. The bounds, because of their simplicity, could be useful in applications. The risk premium of the R&D is close to this upper bound when active development is suspended or delayed, and the project is mothballed, generally early in its life. The risk premium decreases to the lower bound (the risk premium of the completed project) as the project approaches completion. This decline in the risk premium is nonlinear and is particularly pronounced around the time the firm just begins active development. There are also times when it is possible for the risk premium to increase over the life of the project.

We initially restrict attention to the case when the firm’s ability to complete the R&D is known. We obtain a closed-form solution of our model in this case. To our knowledge, this solution is new to the economics literature. We then relax this assumption and assume that the firm does not know its own ability but must learn about it by engaging in R&D. We are able to derive a number of results characterizing the solution to this more general case, even where we are unable to solve in closed form for optimal policies and values. We evaluate the rest of the effects of learning numerically.

Learning by doing also plays an important role in determining the patterns of returns earned over the life of the project. As a result of learning, and endogenous policy responses to it, early resolution of technical uncertainty has a much greater impact on value and expected return than technical success later in the project’s life. The firm will tolerate only a certain number of failures at a given stage of development before it abandons the R&D. This threshold is low in the early stages, so it is possible for two identical firms to work on identical projects with very

different results. One firm might be lucky, in that it achieves positive realizations of uncertainty in the early stages and ultimately completes the project. The other firm, because of bad luck early on, optimally decides to abandon the project and never learns about its underlying profitability.

In the next section we discuss other articles in the literature related to ours. Section 2 introduces the notation and describes the structure of the model. Section 3 presents analytical results concerning the firm's optimal investment policies, the valuation of the R&D venture, and the required risk premium it earns. In Section 4 we solve the model with no learning in closed form. Section 5 explores the effects of learning numerically. Section 6 concludes. More technical derivations and the proofs of the propositions in the article are contained in the appendix.

1. Review of Related Literature

A large economics literature studies dynamic R&D policies for individual firms [see, e.g., Dutta (1997), Grossman and Shapiro (1986), Pakes (1986), Posner and Zuckerman (1990), Weitzman (1979), or Weitzman, Newey, and Rabin (1981)]. Another line of research evaluates socially optimal levels and allocations of investment in research and the effects of patent law and policy [see, e.g., Bhattacharya and Mookherjee (1986), Dasgupta and Maskin (1987), and Gruver (1991)]. The focus of all these models, however, is on optimal policies rather than on valuation. They typically assume, for example, risk neutrality or a concave utility function for the firm. This precludes a study of the determinants of systematic risk and risk premia, and their role in valuation, which is what we focus on.

More recently, a number of articles have studied the R&D process as a contingent claim on the value an underlying asset, which is interpreted as the present value of the cash flows on completion of the R&D. This approach allows for richer specifications than more traditional dynamic programming solutions, and is closer in spirit to our article. The valuation method in these articles differs from ours, however. They take as exogenous a stochastic process for the benefit or *value* of the project to the firm on completion of development. This evolves as a diffusion and the project is valued as a contingent claim on this "underlying security." We take as exogenous the process describing the *cash flows* the project will generate and value these cash flows and the R&D project simultaneously. By deriving the value of the underlying cash flows rather than specifying it exogenously, we are able to focus on the relative systematic risk of R&D and the underlying cash flows and explain the dynamics of the risk premium of the R&D.

Majd and Pindyck (1987) solve an investment problem in which the project requires a fixed total investment to complete, with a maximum instantaneous rate of investment. All the uncertainty in their model comes

from stochastic evolution of the value of the project upon completion. There is no technical risk. The article focuses on how the decision to invest or wait varies with the parameters and the single state variable in the model.

Childs and Triantis (1999) develop and numerically implement a model of dynamic R&D that highlights the interactions across projects. They solve for and interpret optimal policies for a firm with multiple R&D projects, which can run in parallel or sequentially, and calculate the values of the real options such problems present. Their model differs from ours both in function and in purpose. The richness of their specification allows them to analyze in detail the intensity and timing of optimal investment policies. Because our focus is deriving the risk premium of the venture, the range of policies we allow for is more limited. The probabilistic structures of the two models also differ. Investment deterministically yields progress through investment stages in their model, while in our model success is random. The information flows in the two models are also different. In Childs and Triantis (1999), the firm learns about the underlying value only if it invests. Otherwise the diffusion coefficient in the value process is set equal to zero, and the value “holds still.” In our model, investment is required to resolve technical uncertainty. Information about the potential cash flows continues to evolve independently of the firm’s investment policy, and as we will see, this has important consequences for the risk premium of the venture.

Schwartz and Moon (2000) also use continuous-time methods to numerically value R&D as a contingent claim on the value of the benefits created. Their paper is closest to ours in the sense that it allows simultaneously for three types of uncertainty. There is “technical” uncertainty associated with the success of the R&D process itself. There is an exogenous chance for obsolescence, during and after the development process, and there is uncertainty about the value of the project on completion of the R&D. Schwartz and Moon (2000) solve for optimal investment policies, provide comparative statics regarding the option component of the project’s value, and compare the project’s value to the “NPV approach,” which simply discounts expected cash flows given optimal policies. Schwartz and Moon (2000) do not evaluate the relative systematic risk of R&D and the underlying cash flows or study their impact on the dynamics of the risk premia. They also model technical uncertainty differently. In their model, the expected cost to completion of the project is an exogenous stochastic process with drift and diffusion coefficients that depend on its current value and on the current level of investment. Given this process, optimal investment policies are obtained. However, the authors do not show that, given the optimal investment policy, the expected cost to completion of the project has the functional form exogenously postulated. Consequently their model is not dynamically consistent in the rational expectations sense. Our model takes as exogenous

the technology for randomly advancing through stages of the project and then derives optimal investment policies. These will lead to an endogenous process for expected cost to completion, which will have the desirable features of the process assumed by Schwartz and Moon (2000) [as well as Pindyck (1993)]. Specifically, in our model, innovations in the conditional expectation of remaining time to completion are zero when investment is zero, so there is learning by doing.

2. The Model

Our model is structured with several objectives in mind. It is sufficiently tractable to admit closed-form solutions for important cases. It is sufficiently flexible to produce realistic profiles of expected cash flows that could be calibrated to data in an applied setting. Finally, the model involves several distinct sources of uncertainty in order to highlight how systematic and nonsystematic risks interact to determine the magnitude of the risk premium required on the project and its behavior through time. We begin with a brief overview.

2.1 Overview

The firm consists of a single R&D project. When the R&D is successfully completed, the firm will generate a stream of stochastic cash flows, which we model as a geometric random walk that has both idiosyncratic and systematic risk. In order to realize these cash flows, however, the firm must first complete N discrete stages of R&D. In this process the venture is subject to four types of risk, and of these only one involves risk that is systematic.

1. Technical risks pertaining to the successful completion of the R&D itself: At each date an investment is required in order to keep the project “active.” If the investment is made, then with some probability the current stage will be successfully completed and the firm can move on to the next stage. Uncertainty regarding success or failure at each stage is idiosyncratic risk. If the investment is not made, the project is “mothballed” — development remains in the current stage with certainty.
2. Uncertainty about the potential future cash flows the project will produce if completed: At any point in time the firm observes the cash flows the project *would* be producing were it already complete, and this process has both systematic and idiosyncratic risk.
3. The risk of competitive threat or obsolescence: We assume that with a fixed probability in each period the cash flows from the project, whether actual or potential, are extinguished. The uncertainty regarding obsolescence is idiosyncratic.

4. Learning by doing: The probability of success at each stage is unknown to the firm's management, who, based on past outcomes, update their beliefs about it. Uncertainty about the duration and cost of the R&D is resolved by making investments in the process and observing the outcomes. The expected cost to completion for the R&D thus evolves endogenously, reflecting optimal policies.

In making its decisions about whether to continue investing, “mothball” the project, or simply abandon the venture, the firm observes state variables summarizing each of the underlying sources of uncertainty above: (1) the number of stages of R&D still to be completed, (2) the level of cash flow the project would be earning if the R&D were complete, (3) whether the project's potential cash flows have been extinguished through obsolescence, and (4) its history — how long the firm has been actively working on R&D.

With this in mind, we can now proceed to the detailed description of the model. We begin with a description of the stages.

2.2 Staged investment

We consider a firm working in continuous time on an R&D project to bring a new product, process, or technology to the market. Completion of the project involves passing through a sequence of N discrete stages of development successfully. Let $n(t)$ denote the number of stages successfully passed at date t , so that the number remaining is $N - n(t)$. Let $\zeta(t)$ be an indicator function for whether the R&D is complete, that is,

$$\zeta(t) \equiv \begin{cases} 1 & n(t) = N \\ 0 & n(t) < N. \end{cases}$$

At each point in time prior to completion, the firm must decide on the intensity of its investment effort over the next instant, $u(t)$, where $u(t) \in \{0, 1\}$. If the firm invests, so that $u(t) = 1$, the firm incurs a cost of $u(t)I(t)dt$. If the firm does not invest, then the probability that it passes to the next stage is zero. We adopt the convention that $u(t) = 0$ once the project is complete.

2.3 Uncertainty concerning future cash flows

When the R&D is completed the firm receives a project that returns an uncertain cash flow stream, $x(t)$. The source of systematic risk in our model is uncertainty regarding this stream of cash flows. We model these as a standard geometric Brownian motion, denoted $x(t)$, where

$$dx(t) = \mu x(t)dt + \sigma x(t)dz(t). \quad (1)$$

The risk associated with this process is systematic, because we assume $dz(t)$ is correlated with the innovations in the pricing kernel, as described below. Although the firm does not receive $x(t)$ prior to the completion of

the R&D project, it is assumed that the firm's decision makers can still observe $x(t)$ — they know what the firm's cash flows would be were the project complete today. Management's ability to condition their decisions on this variable then imparts systematic risk to all of the endogenous quantities in the model.

2.4 The risk of catastrophic failure

The risk of obsolescence is modeled by assuming that with some probability the cash flow or potential cash flow may be extinguished, and the value of the project therefore becomes zero. The probability of a catastrophic failure over the next instant is ϕdt . We assume the process determining catastrophic failure is independent of $x(t)$ and all other variables in the problem. In particular, it is independent of the pricing kernel, and thus represents nonsystematic risk. Let $\xi(t)$ be an indicator function describing whether catastrophic failure has occurred. That is, $\xi(t)$ starts out at one and becomes zero when failure occurs. The cash flows actually earned by the firm are thus given by the process $x(t)\xi(t)\zeta(t)$.

