Dynamic Equilibrium with Liquidity Constraints

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This article studies an intertemporal economy with liquidity constrained and unconstrained individuals. We use a stopping time approach to solve the finite horizon-constrained consumption portfolio problem with constant relative risk aversion and to examine the structure of equilibrium. The impact of the constraint on the optimal consumption and the financing portfolio is assessed. The equilibrium state price density is related to the exercise boundary of an American-style contingent claim with nonlinear payoff. This stopping time characterization enables us to prove the existence of an equilibrium and can be implemented numerically. Properties of equilibrium bond and stock returns are examined.

The past decade has witnessed strong interest in the study of frictional economies. Motivated by the apparent failure of complete market models to explain even basic properties of asset returns, researchers have devoted extensive efforts to investigate market frictions.[1] Studies focusing on incomplete markets [Scheinkman and Weiss (1986), He and Pearson (1991), Karatzas et al. (1991)] or portfolio constraints [Cvitanic and Karatzas (1993), Cuoco (1997)] have improved our understanding of consumption portfolio choices and equilibrium asset prices. In comparison, the impact of liquidity constraints on investor behavior and equilibrium asset prices is much less understood.

A liquidity constraint arises when borrowing against one's future income is restricted. Labor income is a typical example of an endowment stream that, in general, cannot be marketed because of moral hazard or because

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[1] Fundamental contributions in complete markets settings include Merton (1971, 1973), Breeden (1979), Duffie and Huang (1985), Duffie (1986), Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989, 1991), and Duffie and Zame (1989).

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idiosyncratic payments often cannot be verified, even when they are observable. In essence, a liquidity constraint stipulates that short positions must be fully collateralized with marketable assets. Equivalently it mandates that future inflows of idiosyncratic income cannot become part of liquid wealth until they are effectively cashed. This type of constraint will then matter, when an income stream is received over the course of an individual's life, since it restricts the ability to smooth consumption. As a result, unlike the case of complete markets, the timing of income payments becomes relevant for consumption portfolio choices and asset pricing. It may also be noted that liquidity constraints, of the form considered in this article, are pervasive in practice. For instance, it is standard practice to require that short sales of securities be fully collateralized.[2] This suggests that constraints on liquid wealth may help to improve the empirical fit of asset pricing models. It also bears emphasizing that this type of constraint is substantially distinct from the incomplete markets' or portfolio constraints examined in the prior literature [such as Basak and Cuoco (1998), Constantinides et al. (1997)].

Early results on consumption portfolio optimization with liquidity constraints appear in He and Pages (1993), who prove the existence of an optimal policy from the dual problem. More recently Cuoco (1997) shows existence directly from the primal problem for a general class of constraints which includes minimum capital requirements, such as liquidity constraints. Detemple and Serrat (1998) propose a recursive solution method for the primal problem, which can be implemented in numerical applications. El Karoui and Jeanblanc-Picque (1998; hereafter EJ) relate the dual problem to an American put pricing problem and use the theory of stopping times to demonstrate the existence of an optimal policy.[3] However, while all these efforts are devoted to the study of control issues associated with liquidity constraints, there are only sparse results on equilibrium asset pricing.[4] In addition, while optimal consumption policies have been characterized, little is known about their financing portfolios.

This article contributes on both fronts. First, we complement existing results on consumption portfolio optimization under liquidity constraints by providing closed-form solutions for optimal policies for the case of finite-horizon and constant relative risk aversion. This example yields new economic insights regarding the structure and behavior of optimal consumption.

[2] Standard practices include, principally, stock borrowing restrictions (for equity markets) and reverse repo agreements (for fixed-income markets).

[3] The early article of He and Pages (1993) already characterizes times at which the liquidity constraint binds as stopping times. It also recognizes that, between these times, the investor acts as if unconstrained.

[4] For production economies we have the single-asset economy of Sheinkman and Weiss (1986). For exchange economies, He and Pages (1993: Section 7) provide some intuitions about equilibrium implications assuming that prices are diffusion processes and interest rates are absolutely continuous. Our results in Section 3 provide further insights about the structure of prices in equilibrium and, in particular, document the presence of new (singular) components in prices.

Second, we show existence and analyze the nature of equilibrium in the presence of liquidity constraints. We also provide an example with an explicit solution.

As shown by prior research, the choice problem of a liquidity-constrained individual, taking prices as given, is equivalent to a stopping time problem in which wealth is optimally allocated over an endogenous random time period corresponding to a period of positive liquidity. This is the case for any candidate equilibrium price process. In particular, it is true for the class of state price densities that clear the consumption good's market, that is, state price densities equal to the inverse aggregate demand function evaluated at aggregate consumption and parametrized by the liquidity multiplier of constrained agents. Using these relationships enables us to demonstrate that the equilibrium state price density can be determined from the optimal exercise boundary of a stopping time problem involving a contingent claim with nonlinear payoff. This characterization is interesting for several reasons. One aspect is that it enables us to show the existence of a competitive equilibrium in the setting under consideration. Another motivation is that it provides new insights about the structure of equilibrium prices and allocations. Finally, the characterization has the potential to be used in numerical implementations of the model.

One of our contributions is to show that equilibrium stock and bond returns contain singular components. Specifically the riskless asset's return is comprised of two terms. The first is a locally riskless interest rate which has the familiar structure: it is positively related to the expected consumption growth rate and negatively related to the aggregate consumption risk if the aggregator exhibits positive prudence. The second is a nonincreasing, singular component tied to the occurrence of a binding liquidity constraint. Whenever the constraint binds on an agent, his/her consumption demand experiences a permanent increase. Since at that time the agent is prevented from borrowing, but not lending, the equilibrium riskless return must, in general, be lower than what it would otherwise be. We provide conditions under which it will unambiguously decrease. We also show that while singular components affect stock and bond returns taken individually, the stock's risk premium still satisfies the standard consumption capital asset pricing model. However, the liquidity constraint influences the size of the risk premium by affecting the consumption allocation. In addition, the Sharpe ratio of the stock market unambiguously increases (remains constant) whenever aggregate risk aversion is a decreasing (constant) function of the consumption share of constrained individuals. We provide conditions under which this holds and present a closed-form solution when agents exhibit constant relative risk aversion. Our analysis also suggests that liquidity constraints are unable to rationalize the empirical magnitude of the Sharpe ratio, even though they may help to resolve the risk-free rate puzzle [Weil (1990)].

The next section of the article describes the economy. Section 2 solves the liquidity constrained consumption portfolio problem for CRRA utility and finite horizon. In Section 3 we develop a constructive approach to the determination of equilibrium and investigate the properties of the interest rate, the equity premium, and the stock's volatility. Concluding remarks are formulated in Section 4. Proofs are presented in the appendix.

1. The Economy

We consider an intertemporal economy with finite horizon with the usual assumptions.[5] A stock and a riskless asset are traded in the financial market. There is one share of stock outstanding; the riskless asset is in zero net supply. The stock pays dividends D such that

$$dD_t = D_t[\mu_t^D\, dt + \sigma_t^D\, dW_t], \quad D_0 \text{ given}, \tag{1}$$

where μ^D and σ^D are progressively measurable processes representing the drift and volatility of the dividend process. The stock price satisfies

$$dS_t + D_t\, dt = S_t[dm_t + \sigma_t\, dW_t] \tag{2}$$

$$dm_t = \mu_t\, dt + dA_t \tag{3}$$

subject to some initial value S_0, where m and σ are $\mathcal{F}_{(\cdot)}$-progressively measurable processes. The coefficient m is the cumulative drift of the stock price; it is composed of an instantaneous drift μ and a nonincreasing, singular process A which is null at time 0. As we will show later, the presence of liquidity constraints implies the emergence of such singular terms in equilibrium. The (cumulative) return R^f on the riskless asset satisfies

$$dR_t^f = r_t\, dt + dA_t, \tag{4}$$

where r is an instantaneous rate of interest and A is the singular component introduced above.

The formulations of Equations (2)–(4) are adopted from the start since singular components, in stock and bond prices, will be shown to arise in any equilibrium in which the constraint binds. It will also be shown that these components must be identical across assets.[6] The initial value S_0 and the coefficients (m, μ, A, σ, r) are endogenous processes. To simplify notation,

[5] The uncertainty is represented by a complete probability space $(\Omega, \mathcal{F})$, a probability measure P, a unidimensional Wiener process W defined on $(\Omega, \mathcal{F})$, and a filtration $\mathcal{F}_{(\cdot)}$. We assume that $\mathcal{F}_{(\cdot)}$ is the natural filtration generated by W and take $\mathcal{F}_T = \mathcal{F}$.

[6] The market structure of Equations (2)–(4) is consistent with the absence of arbitrage opportunities [see Karatzas, Lehoczky and Shreve (1991)]. An interpretation of the equilibrium-state price density as the marginal utility of a representative agent in a Pareto-inefficient (constrained) equilibrium follows from our analysis in Section 3.

we define the discount factor $b_t = \exp(-\int_0^t dR_s^f)$. The value of a money market account, or a dollar continuously reinvested at the riskless return, is $1/b_t$. In order to solve the agents' consumption portfolio problems, we impose the following standard condition on candidate equilibrium processes.

Assumption 1. *The market price of risk* $\theta \equiv \sigma^{-1}(\mu - r)$ *satisfies* $E \exp(\frac{1}{2}\int_0^T \theta_t^2\, dt) < \infty$.

As we shall demonstrate, this condition will be satisfied in equilibrium, under mild conditions on the primitives of the model: the class of relevant price processes is then the one satisfying Equations (2)–(4) along with Assumption 1. The corresponding state price density (SPD) process is

$$\xi_t = b_t \exp\left(-\int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t (\theta_s)^2\, ds\right). \tag{5}$$

To simplify notation we also define the relative SPD $\xi_{t,s} = \xi_s/\xi_t$.

Agents in our economy have von Neuman–Morgenstern preferences defined by

$$U^i(c) = E\left(\int_0^T u^i(c_t, t)\, dt\right), \tag{6}$$

$i = 1, \ldots, I$. Let $u_c^i \equiv \partial u^i/\partial x$ denote the marginal utility of consumption. The utility function satisfies standard assumptions: for each i and for each $t \in [0, T]$, $u^i(\cdot, t)$ is strictly increasing, strictly concave and twice continuously differentiable, and the limits $\lim_{x\to 0} u_c^i(x, t) = \infty$, $\lim_{x\to\infty} u_c^i(x, t) = 0$ hold. The consumption space is the set of nonnegative, progressively measurable processes which satisfy $\int_0^T c_t^i\, dt < \infty$, *a.s.*, and such that Equation (6) is finite.

Each agent is endowed with an income process e^i which is a nonnegative and integrable $\mathcal{F}_{(\cdot)}$-progressively measurable process. Initial endowments of shares of the stock are n^i, where $\sum_i n^i = 1$. Initial endowments of the riskless asset are null.

Agent i chooses a consumption process c^i and a portfolio process π^i which represents his/her stock demand. The portfolio is a progressively measurable process which satisfies the integrability condition $\int_0^T \pi_t^i(\mu_t - r_t)\, dt + \int_0^T (\pi_t^i\, \sigma_t)^2\, dt < \infty$, *a.s.* The admissible choice set for c^i and π^i will be specified below. The liquid wealth of an individual is the value of the portfolio of marketed assets: for a policy (c^i, π^i) the liquid wealth process X^i satisfies

$$dX_t^i = X_t^i\, dR_t^f + (e_t^i - c_t^i)\, dt + \pi_t^i[(\mu_t - r_t)\, dt + \sigma_t\, dW_t] \tag{7}$$

subject to the initial condition $X_0^i = n^i S_0$.

Individuals are divided into two categories, depending on whether their income stream can be marketed or not. Recall that by "marketing" the labor income stream, we mean using the future idiosyncratic income as collateral for today's aggregate short position. The set of all agents is $I = U \cup C$, where U is the set of unconstrained agents (type 1) and C is the set of constrained agents (type 2).

A consumption portfolio policy (c^i, π^i) is *admissible* for agent i if and only if the wealth process generated by (c^i, π^i) satisfies

$$\begin{cases} \xi_t X_t^i + E_t\left(\int_t^T \xi_s e_s^i \, ds\right) \geq 0 & \forall t \in [0, T] \quad \text{if} \quad i \in U \\ X_t^i \geq 0 & \forall t \in [0, T] \quad \text{if} \quad i \in C. \end{cases} \tag{8}$$

Let $\mathcal{A}_i$ be the admissible set for $i \in C \cup U$. Equation (8) states that a liquidity-constrained individual is limited to consumption portfolio policies that ensure the nonnegativity of liquid wealth (i.e., the *value of the individual's marketed assets*) at all times and with probability one. On the other hand, an unconstrained individual can run temporary deficits in his/her liquid wealth position as long as total wealth is nonnegative. Here total wealth equals liquid wealth plus the current value of the endowment stream. We can then say that the constrained individual faces a *liquidity* constraint, while the unconstrained individual faces a *solvency* constraint.

