Quantitative Asset Pricing Implications of Endogenous Solvency Constraints

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We study the asset pricing implications of an economy where solvency constraints are endogenously determined to deter agents from defaulting while allowing as much risk sharing as possible. We solve analytically for efficient allocations and for the corresponding asset prices, portfolio holdings, and solvency constraints for a simple example. Then we calibrate a more general model to U.S. aggregate as well as idiosyncratic income processes. We find equity premia, risk premia for long-term bonds, and Sharpe ratios of magnitudes similar to the U.S. data for low risk aversion and a low time-discount factor.

It is difficult to make a debtor pay: debt collection, litigation, and wage garnishment are costly, and the end result is uncertain because the debtor might declare bankruptcy. Motivated by these difficulties, we study a model in which individuals with large financial obligations cannot contract further debts that would make them choose to default. In the model, individual income risks are only incompletely shared and become one of the determinants of asset prices.

In a companion article, Alvarez and Jermann (2000), we present a framework for studying the asset pricing implications when agents can default on their debts, built on earlier work by Kehoe and Levine (1993). In the model, agents default on their debts if defaulting makes them better off. We assume that if agents default, their labor earnings cannot be seized, but they are excluded from asset markets forever. In our equilibrium concept, this lack of commitment results in state- and agent-specific borrowing constraints. These constraints ensure that agents will not default, since they will never owe so much as to make them choose to default. At the same time, the constraints ensure that there is as much risk sharing as possible. In this article we focus

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on the quantitative effects of these endogenous solvency constraints for asset returns.

Our work is related to the asset pricing literature that studies the effects of portfolio restrictions, such as He and Modest (1995) and Luttmer (1996). He and Modest (1995, p. 113) conclude that "none of the market frictions alone—with the possible exception of solvency constraints—can explain the apparent rejection of the first-order equilibrium condition between consumption and asset returns." In this article we use a fully specified theory that endogenously determines both asset prices and solvency constraints.

We start by presenting a set of results about general properties of equilibrium allocations. Since the welfare theorems hold, we characterize equilibria by solving a planning problem. We prove that incomplete risk sharing is possible only for low risk aversion, for a low time preference parameter, and for persistent but not too volatile individual shocks.

We characterize analytically the efficient allocations and the corresponding asset prices, portfolio holdings, and solvency constraints for a simple case. We characterize the type of parameters that deliver different levels of risk sharing, show that the interest rate is lower than in economies without solvency constraints, and illustrate that poor agents face binding solvency constraints. For a calibrated example, we show that with low risk aversion, the pricing kernel can be volatile enough to pass the Hansen–Jagannathan test.

For a detailed quantitative analysis, we specify an endowment process that allows varying degrees of dependence between aggregate and individual income uncertainty. Using moderate values for risk aversion and low time-discount factors, and calibrating the income processes to aggregate and household U.S. data, the model generates risk premia for equity and for long-term bonds similar to the those in the U.S. data. The equilibrium allocations for the calibrated economy exhibit limited risk sharing, which is attained by solvency constraints that, when they bind, allow for small amounts of debt. In this case there is no risk-free rate puzzle, because interest rates are lower than in the corresponding representative agent economy. In fact, to explain risk-free rates on the order of 1% per annum, the model requires a time-discount factor that is lower than those typically used in other studies. Finally, we characterize the deviations from independence of aggregate and individual risks that affect equity and term premia. The equity premium depends on the comovement between individual income risk and the contemporaneous aggregate income growth. This result is driven by solvency constraints that bind frequently, as opposed to the well-known results by Mankiw (1986), Weil (1992), and Constantinides and Duffie (1996) that require convexity of the marginal utility. The term premium depends on the comovement between the forecast of future individual income risk and aggregate income growth.

Compared with several studies on incomplete market economies, our study differs along some of the following dimensions.[1] First, we have complete markets where every claim can be traded and priced; most incomplete market models consider only a very limited set of assets. Second, in our model, the extent of risk sharing depends on the costs and benefits of defaulting. In most incomplete market models, the extent of risk sharing depends primarily on the assumptions about available assets. Third, in our model, agent- and state-specific solvency constraints bind frequently; in most incomplete market models, portfolio constraints rarely bind. Finally, our model is very tractable since we solve a planner's problem and then compute the prices for the corresponding equilibrium—incomplete market models are solved directly as a complicated fixed point over market prices. Given the tractability and the presence of complete markets, we are able to study equity, one-period bonds, and the entire term structure of interest rates.

In Section 1 we present the environment. In Section 2 we present the equilibrium with solvency constraints. In Section 3 we characterize optimal allocations at a general level. In Section 4 we solve analytically a simple example with a two-state income process. Section 5 contains calibration and the quantitative results of the article. Section 6 concludes. An appendix contains the proofs. Readers eager for the quantitative highlights might—in a first reading—skip Sections 3 and 4, and proceed directly from Section 2 to Section 5.

1. The Environment

We consider a pure exchange economy with two (types of) agents. Agents' endowments follow a finite-state Markov process; agents' preferences are identical and are given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarky at any time and for any history.

We use a Markov chain with generic elements $z \in Z$, a set with N elements. We refer to elements of Z as $Z = \{\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_N\}$ and to the time t realization of the process as z_t. We use $z^t = (z_0, z_1, z_2, \ldots, z_t)$ for the length t history of z. We use Π for the matrix containing the transition probabilities and $\pi(z^t|z_0 = z)$ for the corresponding conditional probabilities. We use the notation $\{c_i\}$ and $\{e_i\}$ for the stochastic process of consumption and endowment for each agent; hence, $\{c_i\} = \{c_{i,t}(z^t) : \forall\ t \geq 0, z^t \in z^t\}$. We assume that aggregate endowment $e_t(z^t) \equiv e_{1,t}(z^t) + e_{2,t}(z^t)$ is constant and equal to e.[2]

[1] See, for instance, Mankiw (1986), Telmer (1993), Constrantinides and Duffie (1996), Heaton and Lucas (1996), Krusell and Smith (1997), Zhang (1997), and Marcet and Singleton (1999).

[2] We will show below how to introduce aggregate uncertainty into this notational framework.

Individual endowments are given by a function ϵ_i that depends only on z_t, so that $e_{i,t}(z^t) = \epsilon_i(z_t)$. We assume that $\epsilon_i(z) > 0$ for all i and z. The utility for an agent corresponding to the consumption process $\{c\}$ starting at time t at history z^t is denoted by $U(c)(z^t)$ and is given by

$$U(c)(z^t) \equiv \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^{z^t t+s}} \delta_{t,t+s}(z^{t+s-1}) u(c_{t+s}(z^{t+s})) \pi(z^{t+s} | z_t),$$

where u is the period utility and $\delta_{t,t+s}$ is a time-discount factor. We assume that u: $R_+ \to R$ is strictly increasing, strictly concave, and C^1. The multiperiod time-discount factor $\delta_{t,t+s+1}$ is defined recursively using the one-period state-contingent discount factor β: $Z \to (0, 1)$. Specifically, $\delta_{t,t}$ $(z^{t-1}) \equiv 1$, and for all z^{t+s},

$$\delta_{t,t+s+1}\left(z^{t+s}\right) = \delta_{t,t+s}\left(z^{t+s-1}\right) \cdot \beta(z_{t+s}).$$

Note that the standard case with a constant discount factor $\beta(z) = \bar{\beta}$ corresponds to $\delta_{t,t+s}(z^{t+s-1}) = \bar{\beta}^s$. As we will show below, letting the time-discount factor be state contingent allows us to introduce aggregate stochastic growth.

We assume that the shocks are symmetric across agents in the following sense. Let us denote by $\tilde{\epsilon}$ and $\tilde{\epsilon}'$ two vectors of arbitrary values for the agents' endowments, and let us denote by $\tilde{\beta}$ and $\tilde{\beta}'$ two arbitrary values for the time-discount factor. Then Π, $\epsilon_i(\cdot)$, and $\beta(\cdot)$ are assumed to satisfy

$$\begin{aligned} &\Pr((\epsilon_1(z'), \epsilon_2(z')) = \tilde{\epsilon}', \beta(z') = \tilde{\beta}' | (\epsilon_1(z), \epsilon_2(z)) = \tilde{\epsilon}, \beta(z) = \tilde{\beta}) \\ &= \Pr((\epsilon_2(z'), \epsilon_1(z')) = \tilde{\epsilon}', \beta(z') = \tilde{\beta}' | (\epsilon_2(z), \epsilon_1(z)) = \tilde{\epsilon}, \beta(z) = \tilde{\beta}). \end{aligned}$$

An allocation $\{c_i\}_{i=1,2}$ is resource feasible if

$$c_{1,t}(z^t) + c_{2,t}(z^t) = e_t(z^t) \qquad \text{for all } t \geq 0, z^t \in Z^t. \tag{1}$$

It satisfies the participation constraints if

$$U(c_i)(z^t) \geq U(e_i)(z^t) \equiv U^i(z_t) \qquad \text{for all } t \geq 0, z^t \in Z^t, \tag{2}$$

where we use the notation $U^i(z_t)$ to refer to $U(e_i)(z^t)$ to emphasize that it depends only on z_t. In some cases in which this assumption is not required, we will use $U(e_i)(z^t)$.

Except for the state-contingent time-discount factor, our environment is a special case of the environment studied by Kehoe and Levine (1993). In particular, we consider the case with one good, two agents, and where the participation constraints have autarky as the outside option. Our case is identical to the one studied by Kocherlakota (1996), except that we allow for a stochastic discount factor and a process for the income shocks that is not i.i.d.

1.1 Aggregate uncertainty

Our specification of the time-discount factor accommodates in a very convenient way stochastic aggregate income growth of the type used in Mehra and Prescott (1985) and in much of the asset pricing literature. In particular, an economy with stochastic growth, a constant time-discount factor, and constant relative risk aversion can be expressed as an economy with constant aggregate endowment and a state-contingent discount factor such as the one presented in the preceding subsection.

Let

$$e_{t+1}(z^t, z_{t+1}) = e_t(z^t)\cdot\lambda(z_{t+1}) \qquad \text{and} \qquad e_{i,t}(z^t) = \epsilon_i(z_t)\cdot e_t(z^t) \text{ for } i = 1, 2.$$

Define $\hat{e}_{i,t}(z^t) \equiv e_{i,t}(z^t)/e_t(z^t) = \epsilon_i(z_t)$ for all i so that $\hat{e}_t(z^t) = 1$ for all z^t.

Assuming a constant time-discount factor β and a period utility function of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some positive γ (for simplicity, $\gamma \neq 1$) and defining $\hat{c}_{i,t}(z^t) \equiv c_{i,t}(z^t)/e_t(z^t)$, then $U(\cdot)$ satisfies

$$U(\hat{c}_i)(z^t) = \frac{(\hat{c}_{i,t}(z^t))^{1-\gamma}}{1-\gamma} + \hat{\beta}(z_t)\sum_{z_{t+1}\in Z} U(\hat{c}_i)(z^t, z_{t+1})\hat{\pi}(z_{t+1}|z_t)$$

with probabilities and discount factors

$$\hat{\pi}(z'|z) \equiv \frac{\pi(z'|z)\cdot\lambda(z')^{1-\gamma}}{\sum_{z'}\pi(z'|z)\cdot\lambda(z')^{1-\gamma}} \text{ and } \hat{\beta}(z) \equiv \beta\cdot\sum_{z'}\pi(z'|z)\cdot\lambda(z')^{1-\gamma}.$$

Clearly the resource and participation constraints are satisfied for an allocation $\{c_i\}_{i=1,2}$ in an economy with aggregate growth $\lambda(\cdot)$ and a constant discount factor β if and only if they are satisfied for the corresponding $\{\hat{c}_i\}_{i=1,2}$ allocation in the economy with constant aggregate endowment, discount factors $\hat{\beta}(\cdot)$, and probabilities $\hat{\pi}$. Moreover, the preference orderings are identical in the two corresponding economies.

2. Equilibrium with Endogenous Solvency Constraints

In this section we define a competitive equilibrium with complete markets in Arrow securities and with endogenous solvency constraints. The solvency constraints prohibit agents from holding large amounts of contingent debt, hence preventing default. In general, these solvency constraints will be state-contingent, since the costs and benefits of default vary with the state.

