Rational Beliefs and Security Design

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This article studies the security-design problem of a cash-constrained firm facing investors with diverse beliefs. Investor "rational beliefs" are modeled as varying and yet rational in the sense of Kurz (1994a). With two investors, optimal designs are similar under rational beliefs and rational expectations. With many investors, however, optimal securities under rational beliefs maximize investor differences of opinion, while under rational expectations optimal designs minimize disagreements. We demonstrate that the common practice of issuing multiple securities backed by a single asset is optimal under rational beliefs but not under rational expectations. Researching market beliefs can create substantial value for firms.

This article studies the security-design problem of a cash-constrained firm that faces a set of investors with diverse beliefs. The firm does not possess any useful information about its cash flows that is unavailable to investors. Investors, however, have varying views about the distribution of the firm's future cash flows. The nature of this variability of beliefs affects the firm's security design in a manner that depends on the number of investors. When there are two investors, the firm issues smaller, safer claims to minimize diversity of opinion about its securities. When there are at least three investors, however, the firm maximizes investor diversity of opinion about its claims by issuing risky securities.

We propose that investors have beliefs that are diverse and yet rational in the sense of Kurz (1994a), as discussed at greater length in Section 2. Kurz's notion of a "rational belief" is weaker than the traditional concept of rational expectations on the part of agents, but it captures the idea that the beliefs of rational agents must in some sense conform to the empirical data available to them. Within this constraint, however, the beliefs of agents may differ dramatically.

Our results contrast the optimal security designs when investors have rational beliefs with the optimal designs when they have rational expectations. If investors have rational beliefs, optimal designs are different in the two- and many-investor games. In the rational beliefs two-investor game, diversity of

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opinion is costly to the firm. When there are only two investors, pessimistic investors have a marked influence on the pricing of a firm's securities. They bid only low values themselves, and this in turn encourages optimistic bidders to bid less aggressively. The essential trade-off for the firm in the two-investor model is between the cost of retention and the cost of inducing greatly varying opinions in the investment market. An optimal security design appropriately trades off these two costs. In the rational beliefs many-investor game, the market price of the firm's securities places greater weight on the beliefs of the optimists, and pessimistic investors play only a marginal role. The security will almost always be purchased by an optimistic investor, and optimists essentially compete with each other. As a result, the firm attempts to maximize the optimistic valuation of its securities. Differences of opinion benefit the firm, and optimal securities in the many-investor model exploit these varying views.

If investors have rational expectations, then, irrespective of the number of investors, optimal designs are quite similar to those in the rational beliefs two-investor game. As beliefs grow more diverse, pessimistic investors become more pessimistic. In a rational expectations setting, moreover, optimists are attentive to the views of pessimists, so the caution of the pessimists tempers the enthusiasm of optimists. As a result, diversity of opinion is costly to the firm, and optimal securities are relatively safe, minimizing the difference between the optimistic and pessimistic valuations.

We begin our analysis by showing that any optimal security issued by a cash-constrained firm must pay out fully in some states, for all beliefs types and any number of investors. The remainder of the results in the article highlight the different optimal securities that emerge under rational beliefs and rational expectations. One distinction between rational beliefs and rational expectations is that while the form of optimal designs depends critically on the number of investors under rational beliefs, this is not the case for rational expectations. Optimal securities are very similar under the two belief types when there are two investors.

The following summary of results contrasts optimal designs under rational beliefs and rational expectations for at least three investors. First, when investors have rational beliefs, the firm sells securities to the market even if it is not cash constrained, but this is not true if investors have rational expectations. Second, safe securities are optimal under rational expectations, and risky securities are optimal under rational beliefs. If securities are restricted to be monotone, and to leave the firm a monotone residual, then under reasonable conditions on beliefs either debt or equity is optimal under rational expectations, while equity is optimal under rational beliefs. Third, splitting securities into multiple claims is optimal under rational beliefs but not under rational expectations. Under a restriction on beliefs satisfied by many common distributions, the optimal design under rational beliefs is a prioritized tranche structure similar to that commonly observed in practice.

Our results for optimal security design under rational beliefs also differ from those in the literature on security design in the presence of asymmetric information on the part of the firm. In our model the firm attempts to maximize diversity of opinion by issuing risky securities such as equity. In the asymmetric information literature, the firm minimizes differences in the valuation of its securities by issuing safe claims such as debt.