A pair $(c^i, \pi^i) \in \mathcal{A}_i$ is said to be optimal for agent i if and only if there is no other admissible pair (π', c') such that $U^i(c') > U^i(c^i)$: the dynamic problem of individual i is

$$\sup_{(c, \pi)} U^i(c) \quad s.t. \quad (c, \pi) \in \mathcal{A}_i. \tag{9}$$

Let $\mathcal{A}_i^*$ denote the set of optimal policies. For later purposes, we recall that the liquid wealth process X^i generated by an optimal policy $(c^i, \pi^i) \in \mathcal{A}_i^*$ with $i \in C \cup U$ satisfies the present value formula $\xi_t X_t^i = E_t(\int_t^T \xi_s (c_s^i - e_s^i) \, ds)$. As a result we can focus our attention on policies which satisfy this relation.

Consider now the economy described above with constrained and unconstrained individuals. A competitive equilibrium is a collection of stochastic processes $((c^i, \pi^i)_{i \in C \cup U}, (S, \mu, A, \sigma), r)$ such that (i) (c^i, π^i) is optimal for $U^i, i \in C \cup U$, (ii) markets clear: (a) commodity market: $\sum_i c^i = \sum_i (e^i + n^i D) \equiv e + D$, (b) stock market: $\sum_i \pi^i = S$, (c) riskless asset market: $\sum_i (\pi^i - X^i) = 0$.

2. Consumption Portfolio Choice with a Liquidity Constraint

This section provides background results for the resolution of the constrained consumption portfolio problem (Section 2.1), and presents a closed-form solution for the finite horizon case with CRRA (Section 2.2). To simplify notation we omit superscripts i throughout the section. We also assume that the initial endowment of the stock is null, $n = 0$.

2.1 The stopping time approach

Our analysis relies on the stopping time methodology suggested by He and Pages (1991) and developed by EJ (1998). The core of this approach is the fact that optimal wealth, in the presence of a liquidity constraint, becomes null at endogenous random times at which the liquidity constraint binds. Knowledge of these times determines a set of random subperiods which segment the investment interval. Within each subperiod, the investor is effectively unconstrained and resources can be allocated in an optimal unconstrained manner.

The resolution of the constrained problem then boils down to a stopping time problem, which can be formulated as follows. Let $I(\cdot, v) \equiv u_c^{-1}(\cdot, v)$ be the inverse marginal utility of consumption and recall that the optimal unconstrained consumption policy is given by $c_v = I(z\xi_{t,v}, v)$, where $z \equiv y\xi_t$ and y represents the Lagrange multiplier for the static budget constraint. Since optimal wealth is the present value of future net consumption, it satisfies the optimal stopping problem,

$$X_t(z) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left(\int_t^\tau \xi_{t,v}(I(z\xi_{t,v}, v) - e_v)\, dv\right), \tag{10}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times taking values in $[t, T]$. This expression reflects the intuition that the individual optimally selects the random subperiod over which resources are to be allocated in an unconstrained manner.

Simple arguments also show that optimal liquid wealth [Equation (10)] can be written as

$$X_t(z) = X_t^u(z) + P_t^u(z),$$

where $X_t^u(z)$ represents unconstrained liquid wealth,

$$X_t^u(z) = E_t\left(\int_t^T \xi_{t,v}(I(z\xi_{t,v}, v) - e_v)\, dv\right),$$

and $P_t^u(z)$ is an American put option written on unconstrained liquid wealth with null strike,

$$P_t^u(z) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t[\max\{-\xi_{t,\tau} X_\tau^u(z), 0\}].$$

Thus unconstrained liquid wealth finances unconstrained net consumption $I(z\xi_{t,v}, v) - e_v$ for $v \in [t, T]$. The put option represents the buffer stock of wealth needed to finance the constraint in the future.

Let $B = \{B_t : t \in [0, T]\}$ denote the optimal exercise boundary for the stopping time problem [Equation (10)]. The optimal constrained consumption policy is [see EJ (1998: Theorem 3.3 and Propositions 3.5 and 3.6)],

$$c_t = I((y^* - \gamma_t^*)\xi_t, t),$$

where $\gamma_s^* = (B_0 - \inf_{v \le s}(B_v/\xi_v))^+$ is the multiplier for the liquidity constraint and $y^* = B_0$ is the multiplier for the static budget constraint at the initial date. Thus knowledge of the exercise boundary B completely identifies the constrained consumption policy.

In order to calculate the American put value and the exercise boundary, it is useful to write the early exercise premium representation of the option price. For this purpose, let $\mathcal{E}_s(z) \equiv \{\omega : z \ge B_s\}$ be the set of states of nature in which immediate exercise is optimal at date s and let $1_{\mathcal{E}_s(z)}$ be the indicator of immediate exercise. We have

Theorem 1. *The American put option has the early exercise premium (EEP) representation $P_t^u(z) = \Pi_t(z)$, where*

$$\Pi_t(z) = -E_t\left(\int_t^T \xi_{t,v}(I(z\xi_{t,v}, v) - e_v)1_{\mathcal{E}_s(z)}\, ds\right)$$

is the EEP. The optimal exercise boundary satisfies the recursive integral equation $X_t^u(B_t) + \Pi_t(B_t) = 0$ subject to the boundary condition $I(B_T, T) = e_T$.

Optimal liquid wealth decomposes into two parts. The first component is the present value of net unconstrained consumption until T. The second component is the value of the buffer stock of wealth that must be set aside in order to meet the constraint in the future. This component is simply the value of an American put on unconstrained liquid wealth $X_\tau^u(z)$. Clearly as long as unconstrained liquid wealth is positive, immediate exercise is suboptimal (since exercise at such a time reduces the value of resources $X_t(z)$) and consumption can take place at the initial multiplier z. At times at which immediate exercise is optimal, the multiplier resets and consumption readjusts to a new level. Note also that buffer wealth is the present value of the gains from early exercise: these consist of the net "dividend" yield on unconstrained wealth, which is nothing else than income net of unconstrained consumption.

Theorem 1 also highlights the earlier intuition that the liquidity-constrained problem decomposes in a series of subproblems involving American puts. In fact, the investor's lifetime allocation problem involves a sequence of American put options. At the initial date the investor is endowed with an (unconstrained) wealth process and an American put written on unconstrained wealth. Once unconstrained wealth becomes sufficiently negative, immediate exercise of the put option becomes optimal. At that date the sum of unconstrained wealth and of the put payoff (i.e., constrained wealth) is null. The problem then starts anew: the investor adopts a revised "unconstrained" consumption policy and is endowed with a new American put on the resulting new "unconstrained" wealth process. This pattern repeats until the end of the period T.

The characterization of EJ implies that we only need to identify the optimal exercise boundary B in order to find an explicit formula for the optimal consumption policy. It is in Theorem 1 that we contribute to the existing literature on this particular topic by supplying a practical approach for computing the boundary. Our next section implements the methodology in a concrete environment.

2.2 Consumption portfolio policies for CRRA

In this section we study the constrained consumption/portfolio problem for the CRRA case in a financial market with lognormal prices. While He and Pages (1993) and EJ (1998) solved this model only for the case of an infinite horizon, we provide the solution for the finite horizon case.

Let R denote the coefficient of relative risk aversion. Suppose that (r, θ) are constant and that $A = 0$. The endowment process is $de_t = e_t[\mu^e\, dt + \sigma^e\, dW_t]$, where (μ^e, σ^e) are constants; also assume that there is no endowment of shares of stock $n = 0$. Define the function $G_t(a) = \frac{1}{a}[\exp(a(T-t)) - 1]$. For power, utility function unconstrained optimal wealth is (see proof of Theorem 2 in the appendix)

$$X_t^u(y\xi_t) = e_t[x_t^* G_t(\alpha) - G_t(\beta)], \tag{11}$$

where

$$\alpha = -\left(1 - \frac{1}{R}\right)\left(r + \frac{1}{2}\frac{1}{R}\theta^2\right) \tag{12}$$

$$\beta = \mu^e - r - \sigma^e\theta \tag{13}$$

and the process defined as $x_t^* \equiv (y\xi_t)^{-1/R}/e_t$ satisfies

$$dx_t^*/x_t^* = \mu^*\, dt + \left(\frac{1}{R}\theta - \sigma^e\right) dW_t^e$$

with

$$\mu^* = \alpha - \beta = \frac{1}{R}\left(r + \frac{1}{2}\left(\frac{1}{R} - 1\right)\theta^2\right) - \mu^e + \theta\sigma^e.$$

Here W^e is a Brownian motion under a new measure (see the appendix). The variable x^* represents the ratio of optimal unconstrained consumption to endowment. Its volatility, $\frac{1}{R}\theta - \sigma^e$, is positive (negative) when unconstrained consumption volatility $(\frac{1}{R}\theta)$ exceeds (falls below) the volatility of the endowment (σ^e). From Equation (11) it becomes apparent that the American put on the negative part of unconstrained wealth is equivalent to a random number $e_t G_t(\alpha)$ of American puts written on the unconstrained consumption-to-endowment ratio x_t^* with time-dependent strike $G_t(\beta)/G_t(\alpha)$. The immediate exercise set at date s can then be written in the form $\mathscr{E}_s \equiv \{x_s^* \leq B_s\}$, where B_s is the exercise trigger point.

With this notation we have the following result.

Theorem 2. *For the model outlined above, optimal constrained wealth is, for all* $t \in [0, T]$,

$$X_t = e_t V(x_t^*, t) + e_t \Pi(x_t^*; B(\cdot)),$$

where $e_t V(x_t^*, t)$ *represents the value of unconstrained wealth and* $e_t \Pi(x_t^*; B(\cdot))$, *is the buffer stock of wealth required to meet the constraint (EEP). We have*

$$V(x_t^*, t) = x_t^* G_t(\alpha) - G_t(\beta)$$
$$\Pi(x_t^*; B(\cdot)) = \int_t^T e^{(\mu^e - \sigma^e\theta - r)(v-t)} \Phi^{\pm}(x_t^*, B_v, v-t)\, dv$$

with

$$\Phi^{\pm}(x_t^*, B_v, v-t) \equiv N\left(\mp d(x_t^*, B_v, v-t) \pm \left(\frac{1}{R}\theta - \sigma^e\right)\sqrt{v-t}\right)$$
$$- x_t^* e^{\mu^*(v-t)} N\left(\mp d(x_t^*, B_v, v-t)\right)$$
$$d(x_t^*, B_v, v-t) = \frac{\log(x_t^*/B_v) + (\mu^* + \frac{1}{2}(\frac{1}{R}\theta - \sigma^e)^2)(v-t)}{(\frac{1}{R}\theta - \sigma^e)\sqrt{v-t}},$$

where $\Phi^+(\Phi^-)$ *applies when* $\frac{1}{R}\theta - \sigma^e > 0$ ($\frac{1}{R}\theta - \sigma^e < 0$). *The optimal exercise boundary B is a deterministic function which solves the recursive integral equation* $\Pi(B_t; B(\cdot)) = G_t(\beta) - B_t G_t(\alpha)$; $B_T = 1$. *The liquidity multiplier* γ^* *is*

$$\gamma_t^* = y^* - \inf_{v \in [0, t]} \frac{(B_v e_v)^{-R}}{\xi_v}$$

and $y^* = (B_0 e_0)^{-R}$. *Optimal consumption is* $c_t^* = ((y^* - \gamma_t^*)\xi_t)^{-1/R}$ *with* $y^* - \gamma_t^* = \inf_{v \in [0, t]} \frac{(B_v e_v)^{-R}}{\xi_v}$.

Theorem 2 shows that the exercise boundary B is entirely determined by the coefficients of the model. It can therefore be computed explicitly once these coefficients have been specified. Thus the budget multiplier y^*, the liquidity multiplier γ_t^*, and consequently the consumption policy are obtained in closed form as functions of the parameters of the model. It is also of interest to note that B depends on time but not on the underlying uncertainty (the Brownian motion W). Finally, since the option is equivalent to a random number of puts written on the consumption-to-endowment ratio x^*, the structure of this variable will determine the exercise boundary. In particular,

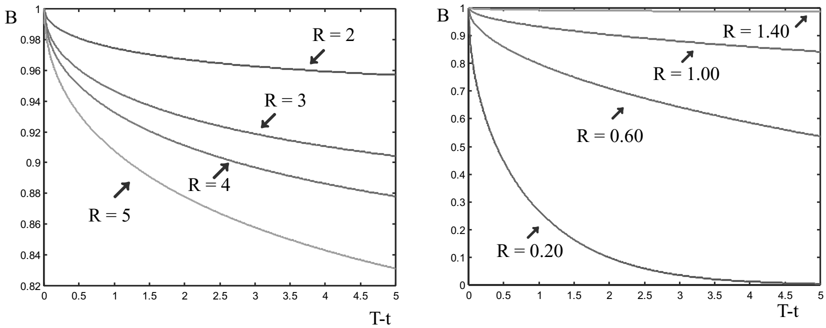

Figure 1
The effects of risk aversion on the optimal exercise boundary
Parameter values are $\mu^e = .10, \sigma^e = .20, \theta = .30, r = 6\%, T = 5, n = 2000$. The boundary is graphed for relative risk aversions equal to $R = 2, 3, 4, 5$ in the left panel and for $R = 0.20, 0.60, 1.00, 1.40$ in the right panel.

the volatility $\frac{1}{R}\theta - \sigma^e$ will affect the properties of the exercise boundary and the put price.