Let $q_t(z^t, z')$ denote the time t, state z^t price of one unit of the consumption good delivered at $t+1$, contingent on the realization of $z_{t+1} = z'$, in terms of time t consumption goods. The holdings of agent i at t of this security are denoted by $a_{i,t+1}(z^t, z')$, and the lower limit on the holdings of agent i is denoted by $B_{i,t+1}(z^t, z')$. Following our notational convention, we use $\{q\}$, $\{a_i\}$, and $\{B_i\}$ for the corresponding stochastic processes. For given $\{q\}$ and $\{B_i\}$, the problem for household i is defined as

$$J_{i,t}(a, z^t) = \max_{\substack{c,\\ \{a_{z'}\}_{z'\in Z}}} \left\{ u(c) + \beta(z_t) \sum_{z'} J_{i,t+1}(a_{z'}, (z^t, z'))\pi(z'|z_t) \right\} \quad (3)$$

$$e_{i,t}(z^t) + a = \sum_{z'\in Z} a_{z'} q_t(z^t, z') + c \quad (4)$$

$$a_{z'} \geq B_{i,t+1}(z^t, z') \quad \text{for all } z' \in Z. \quad (5)$$

Definition 1. *An equilibrium with solvency constraints $\{B_i\}$ for initial conditions $a_{i,0}$ has quantities $\{a_i\}$ and prices $\{q\}$ such that for $i = 1, 2$,*

a. $\{a_{i,t+1}(z^t, z')\}'_{z\in Z}$ achieves the right-hand side of Equation (3) at z^t given $a = a_{i,t}(z^t)$.
b. market clearing, $a_{1,t}(z^t) + a_{2,t}(z^t) = 0$, for all t, all z^t.

From the first-order condition of the agent's problem, we conclude that if an agent's marginal rate of substitution is strictly lower than the corresponding Arrow price, this agent's solvency constraint must bind. In addition, if an agent's solvency constraint does not bind, this agent's marginal rate of substitution must equal the corresponding Arrow price. Consequently, in an equilibrium with solvency constraints, Arrow prices are equal to the highest marginal rate of substitution that is,

$$q_t(z^t, z_{t+1}) = \max_{i=1,2} \beta(z_t) \frac{u'(c_{i,t+1}(z^t, z_{t+1}))}{u'(c_{i,t}(z^t))} \pi(z_{t+1}|z_t). \quad (6)$$

Furthermore, if the solvency constraints do not bind for either agent, the Arrow price [Equation (6)] is equal to the one from the corresponding representative agent economy.

In Equation (3) agents never contemplate the option of default. Now we move to the analysis of the decision of default; this consideration describes our theory of the solvency constraints presented in Alvarez and Jermann (2000). The following condition makes the solvency constraints endogenous.

Definition 2. *An equilibrium with solvency constraints that are not too tight is such that the solvency constraints satisfy*

$$J_{i,t+1}(B_{i,t+1}(z^{t+1}), z^{t+1}) = U(e_i)(z^{t+1}), \quad (7)$$

for all $t = 0, 1, \ldots,$ for all $z^{t+1} \in Z^{t+1}$ and for $i = 1, 2$.

The left-hand side of Equation (7) is the utility of an agent who participates in the market, starting with financial wealth $B_{i,t+1}(z^{t+1})$. The right-hand side of Equation (7) is the agent's utility if he defaults, given our assumption that default is punished by permanent exclusion from asset markets. This condition ensures that solvency constraints prevent default by prohibiting agents from accumulating more contingent debt than they are willing to pay back. At the same time, it allows as much insurance as possible: if the solvency constraint binds and the continuation utility is strictly higher than the value of autarky, the constraint could be relaxed without inducing the agent to accumulate so much debt that he will prefer to default.

To simplify the notation, we state our equilibrium using Arrow prices. The budget set and the incentives to default are the same with any other dynamically complete sets of securities, provided that the solvency constraints are stated in terms of the value of the portfolio at the beginning of the period.[3] Thus the value of any security—not just Arrow securities—is equal to the value of the sum (weighted by the payoffs) of the corresponding Arrow securities given by Equation (6). In fact, the pricing kernel is the highest marginal rate of substitution, so that for the one-period return $R_{t,t+1}$ of any asset, the following must hold:

$$1 = E_t\left[R_{t,t+1} \cdot \left(\max_{i=1,2} \beta_t \frac{u'(c_{i,t+1})}{u'(c_{i,t})}\right)\right].$$

We think equilibria with solvency constraints that are not too tight are interesting, since they restrict endogenously the amount of risk sharing and make a direct connection between asset prices and constraints on borrowing. In Alvarez and Jermann (2000), we show that the first and second welfare theorems hold for our equilibrium definition. Equilibria with solvency constraints that are not too tight and constrained efficient allocations are equivalent because the condition on the solvency constraints [Equation (7)] serves the same purpose as the participation constraints [Equation (2)].

Our interest is in asset prices, but analyzing equilibria directly is difficult. Given the equivalence between efficient allocations and equilibria with solvency constraints that are not too tight, we analyze efficient allocations as a way to characterize equilibrium prices.

3. Characterizing Constrained Efficient Allocations

In this section we provide a recursive formulation of efficient allocations, which we use to establish the following properties of efficient allocations. First, we characterize the parameters that determine the extent of risk sharing, which allows us to concentrate on the cases in which individual risk is important for asset prices. Second, we characterize the decision rules.

Constrained efficient allocations are defined as the processes $\{c_i\}$ that maximize time 0 expected lifetime utility for agent 1, subject to resource balance

[3] In this sense, our use of *solvency constraint* is consistent with other studies, for instance He and Modest (1995).

and the participation constraints for both agents, given some initial (time 0) expected lifetime utility for agent 2. Optimal allocations solve the following maximization problem:

$$V^*(w, z) \equiv \max\{U(c_1)(z)\}$$

subject to Equations (1) and (2), $U(c_2)(z_0) \geq w$, and $z_0 = z$.

A pair (w, z) is in the domain of V^* if and only if $(V^*(w, z), w)$ is in the utility possibility set for $z_0 = z$. Indeed, the function $V^*(w, z)$ describes the utility possibility frontier at time 0 when $z_0 = z$.

Now we restate the preceding problem recursively. We introduce a functional equation and relate its fixed points to the function V^*. The functional equation is not completely standard. The operator T maps a function V defined in a given domain into another function TV. Using TV, a new domain, denoted by $D(z)$, is defined as specified below. Specifically, $V(\cdot, z)$: $D(z) \to R$ for each z, where $D(z) \subset [U^2(z), \infty)$ and the operator T generates TV as

$$TV(w,z) = \max_{\substack{c_{i=1,2,} \\ w'(z'), z' \in Z}} \left\{ u(c_1) + \beta(z) \textstyle\sum_{z' \in Z} V(w'(z'), z') \times \pi(z'|z) \right\} \tag{8}$$

$$c_1 + c_2 \leq e \tag{9}$$

$$u(c_2) + \beta(z) \textstyle\sum_{z' \in Z} w'(z')\ \pi(z'|z) \geq w \tag{10}$$

$$w'(z') \geq U^2(z') \quad \text{for all } z' \in Z \tag{11}$$

$$V(w(z'), z') \geq U^1(z') \quad \text{for all } z' \in Z. \tag{12}$$

For this operator to be well defined, V has to be such that there are c_1, c_2, and $\{w'(z')\}_{z' \in Z}$ for which the constraints of Equations (9)–(12) are satisfied. Consequently the domain of TV for each $z \in Z$ is defined as $D(z) \equiv \{w : (TV)(w, z) \geq U^1(z) \text{ and } w \geq U^2(z)\}$.

It is straightforward to show that the function V^*, defined previously, and its associated domain are a fixed point of T. However, the functional equation [Equation (8)] has more than one fixed point: hence it cannot be a contraction.[4] In particular, autarky is always a fixed point, since it is immediate to verify that the trivial function v defined on the domain given by singleton $\{U^2(z)\}$ and equal to $v(U^2(z), z) = U^1(z)$ is a fixed point of T. Nevertheless, for many parameter values, there are other solutions, namely, V^*.

Even though the operator T is not a contraction, it is useful in computing V^*. Consider the operator $\widetilde{T}$ defined exactly like T in Equation (8) except that the participation constraints for both agents, Equations (11) and (12),

[4] The operator T does not satisfy one of Blackwell's sufficient conditions for a contraction, namely, *discounting*. Indeed, $T(V + a)$ could be bigger than $TV + \beta a$ for a constant a, since the feasible set of choices for $(w(z'))_{z' \in Z}$ is bigger for $V + a$ than for V.

are removed, and denote its unique fixed point by $\widetilde{V}$. The function $\widetilde{V}(\cdot, z)$ is the full risk-sharing frontier when $z_t = z$. Using the approach proposed by Abreu (1988) and Abreu, Pearce, and Stacchetti (1990), one can easily show that $\lim_{n\to\infty} T^n \widetilde{V} = V^*$.

3.1 Risk-sharing regimes

This section describes the types of parameters for which this model has asset pricing implications different from those of the representative agent model. Depending on the parameter values for preferences and endowments, there are three possible regimes for the process for $\{w, z\}$. Independent of the initial condition (w_0, z_0), one can show that (1) full risk sharing forever is possible; (2) only limited risk sharing is possible; or (3) only autarky is possible.

By *full risk sharing* we mean that the allocation is Pareto efficient in the standard sense, ignoring the participation constraints, for some initial condition (w_0, z_0). Parameter values that produce the first case are not interesting to us, since for the purpose of asset pricing, their implications are the same as in the representative agent economy.

We discuss briefly how to verify whether full risk sharing is possible. Kocherlakota (1996) presents sufficient conditions for each case when the shocks are i.i.d. and the discount factor is constant. We consider a slightly different case in the following proposition.

Proposition 1. *Full risk sharing is possible if and only if, for all $z_0 \in Z$,*

$$u(e/2) \sum_{s=0} \sum_{z^t \in Z^t} \beta_{0,t}(z^t)\pi(z^t|z_0) \geq \max_{i=1,2} U^i(z_0). \tag{13}$$

For illustration, consider the case with a constant discount factor β and $Z = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, with $\epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2)$, which by symmetry implies that $\epsilon_1(\mathfrak{z}_2) = \epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2) = \epsilon_1(\mathfrak{z}_1)$.[5] Figure 1 illustrates the case in which full risk sharing is possible for a range of w. When full risk sharing is not possible, there are two cases: one case in which autarky is the only feasible allocation that satisfies the participation constraints and the other in which some other allocations satisfy the participation constraints. This last case is the one that we are interested in, since it is not equivalent to a representative agent economy. Figure 2 illustrates the case in which full risk sharing is not possible.

Which case applies depends on parameter values as explained in the following remark. These parameters determine the attractiveness of autarky relative to some form of risk sharing. Intuitively parameter values that make the punishment less painful also make full risk sharing harder.

[5] We provide further analysis of this case in Section 4.

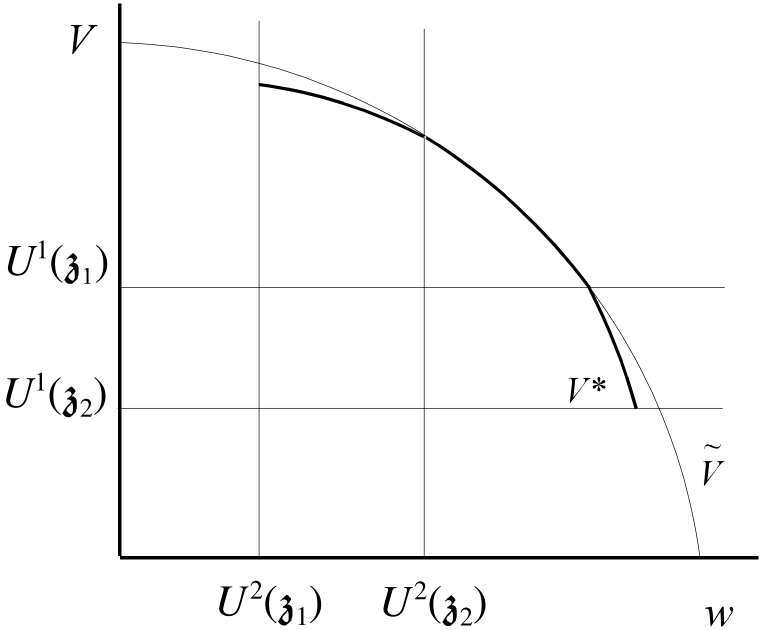

Figure 1
Full risk sharing is possible
Full risk sharing is possible for the region where the utility possibility frontier, V^*, overlaps with the frontier for the Pareto efficient case, $\tilde{V}$.