Underlying this article is the assertion that a large part of the volatility of investment markets in developed economies is due to differences in the beliefs of agents, as opposed to exogenous economic fundamentals. This claim is buttressed by Kurz and Beltratti (1997) and Froot and Frankel (1991), and provides an intuitively appealing explanation for such phenomena as "wild" stock market swings. One implication is that the firm can have a significant impact on its valuation simply by choosing an optimal financial structure. Choosing a security design is presumably easier than finding good projects or creating value in other direct ways. This article argues that researching the state of investor beliefs and choosing an optimal design in light of these beliefs can create substantial value for the firm's original owners.

The article is organized in the following way. In the first section we briefly discuss some related research. Section 2 provides an introduction to the model and to the theory of rational beliefs and describes the solution of the security pricing game under both rational beliefs and rational expectations. In Section 3 we present results on optimal securities. Section 4 analyzes the optimal design when the firm can issue multiple securities backed by the same asset. In Section 5 we discuss the empirical implications of the article. Section 6 concludes.

1. Related Literature

This article belongs to the stream of articles concerning optimal security design that was initiated by Townsend (1979). Our model differs from Townsend's in several important respects. The verifiability of cash flows is not an issue in our model and asymmetric information on the part of the firm does not feature in this article.

The model in this article is similar to Myers and Majluf (1984) in its focus on the importance of differing valuations of a firm's securities. In the Myers–Majluf model, however, it is the firm's private information about valuation that is relevant. In this article, we analyze investors' varying beliefs. The costs of issuing a suboptimal security in the Myers–Majluf article are related to the "lemons" problem introduced by Akerlof (1970). The implications of this problem for security design are developed extensively in the work of Nachman and Noe (1994). The costs in our model, though, are due to the firm's failure to exploit investors' beliefs properly.

Our model is closest in conception to DeMarzo and Duffie (1999). Although DeMarzo and Duffie, like Myers and Majluf, focus on the lemons problem of security design, we adopt from their article the particular formulation of a cash-constrained firm that sells future cash flows to liquid investors. The models differ, however, in several substantive respects. While the articles share some results, such as the optimality of debt, this is only true for the two-investor case. The results in the many-investor game studied in this article are substantially different from those of DeMarzo and Duffie; when the number of investors is large, the firm attempts to maximize investor diversity of opinion, not minimize it as in DeMarzo and Duffie.

This article is also related to two security design articles on asset pooling and tranching. DeMarzo (1999) considers a model in which a firm with asymmetric information prefers to sell assets separately rather than through a pooled sale. In Axelson (1999), an uninformed firm decides whether to pool or split the assets that it sells to investors who receive multidimensional signals. Neither of these articles makes use of the rational beliefs framework that is at the core of this model and, consequently, their results are quite different from those in this article.

2. The Model

2.1 The issuance game

The model we present proposes an economic environment in which a series of firms issues securities through time. At each point in time, one firm issues a security for which investors bid. Firms have identical characteristics through time and investors have identical beliefs. One interpretation of the model would be to view the investors as pension funds or investment houses in which beliefs are passed down through time. At any given point in time, the firm and the investors are only concerned with maximizing their utility in the game that period. Our model may thus be interpreted as a model of dynasties, in which beliefs are inherited but wealth is not. The cash-flow realizations and investments of past periods are available as data to the investors.

At any given period $t-1$, the firm has assets that will produce a cash flow P_t at period t, where $P_t \geq 0$ is a random variable.[1] We assume that P_t has a finite, strictly positive expectation under its associated measure m_t. There are $N \geq 2$ investors who have beliefs over both the realization of the firm's cash flows and the investment of the other investors, and an investor's investment is completely determined by his beliefs. We will presume that in any given period agents believe that $P_t = Q^i$ for some $i \in \{1, \ldots, R\}$ $(R < \infty)$, where the Q^i are random variables. An agent's belief Pr will specify a probability measure over the Q^i. Each agent a receives a private signal s^a which influences his belief.

We now specify the timing in the model, for any given period t.

[1] All random variables in the model are defined on a given probability space $(\Omega, \mathcal{G}, \pi)$.

2.1.1 Timing

1. The firm chooses the security design.
2. Investors' beliefs are realized.
3. Investors pay for the security.
4. Cash flows are realized in the next period.

A security design $X \colon \Re \to \Re$ is a measurable function that specifies the total payoff $X(P_t)$ to investors. A design X must satisfy the feasibility condition $X(p) \leq p$, for any realization p of the next period's cash flow P_t.