This source of risk in the model is intended to account for the possibility that the project fails for reasons that are completely exogenous and unrelated to economy-wide events. Because process $\xi(t)$ is uncorrelated with other variables in the model, it acts in the valuation as a depreciation rate and becomes subsumed by an adjustment to the discount rate. Nevertheless, this source of risk is important in imparting realistic life-cycle effects to the project.

2.5 Technical uncertainty

The technical uncertainty in our model is idiosyncratic risk. Conditional on investment, the probability that in the next instant the firm will complete the current stage is πdt . We can view π as a measure of the firm's productivity in R&D. We allow the possibility that the firm does not know π and instead must infer its value in a Bayesian fashion. Denote $\pi(t)$ as the posterior estimate of π at time t . Given an initial prior distribution, $\pi(t)$ is a function of $n(t)$, the number of completed stages, and $y(t) \equiv \int_0^t u(s)ds$ the cumulative amount of time spent in active development. The following lemma shows how to compute this function.¹

Lemma 1.

$$\pi(t) = \pi(n(t), y(t)) = -\frac{\Phi^{(n(t))}(y(t))}{\Phi^{(n(t)-1)}(y(t))}, \quad (2)$$

where

$$\Phi(y(t)) = E\left[e^{-y(t)z}\right] \quad (3)$$

¹ The authors would like to thank Darrell Duffie for suggesting Lemma 1 together with its proof.

is a generating function (Laplace transform) for the Poisson process with parameter z .

Intuitively the firm updates based on past success. A history of intense investment activity, or high $y(t)$, with limited success, or low $n(t)$, is discouraging news about π . When the firm's prior over possible values of π is a gamma distribution, $\pi(n(t), y(t))$ has a particularly simple form.

Lemma 2. *Suppose the prior distribution of π is a gamma distribution with parameters $\alpha_1 > 0$ and $\alpha_2 > 0$. If the firm has completed n stages with cumulative investment effort $y(t)$, then*

$$\pi(n(t), y(t)) = \frac{\alpha_1 + n(t)}{\alpha_2 + y(t)}. \quad (4)$$

For simplicity, we will employ this parameterization for the rest of the article.

Learning depends only on technical factors, so uncertainty about the firm's productivity is nonsystematic. Nevertheless, within the model, learning has interesting effects on the risk premium because the added variance it imparts alters the option-like claim the firm holds on the underlying cash flows. This can encourage investment in situations where an otherwise identical firm, which knew its R&D productivity, would wait or shut down development. It can also discourage investment in situations where the firm becomes pessimistic early in its experience with the project. This view of learning by doing contrasts with other models, such as Pindyck (1993), and Schwartz and Moon (2000), where the reduction in uncertainty that comes with experience is exogenously specified rather than solved for within the model.

2.6 Investment costs

The instantaneous cost of development activity $u(t)$ is $u(t)I(t)dt$. We parameterize the cost of investment in a manner that allows for both a fixed and a variable component:

$$I(t) = a + bx(t). \quad (5)$$

Most of the important qualitative insights from the basic model can be obtained assuming these costs are fixed, $b = 0$. In this case the costs clearly have no systematic risk. The firm's claim on the project can thus be viewed as an option on a risky stream of cash flows, with an exercise price that is random because of the technical uncertainty, but that has no systematic risk.

Including a variable component allows for increases in scale during the development process that are an involuntary consequence of increased anticipated cash flows. It also allows us to distinguish between decisions to "mothball" the project, or suspend investment, and decisions to abandon it entirely. Our numerical solution procedure for the general model relies

on the presence of at least a small variable component for technical reasons. For the closed-form solutions we provide in Section 4, this is not needed, and all costs can be fixed.

3. Valuation

To analyze the value of the project and the determinants of its risk premium, we must put structure on the pricing kernel in the economy. Following Berk, Green, and Naik (1999) we solve a partial equilibrium problem, taking the pricing kernel as exogenous. We assume the riskless interest rate, r , is constant. This allows us to focus on risk premia attributable to systematic risks that are analogous to “market risk” rather than interest rate risk.

The pricing kernel is given by the process

$$d\nu(t) = -r\nu(t)dt + \varphi\nu(t)dw(t). \quad (6)$$

We denote the market price of risk as

$$\lambda = \sigma\varphi\rho, \quad (7)$$

where $w(t)$ is a standard Brownian motion and ρ is the “local correlation” of the Brownian motion processes $z(t)$ (the cash flow uncertainty) and $w(t)$.

Suppose the firm has passed through $n(t)$ stages of development and is considering whether to invest $u(t)$ over the next instant in an attempt to move to stage $n(t) + 1$. At this time the firm observes the current level of $x(t)$ and whether the project is still alive, $\xi(t) = 1$, or has become obsolete, $\xi(t) = 0$. The firm also knows the cumulative amount of investment effort it has expended in the past, $y(t)$. These constitute the state variables for the firm’s problem.

Denote the project’s value, conditional on $\xi(t) = 1$, as $V^{n(t)}(x(t), y(t))$. (Since the value is trivially zero when $\xi(t) = 0$, we need only solve the valuation problem conditional on the project being alive, and we suppress the dependence on $\xi(t)$.) This value is the maximum of the firm’s dynamic investment problem:

$$\begin{aligned} V^{n(t)}(x(t), y(t)) \equiv & \max_{u(s), s \in (t, T)} E_t \left\{ \frac{\nu(T)}{\nu(t)} V^{n(T)}(x(T), y(T)) \xi(T) \right. \\ & - \int_t^T \frac{\nu(s)}{\nu(t)} u(s) [a + bx(s)] \xi(s) ds \\ & \left. + \int_t^T \frac{\nu(s)}{\nu(t)} x(s) \zeta(s) \xi(s) ds \right\}, \end{aligned} \quad (8)$$

where T is an arbitrary point in the future. The value of the project is the discounted expectation of its future value at T plus the discounted value of

any intermediate cash flows. The intermediate cash flows consist either of possible investments during the R&D phase or cash payments if the R&D is complete (when $u(t) = 0$ and $\zeta(s) = \xi(s) = 1$).

Two standard steps allow us to simplify this problem. First, since $\xi(t)$ is independent of all the other variables in the problem, for all s , $t \leq s \leq T$, we have

$$E_t\{\xi(s) \mid n(s), x(s), \nu(s)\} = e^{-\phi(s-t)}. \quad (9)$$

Using this fact, we can eliminate $\xi(s)$ and replace it with its expectation. The possibility of obsolescence acts like a rate of depreciation in the model. Accordingly, define

$$\hat{r} \equiv r + \phi. \quad (10)$$

Second, it is well known that, with this pricing kernel and the underlying cash flow process, valuation under the risk-neutral measure simply adjusts the drift in $x(t)$ to reflect the market price of risk. Specifically, when

$$\hat{\mu} \equiv \mu - \lambda, \quad (11)$$

the cash flow process is defined to be

$$dx(t) = \hat{\mu}x(t)dt + \sigma x(t)d\hat{z}(t), \quad (12)$$

where $\hat{z}(t)$ is a Brownian motion under the risk-neutral measure. For the project to have a finite value, we must assume that $\hat{r} > \hat{\mu}$. For simplicity, we will further assume that $\sigma^2 > 2\hat{\mu}$.² Under the risk-neutral measure, denoted Q , the firm's value at date t and stage $n(t)$ becomes

$$V^{n(t)}(x(t), y(t)) = \max_{u(s), s \in (t, T)} E_t^Q \left\{ e^{-\hat{r}(T-t)} V^{n(T)}(x(T), y(T)) - \int_t^T e^{-\hat{r}(T-s)} (\zeta(s)x(s) - u(s)[a + bx(s)])ds \right\}. \quad (13)$$

When all stages are complete ($\zeta(t) = 1$), this has a standard solution:

$$V^N(x(t), y(t)) = \frac{x(t)}{\hat{r} - \hat{\mu}}. \quad (14)$$

The value of the completed project is given by the continuous-time version of the Gordon-Williams growth model, with a risk-adjusted growth rate and an interest rate adjusted to reflect the risk of obsolescence.

The solution with $\zeta(t) = 0$, before all stages have been completed, is harder to derive. In the appendix, standard arguments are used to show that at any t , such that $x(t) = x$ and $y(t) = y$, the solution of this problem

² This assumption simplifies the exposition because it allows us to focus attention on only one root in the quadratic form of homogeneous solution to the Bellman equation. If it fails, the solutions in this article would need to be rederived using the other root.

satisfies the Hamilton-Bellman-Jacobi equation:

$$\begin{aligned} & \frac{1}{2}x^2\sigma^2\frac{\partial^2}{\partial x^2}V^n(x, y) + \hat{\mu}x\frac{\partial}{\partial x}V^n(x, y) - \hat{r}V^n(x, y) \\ & + \sup_{u \in \{0,1\}} \left\{ u \left[\pi(n, y)(V^{n+1}(x, y) - V^n(x, y)) \right. \right. \\ & \left. \left. + \frac{\partial}{\partial y}V^n(x, y) - a - bx \right] \right\} = 0. \end{aligned} \quad (15)$$

The value of the control that maximizes Equation (15) is either zero or one. In the region where the firm does not invest ($u = 0$), the evolution of the value is driven entirely by the dynamics of $x(t)$ and the equation of motion is an ordinary differential equation — the terms in the first line of Equation (15). In the region where the firm chooses to invest ($u = 1$), the remaining terms, which capture the consequences of investment, must be included, yielding a partial differential equation (PDE) in x and y . With probability $\pi(n, y)$ over the next instant, the firm will achieve a technical breakthrough to the next stage and the project's value will jump to V^{n+1} . In addition, the value changes as the firm learns about π . Clearly, $u = 1$ if the term in square brackets is positive and $u = 0$ if it is negative. We refer to the region where the firm does not invest ($u = 0$) as the “mothball” region and the region where it does invest ($u = 1$) as the “continuation” region. The following proposition derives the boundary conditions needed for the solution of this PDE.