Remark 1. *Let the probability matrix be given by* $\Pi_\alpha \equiv \alpha I + (1-\alpha)\Pi$ *for* $\alpha \in (0, 1)$. *Then full risk sharing is not possible in any of the following cases:*

(i) The time preference parameter, $\max_z \beta(z)$, is sufficiently small.
(ii) The persistence of Π_α, α, is sufficiently close to one.
(iii) The variance of $\epsilon_i(z)$ is sufficiently close to zero.
(iv) With CRRA utility, the relative risk aversion, γ, is sufficiently small.

The proof of this remark follows by taking the appropriate limit in each of the four cases and verifying that the inequality of Proposition 1 does not hold. In Alvarez and Jermann (2000), we extend this result. In particular, we show that as the parameters approach the limit values mentioned in each of the four cases, autarky is the only feasible allocation.

Mehra and Prescott (1985) and Weil (1992) emphasize that in order to produce a high equity premium and low interest rates, the representative agent model requires values of risk aversion γ and time preference β that are higher than most researchers' prior assessments. By contrast, this model has asset pricing implications different from the representative agent model for *low* values of γ and β; thus our quantitative results cannot rely on high values for γ and β.

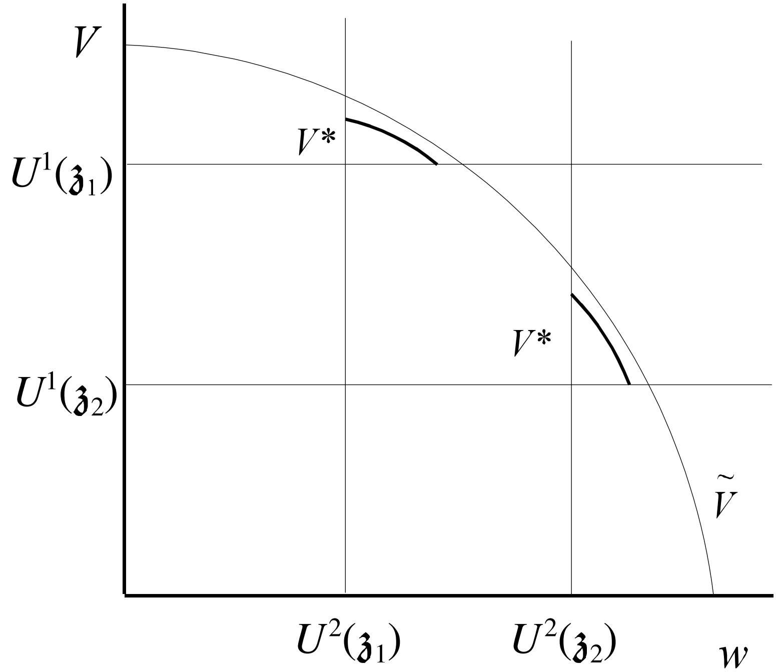

Figure 2
Full risk sharing is not possible
Full risk sharing is not possible because no point on the Pareto efficient utility possibility frontier, $\widetilde{V}$, can be achieved in the economy with participation constraints. For this reason, the frontier, V^*, lies inside the Pareto efficient frontier, $\widetilde{V}$.

3.2 Decision rules for the planning problem

In this subsection we present some results about the properties of the decision rules for consumption $C_i(w, z)$ and for agent 2's continuation utility $W_{z'}(w, z)$. We define $H(z)$ and $L(z)$ as the upper and lower bound of $D(z)$ so that by monotonicity of $V(\cdot, z)$,

$$L(z) \equiv U^2(z) \text{ and } H(z) \equiv V^{-1}(\cdot, z)(U^1(z)) \text{ for all } z \in Z.$$

The time separability of the utility function implies that consumption is increasing in the current continuation utility.

Proposition 2. *Consumption is strictly monotone on z; that is, for all z, $C_2(w, z)$ ($C_1(w, z)$, respectively) is strictly increasing (strictly decreasing) in w.*

Next we derive several properties of the decision rules for future continuation utility as a function of current continuation utility. We will refer to these results in the next section. We find that the decision rules are weakly increasing. Specifically, we consider two cases. If the shock z_t is repeated, then consumption and continuation utility are the same. If the shock is different, then the decision rule is weakly increasing in w.

Proposition 3. [I] *If $z' = z$, then $W_{z'}(w, z) = w$ (the "45°–rule"). [II] If $z' \neq z$ and $(\tilde{w}, w, z, z')$ are such that (i) $w < \tilde{w}$ and (ii) both $W_{z'}(\tilde{w}, z)$ and $W_{z'}(w, z)$ are in the interior of the range of $W_{z'}(\cdot, z)$, that is,*

$$W_{z'}(\tilde{w}, z),\ W_{z'}(w, z) \in (L(z'), H(z')),$$

then $W_{z'}(\tilde{w}, z) > W_{z'}(w, z)$.

4. The Two-State Case

To illustrate how the model works and to get a rough idea about its quantitative potential to explain asset returns, we analyze a simple example. We completely characterize the optimal allocations and all the elements of an equilibrium with solvency constraints that are not too tight, and we compute some numerical cases. This example illustrates, among other things, the circumstances under which agents' solvency constraints bind. Moreover, our examination of the Hansen–Jagannathan bounds demonstrates that we can obtain quantitatively interesting results with low risk aversion.

4.1 Efficient allocations

Consider the case with a constant discount factor β and $Z = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, with $\epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2)$, which by symmetry implies that $\epsilon_1(\mathfrak{z}_2) = \epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2) = \epsilon_1(\mathfrak{z}_1)$. We first show that when full risk sharing is not possible, the decision rules imply a unique ergodic set, where continuation utility w and, hence, consumption depend exclusively on the current value of z. Second, we characterize the values of consumption in the ergodic set.

The decision rules of Equation (8) are completely described by analyzing two cases. If $z' = z$, then the optimal policy is the 45° line as we have shown. For the remaining case of $z' \neq z$, the following proposition, proven in the appendix, shows that the policies rules are constant, a result that we refer to as saying that they are *flat* after reversal.

Proposition 4. *Decision rules that achieve V^* are flat after a reversal of the shock; that is, for all $w \in D(z) = [L(z), H(z)]$,*

$$\begin{aligned} & W_{\mathfrak{z}_2}(w, \mathfrak{z}_1) = L(\mathfrak{z}_2) \equiv \bar{w}(\mathfrak{z}_2) \\ \textit{and} \quad & W_{\mathfrak{z}_1}(w, \mathfrak{z}_2) = H(\mathfrak{z}_1) \equiv \bar{w}(\mathfrak{z}_1). \end{aligned}$$

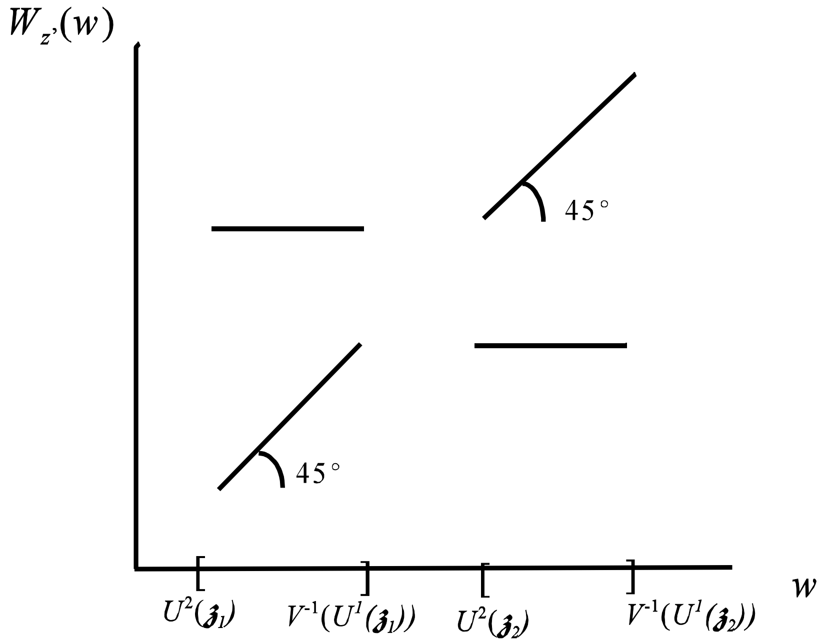

Figure 3
Decision rules for the two-state case when full risk sharing is not possible
Full risk sharing is not possible because the domains associated to the two shocks are disjoint.

Figure 3 plots the decision rules $W_{z'}(w, z)$. For any initial (w, z), after one reversal of the shock z, continuation utility and consumption for agent 2 will attain the values $\bar{w}(z)$ and $\bar{c}(z)$ and will depend only on the current state z. By inspection of these decision rules, if full risk sharing is not possible, the process for $\{w, z\}$ has a unique invariant distribution, with mass on $(\bar{w}(z_1), z_1)$ and $(\bar{w}(z_2), z_2)$. If full risk sharing is possible, then the domains $D(z_1)$ and $D(z_2)$ have a nonempty intersection; thus any constant value of w in that intersection is optimal and constitutes an invariant distribution.

Given our characterization, if full risk sharing is not possible, agent 2's continuation utilities in the ergodic set, $(\bar{w}(z_1), \bar{w}(z_2))$, and the corresponding consumptions, $(\bar{c}(z_1), \bar{c}(z_2))$, have to satisfy the following system of four equations: two promise-keeping conditions,

$$\bar{w}(z_1) = u(\bar{c}(z_1)) + \beta\bar{\pi}\bar{w}(z_1) + \beta(1 - \bar{\pi})\bar{w}(z_2) \text{ and}$$

$$\bar{w}(z_2) = u(\bar{c}(z_2)) + \beta\bar{\pi}\bar{w}(z_2) + \beta(1 - \bar{\pi})\bar{w}(z_1),$$

the boundary condition $\bar{w}(z_2) = U^2(z_2)$, and, due to the symmetry across agents and the resource constraint, $\bar{c}(z_1) = e - \bar{c}(z_2)$. The following proposition, proven in the appendix, determines and characterizes the efficient allocation.

Proposition 5. *The system of four equations has at most two solutions and at least one; autarky is always a solution. The solution different from autarky is efficient if and only if it has a smaller variability than autarky, that is,* $\epsilon_2(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$.

4.2 Equilibrium allocations

In this subsection we construct the elements of an equilibrium with solvency constraints that are not too tight, corresponding to the efficient allocations found in the previous section. This illustrates the second welfare theorem for this environment and clarifies how the equilibrium works. In particular, we show that poor agents are the ones who face binding solvency constraints, and we show that the interest rate is lower and the pricing kernel is more variable than in the corresponding economy without solvency constraints.

We support the efficient allocation found in the preceding subsection as follows. Arrow prices depend exclusively on whether the current state z_t is the same as the state z_{t+1},

$$q_t(z^t, z_{t+1}) = \bar{q}_r \text{ if } z_{t+1} = z_t \text{ and}$$
$$q_t(z^t, z_{t+1}) = \bar{q}_{nr} \text{ if } z_{t+1} \neq z_t, \text{ for all } z^t, z_{t+1}.$$

Using the relationship defining Arrow prices in Equation (6), we obtain that

$$\bar{q}_r = \beta\bar{\pi} \text{ for } z' = z, \tag{14}$$

$$\bar{q}_{nr} = \beta \frac{u'(\bar{c}(\mathfrak{z}_1))}{u'(\bar{c}(\mathfrak{z}_2))}(1 - \bar{\pi}) \text{ for } z' \neq z. \tag{15}$$

Since one-period bond prices are equal to the sum of the Arrow prices, the price of an uncontingent bond and, hence, the interest rate, $1 + i$, is constant and equal to

$$\frac{1}{1+i} = \beta\left[\bar{\pi} + \left(\frac{u'(\bar{c}(\mathfrak{z}_1))}{u'(\bar{c}(\mathfrak{z}_2))}\right)(1 - \bar{\pi})\right]. \tag{16}$$

By inspection of Equation (16), the interest rate for the economy with solvency constraints is lower than the interest rate for the corresponding representative agent economy, which equals β. Also, by inspection of Equations (14) and (15), the pricing kernel, given by $q_t(z^t, z_{t+1})/\pi(z_{t+1}|z_t)$, is more volatile than the pricing kernel of the corresponding representative agent economy, which is constant in this case.