After the firm selects a design X, investors play the pricing game, which is designed as a standard first-price auction. Each investor j must choose an amount b^j to bid. An investor's strategy is a Borel probability measure on the space $[0, \infty)$. The investor with the highest bid wins the auction and pays his bid to the firm. In the case of a tie, one of the highest bids is randomly selected. We denote the signals and bids of all investors other than investor a by s^{-a} and b^{-a}, respectively. Investor a has expected utility

$$U^a(b^a, b^{-a}, s^a, s^{-a}, X, P_t) = \frac{(E^a(X(P_t)|s^a, s^{-a}) - b^a)I_{\{b^a=\max\{b^1,\ldots,b^N\}\}}}{\#\{b^i : b^i = \max\{b^1, \ldots, b^N\}\}}, \quad (1)$$

where E^a denotes expectation under the belief of investor a. The firm has the valuation model

$$V^F(b^1, \ldots, b^N, X, P_t) = E(\max\{b^1, \ldots, b^N\}) + \lambda E(P_t - X(P_t)), \quad (2)$$

where E denotes expectation under the beliefs of the firm and $\lambda \in (0, 1]$ is a discount factor. The firm receives the investment of the investors this period. The firm also receives in the next period the residual cash flows, those not promised by the security to the investors. The discount factor λ describes the degree to which the firm is cash constrained. It is assumed that $1 \geq \lambda$, meaning that the firm is more cash constrained than investors. (We have normalized the investors' discount factor to one.) The issuing firm considers the equilibrium investment levels for different security designs and then chooses the design that maximizes its expected utility.

It is important to note here that the cash flows realized by the firm from the sale of the security do not affect the distribution of P_t. This is in contrast to models such as Nachman and Noe (1994), in which the issuance proceeds are invested and used to generate the next period cash flows. There are two natural interpretations of the model presented here. The first is that a corporation is selling a claim to the cash flows of a subsidiary firm, and that all proceeds from the sale are used by the parent company for other operations. The second is that the firm is selling a claim to an asset-backed security, the cash flows of which will be unaffected by the sale.

2.2 Rational beliefs theory

The issuance game described above is quite general. The focus of this article is to analyze the implications for optimal security design of assuming that the agents in the model have rational beliefs in the sense of Kurz (1994a, 1994b). Kurz's theory of rational beliefs is a weakening of rational expectations that allows agents to have beliefs that are diverse, and yet rational in a specific sense. The theory of rational beliefs is invoked to provide a rigorous grounding for our assumption that equally informed investors may have different beliefs about a firm's prospects. We will make use of some straightforward implications of the theory of rational beliefs, such as the idea that investors may make incorrect forecasts at any point in time, but that their forecasts will be correct on average. Our presumption that agents have heterogeneous beliefs is essential to the results in this article, although other aspects of the theory of rational beliefs are also used in the arguments. The assumption of different priors is not uncommon in the finance literature; here we propose a theoretical underpinning for this idea.

The theory of rational beliefs posits an economic environment in which agents form beliefs about observables in the economy. Agents do not speculate about the underlying structure of the economy or about the economic forces that generate the data they observe. The observables are assumed to have the technical property of *stability*.[2] Informally this requires that agents are able to learn from the data. The stochastic process generating the data must be such that its empirical properties have meaning. For example, all empirical moments must exist. All stationary processes are stable, but the converse is untrue.

It is not possible, in general, to uniquely identify any stable process, even given countably many data points.[3] It is possible, however, to conclude that many processes did *not* generate the data. A rational belief is a distribution over observables that cannot be excluded as possibly being the true data-generating process. We may regard these beliefs as theories or models that the data does not disprove.

When economic observables are stable but not stationary, rational expectations requires that agents possess information about the underlying processes that cannot be derived from historical data. Rational beliefs requires only that their beliefs be consistent with the data. Under rational expectations, agents know the correct distribution of exogenous and endogenous variables; rational beliefs offers a less strenuous definition of rationality.

[2] Let $(\Re^\infty, \mathcal{B}, \Pi, T)$ be a stochastic dynamic system, where $\mathcal{B}$ is the Borel σ-field of $\Re^\infty$, Π is a probability measure, and T is the shift transformation on $\Re^\infty$. The system is stable if for every cylinder B, $\lim_{n\to\infty} \frac{1}{n}\Sigma_{k=0}^{n-1} 1_B(T^k(x))$ exists for Π-almost every $x \in \Re^\infty$.