Figure 1 illustrates that the boundary is a decreasing function of the length of the investment horizon (equivalently the exercise region below the boundary is left connected). This reflects the fact that immediate exercise of a shorter maturity put option will always be optimal if immediate exercise of a longer dated put is optimal. The boundary behavior with respect to risk aversion is perhaps more striking. Indeed, note that an increase in risk aversion uniformly increases (decreases) the boundary when risk aversion is low (high). This dichotomous behavior follows from the fact that (assuming $\theta > 0$) the unconstrained consumption-to-endowment ratio x^* is positively (negatively) related to the underlying source of risk W when $\frac{1}{R}\theta - \sigma^e > 0$ (<0). When $\frac{1}{R}\theta - \sigma^e > 0$, an increase in risk aversion reduces the volatility of x^*, which increases the exercise boundary, since the downside potential of the put, and consequently its value, decreases. The opposite pattern arises when $\frac{1}{R}\theta - \sigma^e < 0$.

The properties of the boundary determine the behavior of the constrained consumption function. The following results are valid.

Proposition 1. *For the setting of Theorem* 2, *the consumption function has the following properties*:

(i) *Suppose that* $\mu^* < 0$. *Under this condition, unconstrained consumption increases with the investment horizon* T. *In contrast, constrained consumption may decrease when the investment horizon increases.*

(ii) *Suppose that* $r > (\frac{1}{2} - 1/R)\theta^2$. *Under this condition, unconstrained consumption increases with risk aversion* R. *In contrast, constrained consumption may be a humped* (*increasing-decreasing*) *function of* *risk aversion. It may become equal to unconstrained consumption at* $\theta = R\sigma^e$.[7]

(*iii*) *The constrained consumption function is homogeneous of degree one in initial endowment and wealth*: $c(kX_t, ke_t, t) = kc(X_t, e_t, t)$.

With CRRA, initial unconstrained consumption can always be expressed as the ratio of the present value of endowment relative to the present value of consumption growth (c_s^u/c_0^u),

$$c_0^u = \frac{E\left[\int_0^T \xi_s e_s ds\right]}{E_0\left[\int_0^T \xi_s \frac{c_s^u}{c_0^u} ds\right]} = \frac{E\left[\int_0^T \xi_s e_s ds\right]}{E_0\left[\int_0^T \xi_s^{1-1/R} ds\right]} = e_0 \frac{\left(\exp(\beta T) - 1\right)/\beta}{\left(\exp(\alpha T) - 1\right)/\alpha}. \tag{14}$$

When the maturity date T increases, the present value of the endowment increases by $\exp(\beta T)$ while the present value of the consumption growth increases by $\exp(\alpha T)$. The ratio of the two, equal to unconstrained consumption, will then increase when $\alpha < \beta$, that is, when $\mu^* = \alpha - \beta < 0$. Under this condition the endowment grows faster, on average, than optimal consumption. An increase in the individual's lifetime will then provide additional resources that can be allocated throughout the life cycle to raise the consumption profile.

As shown in Figure 2 (left panel), constrained consumption may decrease under the same condition. This follows because the exercise boundary decreases with the investment horizon and $c_0^* = B_0 e_0$. In effect, the increased horizon increases the probability of a binding constraint in the future, which promotes a more cautious consumption behavior at the outset. As illustrated, this effect can be sufficiently strong to offset the positive benefits accruing from the additional resources available during the individual's lifetime.

Risk aversion affects the present value of the unconstrained consumption growth rate, but not the present value of the endowment. From Equation (12) we obtain that an increase in R will decrease α if and only if the condition $r > (\frac{1}{2} - 1/R)\theta^2$ holds. Under this condition the present value of unconstrained consumption growth at date s, which equals $E_0[\xi_s^{1-1/R}] = \exp(\alpha s)$, will decrease, and this for all $s \in [0, T]$. It follows that the present value of the unconstrained consumption growth rate decreases, which increases initial consumption c_0^u.

In the presence of a liquidity constraint, initial consumption is seen to increase for low levels of risk aversion. This behavior mimics the behavior of the unconstrained policy because the constraint is either not active or is only active in a limited number of states. The increase in consumption, however, reduces liquid wealth and increases the occurrence of a binding constraint during the individual's lifetime. As a result the period, $[0, T]$ segments

[7] Given the assumption of von Neumann–Morgenstern preferences, the analysis performed here is unable to distinguish between the risk aversion and the intertemporal elasticity of substitution.

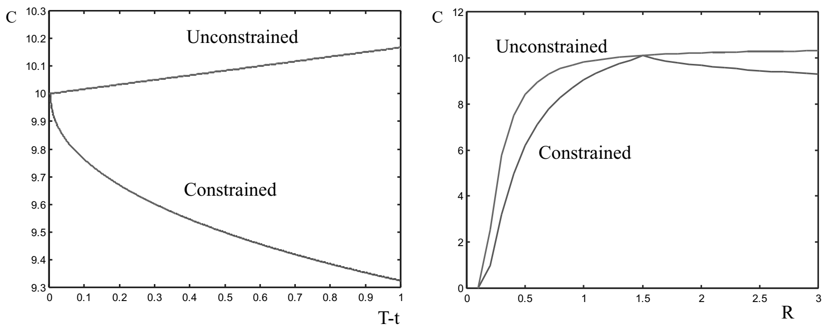

Figure 2
The behavior of the consumption function
Parameter values are $e_0 = 10$, $\mu^e = .10$, $\sigma^e = .20$, $\theta = .30$, $r = 6\%$, $n = 2000$. The left panel shows the effect of the horizon for a risk aversion of $R = 4$. The right panel shows the impact of relative risk aversion for horizon $T = 2$.

into an increasing number of random subperiods and consumption smoothing is increasingly restricted. Initial consumption may eventually fall, in order to allow for reallocations of resources to less favorable states in the first random subperiod. When risk aversion reaches a level where $\theta = R\sigma^e$, the consumption-to-endowment ratio x^* becomes deterministic and the constraint can only bind at deterministic times. The constraint will, in fact, never bind if x^* is sufficiently large and, if so, constrained consumption equals unconstrained consumption. Figure 2 (right panel) illustrates these phenomena.

To conclude the example we also examine the portfolio policy. In the appendix we show that

$$\pi_t^* = \frac{1}{R} E_t\left(\int_t^T \xi_{t,s} c_s^* \, ds \right) \sigma^{-1}\theta - \sigma^{-1}\sigma^e E_t\left(\int_t^T \xi_{t,s} e_s \, ds \right)$$
$$- \frac{1}{R}\sigma^{-1}\left(\theta - R\sigma^e\right) E_t\left(\int_t^T \xi_{t,s} c_s^* 1_{\{v[0,s] \geq t\}} \, ds \right), \qquad (15)$$

where $v[0, s]$ is the last time in $[0, s]$ at which the liquidity constraint binds. This portfolio has three components: a mean-variance term, a hedging term against income fluctuations, and a liquidity hedge. Inspection of this formula shows that the size of the mean-variance component is reduced (increased) at times when wealth is reduced (increased). By the same token, the liquidity hedge is negative if the market price of risk exceeds the volatility of income times risk aversion (i.e., $\theta - R\sigma^e > 0$). Under that condition, an innovation in the stock return has a positive effect on the consumption-to-wealth ratio x^* which tends to reduce the likelihood of a binding constraint. This promotes a reduction in stock holdings, since the stock tends to pay off in unconstrained states. In the opposite situation, when $\theta - R\sigma^e < 0$, a positive innovation in the stock return will decrease the ratio x^* and increase the likelihood of a binding constraint. Since the stock pays off in constrained states, it plays a useful hedging role and will be held to alleviate the impact of the constraint.

3. Asset Pricing with Liquidity Constraints

We now turn to the study of asset prices. Our first task is to extend the stopping time approach to an equilibrium context (Section 3.1). Once this is done we will examine the riskless return and the equity premium (Section 3.2) and provide an explicit solution in a simple setting (Section 3.3).

3.1 Existence and characterization of equilibrium

The setting is the economy of Section 1 with one unconstrained and one constrained agent (agents 1 and 2, respectively). The aggregate endowment is $e = e^1 + e^2$, where e^i follows the Itô process

$$de_t^i = e_t^i[\mu_t^i \, dt + \sigma_t^i \, dW_t]$$

and (μ^i, σ^i) are progressively measurable processes. Aggregate consumption $C = D + e$ has drift μ^c and volatility coefficient σ^c which are given by

$$C\mu_t^c = D\mu_t^D + e^1\mu_t^1 + e^2\mu_t^2 \quad \text{and} \quad C\sigma_t^c = D\sigma_t^D + e^1\sigma_t^1 + e^2\sigma_t^2.$$

In order to establish the existence of an equilibrium we need to solve the equations that describe market clearing and individual rationality. Let y^1 (y^2) denote the Lagrange multiplier for the static budget constraint of the unconstrained (constrained) individual. Let γ^2 be the multiplier for the liquidity constraint of the constrained agent. The consumption demand functions are, respectively, $I_1(y^1\xi_t)$ and $I_2((y^2 - \gamma_t^2)\xi_t)$, where the time argument in $I_i(\cdot, t)$ is omitted to simplify notation. With the renormalization $y \equiv y^2/y^1$, $\gamma_t \equiv \gamma_t^2/y^1$ and the definition $x_t \equiv y - \gamma_t$ we can write the aggregate demand function as $I_1(y^1\xi_t) + I_2(x_t y^1 \xi_t)$. Clearing of the consumption good market mandates that $C_t = I_1(y^1\xi_t) + I_2(x_t y^1\xi_t)$.

Let $g_t(x) \equiv g(C_t, x)$ denote the inverse aggregate demand function with respect to $y^1\xi_t$, that is, the unique solution of $C_t = I_1(g(C_t, x)) + I_2(xg(C_t, x))$. Then $y^1\xi_t = g(C_t, x_t)$ and the cost of the net consumption at date t of the constrained individual is equal to

$$h_t(x_t) \equiv g_t(x_t)[I_2(x_t g_t(x_t)) - e_t^2], \tag{16}$$

up to the multiplier y^1. In the appendix we show that the equilibrium state price density is

$$\xi_t = \frac{g(C_t, y - \gamma_t)}{g(C_0, y)} \equiv \frac{g_t(y - \gamma_t)}{g_0(y)}, \tag{17}$$

where the pair (y, γ) solves the system of equations and inequalities

$$g_t(y-\gamma_t)X_t^2 = E_t\left(\int_t^T h_s(y-\gamma_s)ds\right) \geq 0, \quad t \in [0, T] \tag{18}$$

$$\int_0^T X_t^2 d\gamma_t = 0 \tag{19}$$

$$E\left[\int_0^T h_s(y-\gamma_s)ds\right] = n^2 S_0 g_0(y), \quad y > 0, \tag{20}$$

and γ is a nondecreasing process which is null at 0. Since $\xi_0 = 1$, the budget multiplier of unconstrained agents is $y^1 = g(C_0, y)$.

The first condition, Equation (18), mandates that the equilibrium liquid wealth of the constrained agent be nonnegative at all times. Equation (19) is a complementary slackness condition: when the constraint is slack ($X_t^2 > 0$), the multiplier is flat ($d\gamma_t = 0$); when the multiplier charges ($d\gamma_t > 0$), the constraint must be binding ($X_t^2 = 0$). The third condition, Equation (20), is the static budget constraint of the constrained agent at the initial date: it stipulates that the present value of equilibrium consumption be equal to the value of the agent's endowment of the stock. By contrast with the single-agent problem where the SPD is exogenous, these conditions must now be satisfied, taking account of the reaction of prices. This feedback effect is captured through the inverse aggregate demand function $g_t(y-\gamma_t)$, which determines the SPD and intervenes in the expressions for wealth and for the cost of net consumption of the constrained individual.

Next note that the system of Equations (18)–(20) is similar in structure to the equations characterizing the optimal consumption policy in the static problem with exogenous SPD [see EJ (1998)]. As we show in the appendix, the same methodology can then be applied to characterize the liquidity multiplier γ and prove the existence of a solution. This yields the following stopping time characterization.