Agent 2's consumption depends only on the current state z_t. That is, $c_{2,t}(z^{t-1}, z_t) = \bar{c}(z_t)$, for all $z^{t-1}$and z_t, for the $\bar{c}$ given by the efficient allocation. Agent 2's purchases of Arrow securities depend only on the state in which they pay, $a_{2,t+1}(z^t, z_{t+1}) = \bar{a}(z_{t+1})$, for all z^t and z_{t+1}. Agent 1's consumption and Arrow securities holdings are given by $c_{1,t}(z^t) = e - c_{2,t}(z^t)$ and $a_{1,t+1}(z^{t+1}) = -a_{2,t+1}(z^{t+1})$.

We find the values of $\bar{a}(z)$ using the sequence budget constraints of Equation (4) for agent 2 together with the resource constraint of Equation (1),

$$\bar{a}(\mathfrak{z}_2) = \frac{\bar{c}(\mathfrak{z}_2) - \epsilon_2(\mathfrak{z}_2)}{1 + \bar{q}_{nr} - \bar{q}_r} < 0$$
$$\bar{a}(\mathfrak{z}_1) = -\bar{a}(\mathfrak{z}_2) > 0.$$

The signs of the asset positions are very intuitive. Indeed, $\bar{a}(\mathfrak{z}_1) > 0$ means that agent 2 saves for his low-income state, and $\bar{a}(\mathfrak{z}_2) < 0$ means that agent 2 borrows against his high-income state.

The solvency constraints for the high-income state can be found easily following our definition in Equation (5). In an efficient allocation, the continuation utility corresponding to the high-income shock ($z_{t+1} = \mathfrak{z}_2$ for agent 2) is given by $\underline{U}^i(z_{t+1})$. Then $B_{i,t+1}(z^{t+1}) = a_{i,t+1}(z^{t+1})$, which for agent 2 gives $\overline{B}(\mathfrak{z}_2) \equiv B_{2,t+1}(\mathfrak{z}_2, z^t) = \bar{a}(\mathfrak{z}_2)$.

The solvency constraints corresponding to the low-income shock ($z_{t+1} = \mathfrak{z}_1$ for agent 2) cannot be determined directly from the optimal allocations, because they depend on the solutions to off-equilibrium consumption and portfolio choice problems. We show in the appendix how to determine the constraint in this particular example.

Proposition 6. $\overline{B}(\mathfrak{z}_1) \equiv B_{2,t+1}(z^t, \mathfrak{z}_1) = \left[\frac{\bar{q}_{nr}}{1-\bar{q}_r}\right]\overline{B}(\mathfrak{z}_2) \leq 0.$

In this equilibrium, agents are constrained from borrowing against the future state z_{t+1} where their income will be high, regardless of the current state z_t.[6] But there is an important difference depending on the current state z_t. If the agent has his high-income shock (say, $z_t = \mathfrak{z}_2$ for agent 2), the agent does not want to borrow more against either the bad or the good future state, since for both cases, his marginal rate of substitution is equal to the Arrow price. In this case, he is *at* the constraint, but it is a *false corner*. If the solvency constraint were relaxed a bit, he would not change the optimal choice of consumption and asset holdings. On the other hand, if the current state is the one where the agent has his low income ($z_t = \mathfrak{z}_1$ for agent 2), then the agent wants to borrow against his good state, but he cannot. In this case, the solvency constraint binds, and the marginal rate of substitution between consumption in $z_t = \mathfrak{z}_1$ and in $z_{t+1} = \mathfrak{z}_2$ for agent 2 is strictly lower than the corresponding Arrow price. In this state the solvency constraint cannot be relaxed without changing the choice of consumption and asset holdings. Hence the solvency constraints bind only in the case in which the agent's current income shock is low, that is, poor agents are constrained from borrowing.

[6] Due to symmetry, the asset positions and solvency constraints for agent 1 are trivial, once they are determined for agent 2.

4.3 Calibration: Risk sharing and the Hansen–Jagannathan bounds

We calibrate individual income following Heaton and Lucas (1996) based on a large sample from the PSID. In particular, they find that the log of an agent's income, relative to the aggregate, $\ln \epsilon_{i,t}$, is stationary with a first-order serial correlation of 0.5 and a standard deviation of 0.29 for annual data. We set $\beta = 0.65$ and explore the effect of risk aversion on consumption and on asset pricing implications. We will explore the quantitative effects of β below.[7]

Figure 4 shows the efficient consumption allocation and the corresponding equilibrium elements for values of risk aversion between 0 and 6 using the formulas described above. The top panel displays consumption for agent 2 in each state as a fraction of mean income. As risk aversion increases, efficient allocations go from autarky to partial risk sharing (risk aversion approximately between 2 and 4) and, finally, to full risk sharing. In the bottom panel, we show agent 2's asset holdings and solvency constraints for each state. The figure shows that for low risk aversion, where the equilibrium corresponds to autarky, asset holdings equal solvency constraints, which allow no borrowing at all. For higher risk aversion, agent 2 is constrained in borrowing contingent only on the realization of state 2, when he has relatively high income. For even higher levels of risk aversion, the equilibrium displays full risk sharing, and the solvency constraints do not bind in either of the states.

Hansen and Jagannathan's volatility bounds for stochastic discount factors provide a concise and widely used diagnostic device for asset pricing models.[8] Figure 5 shows the Hansen and Jagannathan volatility bound implied by the data for each mean value of the kernel together with standard deviations and means of the model's pricing kernel for selected values of risk aversion. The model is able to generate kernels that fall inside the Hansen and Jagannathan bound for risk aversion coefficients around 2. As is well known, the standard economy without solvency constraints is very far from passing this test for such low values of risk aversion.[9]

This positive finding also confirms empirical results by He and Modest (1995) that show that solvency constraints can pass a modified volatility bound test for aggregate consumption data. A close look reveals the two-sided role of the risk aversion coefficient. In most of the asset pricing literature, increasing risk aversion increases the volatility of the pricing kernel,

[7] The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\begin{bmatrix} \epsilon_1(\mathfrak{z}_1) \\ \epsilon_1(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.641 \\ 0.359 \end{bmatrix}$, $\begin{bmatrix} \epsilon_2(\mathfrak{z}_1) \\ \epsilon_2(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.359 \\ 0.641 \end{bmatrix}$.

[8] For a detailed survey of applications of this test, see, for example, Cochrane and Hansen (1992).

[9] Introducing aggregate uncertainty into our example would, of course, generally give further volatility to the pricing kernel and help it pass the test.

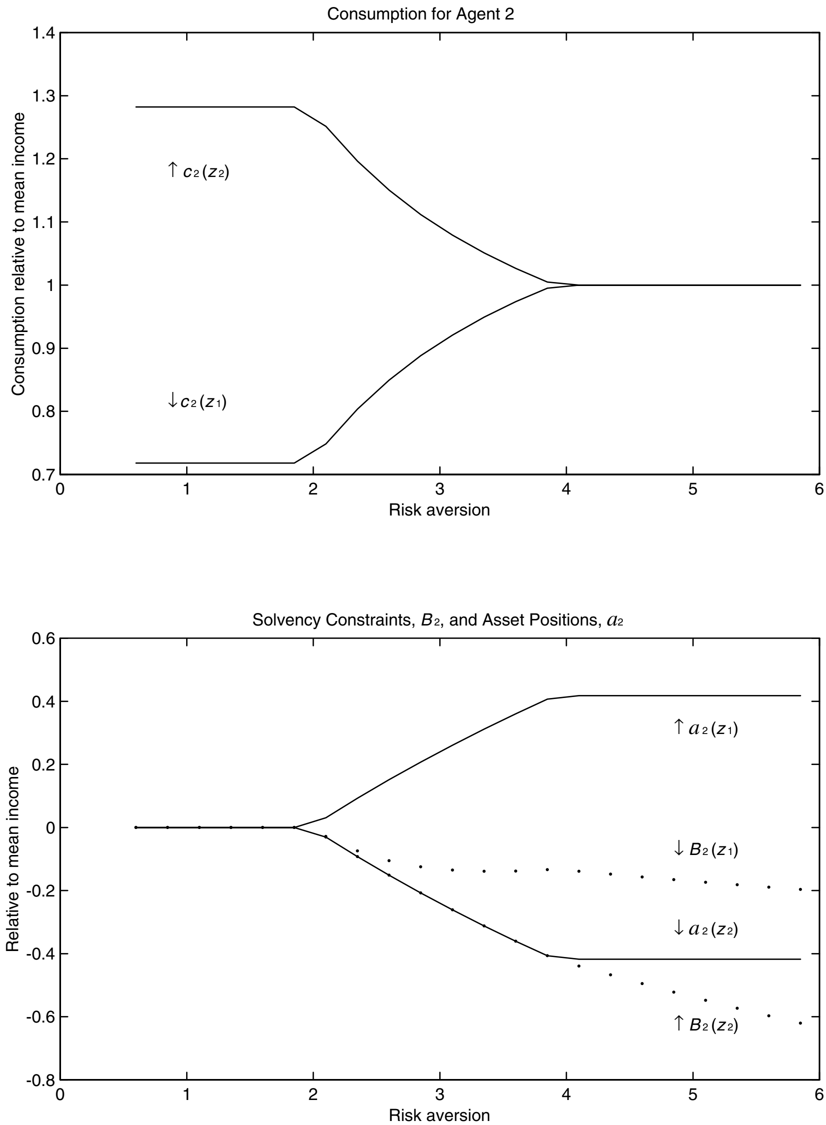

Figure 4
Consumption, solvency constraints, and asset positions as a function of risk aversion

because for a given consumption process marginal utility is more volatile. In our framework, however, the extent of risk sharing and thus the volatility of the consumption process is endogenous. The highest volatility for the pricing kernel is achieved with modest values of risk aversion, which correspond to very limited risk sharing. This can be seen by examining the consumption