Proposition 1. *For $n < N$, let $V_c^n(x, y)$ solve Equation (15) with $u = 1$, and let $V_m^n(x, y)$ solve Equation (15) with $u = 0$. Let these functions satisfy the single crossing property in x at $x_n^*[y]$ for a given n and y with $V_c^n(x, y) > V_m^n(x, y)$ when $x > x_n^*[y]$. Furthermore, let $(\partial/\partial y)V_c^n(x, y) \leq 0$ and $(\partial/\partial y)V_m^n(x, y) \leq 0$. Then for every n , there exists a y_n^* such that for all $y > y_n^*$, $V^n(x, y) = 0$ for every x . When $y \leq y_n^*$,*

$$V^n(x, y) = \begin{cases} V_c^n(x, y) & x \geq x_n^*[y] \\ V_m^n(x, y) & x < x_n^*[y], \end{cases} \quad (16)$$

with

$$V^n(0, y) = 0 \quad (17)$$

$$\lim_{x \rightarrow \infty} V^n(x, y) \propto x. \quad (18)$$

When $x = x_n^*[y]$,

$$V_c^n(x_n^*[y], y) = V_m^n(x_n^*[y], y) \quad (19)$$

$$\frac{\partial}{\partial x}V_c^n(x_n^*[y], y) = \frac{\partial}{\partial x}V_m^n(x_n^*[y], y) \quad (20)$$

$$\frac{\partial^2}{\partial x^2}V_c^n(x_n^*[y], y) = \frac{\partial^2}{\partial x^2}V_m^n(x_n^*[y], y) \quad (21)$$

and when $y = y_n^*$,

$$V_c^n(x, y_n^*) = 0 \quad (22)$$

$$\frac{\partial}{\partial y} V_c^n(x, y_n^*) = 0. \quad (23)$$

3.1 Optimal investment policy

As is clear from Proposition 1, there may be situations where the firm chooses to abandon the project — that is, suspend the project when $y = y_n^*$, with the knowledge that it will never resume the R&D. In this section we provide an explicit characterization for the optimal time to abandon. To derive this, we start with the special case when investment costs are entirely proportional ($a = 0$). The following proposition gives the solution in this case.

Proposition 2. *When investment costs are proportional, $a = 0$, the value is homogeneous in x and has the form*

$$V^n(x, y) = \begin{cases} xH^n(y) & y \leq y_n^* \\ 0 & y > y_n^* \end{cases}$$

where

$$H^n(y) = \int_y^{y_n^*} e^{(\hat{r} - \hat{\mu})(y-u)} \left(\frac{\alpha_2 + y}{\alpha_2 + u} \right)^{\alpha_1 + n} \left(\left(\frac{\alpha_1 + n}{\alpha_2 + u} \right) H^{n+1}(y) - b \right) du \quad (24)$$

and the optimal point to abandon, y_n^* , solves

$$\left(\frac{\alpha_1 + n}{\alpha_2 + y_n^*} \right) H^{n+1}(y_n^*) = b \quad (25)$$

with the boundary condition

$$H^N(y) = \frac{1}{\hat{r} - \hat{\mu}}.$$

It turns out that this trigger strategy holds generally. The same strategy applies even when $a > 0$. To see why, note it is never optimal to abandon the project if a state could arise going forward in which it would be optimal to continue development. That is, it is only optimal to abandon the project if the project would not be developed even when $x \rightarrow \infty$. When

$x \rightarrow \infty$, however, the fixed costs become inconsequential and so the solution in this case must converge to the homogeneous solution. Thus it is only optimal to abandon the project if it is optimal to abandon the same project when the fixed costs are ignored. The following proposition formalizes this argument.

Proposition 3. *The critical value y_n^* defined in Proposition 2 and Proposition 1 are the same. The firm will abandon the R&D project whenever $y > y_n^*$, where y_n^* solves Equation (25).*

So long as $b > 0$, at any stage there is a limited amount of time that the firm is willing to devote to the project. Once this amount of time is reached, the firm chooses to stop investing. This is true for any value of x , in particular, for very large values of x when the fixed costs become inconsequential and the solution converges to the case without fixed costs. When the firm chooses not to invest, $y(t)$ and n cannot change. There is no resolution of technical uncertainty without investment. Since the decision to proceed with development depends only on y and n , it follows that a decision to suspend investment is permanent in this case. It is a decision to abandon the project.

When costs are entirely fixed ($b = 0$), the reverse is true. No matter how low $x(t)$ is relative to the fixed investment costs, there is always positive probability that it will rise to a point where the costs become small relative to the potential gains from resuming development. Thus the firm never abandons the project — in this case $y_n^* = \infty$. The following corollary formalizes this argument.

Corollary 1. *When $x(t) > 0$,*

- (i) *if costs are purely proportional, $a = 0$, investment is never resumed once it is suspended.*
- (ii) *if costs are purely fixed, $b = 0$, the project is never abandoned.*

In the general case, investment policy reflects these two extremes. The firm suspends investment when $x(t)$ falls to a critical level $x_n^*[y(t)]$ that depends on both the current stage of development and learning about its research productivity. If it continues investing without success, however, there will come a point, y_n^* , at which it abandons the project entirely. There are two ways a project can be mothballed — either a negative resolution of uncertainty causes $x(t)$ to drop below $x_n^*[y(t)]$, or repeated failure causes $x_n^*[y(t)]$ to rise above the current $x(t)$. However, so long as $x(t) > 0$, there is only one way a project can be abandoned — through repeated failure.

3.2 Properties of the solution

Proposition 3 leads directly to the following straightforward simplification of Proposition 1 (stated without proof):

Corollary 2. *Let $V_c^n(x, y)$ and $V_m^n(x, y)$ be defined as in Proposition 1. In addition, let y_n^* and $H^n(y)$ be defined in Proposition 2. If $y > y_n^*$, then $V^n(x, y) = 0$ for every x . When $y \leq y_n^*$,*

$$V^n(x, y) = \begin{cases} V_c^n(x, y) & x \geq x_n^*[y] \\ V_m^n(x, y) & x < x_n^*[y], \end{cases} \quad (26)$$

with

$$V^n(0, y) = 0 \quad (27)$$

$$\lim_{x \rightarrow \infty} \frac{V^n(x, y)}{x} = H^n(y). \quad (28)$$

When $x = x_n^*[y]$,

$$V_c^n(x_n^*[y], y) = V_m^n(x_n^*[y], y) \quad (29)$$

$$\frac{\partial}{\partial x} V_c^n(x_n^*[y], y) = \frac{\partial}{\partial x} V_m^n(x_n^*[y], y) \quad (30)$$

$$\frac{\partial^2}{\partial x^2} V_c^n(x_n^*[y], y) = \frac{\partial^2}{\partial x^2} V_m^n(x_n^*[y], y). \quad (31)$$

This corollary shows that the derivation of the solution can be broken down into two distinct steps. First y_n^* and $H^n(y)$ are derived recursively using Equations (24) and (25). Once this step is completed, $V^n(x, y)$ is derived starting with $V^{N-1}(x, y)$ and solving Equation (15) recursively. For each value of n , only the boundary conditions in the above corollary must be imposed. The value of the investment opportunity must be zero when $x(t)$ goes to zero. It must be equal to the value in the case without fixed costs as $x(t) \rightarrow \infty$. On the boundary where the solutions of the two differential equations meet, continuity requires that the values match on the boundary, and smooth pasting requires that the first derivatives match as well. Finally, as is typical in this type of control problem, where investment can be initiated and discontinued instantaneously, a “super-contact” condition also equates the second derivatives on the boundary.

When $u = 0$, the law of motion in the mothball region is an ordinary differential equation, so it is straightforward to derive an analytical form of the solution in this region.

Proposition 4. *In the region where the firm suspends investment, the value of the investment opportunity is*

$$V_m^n(x, y) = A_n(y)x^\beta, \quad (32)$$

where

$$\beta = \frac{(\sigma^2 - 2\hat{\mu}) + \theta}{2\sigma^2} > 1, \quad (33)$$

$$\theta = \sqrt{8\hat{r}\sigma^2 + (\sigma^2 - 2\hat{\mu})^2}, \quad (34)$$

and the function $A_n(y)$ has to be defined from the conditions at the boundary.

An analytical solution to Equation (15) does not exist in the continuation region ($u = 1$) in the general case. A solution does exist, however, in the special case when $\pi(t)$ is only a function of the stage of development, $n(t)$, because in that case the PDE in x and y becomes an ordinary differential equation in x alone. We will derive this solution in the next section and solve the general case numerically. First, however, it is possible to derive a number of general results and provide insight into the sources and magnitude of the risk premia and into the qualitative features of the firm's investment policies.

The risk premium earned on the project at any stage can be derived using standard arguments and is given by

$$\frac{(\partial V^n(x, y)/\partial x)\lambda x}{V^n(x, y)}. \quad (35)$$

The next proposition derives the risk premium of the completed project and the risk premium when the project is mothballed.

Proposition 5. (a) *When the project is completed the risk premium is constant and equal to λ , the market price of risk.* (b) *When the project is in the mothball region, regardless of the number of stages completed, the risk premium is constant and equal to $\beta\lambda$, where β is defined in Proposition 4.*

By Proposition 4, $\beta > 1$, so the risk premium when the project is mothballed always exceeds the risk premium of the completed project. Our solutions in the remaining sections show that the risk premium lies between the two values given by the completed project and the mothballed project. In the mothballed region, the risk premium does not depend on the stage of the project or what the firm has learned about its R&D productivity. The value in this region has constant elasticity with respect to the state variable that carries systematic risk. This makes intuitive sense. Since no investment is occurring in this region, technological

progress and learning are not taking place. The only reason the value changes is due to information the firm receives about the cash flows, as it waits to see if $x(t)$ rises to the point where investment should resume or begin. The relative amount of local volatility this imparts is the same, regardless of n or how deep the firm is in the mothball region.

Myers and Howe (1997) argue that the cost of capital should decrease through the life of the project, due to the higher “leverage” of the project early in its life. At first, much of the present value of the project is associated with the relatively fixed R&D investments, as opposed to the variable net revenues that begin when the project is finished. Analogous to a levered firm, the required return on the “equity” in the project should be higher when it is more levered. As the R&D is brought to completion, and the fixed claims are “paid down,” the required return should decrease.

Our model provides a particularly simple setting in which to address these questions because the risk premium earned by the underlying cash flow process is constant. Thus any variability in the required return must be due to characteristics of the R&D itself. An important special case, chosen not for its realism but because it makes clear the role of operating leverage in the model, is when there are no fixed costs.

Proposition 6. *When there are no fixed costs ($a=0$), the risk premium earned on the R&D project is constant and equal to λ .*

This proposition makes clear that the leverage embedded in the project must be of a specific sort. As is evident from Proposition 2, when costs are proportional to potential cash flow, the optionality in the project, though still present, depends only on technical uncertainty, which is purely idiosyncratic. That is, the decision on when to exercise the option in this case does not depend on the current level of the cash flow process, $x(t)$. This intuition holds generally. Even when there are variable costs, the decision to *abandon* does not depend on $x(t)$. The volatility in returns attributable to the firm’s abandonment option derives entirely from the resolution of idiosyncratic, technical uncertainty.