Theorem 3. *Suppose that the function $h_v(z) \equiv g_v(z)(I_2(zg_v(z)) - e_v^2)$ is strictly decreasing.*[8] *In equilibrium, the liquid wealth of the constrained agent solves the stopping time problem*

$$\begin{aligned} X_t^e(z) &= \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left(\int_t^\tau g_{t,v}(z)(I_2(zg_v(z)) - e_v^2)\, dv\right) \\ &= \frac{1}{g_t(z)} \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left(\int_t^\tau h_v(z)\, dv\right), \end{aligned} \tag{21}$$

[8] If this condition fails the immediate exercise region could be the union of disconnected sets. The characterization of equilibrium would then involve multiple exercise boundaries.

where $g_{t,v}(z) \equiv g_v(z)/g_t(z)$. *Let* $B^e_t = \inf\{z : X^e_t(z) = 0\}$ *denote the optimal (stochastic) exercise boundary. Then, the equilibrium liquidity multiplier equals* $\gamma^e_t(y) = (y - \inf_{s\le t} B^e_s)^+$. *Moreover,* $X^e_t(z) = X^{e,u}_t(z) + P^{e,u}_t(z)$, *where* $X^{e,u}_t(z)$ *is unconstrained wealth evaluated at the constrained-state price density (i.e.,* $g_t(z)X^{e,u}_t(z) \equiv E_t(\int_t^T h_v(z)\,dv)$*) and* $P^{e,u}_t(z)$ *is an American put option written on unconstrained liquid wealth with null strike (i.e., with payoff* $\max\{-X^{e,u}_t(z), 0\}$*). The put has the early exercise premium representation*

$$P^{e,u}_t(z) = \Pi^e_t(z) \equiv -\frac{1}{g_t(z)} E_t\left(\int_t^T h_v(z) 1_{\mathscr{E}^e_v(z)}\,dv\right),$$

where $\Pi^e_t(z)$ *denotes the early exercise premium and* $\mathscr{E}^e_v(z)$ *is the exercise region at date v. The optimal exercise boundary process satisfies the equation* $X^{e,u}_t(B^e_t) + \Pi^e_t(B^e_t) = 0$.

In equilibrium the SPD will reflect the impact of the liquidity constraint on consumption choices. This induced effect feeds back into the constrained individual choice problem and must be consistent with the consumption plan chosen. In other words, wealth must be nonnegative, taking account of the reaction of the equilibrium SPD. This leads to an optimal stopping time problem in which $X^e_t(z)$ represents the value of the constrained individual's net consumption evaluated at the SPD $g_t(z) = g(C_t, z)$.

The theory of optimal stopping ensures the existence of a solution to Equation (21). As in the individual agent problem, the optimal solution involves a stochastic barrier B^e_t which represents the optimal exercise boundary. Immediate exercise is optimal when the boundary process reaches the net multiplier $B^e_t = y - \gamma^e_t$.

In order to complete the demonstration of existence of an equilibrium, it remains to show the existence of a Lagrange multiplier y^e which satisfies the static budget constraint of Equation (20) of the constrained agent. This follows from the properties of the optimal exercise boundary in Theorem 3 and those of the agents' demand functions (see proof in the appendix). In summary, we have

Theorem 4. *Consider the economy of Section* 1 *and suppose that the assumption of Theorem* 3 *holds. A competitive equilibrium* $((c^i, \pi^i)_{i\in C\cup U}, (S, \mu, A, \sigma), r)$ *exists. Let* B^e_t *denote the boundary process of Theorem* 3 *and* $\gamma^e_t(y) = (y - \inf_{s\le t} B^e_s)^+$ *the associated liquidity multiplier. The budget multiplier* y^e *solves the equation*

$$X^{e,u}_0(y) + \Pi^e_0(y) = n^2 E\left(\int_0^T g_v(y - \gamma^e_v(y)) D_v\,dv\right).$$

This equation has a solution, which is unique if the function $h_v(z) - g_v(z)n^2 D_v$ *is strictly decreasing with respect to z. The equilibrium-state price density* ξ^e_t

is given by Equation (17) *evaluated at* $y^e - \gamma_t^e(y^e)$. *The equilibrium consumption allocation is* $c_t^1 = I_1(y^1 \xi_t^e)$ *and* $c_t^2 = I_2((y^e - \gamma_t^e) y^1 \xi_t^e)$, *where* $y^1 = g_0(y^e)$ *and* $\gamma_t^e = \gamma_t^e(y^e)$.

The construction summarized in Theorem 4 expresses the equilibrium state price density of Equation (17) in terms of the inverse aggregate demand function and the equilibrium shadow prices of the budget and liquidity constraints. In the next section we use this characterization to derive the interest rate and the risk premium of the stock.

3.2 Interest rate and equity premium with liquidity constraints

In order to describe the equilibrium interest rate and prices it is convenient to use the risk-aversion coefficient (R^a) and prudence coefficient (P^a) of the representative agent implicit in this equilibrium. Assuming that utilities take the form $u^i(c,t) \equiv \rho_t^{-1} u^i(c)$ with $\rho_t \equiv \exp(\beta t)$, these are, respectively, given by (see proof of Theorem 5 in the appendix)

$$R_t^a = C_t \left[\sum_i \frac{I_i((y^i - \gamma_t^i)\rho_t \xi_t)}{R^i((y^i - \gamma_t^i)\rho_t \xi_t)} \right]^{-1} \equiv C_t \left[\sum_i \frac{c_t^i}{R_t^i} \right]^{-1}$$

$$P_t^a = \frac{(R_t^a)^2}{C_t} \sum_i \frac{P^i((y^i - \gamma_t^i)\rho_t \xi_t) I_i((y^i - \gamma_t^i)\rho_t \xi_t)}{R^i((y^i - \gamma_t^i)\rho_t \xi_t)^2} \equiv \frac{(R_t^a)^2}{C_t} \sum_i \frac{P_t^i c_t^i}{(R_t^i)^2},$$

where R^i and P^i represent the relative risk aversion and relative prudence coefficients of agent i. Our next result details the structure of the competitive equilibrium.

Theorem 5. *Consider the economy of Section* 1 *with* $u^i(c,t) = \exp(-\beta t) \times u^i(c)$ *and suppose that the condition* $E \exp(\frac{1}{2} \int_0^T (R_t^a \sigma_t^c)^2 \, dt) < \infty$ *holds. The equilibrium interest rate and the equity premium are*

$$dR_t^f = r_t \, dt + dA_t \tag{22}$$

$$r_t = \beta + R_t^a \mu^c - \frac{1}{2} R_t^a P_t^a (\sigma^c)^2 \tag{23}$$

$$dA_t = -\frac{R_t^a}{R_t^2} \frac{c_t^2}{C_t} \frac{d\gamma_t^e}{y^e - \gamma_t^e} \tag{24}$$

$$\mu_t - r_t = R_t^a \sigma_t \sigma_t^c \tag{25}$$

where $\gamma_t^e = (y^e - \inf_{s \le t} B_s^e)^+$. *The stock price satisfies the present value formula* $S_t = E_t[\int_t^T g_{t,v} D_v \, dv]$ *where* $g_{t,v} = g(C_v, y^e - \gamma_v^e)/g(C_t, y^e - \gamma_t^e)$.

In equilibrium the riskless asset return has two components. The first is a locally riskless interest rate r which has the familiar structure [Equation (23)]. As usual, r is positively related to the expected consumption growth rate and negatively related to aggregate consumption risk if the aggregator exhibits positive prudence. Unlike models with portfolio constraints or incomplete markets, the cumulative riskless return R^f also has a singular component A, described in Equation (24). This component is tied to the occurrence of a binding liquidity constraint.[9] Whenever the constraint binds the consumption demand (savings demand) of the constrained agent experiences a permanent predictable increase (decrease). The riskless return naturally reflects the increased aggregate demand: its fall is just enough to provide an adequate incentive to reduce consumption immediately. The extent of the decrease is related to the size of the net multiplier $y^e - \gamma^e$. If the net multiplier is small, past consumption deferral has been substantial enough to ensure that the liquidity constraint is satisfied. When the liquidity constraint binds, the marginal value of wealth decreases and the additional consumption increase required to match this decrease becomes large (since marginal utility is very low). The required decrease in the cumulative risk-free return becomes comparatively large. The analysis above suggests that liquidity constraints tend to reduce the cumulative riskless return relative to an unconstrained economy.

The absence of arbitrage opportunities implies that the stock return inherits the same singular component A as the riskless asset. Thus, at times of a binding liquidity constraint, the stock price also experiences sharp decreases. As for the riskless asset return, the severity of the decrease is tied to the consumption share of the constrained population and their risk tolerance relative to the aggregate risk tolerance. An increase in these two factors amplifies the increase in the consumption demand of constrained individuals, which in turn requires a stronger price effect to maintain equilibrium.

Since singular components vanish from the difference between the stock and the riskless asset returns, it is not surprising to realize that the risk premium of the stock [Equation (25)] does not depend directly on the liquidity constraint. Indeed, this risk premium has the usual structure: it satisfies the consumption CAPM. Indirect effects of the liquidity constraint, however, affect the intertemporal properties of the equity premium through the risk aversion of the aggregator which reflects the distribution of consumption across agents and, as illustrated in the next section, through the effects on the structure and properties of volatility. Whenever the constraint binds, the aggregator's weight reallocates consumption across agents and this generates singular components in the evolution of the equity premium.

Can we conclude that liquidity constraints rationalize the empirical magnitude of the Sharpe ratio? The answer is negative. Indeed, the Sharpe ratio has an empirical value of about .37 [using Mehra and Prescott's (1985) numbers]. Since the volatility of the consumption growth rate is .0357, this implies an aggregator's risk aversion of 10.4 [see Equation (25)]. This number seems

[9] Singular components in cumulative returns also arise when marginal utility of agents is bounded at zero [see Karatzas et al. (1991)].

unrealistic since the aggregator's risk aversion is bounded above by the highest risk aversion of the agents populating the economy, as can be verified.

3.3 An example

Suppose now that both agents have a homogeneous utility function with constant relative risk aversion R and null subjective discount rate $\beta = 0$. The aggregator's risk aversion and prudence coefficients are then given by $R_t^a = R$ and $P_t^a = 1 + R$ and the equilibrium Sharpe ratio, $(\mu_t - r_t)/\sigma_t = R\sigma_t^c$, is unaffected by the liquidity constraint. The cumulative riskless return becomes $dR_t^f = r_t\,dt + dA_t$ with

$$\begin{aligned} r_t &= R\mu_t^c - \frac{1}{2}R(1+R)(\sigma_t^c)^2 \\ dA_t &= -\frac{c_t^2}{C_t}\frac{d\gamma_t^e}{y^e - \gamma_t^e}. \end{aligned}$$

Hence the interest rate r is the same while the cumulative interest R^f decreases relative to an economy without the liquidity constraint, but identical in all other respects.

In order to identify the equilibrium liquidity multiplier we assume furthermore that the coefficients of the dividend process D and of the endowment processes (e^1, e^2) are constant (lognormal processes). We also assume logarithmic utility $R = 1$ and suppose that income volatilities are homogeneous across individuals, but differ from dividend volatility (i.e., $\sigma^e \equiv \sigma^1 = \sigma^2 \neq \sigma^D$). For any quadruple (x, y, w, τ_v) define the function

$$\Psi(x, y, w, \tau_v) \equiv \left[x e^{\alpha^D \tau_v + (\sigma^D - \sigma^e) w \sqrt{\tau_v}} + y e^{\alpha^e \tau_v} + 1\right]^{-1}, \tag{26}$$

with $\alpha^D = \mu^D - \mu^2 - \frac{1}{2}((\sigma^D)^2 - (\sigma^e)^2)$ and $\alpha^e = \mu^1 - \mu^2$. With $\tau_v \equiv v - t$, and $w \sim N(0, 1)$ a standard normal random variable, this function represents the endowment-to-consumption ratio e_v^2/C_v at time v given initial values $x = D_t/e_t^2$ and $y = e_t^1/e_t^2$ at time t. With this notation we have

Theorem 6. *Consider the model with constant coefficients and homogeneous logarithmic utility function, and suppose that $\sigma^e = \sigma^1 = \sigma^2 \neq \sigma^D$. The immediate exercise region is $\mathscr{E}^e = \{(D/e^2, t) \in \mathbb{R}_+ \times [0, T] : D/e^2 \leq b_t(z)\}$ where the boundary $b_t^e(z)$ is a deterministic function which solves*

$$\begin{aligned} 0 = &\int_t^T N(-d(b_t^e(z), b_v^e(z), \tau_v))\,dv - (1+z) \\ &\times \int_t^T \int_{d(b_t^e(z), b_v^e(z), \tau_v)}^{\infty} \Psi\left(b_t^e(z), \frac{e_t^1}{e_t^2}, sw, \tau_v\right) n(w)\,dw\,dv \end{aligned} \tag{27}$$

subject to the boundary condition $b_T^e(z) = 1 + z - (e_T^1/e_T^2 + 1)$. In this expression $\tau_v \equiv v - t$, $s \equiv sign(\sigma^D - \sigma^e)$, the function $\Psi(\cdot)$ is defined in

Equation (26) *and*

$$d(x, b_v^e(z), u) = \frac{\log(b_v^e(z)) - \log(x) - \alpha^D u}{|\sigma^D - \sigma^e|\sqrt{u}}.$$

The equilibrium liquidity multiplier is $\gamma_t^e = y^e - \inf_{v\in[0,t]} \Phi_v^e(D_v/e_v^2)$, *where* Φ_v^e *is the inverse of the boundary* $b_v^e(z)$ *with respect to* z *and where* y^e *solves the static budget constraint. The equilibrium-state price density is* $\xi_t^e = g(C_t, y^e - \gamma_t^e)/g(C_0, y^e)$ *with*

$$g(C_t, y^e - \gamma_t^e) = \frac{1 + (y^e - \gamma_t^e)^{-1}}{C_t}. \tag{28}$$

The consumption allocation is $c_t^1 = g(C_t, y^e - \gamma_t^e)^{-1}$ *and* $c_t^2 = (y^e - \gamma_t^e)^{-1} g(C_t, y^e - \gamma_t^e)^{-1}$.