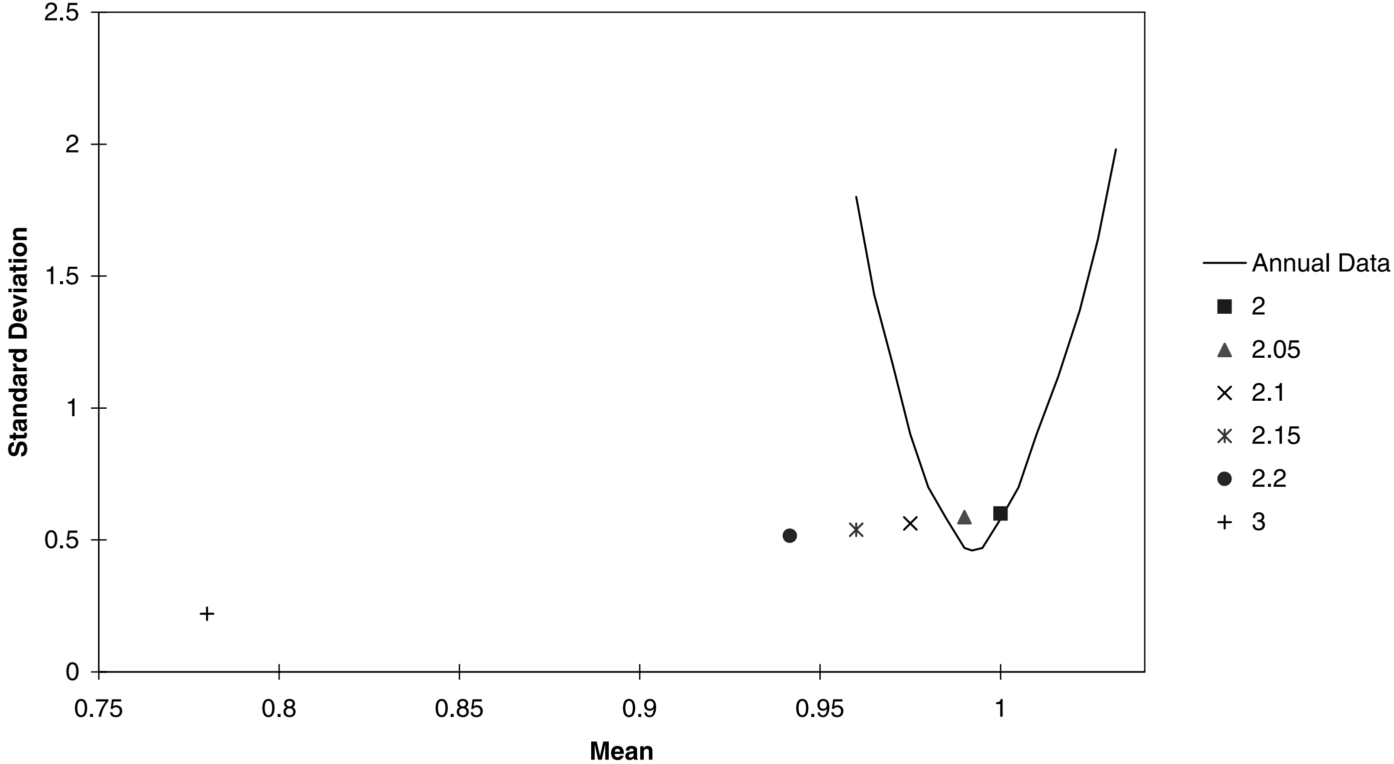

Figure 5
Volatility bound and risk aversion
The figure compares standard deviations of the stochastic discount factor for different levels of risk aversion for the two-state case to a Hansen–Jagannathan volatility bound. The volatility bound is based on data from Cochrane and Hansen (1992).

allocations, asset holdings, and solvency constraints in Figure 4 for risk aversion values close to 2.

5. Quantitative Predictions About Asset Returns

We now consider an endowment process that allows for variable degrees of dependence between individual and aggregate income uncertainty. The form and extent of this dependence turns out to be very important for asset prices. For that reason, our objective in this section is not only to document to what extent a plausibly calibrated model can explain a set of asset pricing moments, but also to derive some qualitative properties about how specific forms of dependence between individual and aggregate income uncertainty generate specific asset return properties.

5.1 Specification of the endowment process

We specify the endowment process with four values for the share of income of each agent, $\epsilon_i(\cdot)$, and two values for the aggregate growth rate, $\lambda(\cdot)$, respecting symmetry across agents. The set Z has four elements: with symmetry, we end up with a total of 10 parameters to be selected, six for Π, two for $\epsilon_2(\cdot)$, and two for $\lambda(\cdot)$.

The subscripts r and e index a recession and an expansion, respectively. The subscripts h and l index a high- and a low-income share for agent 2. With this convention, we have

$$\lambda(\mathfrak{z}_1) = \lambda(\mathfrak{z}_3) = \lambda_r < \lambda_e = \lambda(\mathfrak{z}_2) = \lambda(\mathfrak{z}_4), \text{ and}$$
$$\epsilon_2(\mathfrak{z}_1) = \epsilon_{lr} < \frac{1}{2} < \epsilon_{hr} = \epsilon_2(\mathfrak{z}_3)$$
$$\epsilon_2(\mathfrak{z}_2) = \epsilon_{le} < \frac{1}{2} < \epsilon_{he} = \epsilon_2(\mathfrak{z}_4).$$ [10]

5.2 Calibration of the benchmark endowment process

We use 10 moments, $M1$–$M10$, that describe the aggregate and household income data to select the 10 free parameters of the endowment process. The model period is a year. We focus on four moments of U.S. aggregate output growth in the twentieth century and on six other moments that characterize household income risk. The preference parameters, risk aversion, and time-discount factor are discussed in the next subsection.

M1. First-order serial correlation, $\rho(\lambda) = -0.14$, Mehra and Prescott (1985).

M2. $\Pr(expansion)/\Pr(recession) = 2.65$, NBER business cycle chronology for 1889–1991.

[10] By symmetry, for each z and each value of $\epsilon_2(z)$, there must be another $\tilde{z}$ such that $\epsilon_1(\tilde{z}) = \epsilon_2(z)$. This implies that $\epsilon_{lr} + \epsilon_{hr} = 1$ and $\epsilon_{le} + \epsilon_{he} = 1$.

The first two moments restrict the probability matrix. In particular, they imply a 2×2 matrix for the aggregate state obtained by combining the probabilities for the two recession states, $\mathfrak{z}_1$ and $\mathfrak{z}_3$, and the two expansion states, $\mathfrak{z}_2$ and $\mathfrak{z}_4$. Given this matrix, the next two conditions determine the two values of λ.

M3. $E(\lambda) = 1.83\%$, Mehra and Prescott (1985).
M4. $\mathrm{Std}(\lambda) = 3.57\%$, Mehra and Prescott (1985).

The remaining free parameters are determined jointly. We use the studies by Heaton and Lucas (1996), henceforth HL, and Storesletten, Telmer, and Yaron (1997), henceforth STY, to guide us in determining a benchmark calibration.[11] After defining these moments and our benchmark values, we discuss further below how these choices are related to the two original data studies.

M5. $\mathrm{Std}(\ln \epsilon(z)) = 0.296$.[12]
M6. First-order serial correlation, $\rho(\ln \epsilon(z)) = 0.53$.

The last four moments determine the relationship between idiosyncratic and aggregate income behavior. Consistent with HL and STY we introduce conditional heteroscedasticity of the idiosyncratic incomes exclusively through the probability matrix, as shown by *M7* below.

M7. The cross-sectional dispersion of the shares in recessions relative to expansions, $v_r/v_e = 1$. In a given state, this dispersion is defined as

$$v^2(z_t) \equiv \frac{1}{2} \sum_{i=1,2} \left[\epsilon_i(z_t) - \frac{1}{2}\right]^2,$$

and by symmetry we have $v(\mathfrak{z}_1) = v(\mathfrak{z}_3)$ and $v(\mathfrak{z}_2) = v(\mathfrak{z}_4)$.

M8. Relative standard deviation of individual shares conditional on current and past realizations of the aggregate shock:

$$\frac{\sigma_{r'e}}{\sigma_{e'e}} \equiv \frac{\mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e)}{\mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_e, \lambda_t = \lambda_e)} = 1.5.$$ [13]

M9. Relative standard deviation of individual shares conditional on current and past realizations of the aggregate shock:

$$\frac{\sigma_{r'r}}{\sigma_{e'r}} \equiv \frac{\mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_r)}{\mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_e, \lambda_t = \lambda_r)} = 1.5.$$

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[11] In some cases, two sets of parameters replicate the selected moments. In these cases, we pick the solution that has less variation in the means conditional on whether the destination is a recession or an expansion. This is the process that is closer to the linear processes estimated by HL and STY.

[12] The two studies define their idiosynchratic income variable in a slightly different way. HL use $y = \ln(\epsilon)$, whereas STY use $x = \ln \epsilon_i(z_t) - \frac{1}{I}\sum_{j=1,I} \ln \epsilon_j(z_t) M$. When a first-order log-linear approximation is used, the two measures differ only by a constant, so that the two can be considered, to a first approximation, identical for the moments we consider here.

[13] It can be shown that $\mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e) = \mathrm{std}(\ln \epsilon_i(z_{t+1})|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e, \epsilon_{i,t})$; that is, the conditional standard deviation of the share (and the log) does not depend on the current idiosynchratic shock, only on the aggregate growth state that is fully described by λ_t.

Table 1
Calibration of individual income share process

	HL	HL, our λ	STY, our λ	Benchmark
M5	0.296/0.4	—	0.71	0.296
M6	0.53/0.3	—	0.87	0.53
M7	1.02/0.93	1.03/0.9	[0.93-1.07]	1
M8, M9	0.99/1.19	0.99/1.27	1.88	1.5
M10	1.37/1.01	0.45/0.86	0.9	0.95

The table compares our benchmark calibration for the individual income share process to the processes implied by the studies of Heaton and Lucas (1996) (HL) and Storesletten, Telmer, and Yaron (1997) (STY). In the second and third column, the results from HL and STY are combined with our process for aggregate income growth, λ. For HL, two values are reported, the first is for their entire sample, the second for the subsample of stockholders. *M5* is the standard deviation of the log of the income share, and *M6* the first-order autocorrelation. *M7* is the cross-sectional dispersion of the income shares in recessions relative to expansions. *M8* and *M9* are the relative standard deviations of the log of the income shares conditional on the current state being a recession relative to being an expansion; for *M8* the preceding period's state is an expansion. for *M9* a recession. *M10* is the relative standard deviation of the log of the income shares conditional on the preceding period's state being a recession relative to being an expansion.

M10. Relative standard deviation of individual shares conditional on past realization of the aggregate shock:

$$\frac{\sigma_r}{\sigma_e} \equiv \frac{\text{std}(\ln \epsilon_i(z_{t+1})|\lambda_t = \lambda_r)}{\text{std}(\ln \epsilon_i(z_{t+1})|\lambda_t = \lambda_e)} = 0.95.$$

Table 1 contains the moments implied by the HL and the STY estimations of the household income processes. For the HL calibration there are two values in each cell, the first for the entire sample of 860 households and the second for the subsample of 327 stockholders. Since HL and STY estimate the individual income process conditional on the aggregate income, we combine their estimates of the individual income process with our specification for aggregate income. Our calibration mainly follows the HL calibration, given that they are calibrating a model with the same infinite horizon, two-agent structure. Given that earlier work on asset pricing with incomplete markets has shown that *M8* and *M9* are important, we choose a value slightly higher than HL, but still below STY.

Our model can be solved by iterating on the functional equation [Equation (8)]. For the results in this article we use a guess and verify strategy based on the sufficient conditions presented in Alvarez and Jermann (2000). Efficient allocations have a finite number of consumption values, but consumption takes, in general, more values than the number of states in Z. The main advantage of this approach is that our results are very accurate.

5.3 Quantitative implications for the benchmark case

We document the implications for risk sharing and asset pricing as a function of risk aversion, γ, and the pure time-discount factor, β, for the benchmark endowment process. Figure 6 presents consumption share volatility, the average risk-free rate, the equity premium, and the premium for long-term bonds.

We define *equity* as a claim to aggregate endowment and the *long-term bond* as a real consol (perpetual bond). We find that the extent of risk sharing, as measured by the volatility of consumption shares, is increasing in risk aversion and the time-discount factor. This is a quantitative illustration of the results in Remark 1. With limited risk sharing, there is only a small region, close to the one corresponding to autarky, where the risk-free rate attains reasonably low values. Of interest, the equity premium is highest for values of risk aversion that are higher than what is required to be in autarky. Finally, the shape of the premium for long-term bonds, as a function of γ and β, is similar to the shape for the equity premium, but the values are lower.

To gain intuition about what determines the level of the risk-free rate, we compare it to the representative agent economy. In the latter case,

$$\frac{1}{R^f_{t,t+1}} = E_t\left[\beta\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}\right],$$

and the risk-free rate, $R^f_{t,t+1}$, changes one for one with changes in the time-discount factor β. In the economy with solvency constraints, the risk-free rate satisfies

$$\frac{1}{R^f_{t,\,t+1}} = E_t\left[\beta\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}\max_{i=1,2}\left\{\left(\frac{\hat{c}_{i,t+1}}{\hat{c}_{i,t}}\right)^{-\gamma}\right\}\right],$$

and thus β changes the risk-free rates one for one for a constant consumption allocation, $\hat{c}_{i,t}$ and $\hat{c}_{i,t+1}$. Because the equilibrium allocation also depends on β, there is a second effect on the level of the interest rate. For instance, if β is lowered, there is less risk sharing, and individual consumption shares become more dissimilar. Thus the "max" increases, leading to a reduction in the interest rate. This is an illustration of the result shown in Alvarez and Jermann (2000) that state prices in the solvency constraints economy are higher than the prices in the corresponding representative agent economy.