[3] Kurz (1994a) shows that every stable process is associated with a particular stationary measure. He also shows that many different stable processes give rise to the same associated stationary measure. Historical data can only be used to construct the stationary measure, so it cannot be used to distinguish between multiple stable processes associated with the same stationary measure.

The essential observation undergirding the theory of rational beliefs is that economic agents hold a variety of opinions and interpret data in different ways. Investors have differing theories and ideas.[4] Rational beliefs provides an economic theory for this behavior. The essential component of the theory is the assumption that economic variables are stable but not necessarily stationary. If the strong assumption of stationarity is not made, then multiple theories are supported by the data.

Each agent holds a rational belief, which is a theory that has not been disproved by the data. The beliefs of other agents are uninformative in this framework; each agent knows that other beliefs are rational and may be subscribed to by other agents, but this information does not affect the rationality of his own belief. All agents have access to the same data and do not, therefore, learn from each other. The private signals they receive are not informative signals in the conventional sense, but may rather be thought of as private states of mind that determine which rational beliefs the agents will adopt.

It is commonly observed that approximately equally informed agents often interpret data in different ways. Investors act on the basis of "hunches" or private signals. The theory of rational beliefs allows us to accommodate these behaviors while still giving substance to the meaning of rationality. The technical requirements of rational beliefs will be discussed in the context of the model.

2.3 The issuance game under rational beliefs

We will now specialize the general issuance game described above. In particular, $\{P_1, P_2, \ldots\}$ is presumed to be i.i.d. with associated stationary measure m.[5] (The associated stationary measure, described in note 3, may be thought of as the time-series "average" measure.) It is assumed that $P_t = Q^1$ or $P_t = Q^2$. There are two possible private signals, A and B. Each signal is realized with probability 0.5, and agents' beliefs are given by $\Pr(P_t = Q^2|B) = \Pr(P_t = Q^1|A) = \gamma$, for some $\gamma \in (\frac{1}{2}, 1)$. Let us denote the bid under the signal A by b_A and similarly for b_B. (These may be random variables.) We assume that the signals of the investors are independent. An investor receiving the signal A is said to have belief A and an investor receiving signal B is said to have belief B.

Finally, the main focus of this article is the diversity of investor beliefs about the exogenous variable P_t. We are less concerned with investor beliefs over the investment strategies of other players. For simplicity we therefore maintain the assumption that investors always (correctly) believe that each of their opponents has belief A with probability one-half and belief B otherwise.

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[4] Tagaki (1991) surveys the evidence that participants in the foreign exchange markets have heterogeneous beliefs.

[5] For example, it may be assumed that $\{P_1, P_2, \ldots\}$ is i.i.d. with stationary measure m.

Beliefs over the signals of other investors are assumed to be independent of beliefs over P.

Investors review the data on cash flow realizations and investments in the past. They observe that in the past, agents bid b_A half the time and b_B half the time. Their beliefs about the bids of the other investors will therefore conform to the empirical evidence. The rational beliefs requirement is that investors' beliefs over the firm's cash flows also conform to the data. Let us denote by m_1 and m_2 the one-period probability measures associated with Q^1 and Q^2, respectively. The formal rational beliefs requirement[6] is that $0.5(\gamma m_1+(1-\gamma)m_2)+0.5((1-\gamma)m_1+\gamma m_2) = m$, that is, $0.5(m_1+m_2) = m$. In other words, the agents' beliefs must have the same "average" measure (i.e., associated stationary measure) as the true distribution. This is the sense in which their beliefs are supported by the data. Investors are assumed to be symmetric, with identical distributions over beliefs. Note that there are many beliefs that qualify as rational according to this definition. We assume that the firm has particularly simple rational beliefs. The firm correctly believes that each investor bids b_A with probability 0.5 and b_B otherwise. The firm has a fixed one-period belief over cash flows, given by the probability measure m. The firm's beliefs over (P_t, b_{t-1}), jointly, may remain unspecified since its expected utility is separable in I and P.

That investors are periodically optimistic and pessimistic does not suggest that they will on average lose money. Indeed, an agent's rational belief may correspond precisely to the true distribution of the economic variables. An agent with the correct rational belief accurately calculates his expected profit, and may therefore ensure that it is nonnegative. An agent also may choose the average (associated stationary) measure as his rational belief, but this belief need not be correct and is in no sense preferable to any other rational belief.