With homogeneous income volatilities, all the uncertainty in the evolution of individual wealth is encapsulated in the ratio of dividends relative to the income of the constrained individual. The optimal exercise time is then determined by this ratio and, for the same reason, the optimal exercise boundary is a function of time alone.[10]

From the present value formula in Theorem 5 we also deduce that the volatility of the stock is (see appendix)

$$\begin{aligned}\sigma_t = {} & \sigma^D + \frac{D_t}{C_t}\sigma^D + \sigma^e \frac{e_t}{C_t} - \sigma^D \frac{E_t\left[\int_t^T g(C_v, y^e - \gamma_v^e)\frac{D_v}{C_v} D_v\, dv\right]}{E_t\left[\int_t^T g(C_v, y^e - \gamma_v^e) D_v\, dv\right]} \\ & - \sigma^e \frac{E_t\left[\int_t^T g(C_v, y^e - \gamma_v^e)\frac{e_v}{C_v} D_v\, dv\right]}{E_t\left[\int_t^T g(C_v, y^e - \gamma_v^e) D_v\, dv\right]} \\ & - (\sigma^D - \sigma^e)\frac{E_t\left[\int_t^T \frac{1}{(y^e-\gamma_s^e)^2 C_s}\left(\frac{1}{b_{v[t,s]}^{e\prime}(y^e-\gamma_s^e)} 1_{\{v[0,s]\geq t\}} \frac{D_{v[t,s]}}{e_{v[t,s]}^2}\right) D_s\, ds\right]}{E_t\left[\int_t^T g(C_v, y^e - \gamma_v^e) D_v\, dv\right]},\end{aligned} \tag{29}$$

where $b_v^{e\prime}(z)$ is the (positive) derivative of the boundary function with respect to z and $v[0, s]$ is the last time in $[0, s]$ at which the constraint binds (i.e., the time at which the process $\Phi_v^e(D_v/e_v^2)$ achieves its minimum).

This formula shows that the liquidity multiplier affects volatility at two levels. First, the size of the multiplier conditions the responsiveness of the state price density to fluctuations in aggregate consumption. This effect is

[10] When $\sigma^1 \neq \sigma^2$ the optimal stopping problem has two state variables, D/e^2 and e^1/e^2. The immediate exercise boundary becomes an exercise surface that is parametrized by time and the ratio of endowments across agents; it satisfies an equation similar to Equation (27) with suitably modified components.

implicit in the fourth and fifth terms in the volatility expression, which both depend on the values taken by γ^e. Second, the liquidity multiplier is itself sensitive to innovations in aggregate consumption and this will induce an additional volatility component given by the last term of Equation (29). This effect materializes whenever $\sigma^D \neq \sigma^e$. When $\sigma^D > \sigma^e$ the effect is negative: ceteris paribus the possibility of a binding constraint in the future tends to reduce the volatility of the stock. In this case the ratio of dividends to the endowment of the constrained agent is positively related to innovations in the dividend growth rate. Positive dividend innovations are then associated with a reduction in the likelihood of a binding constraint (i.e., a reduction in future values of the liquidity multiplier). Since the liquidity multiplier is positively related to future state prices, we conclude that the future state price density responds negatively to dividend innovations. The result is that the liquidity constraint dampens the immediate response of the stock price to dividend innovations. When $\sigma^D = \sigma^e$, the liquidity multiplier becomes insensitive to dividend innovations because the ratio of dividends to the endowment of the constrained individual is not affected. In this instance the liquidity multiplier fails to contribute to the volatility of the stock. Finally, the case $\sigma^D < \sigma^e$ is the opposite of $\sigma^D > \sigma^e$.[11]

4. Conclusion

In this article we examined the implications of the presence of liquidity constraints for the behavior of optimal consumption portfolio policies and of equilibrium prices and allocations. At the individual level it was shown that a liquidity constraint has the potential to modify some of the fundamental properties of the demand for consumption. In particular, we illustrated the fact that consumption could decrease with the life span of the individual or with risk aversion, in situations where it increases in the absence of a constraint. At the equilibrium level it was shown that riskless short-term returns have a tendency to decrease in the presence of a liquidity constraint. Liquidity constraints may then help to resolve the risk-free rate puzzle. Somewhat surprisingly we also found that the valuation of fundamental risks does not seem to change much, since the equilibrium Sharpe ratio is a positive and convex combination of the relative risk aversions of the agents in the economy. As a result, liquidity constraints cannot be invoked to explain the empirical magnitude of the Sharpe ratio.

The article also includes a methodological contribution since it provides a constructive approach for the determination of equilibrium in economies with liquidity constraints. This construction is implementable numerically

[11] In more general contexts where the ratio of endowments e^1/e^2 is stochastic, the liquidity multiplier depends on the pair of state variables $(D/e^2, e^1/e^2)$. Dividend innovations will then affect the liquidity multiplier through each of these state variables and this will be reflected in the volatility of the stock price.

and produces new insights about equilibrium quantities such as the riskless asset return and the coefficients of the stock price process. This approach could prove useful in other economic settings where barriers and constraints are important, as well as in environments where default matters.

Appendix: Proofs

Proof of Theorem 1. The process $X_t(z)$ in Equation (10),

$$\begin{aligned}
X_t(z) &= \sup_{\tau\in\mathcal{S}_{t,T}} E_t\left(\int_t^\tau \xi_{t,v}(I(z\xi_{t,v},v)-e_v)\,dv\right)\\
&= E_t\left(\int_t^T \xi_{t,v}(I(z\xi_{t,v},v)-e_v)\,dv\right)+\sup_{\tau\in\mathcal{S}_{t,T}} E_t\left(-\int_\tau^T \xi_{t,v}(I(z\xi_{t,v},v)-e_v)\,dv\right)\\
&= X_t^u(z)+\sup_{\tau\in\mathcal{S}_{t,T}} E_t\left(-\xi_{t,\tau}E_\tau\left(\int_\tau^T \xi_{\tau,v}(I(z\xi_{t,v},v)-e_v)\,dv\right)\right)\\
&= X_t^u(z)+\sup_{\tau\in\mathcal{S}_{t,T}} E_t(-\xi_{t,\tau}X_\tau^u(z)) \equiv X_t^u(z)+\Pi_t(z).
\end{aligned}$$

Note that stopping is never optimal when unconstrained wealth is positive. As a result $\Pi_t(z)=\sup_{\tau\in\mathcal{S}_{t,T}} E_t(\max(-\xi_{t,\tau}X_\tau^u(z),0))$ is a put option on unconstrained wealth with zero strike. Let $x^+\equiv\max\{x,0\}$. In order to prove the early exercise premium (EEP) representation for $\Pi_t(z)$ use the Tanaka–Meyer formula [Karatzas and Shreve (1988), Theorem 3.7.1] to write the put option payoff

$$\begin{aligned}
(-\xi_\tau X_\tau^u(z))^+ &= (-\xi_T X_T^u(z))^+ - \int_\tau^T \xi_v(I(z\xi_{t,v},v)-e_v)1_{\{X_v^u(z)<0\}}\,dv\\
&< -\frac{1}{2}(L(T,0;-\xi X^u(z))-L(\tau,0;-\xi X^u(z)),
\end{aligned}$$

where M is a martingale and $L(\cdot,0;-\xi X^u(z))$ is the local time of $-\xi X^u(z)$ at 0. Since $X_T^u(z)=0$ we obtain $(-\xi_T\times X_T^u(z))^+=0$. The EEP formula then follows from the general representation theorem for Snell envelopes of semimartingales [Rutkowski (1994), Proposition A.1].

To obtain the recursive equation for the boundary process, note that $X_t(B_t)=0$. The boundary condition follows since unconstrained wealth converges to zero as $t\to T$ and since the exercise premium converges to $e_T-I(z,T)$ when evaluated at $z=B_T$. ■

Proof of Theorem 2. Suppose that the coefficients (r,θ) are constant and that $A=0$. For power utility function unconstrained optimal wealth (parametrized by $z=y\xi_t$) is

$$\begin{aligned}
X_t^u(z) &= E_t\Big(\int_t^T \xi_{t,v}((z\xi_{t,v})^{-1/R}-e_v)\,dv\Big)=E_t\Big(\int_t^T (z^{-1/R}\xi_{t,v}^{1-1/R}-e_t\xi_{t,v}e_{t,v})\,dv\Big)\\
&= z^{-1/R}E_t\int_t^T \xi_{t,v}^{1-1/R}\,dv - e_tE_t\int_t^T \xi_{t,v}e_{t,v}\,dv\\
&= z^{-1/R}\int_t^T \exp\left(-(1-1/R)\left(r+\frac{1}{2}\theta^2\right)+\frac{1}{2}(1-1/R)^2\theta^2\right)(v-t)\,dv\\
&\quad - e_t\int_t^T \exp\left(\left(\mu^e-\frac{1}{2}(\sigma^e)^2-r-\frac{1}{2}\theta^2+\frac{1}{2}(\sigma^e-\theta)^2\right)(v-t)\right)dv\\
&= z^{-1/R}\int_t^T \exp(\alpha(v-t))\,dv - e_t\int_t^T \exp(\beta(v-t))\,dv
\end{aligned}$$

$$
\begin{aligned}
&= z^{-1/R}\frac{1}{\alpha}[\exp(\alpha(T-t))-1]-e_t\frac{1}{\beta}[\exp(\beta(T-t)-1]\\
&\equiv z^{-1/R}G_t(\alpha)-e_tG_t(\beta),
\end{aligned}
$$

with α,β as defined in Equations (12) and (13). Under the risk-neutral measure[12] the consumption process $x_t=(y\xi_t)^{-1/R}$ solves $dx_t=(1/R)x_t[(r+\frac{1}{2}(1/R+1)\theta^2)\,dt+\theta(dW_t^*-\theta\,dt)]$ and discounted unconstrained wealth $b_tX_t^u=b_tX_t^u(y\xi_t)$ satisfies

$$
\begin{aligned}
d(b_tX_t^u) &= b_t(G_t(\alpha)dx_t-x_t\exp(\alpha(T-t))\,dt-G_t(\beta)de_t+e_t\exp(\beta(T-t))\,dt)-b_trX_t^u\,dt\\
&\equiv b_t[e_t-x_t]\,dt+b_t\left[x_tG_t(\alpha)\frac{1}{R}\theta-e_tG_t(\beta)\sigma^e\right]dW_t^*.
\end{aligned}
$$

Define $\Pi(x_t,e_t;B(\cdot))\equiv\sup_{\tau\in\mathcal{S}_{t,T}}E_t^*(-b_{t,\tau}X_\tau^u(z))$, where the expectation is under the risk-neutral measure. Since $X_T^u(z)=0$ the EEP representation gives

$$
\Pi(x_t,e_t;B(\cdot))=E_t^*\left[\int_t^T b_{t,v}e_v\left(1-\frac{x_v}{e_v}\right)1_{\{x_v/e_v\le B_v\}}\,dv\right],
$$

where the exercise boundary solves the recursive integral equation $\Pi(e_tB_t,e_t;B(\cdot))=e_t\times\max(G_t\ (\beta)-B_tG_t(\alpha),0);\ B_T=1$. Passing to the new measure

$$
dQ^e=\exp\left(-\frac{1}{2}(\sigma^e)^2+\sigma^eW_T\right)dQ
$$

yields

$$
\Pi(e_tx_t^*,e_t;B(\cdot))=e_tE_t^e\left[\int_t^T e^{(\mu^e-\sigma^e\theta-r)(v-t)}(1-x_v^*)1_{\{x_v^*\le B_v\}}\,dv\right],
$$

where an application of Itô's lemma shows that $x_v^*=x_v/e_v$ satisfies

$$
\begin{aligned}
dx_t^*/x_t^* &= \frac{1}{R}\left[\left(r+\frac{1}{2}\left(\frac{1}{R}-1\right)\theta^2\right)dt+\theta\,dW_t^*\right]-[(\mu^e-\sigma^e\theta)\,dt+\sigma^e\,dW_t^*]\\
&\quad+(\sigma^e)^2\,dt-\frac{1}{R}\theta\sigma^e\,dt\\
&=\left[\frac{1}{R}\left(r+\frac{1}{2}\left(\frac{1}{R}-1\right)\theta^2\right)-(\mu^e-\sigma^e\theta)+(\sigma^e)^2-\frac{1}{R}\theta\sigma^e\right]dt+\left(\frac{1}{R}\theta-\sigma^e\right)dW_t^*.
\end{aligned}
$$