In Tables 2 and 3, for our benchmark case, we set risk aversion $\gamma = 3$ and choose β to match the historical average of the U.S. risk-free rate of 0.80% per year. The implications from our benchmark case are very encouraging. We can generate a sizable equity premium with low risk aversion.[14] The volatility of the risk-free rate and the premium for long-term bonds are close to their empirical counterparts. As suggested by the high volatility of the pricing kernel, $\text{std}(M) = 0.99$, the Sharpe ratio of equity is high. Specifically, it stands at 0.44, slightly higher than the historical average of 0.37. Table 1 illustrates that, compared with the representative agent economy, the restrictions on the portfolio choices implied by the endogenous solvency constraints substantially improve the ability to generate realistic asset pricing implications.

[14] Note that stocks in the model economy do not have leverage; therefore we should compare the model to the unleveraged equity premium in the data, which is lower than the 6.18% of the reported leveraged equity premium.

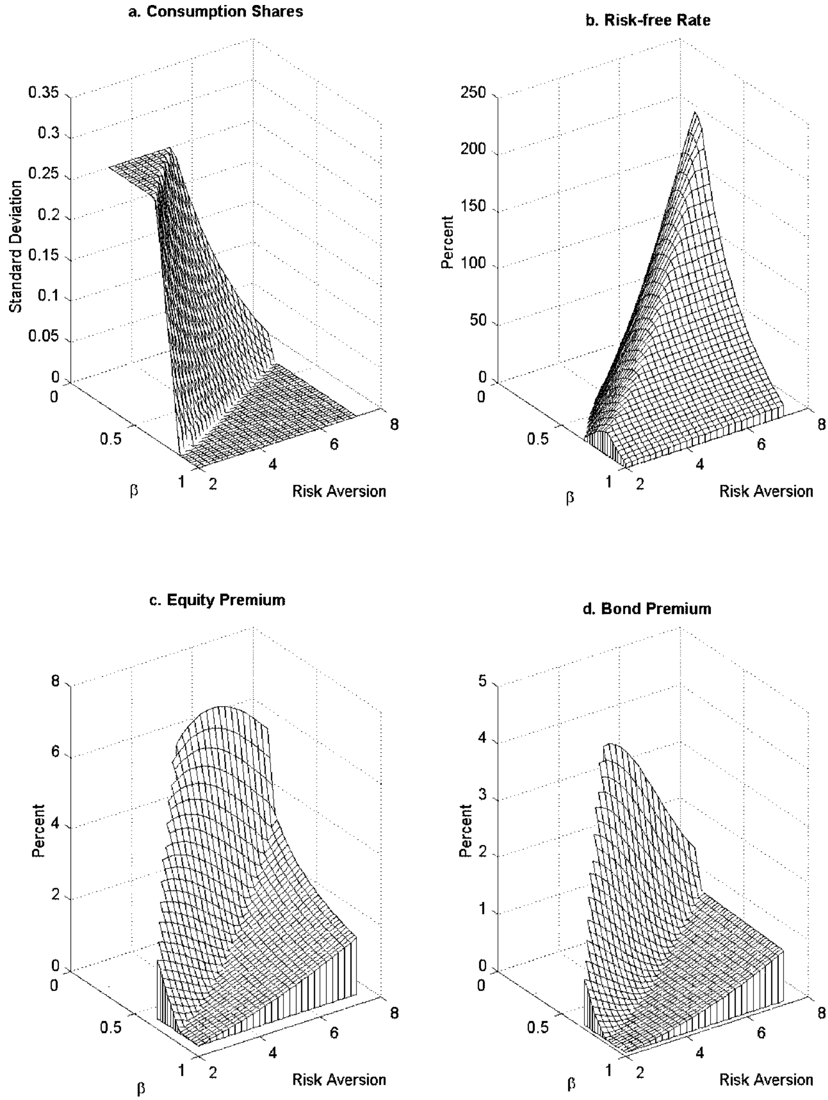

Figure 6
The effects of risk aversion and time-discount factor
The figure shows equilibrium allocations and returns as functions of risk aversion and time-discount factor, β. Panel (a) shows the standard deviation of consumption shares, $\mathrm{std}(\ln \hat{c}_i)$; (b) the unconditional mean of the risk-free rate, $E\left(R^f_{t,t+1}\right)$; (c) the equity premium; and (d) the premium for perpetual bonds.

The implied value of β is lower than the values used in other studies. We have followed the typical procedure used to identify β, that is, we have chosen it to match the average risk-free rate. An alternative way to interpret our calibration is that the value of β is somewhat of a proxy for things that have been abstracted from the model and which influence the attrac-

Table 2
Asset pricing implications from the benchmark economy

	$E(rf)$	$E(rb-rf)$	$E(re-rf)$	Sharpe	std(rf)	std(rb)	std(re)	$E(M)$	std(M)	$\rho(M)$	β
Benchmark	0.80	2.19	3.19	0.44	4.78	6.39	8.83	0.99	0.99	−0.02	0.488
Representative agent	4.00	0.16	0.53	0.11	1.68	2.16	4.81	0.95	0.11	−0.14	1.000
"High β"	0.80	2.03	3.41	0.37	5.56	7.05	9.22	1.00	1.15	−0.02	0.779
U.S. data	0.80	1.68	6.18	0.37	5.67	10.54	16.54				

The table reports asset pricing implications from the benchmark economy. For each case, the time-discounter factor β is set so as to best match the historic mean risk-free rate, *E(rf)*, of 0.80% per year. Returns are defined as follows: *rf* is the one-period risk-free rate; *rb* is the return of a long-term bond (a perpetual bond in the model); *re* is the return to equity. $E(\cdot)$ and std$(\cdot)$ are unconditional means and stands for the first-order serial correlation coefficient. For the "High β" case we use $M6 = 0.9$, $M8.9 = 4$, $M10 = 0.85$, and risk aversion is 3.5. U.S. data for equity and risk-free rates are from Mehra and Prescott (1985); long-term government bond returns are from lbbotson Associates (1998). The model period is one year, returns are in percentages. M stands for the pricing kernel, defined as $M = \max_{i=1,2}\{\beta u(c_{it+1})/u(c_{it})\}$.

tiveness of default relative to the future access to credit markets. The last row of Table 2 presents an alternative "High β" parameterization of the endowment process where individual shocks are more persistent ($M6 = 40.9$) and where recessions and expansions are more asymmetric ($M8 = M9 = 4$ and $M10 = 0.85$). For this parameterization we set $\gamma = 3.5$. In this case, $\beta = 0.78$ is required to match the risk-free rate. Notice that the asset pricing implications are very similar to those in the preceding case.[15]

In Table 3 we document the amount of risk sharing and the tightness of the solvency constraints. An overall measure of the amount of risk sharing is provided by comparing the standard deviation of the shares of consumption to the standard deviation of the income shares. For the benchmark case, consumption volatility, at 0.283, is only slightly lower than income volatility, 0.296, indicating that there is little risk sharing in equilibrium. The rest of Table 3 illustrates the way in which solvency constraints implement this limited amount of risk sharing: constraints bind relatively infrequently, when they bind they are very tight, and they are similar across states. Consider the benchmark case. The value of an agent's portfolio is equal to the solvency constraints 24.7% of the time, and these constraints bind, that is, they have a strictly positive Lagrange multiplier 21.3% of the time.[16] Conditional on agents' portfolio values being equal to their solvency constraints, the constraints are −0.8% of current income and 2.6% of the corresponding values for the economy without constraints. Under the same conditions, they do not vary much across states; the coefficient of variation is 4.4%. For comparison, in the "High β" case, there is somewhat more risk sharing, and the constraints are slightly looser.

[15] Of interest, Ligon, Thomas, and Worral (1997), using a similar model and data from individual consumption and income from poor villages, have estimated values of β in the neigborhood of 1/2.

[16] In Section 4 we discuss in detail the distinction between having a strictly positive Lagrange multiplier and being at a false corner.

Table 3
Risk sharing with solvency constraints

	Benchmark	"High β"
std(ln{εi})	0.296	0.296
std(ln{ci})	0.283	0.276
Fraction of time at constraint	0.247	0.147
Fraction of time with binding constraint	0.213	0.050
Expected value of constraint, conditional on being at constraint:		
as a fraction of current income	−0.008	−0.026
as a fraction of asset holdings for first best	0.026	0.039
Coefficient of variation of constraint, conditional on being at constraint	0.044	0.026

The table documents the amount of risk sharing and the behavior of the solvency constraints. Results are reported for the benchmark calibration and the "High β" calibration. std(ln{ε_i}) and std(ln{c_i}) stand for the standard deviations of the individual income shares and the consumption shares, respectively.

5.4 Comparison with other quantitative studies

Other studies have explored the connection between limited risk sharing, asset prices, and *exogenously* given borrowing constraints. For instance, Telmer (1993) and Heaton and Lucas (1996) show that agents can attain significant risk sharing when faced with transitory income shocks provided that they are allowed to take large short positions. Heaton and Lucas (1996) also displ ay examples in which imposing short-sales constraints on one-period bonds improves their model's asset pricing implications.

Zhang (1997) presents a model with *endogenous* borrowing constraints motivated by a participation constraint like ours. He considers a fixed borrowing constraint, independent of the agent's current income and wealth. Because Zhang's borrowing constraint is fixed, it cannot be *not too tight* in general. In his model, agents are allowed to trade only one-period noncontingent debt, thus restricting the asset pricing analysis to short-term interest rates. In the cases presented in his article, there is considerable risk sharing, and somewhat counterfactual asset pricing implications.[17] In our benchmark case, there is little risk sharing, and asset pricing implications are closer to the data. Motivated by this contrast, we provide more details about the differences in the two articles.

The difference between the results in the two studies lies primarily in the way asset prices are related to marginal rates of substitution and in the way the time preference parameter β is chosen. Denote by $R^{Zf}_{t,t+1}$ and $R^{f}_{t,t+1}$ the one-period return on a noncontingent bond in Zhang's economy and in ours. In Zhang's equilibrium, $1/R^{Zf}_{t,t+1} = \max_{i=1,2} E_t\big[\beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})}\big]$. In our equilibrium, the price of a one-period bond equals the sum of the Arrow securities, $1/R^{f}_{t,t+1} = E_t\big[\max_{i=1,2} \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})}\big]$. Clearly, $R^{f}_{t,t+1} \leq R^{Zf}_{t,t+1}$. Concerning the choice of parameters, Zhang's endowment processes and risk aversion are similar to ours; however, his time preference coefficient β is close to 1, which

[17] For instance, the volatility of the pricing kernel in Zhang's article is well below the Hansen–Jagannathan volatility bound.

is much higher than ours. As a consequence, his examples do not replicate historical mean risk-free interest rates of 0.8%. For instance, with risk aversion equal to 3, as in our benchmark case, the interest rate in his economy is higher than 6% annually. Zhang's finding of extensive risk sharing is consistent with our analysis of the impact of β as shown in Remark 1 and as displayed in Figure 6. In particular, Figure 6 shows that a large value of β results—all else being equal—in much more risk sharing, because autarky is a hard punishment for very patient agents.

5.5 Determinants of the equity premium and the term structure

To understand the determinants of the equity premium and of the term structure, we analyze the behavior of risk premia for different types of dependence between aggregate and individual income uncertainty. We start with the case where aggregate and individual income shocks are independent. Then we document the consequences for asset prices of the types of dependence introduced through heteroscedasticity in the individual risks as described by the moments *M8* to *M10*. We find that the equity premium depends on the comovement between individual income uncertainty and the contemporaneous aggregate income growth. The term premium depends on the comovement between the forecast of future individual income uncertainty and aggregate income growth.

We refer to the case in which the aggregate income growth shock is i.i.d. and is independent of the individual income shock as *independent risks*. The first panel of Table 4 displays the asset pricing implications of the case with independent risks, for which $M1 = 0$ and $M2, M7, M8, M9, M10 = 1$.[18] In all cases presented in Table 4, risk aversion $\gamma = 3$ and the time-discount factor $\beta = 0.5$. With these values, all the equilibria display some, but not complete, risk sharing as can be seen by the fact that $0 < \text{std}(\ln \hat{c}) < 0.296 = \text{std}(\ln \epsilon)$. We summarize our findings in two propositions and two results.