Agents will observe that the private signals of some investors will have done a poor job of predicting outcomes in the past. This does not, however, give them reason to select or abandon any particular rational belief. Investors receive independent private signals each period; in effect, they always seek inspiration from new sources. For convenience, we may loosely refer to these sources of inspiration as oracles. Each rational belief will have associated with it two oracles that correctly and identically selected the signal in each period in the past, but that make opposite predictions in the current period. In holding sequentially independent beliefs, agents may be thought of as always conforming to the views of one of the two oracles with a flawless historical record. In essence, agents may be viewed as always switching to one of the

[6] This is actually the requirement for a weakly rational belief, but we will define all weakly rational beliefs to be rational beliefs. Also, since the agents' beliefs over bids are constant, it can quickly be checked that they have rational beliefs over both the cash flow and the bids. The beliefs described are the simplest example of processes that are stable but not, in general, stationary. These processes, known as simple independently distributed stable (SIDS) processes, were introduced in Nielsen (1996).

oracles that have never made an error. Each rational belief has associated oracles with perfect records, so there is no reason to switch rational beliefs based on the past joint distribution of private signals and payoff realizations.

Our model is idealized in the sense that it is assumed that firms and investors have unlimited historical evidence at hand. We leave aside the question of what the earliest firms and investors did in the absence of data. Since firms and investors are presumed to have access to information of an asymptotic nature, nothing additional is learned from any added finite amount of data. This means that both the extent of information and all agent characteristics remain fixed across periods. We will solve the security design problem for one period; by complete symmetry this solution will hold for all periods. For simplicity, we will neglect time subscripts.

In a model with asymmetric information on the part of the firm, the security design choice of the firm is regarded by investors as a signal about the distribution of the underlying cash flows. If all agents have rational beliefs, as we assume here, investors are not concerned with the beliefs of the firm. Morever, investors are not interested in the beliefs of the other investors. All agents have access to identical historical data and do not regard the beliefs of other agents to be informative about the distribution of the cash flows this period. Under the assumption of rational beliefs on the part of all agents, we can therefore simplify Equation (1) in the following manner:

$$U^a(b^a, b^{-a}, s^a, s^{-a}, X) = \frac{(E^a(X|s^a) - b^a) I_{\{b^a = \max\{b^1,\ldots,b^N\}\}}}{\#\{b^i\colon b^i = \max\{b^1,\ldots,b^N\}\}}, \tag{3}$$

Let a security design X be given. We will set $e_k(X) = E_k[X(P)]$, where this expectation is taken under the measure m_k for P. We assume, without loss of generality, that $e_2(X) \geq e_1(X)$. For convenience, we let $\mu_x = 0.5(e_1(X) + e_2(X))$, the expected value of X under the stationary distribution. We define $\mu = 0.5(e_1(P) + e_2(P))$, the expected value of the whole claim under the stationary distribution.

Definition. A feasible design X is *admissible* if $X \geq 0$.

That is, only limited liability securities are admissible. For the remainder of the article we restrict attention to admissible designs. We now consider the symmetric Bayesian–Nash equilibrium outcome of the first-price auction. We denote the firm's valuation from the sale of security X when investors have rational beliefs by $V^F_{RB}(X, P_t, N)$.

Result 1. There is a unique symmetric equilibrium of the first-price auction when investors have rational beliefs. In this equilibrium

$$\begin{aligned} V^F_{RB}(X, P_t, N) &= e_1(X)\left(1 - \frac{1 + \gamma(2^N - 2N - 2) + N}{2^N}\right) \\ &\quad + e_2(X)\left(\frac{1 + \gamma(2^N - 2N - 2) + N}{2^N}\right) + \lambda(\mu - \mu_x). \end{aligned} \tag{4}$$

A proof is given in the appendix.

In this auction, an investor with a higher valuation for the security will always submit a higher bid than an investor with a lower valuation. Under rational beliefs, investors are not interested in the signals of their competitors or the firm, and the auction is therefore isomorphic to a standard private-value auction. We define $\beta(\gamma, N) = (\frac{1+\gamma(2^N-2N-2)+N}{2^N})$, and note that Equation (4) can be written as

$$V^F_{RB}(X, P_t, N) = e_1(X)(1-\beta(\gamma, N)) + e_2(X)(\beta(\gamma, N)) + \lambda(\mu - \mu_x). \quad (5)$$