Under the equivalent measure Q^e, the process $W_t^e=W_t^*-\sigma^et$ is a Brownian motion process and

$$
dx_t^*/x_t^*=\mu^*\,dt+\left(\frac{1}{R}\theta-\sigma^e\right)dW_t^e
$$

with

$$
\mu^*=\frac{1}{R}\left(r+\frac{1}{2}\left(\frac{1}{R}-1\right)\theta^2\right)-\mu^e+\theta\sigma^e
$$

The event $\{x_v^*\le B_v\}$ is equivalent to

$$
\begin{aligned}
\frac{W_v^e-W_t^e}{\sqrt{v-t}} &\le -\frac{\log(x_t^*/B_v)+(\mu^*-\frac{1}{2}(\frac{1}{R}\theta-\sigma^e)^2)(v-t)}{(\frac{1}{R}\theta-\sigma^e)\sqrt{v-t}}\\
&\equiv -d(x_t^*,B_v,v-t)+\left(\frac{1}{R}\theta-\sigma^e\right)\sqrt{v-t}
\end{aligned}
$$

---

[12] The risk-neutral measure is $dQ=b_T^{-1}\xi_T\,dP$, where $b_T=\exp(-rT)$. Under Q the process $dW_t^*=dW_t+\theta\,dt$ is a Brownian motion [see Karatzas and Shreve (1988), Theorem 5.1, p. 191].

if $\frac{1}{R}\theta-\sigma^e>0$ and the reverse inequality otherwise. Substituting in the expression for $\Pi(x_t, e_t; B(\cdot))$ and computing the expectation gives

$$\Pi(e_t x_t^*, e_t; B(\cdot)) = e_t\left[\int_t^T e^{(\mu^e-\sigma^e\theta-r)(v-t)}\Phi^{\pm}(x_t^*, B_v, v-t)\,dv\right],$$

where

$$\Phi^{\pm}(x_t^*, B_v, v-t) = N\left(\mp d(x_t^*, B_v, v-t) \pm \left(\frac{1}{R}\theta-\sigma^e\right)\sqrt{v-t}\right) - x_t^* e^{\mu^*(v-t)} N(\mp d(x_t^*, B_v, v-t)).$$

Here Φ^+ (Φ^-) applies when $\frac{1}{R}\theta-\sigma^e>0$ ($\frac{1}{R}\theta-\sigma^e<0$). The exercise boundary satisfies $G_t(\beta)-B_tG_t(\alpha)=\Pi(B_t,1;B(\cdot))$ subject to the boundary condition $B_T=1$. The solution of this equation is a function of time alone. With the notation $\Pi(x_t^*; B(\cdot))=\Pi(x_t^*, 1; B(\cdot))$ we obtain the expressions in the theorem. Theorem 3.3 and Propositions 3.5 and 3.6 in El Karoui and Jeanblanc-Picque (1998) imply that the liquidity multiplier γ^* is $\gamma_t^* = y^* - \inf_{v\in[0,t]}((B_v e_v)^{-R}/\xi_v)$ and $y^*=(B_0e_0)^{-R}$. ■

Proof of Proposition 1. We first establish the properties of unconstrained consumption. Unconstrained consumption equals

$$c_0^u = y^{-1/R} = \frac{E\left[\int_0^T \xi_s e_s\,ds\right]}{E\left[\int_0^T \xi_s^{1-1/R}\,ds\right]} = e_0\frac{\alpha}{\beta}\frac{(\exp(\beta T)-1)}{(\exp(\alpha T)-1)},$$

where $\alpha=-(1-\frac{1}{R})(r+\frac{1}{2}\frac{1}{R}\theta^2)$ and $\beta=\mu^e-r-\sigma^e\theta$. After some calculation we obtain

$$\frac{\partial c_0^u}{\partial T} = e_0\frac{\alpha}{\beta}\exp((\alpha+\beta)T)\frac{\beta-\alpha-\beta\exp(-\alpha T)+\alpha\exp(-\beta T)}{(\exp(\alpha T)-1)^2}$$

$$\frac{\partial c_0^u}{\partial R} = e_0\frac{(\exp(\beta T)-1)}{\beta}\frac{(\exp(\alpha T)(1-\alpha T)-1)}{(\exp(\alpha T)-1)^2}\frac{\partial\alpha}{\partial R}.$$

From the first expression it follows that $\partial c_0^u/\partial T>0$ if and only if

$$\psi(T)\equiv\frac{\alpha}{\beta}(\beta(1-\exp(-\alpha T))-\alpha(1-\exp(-\beta T)))>0.$$

Note that $\psi(0)=0$ and that $\psi'(T)=\alpha^2((\exp(-\alpha T))-\exp(-\beta T))\gtrless 0$ if and only if $\beta\gtrless\alpha$. We conclude that $\psi(T)>0$ if and only if $\beta>\alpha$, that is, if and only if $\mu^*=\alpha-\beta<0$.

To identify the sign of $\partial c_0^u/\partial R$ note that $\varphi(T)\equiv\exp(\alpha T)(1-\alpha T)-1\le 0$ for all T. Indeed, we have $\varphi(0)=0$ and $\varphi'(T)=-\alpha^2T\exp(\alpha T)\le 0$ for all $T\ge 0$. It follows that $\partial c_0^u/\partial R>0$ if and only if

$$\frac{\partial\alpha}{\partial R}=\left(-\frac{1}{R^2}\right)\left(r+\left(\frac{1}{R}-\frac{1}{2}\right)\theta^2\right)<0,$$

that is, if and only if $r>(\frac{1}{2}-1/R)\theta^2$.

Properties (i) and (ii) of constrained consumption are established through the numerical example. For (iii) note that the optimal liquid wealth process for an initial condition (kX_0, ke_0) is

$$\xi_t X_t^{(k)} = E_t\left[\int_t^T [\xi_s^{1-1/R}(y^{(k)}-\gamma_s^{(k)})^{-1/R}-\xi_s k e_s]\,ds\right].$$

It is easy to verify that $X_t^{(k)}=kX_t$, $y^{(k)}=k^{-R}y$, and $\gamma_s^{(k)}=k^{-R}\gamma_s$ satisfies the required constraints. Homogeneity of the consumption function follows. ■

Proof of Equation (15). In the case of CRRA liquid wealth equals

$$\xi_t X_t = E_t\left[\int_t^T \xi_s^\rho (y-\gamma_s)^{-1/R}\, ds\right] - E_t\left[\int_t^T \xi_s e_s\, ds\right],$$

where $\rho \equiv 1-1/R$. Itô's lemma shows that the volatility of the left-hand side of this equation is $\xi_t(-\theta X_t + \pi_t\sigma)$. The Clark–Ocone formula [Nualart (1995), Proposition 1.3.5] gives the volatility of the right-hand side,

$$-\rho\theta E_t\left[\int_t^T \xi_s^\rho (y-\gamma_s)^{-1/R}\, ds\right] + (\theta-\sigma^e)E_t\left[\int_t^T \xi_s e_s\, ds\right] + \frac{1}{R}E_t\left[\int_t^T \xi_s^\rho (y-\gamma_s)^{-(1+1/R)}(\mathcal{D}_t\gamma_s)\, ds\right],$$

where $\mathcal{D}_t\gamma_s$ is the Malliavin derivative of the liquidity multiplier. Combining these two expressions and using the definition of liquid wealth shows that

$$\begin{aligned}\pi_t &= \frac{1}{R}\sigma^{-1}\theta X_t + \left(\frac{1}{R}\theta - \sigma^e\right)\sigma^{-1}E_t\left[\int_t^T \xi_{t,s} e_s\, ds\right] \\ &\quad + \frac{1}{R}\sigma^{-1}\xi_t^{-1}E_t\left[\int_t^T \xi_s^\rho (y-\gamma_s)^{-(1+1/R)}(\mathcal{D}_t\gamma_s)\, ds\right],\end{aligned}$$

where the expectation in the last term can also be written as $E_t[\int_t^T \xi_s c_s^*(y-\gamma_s)^{-1}(\mathcal{D}_t\gamma_s)\, ds]$.

Now recall that

$$\gamma_s^* = y^* - \inf_{v\in[0,s]} \frac{(B_v e_v)^{-R}}{\xi_v} = y^*\left(1 - \inf_{v\in[0,s]} B_v^{-R}(x_v^*)^R\right).$$

Let $v[0,s] \equiv \inf\{v \in [0,s] : \gamma_s^* = y^*(1 - B_v^{-R}(x_v^*)^R)\}$. Exercise 1.2.11 and the proof of Proposition 2.1.3 in Nualart (1995) show that

$$\begin{aligned}\mathcal{D}_t\gamma_s^* &= -y^* B_{v[0,s]}^{-R} 1_{\{v[0,s]\geq t\}} \mathcal{D}_t(x_{v[0,s]}^*)^R \\ &= -y^* B_{v[0,s]}^{-R} 1_{\{v[0,s]\geq t\}} R(x_{v[0,s]}^*)^{R-1}\mathcal{D}_t(x_{v[0,s]}^*) \\ &= -y^* B_{v[0,s]}^{-R} 1_{\{v[0,s]\geq t\}} R(x_{v[0,s]}^*)^R\left(\frac{1}{R}\theta - \sigma^e\right),\end{aligned}$$

where in the second line we used the chain rule of Malliavin calculus and in the third line the formula $\mathcal{D}_t x_{v[0,s]}^* = x_{v[0,s]}^*(\frac{1}{R}\theta - \sigma^e)$, on the event $\{v[0,s]\geq t\}$, which follows since x^* is a lognormal process. The identity

$$-y^* B_{v[0,s]}^{-R} 1_{\{v[0,s]\geq t\}}(x_{v[0,s]}^*)^R = (\gamma_{v[0,s]}^* - y^*)1_{\{v[0,s]\geq t\}} = (\gamma_s^* - y^*)1_{\{v[0,s]\geq t\}}$$

(i.e. the definition of $\gamma_{v[0,s]}^*$) then implies

$$\mathcal{D}_t\gamma_s^* = -(y^* - \gamma_s^*)1_{\{v[0,s]\geq t\}}(\theta - R\sigma^e).$$

Substituting this expression in the formula for π above now gives the formula in the text. ■

Remark

An alternative approach for deriving the optimal portfolio would be to calculate the volatility of the wealth process $X_t = e_t V(x_t^*, t) + e_t\Pi(x_t^*; B(\cdot))$. By Itô's lemma we get

$$\pi_t\sigma = X_t\sigma^e + e_t(V_x(x_t^*, t) + \Pi_x(x_t^*; B(\cdot)))x_t^*\left(\frac{1}{R}\theta - \sigma^e\right)$$

where

$$V_x(x_t^*, t) = G_t(\alpha)$$
$$\Pi_x(x_t^*; B(\cdot)) = \int_t^T e^{(\mu^e - \sigma^e\theta - r)(v-t)} \Phi_x^{\pm}(x_t^*, B_v, v-t)\, dv$$

and

$$\Phi_x^{\pm}(x_t^*, B_v, v-t) \equiv m(x_t^*, B_v, v-t)(\mp d_x(x_t^*, B_v, v-t)) - e^{\mu^*(v-t)} N(\mp d(x_t^*, B_v, v-t))$$

with

$$m^{\mp}(x_t^*, B_v, v-t) \equiv n\left(\mp d(x_t^*, B_v, v-t) \pm \left(\frac{1}{R}\theta - \sigma^e\right)\sqrt{v-t}\right) - x_t^* e^{\mu^*(v-t)} n(\mp d(x_t^*, B_v, v-t))$$
$$d_x(x_t^*, B_v, v-t) = \frac{1}{x_t^*(\frac{1}{R}\theta - \sigma^e)\sqrt{v-t}}.$$

Using the relations

$$V(x_t^*, t) = V_x(x_t^*, t)x_t^* - G(\beta)$$
$$\Pi(x_t^*; B(\cdot)) = \Pi_x(x_t^*; B(\cdot))x_t^* + H^{\mp}(x_t^*, B(\cdot), v-t)$$

with

$$H^{\mp}(x_t^*, B(\cdot), v-t) \equiv \int_t^T e^{(\mu^e - \sigma^e\theta - r)(v-t)} N\left(\mp d(x_t^*, B_v, v-t) \pm \left(\frac{1}{R}\theta - \sigma^e\right)\sqrt{v-t}\right) dv$$
$$- x_t^* \int_t^T e^{(\mu^e - \sigma^e\theta - r)(v-t)} m^{\mp}(x_t^*, B_v, v-t)(\mp d_x(x_t^*, B_v, v-t))\, dv$$

enables us to rewrite the optimal portfolio as

$$\pi_t\sigma = X_t\sigma^e + e_t(V(x_t^*, t) + \Pi(x_t^*; B(\cdot)))\left(\frac{1}{R}\theta - \sigma^e\right)$$
$$+ e_t(G(\beta) - H^{\mp}(x_t^*, B_v, v-t))\left(\frac{1}{R}\theta - \sigma^e\right)$$

or, equivalently,

$$\pi_t\sigma = \frac{1}{R}\theta X_t + e_t G(\beta)\left(\frac{1}{R}\theta - \sigma^e\right) - e_t H^{\mp}(x_t^*, B_v, v-t)\left(\frac{1}{R}\theta - \sigma^e\right).$$

Since $X_t = E_t(\int_t^T \xi_{t,s}(c_s^* - e_s)\, ds)$ and $E_t(\int_t^T \xi_{t,s} e_s\, ds) = e_t G(\beta)$ we can also write

$$\pi_t\sigma = \frac{1}{R}\theta E_t\left(\int_t^T \xi_{t,s} c_s^*\, ds\right) - \sigma^e E_t\left(\int_t^T \xi_{t,s} e_s\, ds\right) - e_t H^{\mp}(x_t^*, B_v, v-t)\left(\frac{1}{R}\theta - \sigma^e\right).$$

The last term in this expression is an alternative formula for the impact of the liquidity constraint on the optimal portfolio. By the formula based on the Malliavin derivative, this term has the sign of $\theta - R\sigma^e$.