Proposition 7. *With independent risks, interest rates are constant; thus there is no risk premium for bonds.*

As shown in the appendix, the assumption of independent risks is not sufficient to generate this result in general. This result also depends on the particular number of states and agents chosen in this calibration. Nevertheless, we think this is a useful benchmark, because starting with a case that has constant interest rates will allow us to consider separately what determines term premia and premia for payout uncertainty. In the appendix, we show the following:

[18] The economy also displays symmetry between expansions and recessions, that is, $M2 = 1$. This is not necessary for the results presented here. Even with $M2 \neq 1$, the same qualitative properties hold. For comparison, Table 4 also displays a "no heteroscedasticity" case, where $M1 = -0.14$ and $M2 = 2.65$, as in the benchmark calibration.

Table 4
Determinants of the equity premium and the term structure of interest rates

	E(rf)	E($rb - rf$)	E($re - rf$)	Sharpe	std(rf)	std(rb)	std(re)	std($\ln\{c_i\}$)	β	$\frac{E(1+re)}{E(1+rf)} - 1$	Calibration
Panel 1: Solvency constraints economy											
Independent risks	5.62	0.00	0.39	0.10	0.00	0.00	3.72	0.27	0.50	0.3689	*M2, 8, 9, 10* = 1, *M1* = 0
No heteroscedasticity	5.49	0.16	0.51	0.11	1.69	2.15	4.83	0.27	0.50	—	*M8, 9, 10* = 1
Representative agent	109.63	0.00	0.77	0.10	0.00	0.00	7.38	0.00	0.50	0.3689	*M1* = 0, *M2* = 1
Panel 2: Solvency constraints economy: the equity premium											
$\text{cov}(\lambda, \text{std}(\ln\{\varepsilon_i\})) < 0$	6.63	0.00	**1.29**	0.34	0.00	0.00	3.78	0.26	0.50	—	*M8* = *M9* = 1.5
$\text{cov}(\lambda, \text{std}(\ln\{\varepsilon_i\})) > 0$	4.85	0.00	**−0.34**	−0.09	0.00	0.00	3.66	0.28	0.50	—	*M8* = *M9* = 0.75
Panel 3: Solvency constraints economy: the term structure											
$\text{cov}(\lambda, \text{std}(\ln\{\varepsilon_i\})) > 0$	5.15	**0.42**	0.83	0.10	3.85	5.32	8.39	0.29	0.50	—	*M10* = 0.95
$\text{cov}(\lambda, \text{std}(\ln\{\varepsilon_i\})) < 0$	6.22	**−0.31**	0.06	0.02	3.65	5.01	3.67	0.27	0.50	—	*M10* = 1.05

The table documents asset pricing implications for different calibrations. These calibrations illustrate how relationship between aggregate and individual income uncertainty determines the equity premium, see panel 2, and the term structure of interest rates, see panel 3. Returns are defined as follows: *rf* is the one-period risk-free rate; *rb* is the return of a long-term bond (a perpetual bond in the model); *re* is the return to equity. $E(\cdot)$ and std$(\cdot)$ are unconditional means and standard deviations. $\ln\{\varepsilon_i\}$ and $\ln\{c_i\}$ stand for the longs of the individual income share and consumption share, respectively. Returns are in percentages.

Proposition 8. *With independent risks, the (multiplicative) equity premium in the solvency constraints economy is identical to the one in the corresponding representative agent economy.*

As the reader can see in panel 1 of Table 4, the multiplicative risk premium for both economies is equal to 0.3689%.[19] It is true in general that with independent risks a one-period risky claim with payout contingent on the aggregate endowment has the same risk premium in the solvency constraints economy as it has in the corresponding representative agent economy. As shown in the appendix, this equivalence result extends here to the equity premium because interest rates are constant.

The following result illustrates a departure from independent risks that produces a different equity premium in the solvency constraints economy.

Result 1. *A negative covariance of the individual income variance with the contemporaneous aggregate income growth ($M8 = M9 > 1$) increases the equity premium, and a positive covariance ($M8 = M9 < 1$) reduces the equity premium relative to the corresponding representative agent economy.*

With $M8 = M9 > 1$, recessions are associated with higher individual income risk. Therefore, given the limited amount of risk sharing, the agents end up absorbing more risk in their consumption in recessions. Thus, in recessions the pricing kernel is higher, because volatility in idiosyncratic consumption translates into higher values through the max operator, as is suggested by Equation (6). This makes equity more risky, as illustrated in the second panel of Table 4. Result 1 is related to a well-known result by Mankiw (1986), who shows that the price of a risky strip will be lower in a static model with exogenous incomplete markets. In his environment, the convexity of the marginal utility is a necessary condition for a higher premium. Constantinides and Duffie (1996) use an assumption analogous to $M8 = M9 > 1$ in a model with exogenous incomplete markets and permanent individual income shocks to show that the pricing kernel is identical to the one in an economy with complete markets, but for an agent with higher risk aversion. In both cases, the results follow from the assumption that agents have convex marginal utility and that there are no binding portfolio constraints: hence all agents have the same valuation of assets. Instead, in the economy with solvency constraints, Result 1 follows from the fact that marginal valuations across agents differ, and prices are equal to the highest marginal rate of substitution.

Note that interest rates remain constant even if we introduce dependence of aggregate and individual risks such that $M8 = M9 \neq 1$, as can be seen in the

[19] The *multiplicative equity premium* is defined as $E_t(R^e_{t,t+1})/E_t(R^f_{t,t+1}) - 1$. We define the *equity premium* in the standard way as $E(R^e_{t,t+1}) - E(R^f_{t,t+1})$. In the case considered for Proposition 8, the multiplicative equity premium is constant across states and the conditional expectations of the two returns are constant.

second panel of Table 4. For this case, the dependence introduced between the aggregate and the individual shocks does not introduce any predictable changes in the pricing kernel, which is why interest rates remain constant.

The next result isolates a feature that explains volatility in interest rates and the existence of nonzero risk premia for bonds.

Result 2. *Introducing dependence between the aggregate income growth and next period's individual income risk makes the risk-free rate variable. A positive covariance of the aggregate income growth and next period's individual income risk, $M10 < 1$, creates positive term premia. A negative covariance of the aggregate income growth and next period's individual income risk, $M10 > 1$, creates negative term premia.*

Predictable movements in the pricing kernel are required for nonconstant interest rates. In this case, the predictable changes are introduced through the conditional variance of the individual's income.[20] For instance, if in recessions the variance for next period's consumption share is expected to be lower than in expansions, then prices of bonds are lower in recessions and higher in expansions. Thus a positive term premium is required to compensate bondholders. In other words, with countercyclical interest rates, capital gains accrue to bondholders when they are valued relatively less—requiring positive term premia. Panel 3 of Table 4 illustrates this case.

6. Conclusion

The objective of this article was to explore the asset pricing implications of a model with endogenously determined solvency constraints. We showed under which circumstances the endogenous solvency constraints bind, and we characterized the pricing kernel. We found that for plausibly calibrated income processes and for low values of risk aversion, the model produces solvency constraints that bind often and, as a consequence, individual consumption is volatile enough so that the resulting pricing kernel passes the Hansen–Jagannathan test. In addition, given the calibrated correlations between the individual income risk and the aggregate income, the model produces sizable premia for equity and long-term bonds. We also provide characterizations of how the dependence between individual and aggregate risks determines both equity and bond premia.

We think that this model improves on the standard representative agent economy and on the—arbitrary—incomplete market economies, since it makes the portfolio constraints endogenous and simultaneously obtains asset pricing implications that are closer to the data. Nevertheless, our assumptions

[20] By contrast, in the standard representative agent model of Campbell (1986), this happens through serial correlation in the growth rate of aggregate consumption.

about punishment for defaulting is also arbitrary. We assume that default is punished by permanent exclusion of all asset markets, but entails no garnishment of labor income. Instead, if the exclusion from asset markets is temporary, it will make default more attractive. On the other hand, if only a fraction of the labor income is garnished, it will make default less attractive. Incorporating these more realistic alternatives may produce similar results, since they compensate each other. We leave the investigation of these alternatives, as well as the introduction of endogenous costly punishment, as a topic for future research.

A shortcoming of our approach is that default does not occur in equilibrium; thus we cannot analyze interesting issues such as the pricing of securities with positive probability of default or the incentives and characteristics of agents who do default. With respect to the incentives to default, the parameterization in this article displays two properties. First, agents who are currently poor find themselves unable to borrow. Second, the solvency constraints prevent agents with high income realizations from defaulting. While the first property is realistic, the second is not.

Another open issue pertains to the amount of borrowing—measured by the frequency and the size of the binding solvency constraints. In our benchmark calibration, the constraints do not bind very frequently, but when they do, introspection suggests that they allow for too little borrowing. An alternative to examining the solvency constraints directly is to compare the extent of risk sharing in the model with the empirical evidence. In our benchmark case, there is very limited risk sharing. Our reading of the literature is that the available empirical evidence on this issue is mixed. For high-frequency income changes, there is evidence for considerable, but far from perfect, risk sharing [Mace (1991), Attanasio and Weber (1992), Attanasio and Davis (1996)]. For low-frequency income changes, the evidence indicates almost no risk sharing [Cochrane (1991), Attanasio and Davis (1996)]. Our analysis in Remark 1, part (b) shows that persistent shocks are more difficult to share. Thus modeling the income process as composed of a transitory and a permanent component, observed separately, will likely help to close this gap between the model and the evidence. We leave this task for future research.

Finally, the quantitative results in this article, like related work on asset pricing models with heterogeneous agents, relies on the interaction between idiosyncratic and aggregate risk. Further empirical research about this interaction should usefully complement the construction of new models.

Appendix

Proof of Proposition 1. Full risk sharing is characterized by resource feasibility and constancy of the ratio of the marginal utilities across agents. If Equation (13) is satisfied, then the participation constraints are satisfied for the allocation $c_i = e/2$. Conversely, if full risk sharing is feasible, clearly Equation (13) is satisfied.

Proof of Proposition 2. By using Benveniste and Scheinkman, one shows that

$$\frac{\partial V(w,z)}{\partial w} = -\frac{u'(e - C_2(w,z))}{u'(C_2(w,z))},$$

which is strictly decreasing in w, since the value function is strictly concave. Thus $C_2(w,z)$ is increasing.

Proof of Proposition 3. The proof of [I] follows from examining the case in which neither agent is constrained in the future. In this case,

$$\begin{aligned}\frac{\partial V(w,z)}{\partial w} &= -\frac{u'(e - C_2(w,z))}{u'(C_2(w,z))}\\ &= -\frac{u'(e - C_2(W_z(w,z),z))}{u'(C_2(W_z(w,z),z))} = \frac{\partial V(W_z(w,z),z)}{\partial w}.\end{aligned}$$

Finally, using Proposition (2) [I] is obtained. The proof of [II] follows from a variation of the preceding argument.

Proof of Proposition 4. To demonstrate this property, we state the following two lemmas.

Lemma 1. *If $\partial V(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1)/\partial w > \partial V(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2)/\partial w$, then the decision rules that achieve TV are flat after reversal.*

Proof. The proof follows immediately from the first-order conditions of the problem defined by the right-hand side of Equation (8).

Lemma 2. *If the decision rules after reversal are flat, then $\partial TV(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1)/\ \partial w > \partial TV(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2)/\partial w$.*

Proof. Using the "45° rule" result from Proposition (3) and the assumption that the decision rules are flat after reversal in the two promise-keeping equations [Equation (10)] evaluated at $\bar{w}(\mathfrak{z}_1)$ and $\bar{w}(\mathfrak{z}_2)$, we obtain

$$\bar{w}(\mathfrak{z}_1) - \bar{w}(\mathfrak{z}_2) = \frac{u(e - C_2(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1)) - u(C_2(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2))}{1 - \beta\bar{\pi} + \beta(1-\bar{\pi})}.$$

The desired result now follows by using the envelope condition.