The fact that the decision to abandon does not affect the risk premium illustrates an important point. The factors that impart systematic risk to a venture are naturally viewed as exogenous to the R&D process itself. Information about these exogenous factors continues to evolve independently of the firm’s investment choices. For example, macroeconomic conditions can be observed without direct investment by the firm. Thus the response to bad news about such factors is to delay investment and see if conditions improve. The decision to walk away from a project is much more likely to be driven by a determination that it is technologically infeasible or prohibitively costly, and the factors leading to such a determination are likely to be idiosyncratic to the project, firm, or industry.

4. Solution without Learning

When the success intensity is a known function of the stage of development, the model admits a closed-form solution. Generally the value functions $V^n(x, y)$ only depend on y through $\pi(t)$, which is a function of y . This is obvious from inspection of Equation (15). It is also economically obvious. The length of time already spent working on the project is a sunk cost. It is only relevant for future decisions if it is informative about the probability of success going forward. When this probability is known, the time spent working on the project cannot affect the decision on whether to keep working, so V^n is not a function of y .

When the success intensity is constant, $\pi(t) = \pi$,³ Equation (15) becomes

$$\begin{aligned} \frac{1}{2}x^2\sigma^2\frac{\partial^2}{\partial x^2}V^n(x) + \hat{\mu}x\frac{\partial}{\partial x}V^n(x) - \hat{r}V^n(x) \\ + u_n^*(x)(\pi[V^{n+1}(x) - V^n(x)] - a - bx) \\ = 0, \end{aligned} \quad (36)$$

where $u_n^*(x)$ denotes the optimal investment rule. This recursive system of ordinary differential equations is solved by starting with the terminal value and working backwards. At each stage the solution involves several constant parameters and a number of constants for stage n that are defined recursively given the constants obtained solving the previous stage and the boundary conditions in Corollary 2. The next proposition states this solution.

Proposition 7. *When the success intensity π is known and constant, the value of the R&D project, $V^n(x)$, is*

$$V^n(x) = \begin{cases} \sum_{i=n}^{N-1} F_n^i x^\gamma (\log x)^{i-n} + B_n x + G_n & x \geq x_n^* \\ A_n x^\beta & x < x_n^* \end{cases} \quad (37)$$

The constants that depend on the stage are

$$G_n = \frac{G_{n+1}\pi - a}{\hat{r} + \pi} = -a \sum_{i=1}^{N-n} \frac{\pi^{i-1}}{(\pi + \hat{r})^i} \quad (38)$$

$$B_n = \frac{B_{n+1}\pi - b}{\hat{r} + \pi - \hat{\mu}} = \frac{1}{\hat{r} - \hat{\mu}} \left(\frac{\pi}{\pi + \hat{r} - \hat{\mu}} \right)^{N-n} - b \sum_{i=1}^{N-n} \frac{\pi^{i-1}}{(\pi + \hat{r} - \hat{\mu})^i} \quad (39)$$

³ The solution in the case when $\pi(t)$ is not constant but is a known function of the current stage has a different functional form and is available at <http://faculty.haas.berkeley.edu/berk/rd.html>.

$$F_n^i = \begin{cases} 0 & i = N \\ \frac{2\pi}{(i-n)\psi} F_{n+1}^i + \frac{\sigma^2(i-n+1)}{\psi} F_n^{i+1} & n < i < N \\ \frac{2\sigma^2(x_n^*)^{-\gamma}}{\psi + \theta} ((1-\beta)B_n x_n^* - \beta G_n) - \sum_{i=n+1}^{N-1} F_n^i \left((\log x_n^*)^{i-n} - \frac{2\sigma^2(i-n)}{\psi + \theta} (\log x_n^*)^{i-n-1} \right) & i = n \end{cases} \quad (40)$$

$$A_n = \left(B_n x_n^* + G_n + \sum_{i=n}^{N-1} F_n^i (x_n^*)^\gamma (\log x_n^*)^{i-n} \right) (x_n^*)^{-\beta} \quad (41)$$

and x_n^* solves the following equation:

$$\begin{aligned} & (\beta - 1)(1 - \gamma)B_n x_n^* - \beta\gamma G_n \\ & = \frac{(x_n^*)^\gamma}{2\sigma^2} \sum_{i=n+1}^{N-1} \left((i-n)(\psi - \theta)F_n^i - 4\pi F_{n+1}^i \right) (\log x_n^*)^{i-n-1}. \end{aligned} \quad (42)$$

The constants β and θ are given by Equations (33) and (34), respectively, from Proposition 4, while the remaining constants are

$$\psi = \sqrt{8(\hat{r} + \pi)\sigma^2 + (\sigma^2 - 2\hat{\mu})^2} \quad (43)$$

and

$$\gamma = \frac{(\sigma^2 - 2\hat{\mu}) - \psi}{2\sigma^2}. \quad (44)$$

This closed-form solution allows us to calculate the risk premium explicitly. The following corollary does this.

Corollary 3. *The risk premium (the instantaneous expected rate of return is excess of the riskless rate) is equal to*

$$\begin{cases} \lambda \frac{B_n x + x^\gamma \sum_{i=n}^{N-1} (\gamma(\log x)^{i-n} + i - n) F_n^i}{B_n x + G_n + x^\gamma \sum_{i=n}^{N-1} (\log x)^{i-n} F_n^i} & x \geq x_n^* \\ \lambda\beta & x < x_n^* \end{cases} \quad (45)$$

which is once continuously differentiable everywhere.

To better understand the determinants of the risk premia associated with the project, it is useful to distinguish between the effects of voluntary exercise of the options associated with the project and its operating leverage. That is, even if investment cannot be suspended, the project can be viewed as a position that is short the fixed costs associated with investment and long the cash flows which have systematic risk. To

distinguish these cases, the next lemma gives the solution for the value of the firm and risk premium when investment cannot be suspended.

Lemma 3. *When the firm cannot suspend investment, the value of the project is*

$$V_n(x) = B_n x + G_n. \quad (46)$$

The risk premium (instantaneous rate of return in excess of the riskless rate) of the project is

$$\lambda \frac{B_n x}{B_n x + G_n}. \quad (47)$$

It is obvious from Equation (47) that the risk premium grows larger as one moves back from the terminal stage N simply due to the operating leverage in the project. From Equation (38) it is clear that G_n is negative and grows in absolute value with $N-n$. Thus the denominator becomes smaller relative to the numerator earlier in the life of the development project. The consequences of the optionality for the value and the risk premium are evident in the nonlinear terms in Equation (45). Our examples will show that these terms tend to mitigate the increases in the risk premium. Indeed, it is evident that when investment is involuntary, the value can approach zero, driving the risk premium in Equation (47) to infinity. The option to suspend investment limits this behavior, with the result that the risk premium is bounded above by its value in the mothball region.

To illustrate these effects, we parameterize the model using the values in Table 1. Time is measured in years. The cash flow process grows at 3% per year on average, with an annual standard deviation of 40% per year, so its innovations are quite variable relative to this growth rate. The success intensity at any point is given by $\pi = 2.0$, which translates into an 86% probability of completing at least one stage in a year. The probability of surviving one year without obsolescence, $e^{-0.1054}$, is 90%. The fixed costs are assumed to flow at rate 1.0, while the variable costs are 10% of the

Table 1
Parameter values for numerical examples

Parameter	Value
N	20
μ	0.03
σ	0.4
r	0.07
φ	0.4
π	2.0
ϕ	0.1054
a	1.0
b	0.1
λ	0.08
$\beta\lambda$	0.2

Table 2
Value and risk premium of the R&D venture when π is known

$x(t)$		Number of completed stages (n)					
		0	4	8	12	16	20
5	Value	0.03 (−15.9)	0.68 (−8.06)	5.3 (3.74)	21.56 (21.51)	48.3 (48.3)	88.74 (88.74)
	Risk premium	20.00 (—)	20.00 (—)	20.00 (38.99)	12.04 (12.15)	9.08 (9.08)	8.00 (8.00)
10	Value	0.16 (−13.24)	3.87 (0.75)	22.14 (21.97)	54.18 (54.18)	103.1 (103.1)	177.48 (177.48)
	Risk premium	20.00 (—)	20.00 (187.5)	12.98 (13.28)	9.65 (9.65)	8.51 (8.51)	8.00 (8.00)
15	Value	0.45 (−10.58)	10.46 (9.56)	40.24 (40.2)	86.85 (86.85)	157.91 (157.91)	266.22 (266.22)
	Risk premium	20.00 (—)	18.03 (22.11)	10.84 (10.88)	9.03 (9.03)	8.33 (8.33)	8.00 (8.00)
20	Value	0.93 (−7.91)	18.73 (18.37)	58.45 (58.43)	119.52 (119.52)	212.72 (212.72)	354.96 (354.96)
	Risk premium	20.00 (—)	14.57 (15.35)	9.97 (9.98)	8.75 (8.75)	8.24 (8.24)	8.00 (8.00)

The cells in the table show the value and (annualized) risk premium (in percent) of a venture that requires 20 stages to complete for a given $x(t)$ and the number of completed stages. Bold-face values correspond to mothballed projects. The numbers in parentheses give the value and risk premium (given by Lemma 3) of the venture assuming development must be undertaken. The risk premium is the instantaneous expected return minus the risk-free rate and is annualized.

value of $x(t)$. The value of the standard deviation in the pricing kernel (40% per year) is chosen, together with the correlation between the pricing kernel's innovations and those of the cash flow process (0.5) to provide a risk premium (λ) of 8% for the cash flows of the completed project. The final line in Table 1 gives the upper bound ($\beta\lambda$) for the risk premium, derived in Proposition 4. This is 20%.

Table 2 shows the value of the R&D project for different levels of $x(t)$ at different stages of development. The values in parentheses show the value of the venture assuming investment is nonvoluntary. Thus the difference between the actual values and the values in parentheses is a measure of the importance of the *option*, as opposed to the operating leverage associated with the fixed costs. The value of this option is most important in the early stages of development and for lower values of $x(t)$. The importance of the convexity that this imparts can be gauged by comparing the price appreciation for an increase in $x(t)$ when the option is important (e.g., 4 completed stages) to when it is not (e.g., 16 completed stages).

Table 2 also gives the risk premium at every stage. In Table 2, the bottom two numbers in each cell are the annualized risk premium on the project and, in parentheses, what the annualized risk premium would be if

there were no option to abandon or mothball. Comparing these numbers allows us to separately evaluate the contribution of operating leverage and the voluntary exercise decision to the risk premium. For high values of $x(t)$ and when the project is close to completion, these values are close together. The risk premium in these cases is attributable to the operating leverage associated with the fixed R&D expenses rather than the voluntary nature of those expenses. On the other hand, in the region where the firm is close to exercising the option to mothball, the risk premium on the project is significantly lower than that on the analogous venture with deterministic expenditures. The option to suspend investment truncates the down side of the distribution, lowering both systematic and unsystematic risk. Thus the “leverage” and the “optionality” embedded in the R&D venture operate in opposite directions. This is not surprising when one considers an analogy from traditional option pricing theory. An option that is in the money earns a lower risk premium than a forward contract on the same underlying asset when the strike price and forward price are equal.