Direct calculation shows that $\beta(\gamma, 2) < \frac{1}{2}$, $\beta(\gamma, 3) = \frac{1}{2}$, and $\beta(\gamma, N) > \frac{1}{2}$ for $N \geq 4$. There is thus a qualitative difference in the outcomes of the issuance game when there are two investors and when there are three investors or more. When there are only two investors, those with low valuations of the security (pessimists) exert a strong influence on the price since they win the auction with reasonable frequency and allow investors with high valuations (optimists) to bid less aggressively. As a result, issuing a security about which there are strong differences of opinion is costly to the firm, since the pessimists have a low very valuation of the claim. The firm prefers to issue securities that minimize the diversity of beliefs. As the number of investors grows, pessimists become less relevant. They rarely win the auction, and optimists must make higher bids since they are now competing essentially only with other optimists. Since the optimistic valuation is now more important in determining the price, the firm will seek to issue securities that maximize differences of opinion and raise the optimistic bids.

2.4 The issuance game under rational expectations

In this section we consider the outcome of the issuance game when investors have rational expectations. In this model, investors are concerned with the signals of other investors. We presume that $\{P_1, P_2, \ldots\}$ is i.i.d. with stationary measure m, and that $P_t = Q^1$ or $P_t = Q^2$. As in the rational beliefs model, there are two possible private signals, A and B. Each signal is realized with probability 0.5, and agents interpret signals in the following way: $\Pr(B|P_t = Q^2) = \Pr(A|P_t = Q^1) = \gamma$, for some $\gamma \in (\frac{1}{2}, 1)$. The rational expectations model differs from the rational beliefs model in two respects. First, agents regard all signals to be conditionally independent (conditional on P_t). Second, their beliefs over the probability of the signals given the state of P_t are assumed to be correct. Agents thus regard all signals to be informative, and their interpretations of the signals correspond to the objective conditional measure.

We denote the firm's valuation from the sale of security X when investors have rational expectations by $V^F_{RE}(X, P_t, N)$. We again assume, without loss of generality, that $e_2(X) \geq e_1(X)$.

Result 2. There is a unique symmetric equilibrium of the first-price auction when investors have rational expectations. In this equilibrium

$$V_{RE}^{F}(X, P_t, N) = e_1(X)\left(\frac{1}{2} + \frac{(1-\gamma)^{N-1}\gamma^{N-1}(2\gamma-1)N}{2(\gamma^N + (1-\gamma)^N)}\right) + e_2(X)\left(\frac{1}{2} - \frac{(1-\gamma)^{N-1}\gamma^{N-1}(2\gamma-1)N}{2(\gamma^N + (1-\gamma)^N)}\right) + \lambda(\mu - \mu_x). \quad (6)$$

A proof is given in the appendix. In this auction as well, optimists will always submit a higher bid than pessimists. We define $\kappa(\gamma, N) = (\frac{1}{2} - \frac{(1-\gamma)^{N-1}\gamma^{N-1}(2\gamma-1)N}{2(\gamma^N+(1-\gamma)^N)}) \in (0, \frac{1}{2})$, and note that Equation (6) can be written as

$$V_{RE}^{F}(X, P_t, N) = e_1(X)(1 - \kappa(\gamma, N)) + e_2(X)(\kappa(\gamma, N)) + \lambda(\mu - \mu_x). \quad (7)$$

Equation (7) makes clear that diversity of opinion is costly to the firm under rational expectations, as it is under rational beliefs in the two-investor case. If investors have rational expectations, optimists place weight on the views of pessimists. As beliefs grow more diverse, the pessimists become very pessimistic. This reduces their bids dramatically, particularly since they are aware that they only win the auction if all investors are pessimistic. Under rational expectations, moreover, optimists are concerned that in winning the auction they have likely beaten a number of pessimists. This winner's curse induces optimists to bid somewhat conservatively. The combination of these two effects makes it unattractive for the firm to issue securities about which investors strongly disagree.

Results 1 and 2 imply that if investors have rational expectations, or if $N = 2$ and investors have rational beliefs, then

$$V^F(X, P_t, N) = \mu_x(1-\lambda) + k_1(\gamma, N)(\min\{e_1(X), e_2(X)\} - \max\{e_1(X), e_2(X)\}) + \lambda\mu \quad (8)$$

for some $k_1(\gamma, N) > 0$. Result 1 implies that if investors have rational beliefs and $N \geq 3$ then

$$V^F(X, P_t, N) = \mu_x(1-\lambda) + k_2(\gamma, N)(\max\{e_1(X), e_2(X)\} - \min\{e_1(X), e_2(X)\}) + \lambda\mu \quad (9)$$

for some $k_2(\gamma, N) \geq 0$. We use of the formulations given in the above two equations to prove the results in the remainder of the article.[7]

[7] While these formulas have been derived for the first-price auction, the general implications of the pricing game for security design are robust to the use of second-price auction and Shapley–Shubik mechanisms [see Garmaise (1999)].