Derivation of Equations (18)–(20)
To prove the existence of an equilibrium we need to solve the equations that describe market clearing and individual rationality, namely,

$$C_t = I_1(y^1\xi_t) + I_2((y^2 - \gamma_t^2)\xi_t) \tag{30}$$

$$X_t^2 = E_t\left(\int_t^T \xi_{t,s}[I_2((y^2-\gamma_s^2)\xi_s) - e_s^2]\,ds\right) \geq 0, \quad t \in [0,T] \tag{31}$$

$$\int_0^T X_t^2 d\gamma_t^2 = 0, \text{ with } \gamma^2 \text{ a nondecreasing process, null at 0} \tag{32}$$

$$E\int_0^T \xi_s[I_1(y^1\xi_s) - e_s^1]\,ds = n^1 S_0, \quad y^1 > 0 \tag{33}$$

$$E\int_0^T \xi_s[I_2((y^2-\gamma_s^2)\xi_s) - e_s^2]\,ds = n^2 S_0, \quad y^2 > 0. \tag{34}$$

By Walras's law, one of the budget constraints of Equations (33) and (34) is redundant. We can then normalize the price system by setting the initial state price density to $\xi_0 = 1$, define $y \equiv y^2/y^1$ and $\gamma \equiv \gamma^2/y^1$, and solve the remaining system for (y_1, y, γ, ξ). Solving the market-clearing condition of Equation (30) yields the state price density

$$\xi_t = \frac{g(C_t, y-\gamma_t)}{g(C_0, y)} \equiv \frac{g_t(y-\gamma_t)}{g_0(y)} \tag{35}$$

and the multiplier $y^1 = g(C_0, y-\gamma_0) = g(C_0, y)$. Here $g(\cdot)$ is the inverse aggregate demand function with respect to $y^1\xi$: it uniquely solves $C_t = I_1(g(C_t, y-\gamma_t)) + I_2((y-\gamma_t)g(C_t, y-\gamma_t))$. Note that $g(\cdot)$ depends on the net shadow price of the constraint $x \equiv y - \gamma$. Substituting in Equations (31), (32), and (34) and using the definition of $h_t(x)$ leads to the conditions stated in the text. ■

Proof of Theorem 3. The proof uses the same ingredients as for the individual problem [see El Karoui and Jeanblanc-Picque (1998) for details]. Define a nonincreasing process $\hat{z}_t$ and consider the class of candidate equilibrium process $g_t(\hat{z}_t)$. Because of competitive behavior the process $g_t(\hat{z}_t)$ is taken as exogenously given in the derivations below. The dual problem of the constrained individual leads to the following singular control problem

$$J(y-\gamma^*) = \inf_\gamma E\left[\int_0^T (u^2(I_2((y-\gamma_t)g_t(\hat{z}_t))) - (y-\gamma_t)h_t(y-\gamma_t, g_t(\hat{z}_t)))\,dt\right],$$

where $h_t(z, g_t(\hat{z}_t)) \equiv g_t(\hat{z}_t)[I_2(zg_t(\hat{z}_t)) - e_t^2]$ and γ is a nondecreasing process. The process γ has a right continuous, nondecreasing, inverse $\tau_k \equiv \inf\{s : \gamma_s \geq k\}$. The process τ_k is a stopping time and the following events coincide: $\{\gamma_s \geq k\} = \{\tau_k \leq s\}$. The following result holds,

Lemma 1. *For any differentiable function $f(z)$ we have*

$$\inf_\gamma E\int_0^T f(y-\gamma_t)\,dt = E\int_0^T f(y)\,dt - \int_0^y \sup_\tau E\left(\int_0^T 1_{\{\tau_u \leq t\}} f'(y-u)\,dt\right)du$$

Proof of Lemma 1. Consider a differentiable function $f(z)$ and suppose that $\gamma_t \in [0, y]$. Using $\{\gamma_t \geq u\} = \{\tau_u \leq t\}$ enables us to write

$$\begin{aligned} f(y-\gamma_t) - f(y) &= -\int_0^{\gamma_t} f'(y-u)\,du \\ &= -\int_0^y 1_{\{\gamma_t \geq u\}} f'(y-u)\,du \\ &= -\int_0^y 1_{\{\tau_u \leq t\}} f'(y-u)\,du. \end{aligned}$$

Integrating over time, taking the expectation and minimizing with respect to the multiplier γ yields,

$$\inf_{\gamma} E\int_0^T f(y-\gamma_t)\,dt = E\int_0^T f(y)\,dt - \sup_{\tau} E\int_0^T\int_0^y 1_{\{\tau_u\le t\}} f'(y-u)\,du\,dt.$$

Rearranging the last term gives the expression in the lemma. ■

Consider now the function $f(z)=u^2(I_2(zg_t(\hat{z}_t))-zh_t(z,g_t(\hat{z}_t))$. Taking its derivative and using $u_c^2(I_2(zg_t(\hat{z}_t))=zg_t(\hat{z}_t)$ leads to

$$\begin{aligned} f'(z) &= [[u_c^2(I_2(zg_t(\hat{z}_t))-zg_t(\hat{z}_t)]I_2'(zg_t(\hat{z}_t))-I_2(zg_t(\hat{z}_t))+e_t^2]g_t(\hat{z}_t)\\ &= [-I_2(zg_t(\hat{z}_t))+e_t^2]g_t(\hat{z}_t). \end{aligned}$$

An application of lemma 1 then gives

$$J(y-\gamma^*) = J(y) - \int_0^y \left[\sup_{\tau} E\int_0^T 1_{\{\tau_u\le t\}}[-I_2((y-u)g_t(\hat{z}_t))+e_t^2]g_t(\hat{z}_t)\,dt\right]du, \tag{36}$$

where

$$J(y)\equiv E\int_0^T [u^2(I_2(yg_t(\hat{z}_t))-yh_t(y,g_t(\hat{z}_t))]\,dt.$$

Let $\tau_z^0\equiv \inf\{s : y-\gamma_s\le z\}$ be the (nonincreasing) inverse of the nonincreasing process $y-\gamma_t$, and note that $\{\tau_u\le t\}=\{u\le\gamma_t\}=\{y-u\ge y-\gamma_t\}=\{\tau_{y-u}^0\le t\}$. With the change of variables $z=y-u$ we obtain $\{\tau_u\le t\}=\{\tau_z^0\le t\}$. Performing this change of variables in Equation (36) then gives

$$J(y-\gamma^*) = J(y) - \int_0^y \left[\sup_{\tau^0} E\int_{\tau_z^0}^T [-I_2(zg_t(\hat{z}_t))+e_t^2]g_t(\hat{z}_t)\,dt\right]dz.$$

The right-hand side of this equality can also be written as

$$J(y) - \int_0^y \left[\sup_{\tau^0} E\left(\int_0^{\tau_z^0} [I_2(zg_t(\hat{z}_t))-e_t^2]g_t(\hat{z}_t)\,dt\right) - E\left(\int_0^T [I_2(zg_t(\hat{z}_t))-e_t^2]g_t(\hat{z}_t)\,dt\right)\right]dz.$$

But at equilibrium we need $g_t(z)=g_t(\hat{z}_t)$, that is, $\hat{z}_t=z$. Recalling the definition of the process $X_t^e(z)$ in the theorem gives $J(y-\gamma^*)=J(y)-\int_0^y g_0(z)(X_0^e(z)-X_0^{e,u}(z))\,dz$, where $X_0^{e,u}(z)\equiv E(\int_0^T g_{0,t}(z)[I_2(zg_t(z))-e_t^2]\,dt)$.

Thus, to solve the singular control problem (at equilibrium prices), we need to solve the stopping time problem with value $X_0^e(z)$. Under the assumption of the theorem we can define the optimal exercise boundary as $B_t^e=\inf\{z : X_t^e(z)=0\}$. By construction $\gamma_t^e(y)=(y-\inf_{s\le t}B_s^e)^+$. ■

Proof of Theorem 4. To show the existence of equilibrium we need to demonstrate the existence of a multiplier y^e that satisfies the static budget constraint of Equation (20) of constrained agents.

Consider the function $g_t(x)$ which solves $C_t=I_1(g_t(x))+I_2(xg_t(x))$. Clearly $g_t(x)$ is strictly decreasing in x and satisfies the limiting conditions $\lim_{x\to 0}g_t(x)=\infty$, $\lim_{x\to 0}(xg_t(x))=u_c^2(C_t,t)$, and $\lim_{x\to\infty}g_t(x)=u_c^1(C_t,t)$.

These properties of $g_t(x)$ imply that the continuous function $h_t(x)=g_t(x)[I_2(xg_t(x))-e_t^2]$ has the limits $\lim_{x\to 0}h_t(x)=\infty$ and $\lim_{x\to\infty}h_t(x)<0$. Also $\lim_{x\to 0}[I_2(xg_t(x))-e_t^2]=C_t-e_t^2=D_t+e_t^1$.

By definition $B^e = \{B^e_v : v \in [0,T]\}$ represents the stochastic boundary at which immediate exercise of the put option is optimal (i.e., liquid wealth is null). This is a nonnegative process that provides an upper bound on the net multiplier $y-\gamma$ for wealth to be nonnegative. Since the endowment of agents (at date 0) is nonnegative and wealth is a decreasing function of y, by assumption, we can restrict attention to constants y such that $y \le B^e_0$. The associated liquidity multiplier is $\gamma^e_t(y) = (y - \inf_{s\le t} B^e_s)^+$ with $\gamma^e_0(y) = 0$.

Using Theorem 3 enables us to write the budget constraint of Equation (20) as

$$E\left[\int_0^T h_s\left(y-\left(y-\inf_{v\le s} B^e_v\right)^+\right)ds\right] = n^2 E\left[\int_0^T g_s\left(y-\left(y-\inf_{v\le s} B^e_v\right)^+\right)D_s\,ds\right]. \tag{37}$$

The limiting properties of $g_t(x)$ and $h_t(x)$, and the definition of the boundary B^e, then imply

$$E\left[\int_0^T h_s\left(y-\left(y-\inf_{v\le s} B^e_v\right)^+\right)ds\right] \to 0 < n^2 E\left[\int_0^T g_s\left(\inf_{v\le s} B^e_v\right)D_s\,ds\right]$$

as $y \to B^e_0$, and

$$\begin{aligned} E\left[\int_0^T h_s\left(y-\left(y-\inf_{v\le s} B^e_v\right)^+\right)ds\right] &\to \lim_{y\longrightarrow 0} E\left[\int_0^T g_s(y)(D_s+e^1_s)\,ds\right] \\ &> n^2 \lim_{y\longrightarrow 0} E\left[\int_0^T g_s(y)D_s\,ds\right] \end{aligned}$$

as $y \to 0$ (recall that $n^2 < 1$). By continuity, we conclude that there exists a constant y^e which satisfies the static budget constraint. Moreover, when the function $h_v(z) - g_v(z)n^2 D_v = g_v(x)[I_2(xg_v(x)) - (n^2 D_v + e^2_v)]$ is strictly decreasing, the difference between the left- and right-hand sides of Equation (37) is a strictly decreasing function. It follows that the solution, under this condition, is unique. ■