An interesting feature Table 2 illustrates is that even though the firm's investment experience is idiosyncratic, resolution of this uncertainty still affects the risk premium. One can think of the R&D venture as a series of compound options on the underlying cash flows. The “strike price” of this option is the expected future investment required to complete the R&D. The risk premium of an option is decreasing in its “in the moneyness.” When a stage is completed, the expected future investment is decreased, thus reducing the “strike price” and increasing the “in the moneyness.” The net result is a decrease in the risk premium.

The same intuition explains Figure 1, which plots the risk premium against the number of stages completed, n , and the value of the cash flow process, $x(t)$, for different parameter choices. The dark line is x_m^* , the highest $x(t)$ for which the project is still mothballed. In all cases the risk premium is highest when the project is mothballed. The area below the flat region is the region where active development occurs. It is clear from the plots that the risk premium drops dramatically as soon as economic conditions improve to the point that the R&D is undertaken. It continues to fall toward the risk premium earned by the completed project as circumstances improve and progress continues. The risk premium decreases monotonically and nonlinearly as stages are completed, consistent with the arguments of Myers and Howe (1997) that the implicit leverage in the project falls as it proceeds, and with the intuition that the project is a compound option on the underlying cash flow process.

The lower bound the risk premium approaches as the project matures is the risk premium earned by the cash flow process itself. As $x(t)$ rises, the risk premium approaches this level at earlier points in the project's life, since the value of the R&D project approaches proportionality in $x(t)$ as

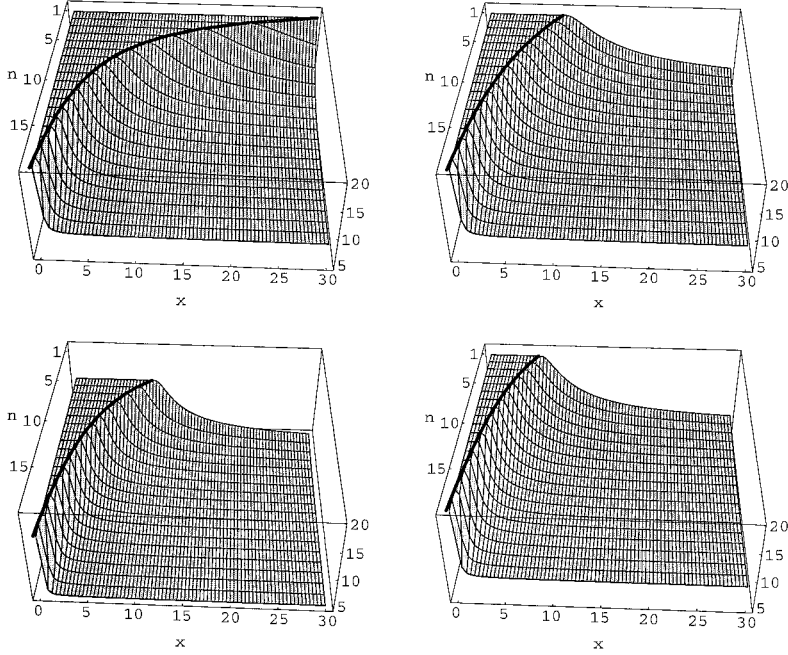


Figure 1

Risk premium of the R&D venture when π is known

The risk premium (in percent per annum) of a project that requires 20 stages to complete is plotted as a function of $x(t)$ and n , the number of completed stages, for different parameter values. The dark line marks x_n^* , so the mothball region is on the left side of the line. The top left plot is for the parameters in Table 1. All the other plots change only one parameter. For the top right plot, $b = 0$. The bottom left plot has $\sigma = 0.2$ and the bottom right plot has $\pi = 3.0$. The risk premium is the instantaneous expected return minus the risk-free rate.

the fixed costs and the mothball option become less important. Finally, note the higher sensitivity of x_n^* , the point just before active development begins, to the number of completed stages early in development. In the base case, completing the second stage decreases x_n^* from 30.0 to 21.2. In contrast, completing the 14th stage decreases x_n^* from 1.77 to 1.39.

The risk premium reaches an upper bound in the mothballed region, $\lambda\beta$, as $x(t)$ and n become small. Thus the risk premium on the project always lies between $\lambda\beta$ and λ . These bounds, which are simple to compute, may be useful in applications, since they could potentially provide conservative and liberal “hurdle rates” for the project.

5. Learning

We now turn to the solution for the general model. Analytical solutions to the PDE are not available. Instead, we first use Equations (24) and (25) in Proposition 2 to derive $H^n(y)$ and y_n^* , the contribution of past investment

to value and the critical level of past investment at which the project is abandoned. Then we numerically solve the model using Corollary 2. Further details can be found in the appendix.

We use the same parameterization in this section as previously (see Table 1). In addition, we also need to specify the initial prior on π , the success intensity. We assume that this prior is a Gamma distribution (see Lemma 2) with parameters $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$. This implies that $\pi(n, 0) = 2.0$, which means that the prior matches the known success intensity used in the last section. Furthermore, from Equation (4), the posterior estimate of π will again match this intensity any time $y = \frac{n}{2}$.

The importance of uncertainty in R&D productivity can be gauged by contrasting the case with learning to the case with no learning. To facilitate this comparison, Table 3 only provides the value of the project when $\pi(n, y) = 2.0$, at different stages of development. Thus, although the estimate of π is held fixed throughout the table, the firm's confidence in the estimate varies because this depends on n and y . Two facts stand out in the table. First, the value of the project when π is unknown is orders of magnitude larger in the early stages of development. Second, the risk premium is lower in the presence of the additional uncertainty over π . The lower risk premium may seem counterintuitive, given the higher levels of uncertainty. It is attributable to the way the optimal investment policy responds.

Table 3
Value of the R&D venture when $\pi(n, y) = 2.0$ (the prior)

		Number of completed stages ($y(t) = n/2$)					
$x(t)$		0	4	8	12	16	20
5	Value	0.72 (0.03)	0.93 (0.68)	5.34 (5.30)	20.91 (21.56)	47.5 (48.3)	88.74 (88.74)
	Risk premium	20.00 (20.00)	20.00 (20.00)	17.07 (20.00)	11.37 (12.04)	8.87 (9.08)	8.00 (8.00)
10	Value	4.13 (0.16)	5.34 (3.87)	21.48 (22.14)	52.44 (54.18)	101.3 (103.1)	177.5 (177.5)
	Risk premium	14.19 (20.00)	15.88 (20.00)	11.98 (12.98)	9.43 (9.65)	8.41 (8.51)	8.00 (8.00)
15	Value	9.22 (0.45)	12.72 (10.46)	38.74 (40.24)	84.03 (86.85)	155.2 (157.9)	266.2 (266.2)
	Risk premium	11.91 (20.00)	13.29 (18.03)	10.4 (10.84)	8.9 (9.03)	8.27 (8.33)	8.00 (8.00)

The cells in the table show the value and risk premium (in percent) of a venture that requires 20 stages to complete for a given $x(t)$. The time spent in development is half the number of completed stages so that $\pi(n, y) = 2.0$. Bold-face values correspond to mothballed projects. The numbers in parentheses give the value and risk premium when the probability of success is known ($\pi = 2.0$) and so there is no learning (i.e., the numbers in Table 2). The risk premium is the instantaneous expected return minus the risk-free rate and is annualized.

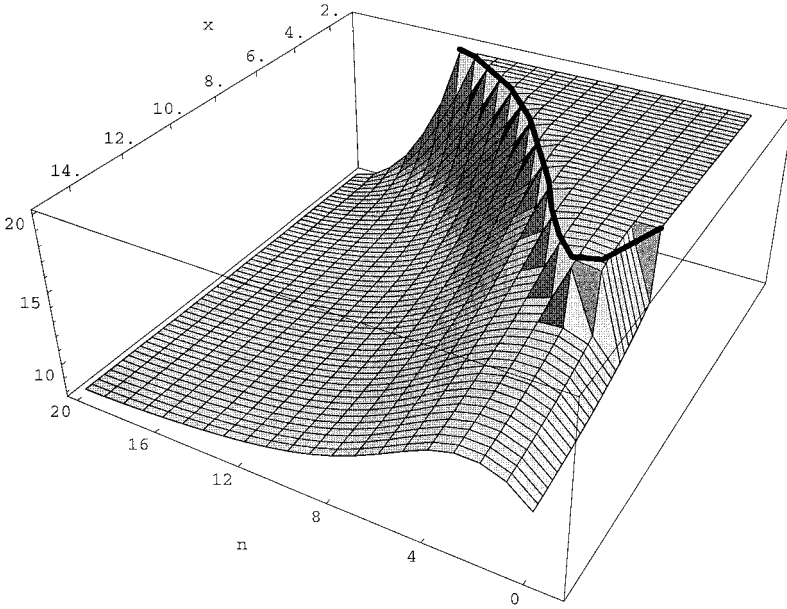


Figure 2
Risk premium of the R&D venture when π is unknown

The risk premium of the project that requires 20 stages to complete is plotted as a function of $x(t)$ and the number of completed stages (n) when $\pi(n, y) = 2.0$ (i.e., $y(t) = n/2$). The dark line marks $x_n^*[n/2]$, so the mothball region is the region “behind” the dark line. The risk premium is the expected return of the project less the riskless rate.

The extra uncertainty adds considerable value to the project in its early stages because it increases the value of the growth option. It also alters the optimal investment policy. For example, at the outset, when $x(t) = 10.0$, the project is mothballed when π is known, but is undertaken when it is unknown. In the latter case, undertaking the research has added value because it allows the firm to obtain a better estimate of π . Because the firm has an option-like claim, a positive realization of uncertainty about π is very valuable, so the firm attempts the research initially in the hopes that the true π is greater than 2.0. This, in turn, lowers the risk premium, since the risk premium of a mothballed project exceeds the risk premium of a project under development.