If full risk sharing is not possible, then $\partial \widetilde{V}(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1)/\partial w > \partial \widetilde{V}(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2)\ /\partial w$. The result that the optimal decision rules are flat after reversal for the fixed point V^* follows from the combination of Lemmas 1 and 2 with the result that $\lim_{n\to\infty} T^n\widetilde{V} = V^*$. At each iteration, the two lemmas are applied sequentially. Then the domains are computed for TV. The fact that the described property is preserved for the limit follows directly from the fact that the limit is differentiable.

Proof of Proposition 5. By repeated substitution, this system reduces to one equation, $u_2 = h(u_2)$, in one unknown, $u_2 \equiv u(\bar{c}(\mathfrak{z}_2))$, where the equation is defined as

$$u_2 = h(u_2) \equiv \frac{(1-\beta)(1-2\beta\bar{\pi}+\beta)}{1-\beta\bar{\pi}}U^2(\mathfrak{z}_2) - \frac{\beta(1-\bar{\pi})}{1-\beta\bar{\pi}}f(u_2), \tag{A.1}$$

$f(u_2) \equiv u(e - u^{-1}(u_2))$. Since u is strictly increasing and strictly concave, the function h is strictly increasing and strictly convex; hence Equation (A.1) has at most two solutions. Notice that autarky, that is, $\bar{w}(\mathfrak{z}_i) = U^2(\mathfrak{z}_i)$, $\bar{c}(\mathfrak{z}_i) = \epsilon_2(\mathfrak{z}_i)$ for $i = 1, 2$ satisfies Equation (10), resource feasibility, and the boundary condition; hence, Equation (A.1) has at least one solution. We now describe which of the two solutions characterizes the efficient allocation, which we denote by $\bar{u}_2$.

Consider the two solutions to Equation (A.1), u_{2l} and u_{2h}. Ordering them as $u_{2l} \leq u_{2h}$, we show that u_{2l} is the efficient allocation. First, if u_{2l} equals autarky, then u_{2h} corresponds to an allocation that is more volatile than the allocation at u_{2l}, with the same mean. Thus, by concavity, u_{2l} gives higher utility. Second, we consider the case in which u_{2h} equals autarky. By assumption, $U^2(\mathfrak{z}_2) > \frac{u^*}{1-\beta}$ (that is, there is no full risk sharing), which by direct computation implies that $h(u^*) > u^*$, and by convexity of h, $u^* < u_{2l}$ or $u_{2h} < u^*$. Because $u_{2h} < u^*$ is ruled out if u_{2h} equals autarky, we have $u^* < u_{2l}$; that is, $\frac{e}{2} < \bar{c}_l(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$. Thus u_{2l} has higher utility than autarky u_{2h}, and by symmetry $\epsilon_2(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$. Finally, because of concavity, the participation constraint for $\mathfrak{z}_1$ is satisfied.

Proof of Proposition 6. For the given prices $\{\bar{q}\}$, we directly check the definition for solvency constraints that are not too tight for the postulated solvency constraints for agent 2, $\{B_2\}$. For $z_{t+1} = \mathfrak{z}_2$, with initial wealth $\underline{a}_{t+1} = \overline{B}(\mathfrak{z}_2)$, the efficient choice for consumption and asset holdings from the equilibrium is still feasible with these constraints, and $J_{2,t+1}(\overline{B}(\mathfrak{z}_2), (z^t, \mathfrak{z}_2)) = U^2(\mathfrak{z}_2)$ as is required for solvency constraints that are not too tight. For $z_{t+1} = \mathfrak{z}_1$, with initial wealth $\underline{a}_{t+1} = \overline{B}(\mathfrak{z}_1)$, for any future path for which state $\mathfrak{z}_1$ continues to repeat itself without interruption for the next k periods, the Euler equation and the price $\bar{q}_r$ imply optimal individual choices such that $\underline{c}(z_{t+1} = \mathfrak{z}_1) = \underline{c}(z_{t+k} = \mathfrak{z}_1) = \underline{c}_0$, $\underline{a}(z_{t+k} = \mathfrak{z}_1) = \overline{B}(\mathfrak{z}_1)$, and $J_{2,t+k}(\overline{B}(\mathfrak{z}_1), (z^t, \mathfrak{z}_1)) = U^2(\mathfrak{z}_1)$. For $z_{t+1} = \mathfrak{z}_1$, with initial wealth $\underline{a}_{t+1} = \overline{B}(\mathfrak{z}_1)$, followed by $z_{t+2} = \mathfrak{z}_2$, the solvency constraint will bind at $\overline{B}(\mathfrak{z}_2)$, and from there on optimal consumption and portfolio choices will be identical to those in the equilibrium, given that prices are unchanged, initial wealth levels are identical and constraints are such that these choices are feasible. With this characterization, we have the following two equations for $\underline{c}_0$ and $\overline{B}(\mathfrak{z}_1)$, a budget constraint and the equation defining implicitly the solvency constraints:

$$\epsilon_2(\mathfrak{z}_1) + \overline{B}(\mathfrak{z}_1) = \bar{q}_r \overline{B}(\mathfrak{z}_1) + \bar{q}_{nr} \overline{B}(\mathfrak{z}_2) + \underline{c}_0, \text{ and}$$
$$J_{2,t+1}(\overline{B}(\mathfrak{z}_1), (z^t, \mathfrak{z}_1)) = U^2(\mathfrak{z}_1) = u(\underline{c}_0) + \beta\bar{\pi} U^2(\mathfrak{z}_1) + \beta(1-\bar{\pi})U^2(\mathfrak{z}_2),$$

which give that $\underline{c}_0 = \epsilon_2(\mathfrak{z}_1)$ and $\overline{B}(\mathfrak{z}_1) = [\bar{q}_{nr}/(1-\bar{q}_r)]\overline{B}(\mathfrak{z}_2) \leq 0$. The same argument can be applied for any time period; for agent 1, the solvency constraints are obtained by symmetry.

Proof of Proposition 7. Assume that z can be decomposed as $z = (x, y) \in Z = X \times Y$ and λ and ϵ_i are functions of y and x, respectively. Specifically, λ: $Y \to R_+$ and ϵ_i: $X \to (0, 1)$. By *independent risks* we mean that there is a probability distribution ϕ and a stochastic matrix ψ such that $\pi(z'|z) \equiv \pi((x', y')|(x, y)) = \phi(y') \cdot \psi(x'|x)$ for all z, z'.

Now consider the price of a one-period bond,

$$\beta \sum_{x_{t+1}} \sum_{y_{t+1}} \max_{i=1,2} \left[\left(\frac{\hat{c}_{i,t+1}}{\hat{c}_{i,t}} \right)^{-\gamma} \right] \lambda(y_{t+1})^{-\gamma} \phi(y_{t+1}) \cdot \psi(x_{t+1}|x_t).$$

In Alvarez and Jermann (2000), we show in Proposition 4.18 that for an economy with independent risks, the consumption shares $\hat{c}_i$ do not depend on the aggregate state y (otherwise, unnecessary volatility in consumption would be introduced). Thus the price of a one-period bond can be written as

$$\beta \left\{ \sum_{y_{t+1}} \lambda(y_{t+1})^{-\gamma} \phi(y_{t+1}) \right\} \left\{ \sum_{x_{t+1}} \max_{i=1,2} \left[\left(\frac{\hat{c}_{i,t+1}}{\hat{c}_{i,t}} \right)^{-\gamma} \right] \cdot \psi(x_{t+1}|x_t) \right\},$$

and each of the two terms in curly braces can be shown to be constant, due to the independent risks assumption. By no-arbitrage, the risk premia for bonds of any maturity is zero.

Proof of Proposition 8. We start by rewriting the equity premium as a weighted average of risk premia to individual dividends. Let D_{t+k} be the dividend of a stock at $t+k$, and let

$\{D_{t+k}\}_{k=1}^{\infty}$ be the dividend process and $V_t[\cdot]$ be the value at t of D_{t+k}. Define the value of the stock as $V_t[\{D_{t+k}\}_{k=1}^{\infty}] = \sum_{k=1}^{\infty} V_t[D_{t+k}]$, so that the price of a stock can be seen to be the price of a portfolio containing claims to each dividend, that is, a portfolio of *dividend strips*. The one-period holding return of equity is

$$R_{t,t+1}[\{D_{t+k}\}_{k=1}^{\infty}] = \sum_{k=1}^{\infty} \left(\frac{V_t[D_{t+k}]}{V_t[\{D_{t+k}\}_{k=1}^{\infty}]} \right) \cdot R_{t,t+1}[D_{t+k}],$$

where the one-period return of a dividend strip is $R_{t,t+1}[D_{t+k}] = V_{t+1}[D_{t+k}]/\ V_t[D_{t+k}]$. With a constant dividend denoted by $1_{t,t+1}$, the conditional multiplicative equity premium is

$$\frac{E_t(R_{t,t+1}[\{D_{t+k}\}_{k=1}^{\infty}])}{R_{t,t+1}[1_{t,t+1}]} = \sum_{k=1}^{\infty} w_t[D_{t+k}] \cdot \frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]}, \tag{A.2}$$

where w_t are nonnegative weights $w_t[D_{t+k}] = V_t[D_{t+k}]/V_t[\{D_{t+k}\}_{k=1}^{\infty}]$ and where $E_t(R_{t,t+1}[D_{t+k}])/R_{t,t+1}[1_{t,t+1}]$ is the conditional multiplicative risk premium of a strip paying D_{t+k} at $t+k$.

Equation (A.2) shows that the equity premium is equal to a weighted average of all the strip premia. We have shown in Alvarez and Jermann (2000) in Proposition 4.18 that with independent risks, the risk premium for a one-period dividend strip, that is, $E_t(R_{t,t+1}[D_{t+1}])/R_{t,t+1}[1_{t,t+1}]$, is identical in the solvency constraint economy to the one in the corresponding representative agent economy. Thus in order to show that the multiplicative equity premium is equal across the two economies, it is sufficient to show that risk premia for strips with different maturity dates k are equal for each economy separately.

We first show that for a representative agent economy, risk premia for strips with different maturities are equal for all k. Given i.i.d. aggregate growth rates, $V_t[D_{t+k}] = [\beta E(\lambda^{1-\gamma})]^k$, and we have $E_t(R_{t,t+1}[D_{t+k}]) = 1/[\beta E(\lambda^{1-\gamma})]$. Thus

$$\frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]} = \frac{E(\lambda^{-\gamma})}{E(\lambda^{1-\gamma})}.$$

We now show that for the solvency constraints economy considered here, risk premia for strips with different maturities are equal for all k. In Alvarez and Jermann (2000), we show in Proposition 4.18 that for an economy with independent risks, the consumption shares $\hat{c}_i$ do not depend on the aggregate state y (otherwise, unnecessary volatility in consumption would be introduced). Thus the pricing kernel can be written as $\beta\{\lambda(y_{t+1})^{-\gamma}\phi(y_{t+1})\}\ \{m(x_{t+1}|x_t) \cdot \psi(x_{t+1}|x_t)\}$, where the first part in curly braces depends only on the aggregate shock and the second part in curly braces depends only on the individual shock. Using this notation we can write that $V_t[D_{t+1}] = \beta E(\lambda^{1-\gamma})E(m)$, where we have used the facts that aggregate growth rates are i.i.d., that interest rates are constant so that $E_t(m(x_{t+1}|x_t)) = E(m)$, and that the probabilities are separable for y and x so that $\text{cov}_t(\lambda(y_{t+1})^{-\gamma}, m(x_{t+1}|x_t)) = 0$. More generally we have that $V_t[D_{t+k}] = [\beta E(\lambda^{1-\gamma})E(m)]^k$, where we have used the additional result that $\text{cov}_t(m(x_{t+1}|x_t), m(x_{t+2}|x_{t+1})) = 0$. This last result follows directly from the fact that with constant interest rates, term premia are all equal to zero. We then have $E_t(R_{t,t+1}[D_{t+k}]) = 1/[\beta E(\lambda^{1-\gamma})E(m)]$; thus

$$\frac{E_t(R_{t,\,t+1}[D_{t+k}])}{R_{t,\,t+1}[1_{t,\,t+1}]} = \frac{E(\lambda^{-\gamma})}{E(\lambda^{1-\gamma})}.$$

As before, the term structure for strip premia is flat; thus it is at the same level as the one for the representative agent economy.

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