Proof of Theorem 5. To simplify notation we assume that the unconstrained agent has a liquidity constraint multiplier $\gamma^1 \equiv 0$. Agent i has utility $u^i(c,t) = \rho_t^{-1}u^i(c)$, where $\rho_t = \exp(\beta t)$; let $I_i(y)$ be the inverse of u^i_c. The goods market clearing condition is $C_t = \sum_i I_i((y^i - \gamma^i_t)\rho_t\xi_t)$. Applying Itô's lemma on both sides of this equation yields

$$dC_t = \sum_i I'_i \times (y^i - \gamma^i_t)\rho_t\xi_t\left[\frac{d\rho_t}{\rho_t} + \frac{d\xi_t}{\xi_t} - \frac{d\gamma^i_t}{y^i - \gamma^i_t}\right] + \frac{1}{2}\sum_i I''_i \times ((y^i - \gamma^i_t)\rho_t\xi_t)^2 \frac{d[\xi]_t}{(\xi_t)^2},$$

where $[\xi]_t$ denotes the quadratic variation process. Dividing by C

$$\frac{dC_t}{C_t} = -\frac{1}{R^a_t}\left[\frac{d\rho_t}{\rho_t} + \frac{d\xi_t}{\xi_t}\right] + \frac{1}{R^2_t}\frac{c^2_t}{C_t}\frac{d\gamma^e_t}{y^e - \gamma^e_t} + \frac{1}{2}\frac{P^a_t}{(R^a_t)^2}\frac{d[\xi]_t}{(\xi_t)^2},$$

where

$$\begin{aligned} R^a_t &\equiv -\left[\frac{\sum_i I'_i((y^i - \gamma^i_t)\rho_t\xi_t)(y^i - \gamma^i_t)\rho_t\xi_t}{\sum_i I_i((y^i - \gamma^i_t)\rho_t\xi_t)}\right]^{-1} \\ P^a_t &\equiv (R^a_t)^2 \frac{\sum_i I''_i((y^i - \gamma^i_t)\rho_t\xi_t)((y^i - \gamma^i_t)\rho_t\xi_t)^2}{\sum_i I_i((y^i - \gamma^i_t)\rho_t\xi_t)}, \end{aligned}$$

and R^2_t is the coefficient of risk aversion of the constrained agent. By definition $I_i(y)$ satisfies $u^i_c(I_i(y)) = y$. Substituting $d\rho_t = \rho_t\beta\,dt$ and $d\xi_t = -\xi_t[dR^f_t + \theta_t\,dW_t]$ gives the equilibrium formulas $\theta_t = R_t^a \sigma^c$ and

$$dR_t^f = \left(\beta + R_t^a \mu^c - \frac{1}{2} R_t^a P_t^a (\sigma^c)^2\right) dt - \frac{R_t^a}{R_t^2} \frac{c_t^2}{C_t} \frac{d\gamma_t^e}{y^e - \gamma_t^e}.$$

The formulas in the theorem follow since $\theta_t = \sigma_t^{-1}(\mu_t - r_t)$ and $dR_t^f = r_t\, dt + dA_t$. ■

Proof of Theorem 6. In this economy the state price density is $\xi_t = g(C_t, y^e - \gamma_t^e)/g(C_0, y^e)$, where $g(C_t, z) = (1 + z^{-1})/C_t$. The stopping time problem in Theorem 3 becomes

$$Y_t^e(z) \equiv g_t(z) X_t^e(z) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left(\int_t^\tau \left(z^{-1} - (1 + z^{-1}) \frac{e_v^2}{C_v}\right) dv\right),$$

which can also be reformulated as

$$\begin{aligned} Y_t^e(z) &= E_t\left(\int_t^T \left(z^{-1} - (1 + z^{-1}) \frac{e_v^2}{C_v}\right) dv\right) + \sup_{\tau \in \mathcal{S}_{t,T}} -E_t\left(\int_\tau^T \left(z^{-1} - (1 + z^{-1}) \frac{e_v^2}{C_v}\right) dv\right) \\ &= E_t\left(\int_t^T \left(z^{-1} - (1 + z^{-1}) \frac{e_v^2}{C_v}\right) dv\right) + \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left(-E_\tau\left(\int_\tau^T \left(z^{-1} - (1 + z^{-1}) \frac{e_v^2}{C_v}\right) dv\right)\right)^+. \end{aligned}$$

Let V_t be the first term of $Y_t^e(z)$. Evaluating the integrals gives

$$V_t = z^{-1}(T - t) - \left(1 + z^{-1}\right) \int_t^T \int_{-\infty}^{+\infty} \Psi\left(\frac{D_t}{e_t^2}, \frac{e_t^1}{e_t^2}, w, v - t\right) n(w)\, dw\, dv,$$

where $\Psi(\cdot)$ is defined in Equation (26). Using Itô's lemma shows that

$$dV_t = -\left(z^{-1} - (1 + z^{-1}) \frac{e_t^2}{C_t}\right) dt + \phi_t\, dW_t$$

for some volatility process ϕ.

The second term of $Y_t^e(z)$ is an American put option with payoff $\max\{-V_t, 0\}$, which is a strictly decreasing function of the ratio D_t/e_t^2. Let $b^e(\cdot)$ be the optimal exercise boundary and define the set

$$A_v(w) \equiv \left\{w : \frac{D_v}{e_v^2} \leq b_v^e\right\}$$

of realizations w for which immediate exercise at date v is optimal. For $s \equiv sgn(\sigma^D - \sigma^2) > 0$ we have

$$A_v\big(w\big) = \left\{w : w \leq \frac{\log\big(b_v^e\big) - \log\big(\frac{D_t}{e_t^2}\big) - \alpha^D(v - t)}{(\sigma^D - \sigma^2)\sqrt{v - t}}\right\},$$

and with the definition

$$d\left(\frac{D_t}{e_t^2}, b_v^e, v - t\right) = \frac{\log(b_v^e) - \log\big(\frac{D_t}{e_t^2}\big) - \alpha^D(v - t)}{\big|\sigma^D - \sigma^2\big|\sqrt{v - t}}$$

we can also write

$$A_v\big(w\big) = \left\{w : w \leq d\left(\frac{D_t}{e_t^2}, b_v^e, v - t\right)\right\}.$$

For $s = sgn(\sigma^D - \sigma^e) < 0$ we get $A_v(w) = \big\{w : w \geq -d\big(\frac{D_t}{e_t^2}, b_v^e, v - t\big)\big\}$.

Since $V_T = 0$ the EEP representation of the American put option gives

$$\begin{aligned}
\sup_{\tau\in\mathcal{S}_{t,T}} E_t\big(-V_\tau\big)^+ &= -E_t\left(\int_t^T \left(z^{-1} - (1+z^{-1})\left[\frac{D_v}{e_v^2} + \frac{e_v^1}{e_v^2} + 1\right]^{-1}\right) 1_{A_v(w)}\, dv\right)\\
&= -\int_t^T \int_{A_v(w)} \left(z^{-1} - (1+z^{-1})\Psi\left(\frac{D_t}{e_t^2}, \frac{e_t^1}{e_t^2}, w, v-t\right)\right) n(w)\, dw\, dv\\
&= -z^{-1}\int_t^T P\big(A_v(w)\big)\, dv + (1+z^{-1})\\
&\quad\times \int_t^T \int_{A_v(w)} \Psi\left(\frac{D_t}{e_t^2}, \frac{e_t^1}{e_t^2}, w, v-t\right) n(w)\, dw\, dv.
\end{aligned}$$

Standard manipulations now show that the optimal exercise boundary b^e solves

$$\begin{aligned}
0 &= z^{-1}(T-t) - (1+z^{-1})\int_t^T \int_{-\infty}^{+\infty} \Psi\left(b_t^e, \frac{e_t^1}{e_t^2}, w, v-t\right) n(w)\, dw\, dv\\
&\quad - z^{-1}\int_t^T N(d(b_t^e, b_v^e, v-t))\, dv + (1+z^{-1})\\
&\quad\times \int_t^T \int_{-\infty}^{d(b_t^e, b_v^e, v-t)} \Psi\left(b_t^e, \frac{e_t^1}{e_t^2}, sw, v-t\right) n(w)\, dw\, dv,
\end{aligned}$$

or, equivalently,

$$0 = \int_t^T N\big(-d(b_t^e, b_v^e, v-t)\big)\, dv - (1+z)\int_t^T \int_{d(b_t^e, b_v^e, v-t)}^{\infty} \Psi\left(b_t^e, \frac{e_t^1}{e_t^2}, sw, v-t\right) n(w)\, dw\, dv.$$

Inspection of this equation reveals that b^e is a function of time parametrized by z (i.e., $b_t^e = b_t^e(z)$). Moreover, using the fact that $z^{-1} - (1+z^{-1})(e_t^2/C_t)$ is a decreasing function of z, we can show that $b_t^e(z)$ is increasing in z. It follows that a unique inverse, $\Phi_t^e(\cdot)$, exists. Since the inverse is an increasing function we can write the continuation region in the form

$$\big\{D_t/e_t^2 > b_t^e(y^e - \gamma_t^e)\big\} = \big\{\Phi_t^e\big(D_t/e_t^2\big) > y^e - \gamma_t^e\big\}.$$

In this region the multiplier stays flat; it increases at times when immediate exercise is optimal. Thus $y^e - \gamma_t^e = \inf_{v\in[0,t]} \Phi_v^e\big(D_v/e_v^2\big)$. The initial multiplier y^e is the constant which satisfies the static budget constraint. If $n^2 = 0$, this is $y^e = \Phi_0^e\big(D_0/e_0^2\big)$. If $n^2 > 0$ we have $y^e < \Phi_0^e\big(D_0/e_0^2\big)$. The pricing kernel and the consumption allocation follow by substituting $y^e - \gamma^e$ back into the relevant formulas. ■

Derivation of Equation (29). The proof uses the same formulas from Nualart (1995) that were used to derive Equation (15). Since $\gamma_s^e = y^e - \inf_{v\in[0,s]} \Phi_v^e(D_v/e_v^2)$ we obtain

$$\begin{aligned}
\mathcal{D}_t \gamma_s^e &= -\Phi_{v[0,s]}^{e\prime}\left(\frac{D_{v[t,s]}}{e_{v[t,s]}^2}\right) 1_{\{v[0,s]\ge t\}} \mathcal{D}_t\left(\frac{D_{v[t,s]}}{e_{v[t,s]}^2}\right)\\
&= -\frac{1}{b_{v[t,s]}^{e\prime}\big(y^e - \gamma_s^e\big)} 1_{\{v[0,s]\ge t\}} \frac{D_{v[t,s]}}{e_{v[t,s]}^2}(\sigma^D - \sigma^e).
\end{aligned}$$

In the second line we used the facts that the derivative $\Phi_v^{e\prime}$ is given by

$$\Phi_v^{e\prime}(x) = 1/b_v^{e\prime}(\Phi_v^e(x))$$

and that $\Phi^e_{v[0,s]}(D_{v[0,s]}/e^2_{v[0,s]}) = y - \gamma^e_s$. As before

$$v[0,s] = \inf\left\{v \in [0,s] : \Phi^e_v\big(D_v/e^2_v\big) = \inf_{u\in[0,s]} \Phi^e_u\big(D_u/e^2_u\big)\right\}$$

is the last time in $[0, s]$ at which the constraint binds (i.e., the first time at which Φ^e_u reaches its minimum).

From Theorem 5, recall that the stock price is given by

$$g(C_t, y^e - \gamma^e_t)S_t = E_t\left[\int_t^T g(C_v, y^e - \gamma^e_v)D_v\, dv\right].$$

The Clark–Ocone formula [Nualart (1995), Proposition 1.3.5] implies that the volatility of the right-hand side is

$$E_t\left[\int_t^T g_1(C_v, y^e - \gamma^e_v)\big(\sigma^D D_v + \sigma^1 e_{1v} + \sigma^2 e_{2v}\big)D_v\, dv\right]$$
$$- E_t\left[\int_t^T g_2(C_v, y^e - \gamma^e_v)(\mathscr{D}_t\gamma^e_v)D_v\, dv\right] + \sigma^D E_t\left[\int_t^T g(C_v, y^e - \gamma^e_v)D_v\, dv\right],$$

where $\mathscr{D}_t\gamma^e_v$ is given by the expression above. Itô's lemma gives the volatility coefficient of the left-hand side as $g_1(C_t, y^e - \gamma^e_t)C_t S_t \sigma^C_t + g(C_t, y^e - \gamma^e_t)S_t\sigma_t$. Combining these two expressions and simplifying leads to

$$\begin{aligned}\sigma_t = \sigma^D &- \frac{g_1(C_t, y^e - \gamma^e_t)}{g(C_t, y^e - \gamma^e_t)}\big(D_t\sigma^D + \sigma^1 e_{1t} + \sigma^2 e_{2t}\big)\\ &+ \frac{E_t\big[\int_t^T g_1(C_v, y^e - \gamma^e_v)\big(\sigma^D D_v + \sigma^1 e_{1v} + \sigma^2 e_{2v}\big)D_v\, dv\big]}{E_t\big[\int_t^T g(C_v, y^e - \gamma^e_v)D_v\, dv\big]}\\ &- \frac{E_t\big[\int_t^T g_2(C_v, y^e - \gamma^e_v)(\mathscr{D}_t\gamma^e_v)D_v\, dv\big]}{E_t\big[\int_t^T g(C_v, y^e - \gamma^e_v)D_v\, dv\big]}.\end{aligned}$$

Since $g(C_t, y^e - \gamma^e_t) = \frac{1+(y^e-\gamma^e_t)^{-1}}{C_t}$ we obtain

$$g_1(C_t, y^e - \gamma^e_t) = -g(C_t, y^e - \gamma^e_t)/C_t$$
$$g_2(C_t, y^e - \gamma^e_t) = -(y^e - \gamma^e_t)^{-2}/C_t.$$

Substituting these expressions in the volatility formula above and using $\sigma^1 = \sigma^2$ produces Equation (29). ■

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