The added benefit of experimentation is short lived, as experience quickly provides a better estimate of π . This increases the likelihood of being mothballed. Consequently, and in contrast to the intuition in Myers and Howe (1997), it is possible for the risk premium to actually increase over the life of the project (see Figure 2 or the second to last row of Table 3). In fact, as Figure 2 shows, holding $x(t)$ and the estimate of π constant, a project that initially is under development when no stages are completed may later be mothballed even if stages are successfully completed in the

Table 4
Accumulated time in development that triggers abandonment, y_n^*

Number of completed stages or past successes (n)																			
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\frac{1}{2}$	1	$1\frac{3}{4}$	$2\frac{3}{4}$	$3\frac{3}{4}$	$4\frac{3}{4}$	6	$7\frac{1}{2}$	$9\frac{1}{4}$	$11\frac{1}{4}$	13	17	$21\frac{1}{4}$	$26\frac{3}{4}$	$34\frac{1}{2}$	$46\frac{1}{4}$	$65\frac{1}{2}$	$103\frac{1}{2}$	$207\frac{1}{4}$	$964\frac{3}{4}$

The cells in the table show, for a given number of completed stages, the accumulated time (in years) in development at which it is optimal to abandon the 20-stage venture. The numerical procedure uses a time step of 3 months, so the table shows y_n^* rounded down to the next lowest quarter.

interim. As the firm's estimate of π becomes sufficiently precise, however, the solution converges to the case when π is known, and completing a stage unambiguously reduces the probability of being mothballed and therefore the risk premium.⁴

The importance of early learning can be seen in Table 4, which gives the length of time that triggers the abandonment decision. Early in the life of an R&D project, technical failure is very costly. Initially just six months with no successes triggers abandonment. So, for example, for a project in which the true success probability equals the prior, about 37% of the time the project will be abandoned before the first stage is complete. In contrast, later in the life of an R&D project, failure is much less costly. With 19 stages complete, it takes 964 years before the abandonment option is exercised. This reflects both the effect of past success on posteriors and the lower expected cost to completion when only a few stages remain. Nevertheless, it is clear that luck plays an important role in which projects firms ultimately develop. Two identical firms doing R&D on identical projects can, because of small but early differences in the resolution of the technological uncertainty, come to completely different outcomes.

The optimal exercise of the option to mothball depends on $n(t)$ and $y(t)$. Changes in either $n(t)$ or $y(t)$ will affect the "in the moneyness" of the option and its risk premium. As before, the amount by which the risk premium of the incomplete project exceeds λ depends on the resolution of technical uncertainty, even though this risk is purely idiosyncratic and so is not priced directly.

Figure 3 plots the risk premium as a function of the current $x(t)$ and $y(t)$ at four different stages of development. For example, when $x(t) = 10$, it is optimal to develop initially and the risk premium is 14.2% per year

⁴ The value when π is unknown actually drops below the case with known π when n is large (e.g., $n = 16$ in Table 3). This results, presumably, from the asymmetry in the posterior distribution of π induced by the fact that in the latter stages, $N \ll y_n^*$. So, for example, when $n = 16$ (so $y(t) = 8$), the absolute most that π can depart from its current value of 2.0 in the positive direction is if the remaining stages are instantaneously completed, so $(18 + 4)/9 = 2.44$. On the other hand, the worst negative departure is that no further stages are completed until the project is abandoned (in this case after 65.5 years — see Table 4), so $18/(9 + 65.5) = 0.2416$.

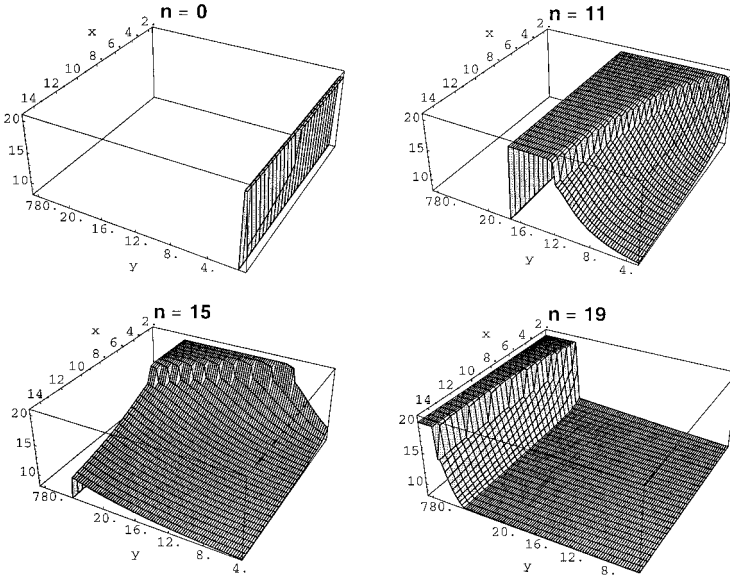


Figure 3
Risk premium of the R&D venture when π is unknown

For four different stages of completion ($n = 0, 11, 15$, and 19), the risk premium of a project that requires 20 stages to complete is plotted as a function of $x(t)$ and $y(t)$, the time spent (in years) in development. Note there is a change of scale (190-fold increase) on the y axis for values greater than 24 years. Blank areas correspond to values for which the venture is abandoned.

(Figure 3, upper left panel). After just one quarter without a success, the project is mothballed and the risk premium jumps to the upper bound of 20%. (Areas below the flat parts of the plots are regions with active development and the empty areas in the plots correspond to regions where the venture is abandoned.) On the other hand when 11 stages are completed after 2.75 years (Figure 3, upper right panel), for the same value of $x(t)$, the risk premium is only 8.57%, close to the lower bound. At this stage, going one quarter without a success changes the premium only marginally, to 8.65%. The situation is quite different if the amount of time already spent on the project is large. After 11 years of work with 11 stages completed, the risk premium is 16.8%, and it jumps to 20% if one more quarter goes by without a success and the project is mothballed. With 19 stages complete (Figure 3, lower right panel), the project is so far along that unless time spent is very large (in excess of a couple of centuries — note the scale change on the plot), the number of past failures makes very little difference and the risk premium is within 1% of the lower bound.

The case with 15 completed stages in Figure 3 (lower left panel) provides a nice illustration of the difference between the decision to mothball

and abandon. Recall that the resolution of technological uncertainty is idiosyncratic. The decision to mothball affects the risk premium because the optimal exercise policy depends on the current level of $x(t)$. The decision to abandon, however, depends only on how long the project has been under development, so this option does not command a risk premium. This can be clearly discerned from the plot. When $x(t)$ is small, it is possible that after enough time, the project will be mothballed. So as the time under active development increases, the risk premium increases dramatically to the upper bound. When $x(t)$ is large, it is never optimal to mothball, although for a large enough number of failures, it might be optimal to abandon. Since the abandonment option does not command a risk premium, the risk premium increases only slightly⁵ as the number of failures approaches $y_n^* = 46\frac{1}{4}$ years.

6. Conclusion

Research and development and similar investment projects have the property that much of the value of the investment is associated with future cash flows that are contingent on intermediate decisions. The uncertainty embedded in these projects is of two distinct types. There is purely idiosyncratic risk associated with the resolution of technical uncertainty. There is also risk associated with cash flows after development is complete, which will have a systematic component.

Our analysis highlights the importance of the nature of the information decision makers condition on when they make the intermediate investment decisions to continue, expand, contract, delay, or abandon the project. Whenever that information includes variables, such as forecasts of postdevelopment cash flows, that have a systematic component, this will impart considerable systematic risk to the project, even when the development process itself involves only technical risk. Our results show that the systematic risk, and the required risk premium, of the venture are likely highest early in its life, and most often decrease as it approaches completion. There are times when the risk premium can increase over the life of the venture, but such times are limited to the early stages of the venture. The option-like characteristics of the venture also determine the nature of the dependence of its value on the underlying state variables. At certain times during the project's life, its value is markedly responsive to information about technical progress or potential profitability, and at others much less so.

⁵ This small increase results from the option to mothball, since $x_{15}^*[y]$ is increasing in y , as is evident from the plot.

Appendix A: Derivation of the Hamilton-Bellman-Jacobi Equation

The derivation of Equation (15) relies on a number of technical, but standard, conditions. Interested readers can consult a standard text [see, e.g., Duffie (1996, Appendix E)]. Here we will assume that they are satisfied. Using Ito's lemma,

$$\begin{aligned} e^{-\hat{r}(T-t)} V^{n(T)}(x(T), y(T)) &= V^{n(t)}(x(t), y(t)) + \int_t^T e^{-\hat{r}(s-t)} \left[\mathcal{D}V^{n(s)}(x(s), y(s)) \right. \\ &\quad \left. - \hat{r}V^{n(s)}(x(s), y(s)) \right] ds + \int_t^T e^{-\hat{r}(s-t)} V_x^{n(s)}(x(s), y(s)) \sigma dz_s \\ &\quad + \int_t^T e^{-\hat{r}(s-t)} \left[V^{n(s)+1}(x(s), y(s)) - V^{n(s)}(x(s), y(s)) \right] dq_s, \end{aligned} \quad (48)$$

where

$$\mathcal{D}V^n(x, y) = \frac{1}{2}x^2\sigma^2 V_{xx}^n(x, y) + \mu x V_x^n(x, y) + u V_y^n(x, y),$$

and q_t denotes a unit jump process that jumps on success and whose intensity process is given by $\pi(n(t), y(t))u(t)$. Taking the expectation (under the risk neutral measure) of both sides of Equation (48) and rearranging terms provides

$$\begin{aligned} V^{n(t)}(x(t), y(t)) &= E_t^Q \left\{ e^{-\hat{r}(T-t)} V^{n(T)}(x(T), y(T)) \right. \\ &\quad \left. - \int_t^T e^{-\hat{r}(s-t)} \left(\mathcal{D}V^{n(s)}(x(s), y(s)) - \hat{r}V^{n(s)}(x(s), y(s)) \right) \right. \\ &\quad \left. + u(s)\pi(n(s), y(s)) \left[V^{n(s)+1}(x(s), y(s)) - V^{n(s)}(x(s), y(s)) \right] \right\} ds. \end{aligned}$$

Comparing this expression for $V^{n(t)}(x(t), y(t))$ to the expression in Equation (13), for the optimal choice of the function $u(t)$,

$$\begin{aligned} \zeta(s)x(s) - u(s)[a + bx(s)] &= \mathcal{D}V^{n(s)}(x(s), y(s)) - \hat{r}V^{n(s)}(x(s), y(s)) \\ &\quad + u(s)\pi(n(s), y(s)) \left[V^{n(s)+1}(x(s), y(s)) - V^{n(s)}(x(s), y(s)) \right], \end{aligned}$$

which is Equation (15) when $\zeta(s)=0$. When $\zeta(s)=1$ (so that $u(s)=0$), the differential equation reduces to

$$x(s) = \mathcal{D}V^N(x(s)) - \hat{r}V^N(x(s)), \quad (49)$$

which is well known to have Equation (14) as the solution.