3. Security Design

We will now make use of our results from the previous section on the outcomes of the issuance game to derive implications for optimal security design. We will begin by introducing two securities. The strategy in which the firm issues the security $X = 0$ is referred to as *no-issuance*. The security X such that $X(p) = p$ for all p is called a *full-equity claim*.

Definition. Security X *dominates* security Y if the firm's equilibrium utility from issuing X is weakly higher than its equilibrium utility from issuing Y.

Definition. An admissible security X is *optimal* if X dominates all admissible securities.

Prior to stating our results, we make the simplifying assumption that m_1 and m_2 have associated probability density functions f_1 and f_2, respectively. We define F_1 and F_2 to be the respective associated cumulative distribution functions. We begin with a result that holds both for investors with rational beliefs and for investors with rational expectations. The result highlights a general feature of all optimal designs.

Result 3. If $\lambda < 1$, then any optimal design X exhausts all cash flows with strictly positive probability. That is, there must exist a measurable set $R \subset \Re$ with $m(R) > 0$ such that $X(p) = p$ for all $p \in R$.

This result suggests that optimal designs completely liquidate the firm (i.e., pay out all cash flows) in at least some states of the world. The fact that liquidation does not incur any penalty, such as bankruptcy costs, is crucial here. It is clear that in the presence of bankruptcy costs liquidation would be avoided; the result states that in the absence of such costs, liquidation in some states is an optimal strategy for the firm.

The following argument outlines the intuition underlying this result. Suppose that $e_1(X) < e_2(X)$ for the optimal design X. First assume that Equation (8) obtains. In this case, it must be that $X(p) = p$ wherever $f_1(p) > f_2(p)$. Otherwise we could transfer payments from states in which $f_2(p) > f_1(p)$ to states in which $f_1(p) > f_2(p)$ and keep μ_x constant. This transfer would increase $e_1(X)$ and decrease $e_2(X)$, and thereby increase the firm's valuation. If Equation (9) holds, then a similar argument shows that $X(p) = p$ wherever $f_2(p) > f_1(p)$. Now suppose that $e_1(X) = e_2(X)$ for the optimal X. It must be that $X(p) = p$ for all p such that $f_1(p) = f_2(p)$. This region may have measure zero. If there is slack both in the region such that $f_1(p) > f_2(p)$ and in the region such that $f_1(p) < f_2(p)$, then by increasing payments judiciously in both regions we are able to increase μ_x while keeping $e_1(X) = e_2(X)$. This would improve the firm's valuation under both Equation (8) and Equation (9). Since X is optimal, this is impossible, so that without loss of generality there can be no slack in the region $f_1(p) \geq f_2(p)$.

A proof of Result 3 is given in Garmaise (1999).

The security-design implications of investor rational expectations and investor rational beliefs are, in general, quite dissimilar. The following results analyze several important differences.

Result 4. Suppose $\lambda = 1$ (no liquidity constraints). If investors have rational expectations, or if $N = 2$ and investors have rational beliefs, then no-issuance is optimal. If $N \geq 3$ and investors have rational beliefs, then issuing any admissible security dominates no-issuance.

Proof. Let an admissible security X be given. Assume, without loss of generality, that $e_2(X) \geq e_1(X)$. The firm's valuation from no-issuance is μ. If investors have rational expectations, or if $N = 2$ and investors have rational beliefs, Equation (8) shows that

$$V^F(X, P_t, N) = k_1(\gamma, N)(e_1(X) - e_2(X)) + \mu \leq \mu. \tag{10}$$

If investors have rational beliefs and $N \geq 3$, however, then $V^F(X, P_t, N) = k_2(\gamma, N)(e_2(X) - e_1(X)) + \mu \geq \mu$.