Appendix B: Proofs

Proof of Lemma 1. Assume that it is optimal to try for the next stage at time t . Let $\tau_n < t$ be the time of the n^{th} success. Then the probability that another success will occur in the next instant is

$$\pi_n(y(t)) \equiv \lim_{\Delta \rightarrow 0} \frac{P[\tau_{n+1} \in [t, t + \Delta] | \tau_{n+1} > t]}{\Delta} \quad (50)$$

$$\begin{aligned} P[\tau_{n+1} \in [t, t + \Delta] | \tau_{n+1} > t] &= \frac{P[\tau_{n+1} \in [t, t + \Delta] \wedge \tau_{n+1} > t]}{P[\tau_{n+1} > t]} \\ &= \frac{\int (e^{-zy}(zy)^n (z\Delta + o(\Delta)) / n!) f(z) dz}{\int (e^{-zy}(zy)^n / n!) f(z) dz}, \end{aligned} \quad (51)$$

where, for convenience, we have suppressed the dependence of y on t . Substituting Equation (51) into Equation (50) and taking the limit gives

$$\pi_n(y) = \frac{\int \frac{e^{-zy}(zy)^n}{n!} f(z) dz}{\int \frac{e^{-zy}(zy)^n}{n!} f(z) dz} = \frac{\int e^{-zy} z^{n+1} f(z) dz}{\int e^{-zy} z^n f(z) dz}. \quad (52)$$

The integrand in the numerator is the probability of observing the $n+1$ success in the next instant times the probability of having already observed n successes in the time interval y . The integrand in the denominator is the probability of having n success in the time interval y . The integrals are over the possible different values for π . Given a Poisson process with intensity z , the Laplace transform of z is

$$\Phi(y) = E[e^{-zy}]. \quad (53)$$

The n th derivative of $\Phi(y)$ is

$$\Phi^{(n)}(y) = (-1)^n E[z^n e^{-zy}] = (-1)^n \int e^{-zy} z^n f(z) dz. \quad (54)$$

Substituting this into Equation (52) provides

$$\pi_n(y) = \frac{\int e^{-zy} z^{n+1} f(z) dz}{\int e^{-zy} z^n f(z) dz} = \frac{-\Phi^{(n+1)}(y)}{\Phi^{(n)}(y)}. \quad (55)$$

Proof of Lemma 2. This follows directly from Lemma 1 by letting $f(z)$ in Equation (55) be a gamma distribution. ■

Proof of Proposition 1. We begin by first justifying the boundary conditions of Equations (22) and (23). Note that $\partial V_n(x, y)/\partial y \leq 0$, and $V_n(x, y) \geq 0$, so we have

$$\begin{aligned} & \left(\pi(n, y) [V^{n+1}(x, y) - V^n(x, y)] + \frac{\partial}{\partial y} V^n(x, y) - a - bx \right) \leq \left(\pi(n, y) V^{n+1}(x, y) - bx \right) \\ & \leq x \left(\frac{\pi(n, y)}{\hat{r} - \hat{\mu}} - b \right). \end{aligned}$$

By Lemma 2, $\lim_{y \rightarrow \infty} \pi(n, y) = 0$, which implies from the above expression that there exists a y^* such that for all $y > y^*$ and any x ,

$$\left(\pi(n, y) [V^{n+1}(x, y) - V^n(x, y)] + \frac{\partial}{\partial y} V^n(x, y) - a - bx \right) < 0.$$

From Equation (15), this implies that for all $y > y^*$ and any x , $u(t) = 0$, which in turn implies that for all $y > y^*$ and any x , $V^n(x, y) = 0$, which provides Equation (22). Equation (23) follows from smooth pasting.

The boundary conditions of Equations (17) and (18) are standard. Equation (17) requires the project to be worthless when $x = 0$ and Equation (18) rules out bubbles. The boundary conditions of Equations (19) and (20) are the standard value-matching and smooth-pasting conditions. To see that the second derivatives match as well, note that $u = 1$ if and only if

$$\pi(n, y) [V^{n+1}(x, y) - V^n(x, y)] + \frac{\partial}{\partial y} V^n(x, y) > a - bx.$$

However, since V is continuous, this implies that on the boundary when $x = x_n^*[y]$,

$$\pi(n, y) [V^{n+1}(x_n^*[y], y) - V^n(x_n^*[y], y)] + \frac{\partial}{\partial y} V^n(x_n^*[y], y) = a + bx_n^*[y]. \quad (56)$$

On the boundary, V_m and V_c both solve Equation (15):

$$\begin{aligned} & \frac{1}{2} (x_n^*[y])^2 \sigma^2 \frac{\partial^2}{\partial x^2} V_c^n(x_n^*[y], y) + \hat{\mu} x_n^*[y] \frac{\partial}{\partial x} V_c^n(x_n^*[y], y) - \hat{r} V_c^n(x_n^*[y], y) \\ & + \pi(n, y) [V_c^{n+1}(x_n^*[y], y) - V_c^n(x_n^*[y], y)] + \frac{\partial}{\partial y} V_c^n(x_n^*[y], y) - a - b x_n^*[y] = 0 \end{aligned} \quad (57)$$

$$\frac{1}{2} (x_n^*[y])^2 \sigma^2 \frac{\partial^2}{\partial x^2} V_m^n(x_n^*[y], y) + \hat{\mu} x_n^*[y] \frac{\partial}{\partial x} V_m^n(x_n^*[y], y) - \hat{r} V_m^n(x_n^*[y], y) = 0 \quad (58)$$

Subtracting these two equations from one another and using the fact that the solutions and their first derivatives match on the boundary leaves

$$\begin{aligned} & \frac{1}{2} (x_n^*[y])^2 \sigma^2 \frac{\partial^2}{\partial x^2} V_c^n(x_n^*[y], y) - \frac{1}{2} (x_n^*[y])^2 \sigma^2 \frac{\partial^2}{\partial x^2} V_m^n(x_n^*[y], y) \\ & = -\pi(n, y) [V_c^{n+1}(x_n^*[y], y) - V_c^n(x_n^*[y], y)] - \frac{\partial}{\partial y} V_c^n(x_n^*[y], y) + a + b x_n^*[y] \\ & = 0, \end{aligned}$$

where the last equality follows from Equation (56). So we have

$$\frac{\partial^2}{\partial x^2} V_c^n(x_n^*[y], y) = \frac{\partial^2}{\partial x^2} V_m^n(x_n^*[y], y). \quad \blacksquare$$

Proof of Proposition 2. Substituting $V^n(x, y) = xH^n(y)$ into Equation (15) taking derivatives and simplifying provides

$$(\hat{\mu} - \hat{r})H^n(y) + \sup_{u \in \{0, 1\}} \left\{ u \left(\pi(n, y) [H^{n+1}(y) - H^n(y)] + \frac{d}{dy} H^n(y) - b \right) \right\} = 0. \quad (59)$$

It is straightforward to check that the functional form for $H_n(y)$ satisfies this differential equation subject to the boundary condition that $V^N(x, y) = x/(\hat{r} - \hat{\mu})$, which implies $H^N(y) = 1/(\hat{r} - \hat{\mu})$. This means that the optimal decision rule is independent of x .

Let y_n^* denote the boundary at stage n . From Proposition 1, the value-matching and smooth-pasting conditions are

$$H^n(y_n^*) = 0 \quad (60)$$

$$\frac{d}{dy} H^n(y_n^*) = 0. \quad (61)$$

The fact that Equation (60) is satisfied can be verified by inspection. Substituting these two conditions into Equation (59) and simplifying provides

$$\pi(n, y_n^*) H^{n+1}(y_n^*) = b,$$

where, by Lemma 2, $\pi(n, y_n^*) = (\alpha_1 + n/\alpha_2 + y_n^*)$. This expression can now be used to prove that Equation (61) is satisfied as well. $\blacksquare$

Proof of Proposition 3. We first establish that for $y > y_n^*$ the project is abandoned, that is, $V^n(x, y) = 0$. When $a = 0$, we know from Proposition 2 that $V^n(x, y) = xH^n(y)$. From the same proposition we know that for $y \geq y_n^*$, $H^n(y) = 0$. The result then follows from the fact that when $a > 0$, $0 \leq V^n(x, y) \leq x H^n(y)$ for any x, y .

Next, we show that if $y < y_n^*$ and $x > 0$, the project is never abandoned — $V^n(x, y) > 0$. Rewriting Equation (15) for the optimal choice of $u(t)$ gives

$$\frac{1}{2} x \sigma^2 \frac{\partial^2}{\partial x^2} V^n(x, y) + \hat{\mu} \frac{\partial}{\partial x} V^n(x, y) - \hat{r} \frac{V^n(x, y)}{x} + u \left(\pi(n, y) \left[\frac{V^{n+1}(x, y)}{x} - \frac{V^n(x, y)}{x} \right] + \frac{\partial}{\partial y} \frac{V^n(x, y)}{x} - \frac{a}{x} - b \right) = 0. \quad (62)$$

Note that as $x \rightarrow \infty$, this equation converges to the case when $a = 0$. So in the limit the solution must converge to the homogeneous solution:

$$\lim_{x \rightarrow \infty} \frac{V^n(x, y)}{x} = H^n(y) > 0. \quad (63)$$

Thus there must always exist an x large enough so that for any $y < y_n^*$, $V^n(x, y) > 0$. ■

Proof of Corollary 1. The first part of the corollary follows directly from Proposition 3 using the arguments in the text. The second part of the corollary follows by noting from Equation (25), when $b = 0$, $y_n^* = \infty$. ■

Proof of Proposition 4. Differentiating Equation (32) and substituting the resulting expressions into the PDE immediately verifies that the solution has the conjectured form. Imposing the boundary condition that $V^n(0, y) = 0$, gives the functional form for β . To show that $\beta > 1$, note that

$$\begin{aligned} \theta^2 &= 8\hat{r}\sigma^2 + (\sigma^2 - 2\hat{\mu})^2 \\ &= 8\hat{r}\sigma^2 + \sigma^4 - 4\hat{\mu}\sigma^2 + 4\hat{\mu}^2 \\ &> \sigma^4 + 4\hat{\mu}\sigma^2 + 4\hat{\mu}^2 = (\sigma^2 + 2\hat{\mu})^2, \end{aligned}$$

where we used the assumption that $\hat{r} > \hat{\mu}$. Substituting this expression into Equation (33) and using the assumption that $\sigma^2 > 2\hat{\mu}$ provides the result. ■

Proof of Proposition 5. In each case the result follows immediately from substituting the expressions for the value, Equations (14) and (32) into the expression for the risk premium, Equation (35).

Proof of Proposition 6. Substituting the solution when $a = 0$, $V^n(x, y) = xH^n(y)$, into Equation (35), shows that the risk premium is λ . ■

Proof of Proposition 7 ⁶. By direct substitution, it is straightforward to show that the solution satisfies the differential equation for every n . That leaves showing that the solution satisfies the boundary conditions. Since $\beta > 1$ and $\gamma < ((\sigma^2 - 2\hat{\mu}) - |\sigma^2 - 2\hat{\mu}|)/2\sigma^2 \leq 0$, the boundary conditions of Equations (27) and (28) are satisfied. Continuity at x_n^* follows directly by substituting for A_n in Equation (37), using Equation (41). To show that the first derivatives are matched on both sides of x_n^* , take the first derivative of the solution and substitute for A_n , using Equation (41), and F_n^n , using Equation (40). To verify that the second derivatives are matched as well, take the second derivative of Equation (37) and make the same two substitutions. Equation (42) can then be used to establish equality at x_n^* . ■

⁶ We are indebted to Alex Boulatov for suggesting the form of the solution to the differential equation in the case when $\pi(t)$ is constant (i.e., not a function of the stage).

Proof of Corollary 3. The result in the mothball region follows from Proposition 5. In the continuation region, the result follows from substituting Equation (37) in the continuation region into Equation (35) and simplifying. Continuity and differentiability follow by inspection at all points except for x_n^* , where the results follow from Equations (29)–(31). ■

Proof of Lemma 3. The solution of the PDE with $u_n^*(x)$ fixed at unity is given by Equation (37) in Proposition 7 in the continuation region. The boundary condition $V''(0) = 0$ then requires that $F_n^i = 0$ for all n and i . The risk premium is obtained by substituting the solution into Equation (35) and simplifying. ■

Appendix C: Numerical Details

Generating numerical solutions to the PDE in the general case is computationally very intensive. We experimented with different approaches, including a finite element approximation for the differential equation. In the end we settled on approximating the continuous-time problem with the corresponding discrete-time problem. The exact specification of the discrete problem is described in an earlier draft of this article, [Berk, Green, and Naik (1998)]. This approach affords the advantage that the solution is guaranteed to be the exact solution of the discrete-time problem. Thus the reader can readily interpret the source of numerical error as the difference between the discrete and continuous model. Such an interpretation is not available when the continuous-time model is approximated directly. Although the numerical errors are small, it is not computationally feasible to eliminate them entirely. Given this reality, the discrete approximation has a significant advantage because the differences between a continuous and discrete model are well understood.

In the discrete-time model we divide up time into periods of length $\Delta \equiv 1/M$, where M is the number of discrete time intervals in a year. All cash flows are assumed to occur at the beginning of the time period. In our case it is computationally infeasible to increase M beyond 4, so this is what is used. The parameters used in the discrete-time model are the corresponding continuous-time equivalents. Specifically, all intensities (i.e., r , μ , λ , and ϕ) are multiplied by Δ . So, for example, the periodic interest rate in the discrete-time model is given by $r\Delta = r/M$. The standard deviations are multiplied by $\sqrt{\Delta}$ so, for example, the periodic standard deviation of the cash flow is $\sigma/\sqrt{M}$. In discrete time, the probability of completing stage $n + 1$ in the next unit of time, Δ , is $\pi(n, y)\Delta$. If we let m be the number of past periods in which there were no successes, so that $y = (m + n)\Delta$, then (using Lemma 2) we have

$$\pi(n, y)\Delta = \frac{\alpha_1 + n}{\alpha_2 + (m + n)\Delta} \Delta = \frac{\alpha_1 + n}{\alpha_2 M + (m + n)}.$$

This is the updating rule used in the discrete-time model.

The fact that the cash flows occur at the beginning of each discrete period rather than continuously over the period introduces a bias into the valuation procedure. In discrete time the value of the complete project is given by

$$\frac{x\Delta}{1 - e^{(\hat{r} - \hat{\mu})\Delta}}.$$

This is close to, but not the same as Equation (14), the value in continuous time. When $M = 4$, the difference is small but not undetectable. We therefore decided to correct for this error by adjusting the periodic cash flow. Thus the periodic cash flow of the completed project in the discrete-time model is taken to be

$$\frac{x}{M} \left(\frac{1 - e^{(\hat{\mu} - \hat{r})/M}}{(\hat{r} - \hat{\mu})/M} \right) = x \left(\frac{1 - e^{(\hat{\mu} - \hat{r})/M}}{(\hat{r} - \hat{\mu})} \right).$$

A similar problem exists for the cash flows prior to completion. To make sure the present value of the expected costs to complete a stage in the two models is the same, we did a similar adjustment. In the continuous-time model, it is simple to show that the present value of the expected fixed costs incurred to complete the next stage (assuming you never choose to mothball) is given by

$$a \frac{\pi(n, y)}{(\hat{r} + \pi(n, y))^2}.$$

The corresponding present value of the expected fixed costs incurred to complete the next stage in the discrete-time model is given by

$$ae^{2\hat{r}/M} \frac{\pi(n, y)}{(\hat{r} + \pi(n, y))^2}.$$

To correct for the difference between these two expressions, the fixed costs that are incurred in each discrete period, Δ , are taken to be

$$\frac{a}{M} e^{-2\hat{r}/M}.$$

Using the same logic, the variable costs in each discrete period are taken to be

$$be^{-2(\hat{r}-\hat{\mu})/M} \left(\frac{x}{M} \right).$$

None of these adjustments alter the qualitative conclusions in the article.

Our programs solve numerically for the value of the R&D venture at each n, m (the number of past time intervals with no successes), and over a range of discrete values for x . The solution technique we use implements Corollary 2. We begin by using the discretized version of Proposition 2 [see Berk, Green, and Naik (1998, Proposition 6)] to solve analytically for $H^n(m)$ and $m^*(n)$, the corresponding discrete versions of $H^n(y)$ and y_n^* . Letting $V(n, m, x)$ be the corresponding value for $V^n(x, y)$ in the discrete-time model, we have for any $m > m^*(n)$ and $n < N$, $V(n, m, x) = 0$. Note that $V(N-1, m^*(N-1), x)$ is a function of itself and the value when the last stage is completed, which is known. It therefore can be derived numerically by first initially guessing the value of $V(N-1, m^*(N-1), x)$ over a discrete set of values (grid points) for x . Values between these grid points are derived by quadratic interpolation. Using this guess, a new value of $V(N-1, m^*(N-1), x)$ at each grid point is derived and this value is used as the new input. This process is repeated until the value over the grid points converge (the new value is no more than 0.001% different from the old value anywhere in the grid).⁷ This process is then repeated for all remaining values of $m < m^*(N-1)$. Once this is complete, one more step back can be taken and the value for $V(N-2, m^*(N-2), x)$ can be derived using the same process. This is continued until V is derived for all n and m at every grid point. The only remaining issue is what is done when the value is needed for an x outside of the grid. Since our grids start at zero, this only occurs for values of x , larger than the maximum grid value. Since for large x , the mothball option value is small, we use the value of the project with no mothball option, which is solved for analytically [see Berk, Green, and Naik (1998, Proposition 14)]. Further details can be found in Berk, Green, and Naik (1998, Appendix B). The program itself was written in C and is available on request.

In our simulations we use a 200-point grid between the points 0 and 100. The spacing between the points was not equal; more points were used at lower values of x . The ratio between the largest spacing between grid points and the smallest is 10.

⁷ The proof that $V(n, \cdot, \cdot)$ is a contraction mapping can be found in Berk, Green, and Naik (1998).

References

- Berk, J. B., R. C. Green, and V. Naik, 1998, "Valuation and Return Dynamics of New Ventures," Working Paper 7645, National Bureau of Economic Research.
- Berk, J. B., R. C. Green, and V. Naik, 1999, "Optimal Investment, Growth Options and Security Returns," *Journal of Finance*, 54, 1153–1607.
- Bhattacharya, S., and D. Mookherjee, 1986, "Portfolio Choice in Research and Development," *Rand Journal of Economics*, 17, 594–605.
- Brealey, R. A., and S. C. Myers, 1996, *Principles of Corporate Finance* (5th ed.), McGraw-Hill, Boston.
- Brealey, R. A., S. C. Myers, and A. J. Marcus, 1995, *Fundamentals of Corporate Finance*, McGraw-Hill, Boston.
- Childs, P. D., and A. J. Triantis, 1999, "Dynamic R&D Investment Policies," *Management Science*, 45, 1359–1377.
- Dasgupta, P., and E. Maskin, 1987, "The Simple Economics of Research Portfolios," *Economic Journal*, 97, 581–595.
- Dutta, P. K., 1997, "Optimal Management of an R&D Budget," *Journal of Economic Dynamics and Control*, 21, 575–602.
- Duffie, D., 1996, *Dynamic Asset Pricing Theory* (2nd ed.), Princeton University Press, Princeton, NJ.
- Grossman, G. M., and C. Shapiro, 1986, "Optimal Dynamic R&D Programs," *Rand Journal of Economics*, 17, 581–593.
- Gruver, G. W., 1991, "Optimal R&D Policy for a Patent Race with Uncertain Duration," *Mathematical Social Sciences*, 22, 69–85.
- Majd, S., and R. S. Pindyck, 1987, "Time to Build, Option Value, and Investment Decisions," *Journal of Financial Economics*, 18, 7–27.
- Myers, S. C., and C. D. Howe, 1997, "A Life-Cycle Model of Pharmaceutical R&D," technical report, Massachusetts Institute of Technology.
- Pakes, A., 1986, "Patents as Options: Some Estimates of the Value of Holding European Patent Stocks," *Econometrica*, 54, 755–784.
- Pindyck, R. S., 1993, "Investments of Uncertain Cost," *Journal of Financial Economics*, 34, 53–76.
- Posner, M. J. M., and D. Zuckerman, 1990, "Optimal R&D Programs in a Random Environment," *Journal of Applied Probability*, 27, 343–350.
- Schwartz, E. S., and M. Moon, 2000, "Evaluating Research and Development Investments," in M. J. Brennan and L. Trigeorgis (eds.), *Innovation, Infrastructure, and Strategic Options*, Oxford University Press, Oxford.
- Weitzman, M., W. Newey, and M. Rabin, 1981, "Sequential R&D Strategy for Synfuels," *Bell Journal of Economics*, 12, 574–590.
- Weitzman, M. L., 1979, "Optimal Search for the Best Alternative," *Econometrica*, 47, 641–654.

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