This result provides the first stark contrast between optimal securities under rational beliefs and under rational expectations. If investors have rational expectations, Result 4 shows that a firm's motivation for the sale of securities must be that it is suffering under liquidity constraints. It is the case, however, that firms will sometimes seek external financing even when they possess sufficient resources to fund a project internally.[8] The rational beliefs of investors can provide a rationale for this behavior. If there are more than two investors, then Result 4 shows that even firms that are not liquidity constrained may seek outside financing. They can exploit differences of opinion with investors to sell their securities at prices that the firms deem attractive.

Result 5. Suppose that $e_1(P) = e_2(P)$. If investors have rational expectations, or if $N = 2$ and investors have rational beliefs, then full equity is optimal.

A proof is given in the appendix.

In this case the agents' beliefs are very similar, but not necessarily identical. The firm, in general, prefers to issue more cash flows. The cost of doing so is that it may not receive a sufficiently high price from investors. If investors have rational expectations, or if $N = 2$ and investors have rational beliefs, it is costly for the firm to issue securities that are valued very differently by optimists and pessimists. Result 5 shows that when investor beliefs are reasonably uniform, the firm can simply sell the largest claim possible (an equity claim). Result 5 also applies in the special case that investors have rational expectations and symmetric information.

[8] See Lerner and Tufano (1995) for an example. Pulvino (1998) details asset sales by financially unconstrained firms.

If investors have rational beliefs and $N \geq 3$, then Result 5 need not apply. The firm will issue securities that seek to maximize the variability of investor opinions. A security promising large payments in states most expected by some investors while promising no payments in states most expected by other investors may be preferred to full equity.

The next two results describe the optimal security in a wide set of circumstances.

Result 6. Suppose that investors have rational expectations, or that $N = 2$ and investors have rational beliefs. Assume that $e_2(P) > e_1(P)$. Consider the class of securities consisting of any Z such that $Z(P) \geq 0$. If there is an optimal security within this class, then some security X with the following characteristics must be optimal: $X(p) = p$ for all p such that $\frac{f_2(p)}{f_1(p)} < a$, $X(p) = bp$ for all p such that $\frac{f_2(p)}{f_1(p)} = a$, $X(p) = 0$ otherwise, for a given $a \in [1, \infty]$ and a given $b \in [0, 1]$. Denote this security[9] by $X_{a,b}$. Security $X_{a,b}$ will satisfy $e_2(X_{a,b}) \geq e_1(X_{a,b})$,

A proof is given in the appendix.

The intuition behind Result 6 is as follows. The firm would prefer to issue a high-expected-payoff security about which there is little diversity of opinion. However, under the assumptions of the result, the firm is unable to issue a full-equity claim without causing investors to have varying beliefs about its value. The best that the firm can do is to try to make $e_1(X)$ and $e_2(X)$ as close as possible. The challenge is to try to raise the pessimistic value of the security (under f_1), since the security will usually have a high value under the optimistic density (f_2). This is done by concentrating payments in the states expected to occur under the pessimistic density, which is exactly what the above specified design does.

The optimal securities described in Result 6 are extreme in the sense that in some states they pay out fully, while in others they do not pay out at all. (As the statement of the result makes clear, there may be a third range in which partial payments are made.) In models which include differing clienteles of investors [Allen and Gale (1994), for example], it is typically optimal to offer extreme securities because they concentrate payments in the states most valued by a given clientele. A novel aspect of Result 6 is that the securities described in the result are meant to be valued under both the pessimistic and optimistic densities, and yet they still take an extreme form. The result suggests, perhaps surprisingly, that extreme securities are optimal for achieving the two objectives of minimizing differences of opinion and maximizing the size of the issue.

[9] Set $X(p) = p$ for all p such that $f_1(p) = 0 = f_2(p)$. In any case, payments in these states will affect neither $e_1(X)$ nor $e_2(X)$.

Result 7. Suppose that $N \geq 3$ and investors have rational beliefs. Consider the class of securities consisting of any Z such that $Z(P) \geq 0$. If there is an optimal security within this class, then some security W with the following characteristics must be optimal: $W(p) = p$ for all p such that $\frac{f_{3-i}(p)}{f_i(p)} < a$, $W(p) = bp$ for all p such that $\frac{f_{3-i}(p)}{f_i(p)} = a$, $W(p) = 0$ otherwise, for a given $i \in \{1, 2\}$, $a \in [1, \infty]$, and $b \in [0, 1]$. Denote this security[10] by $W_{i,a,b}$. Security $W_{i,a,b}$ will satisfy $e_i(W_{i,a,b}) \geq e_{3-i}(W_{i,a,b})$.

A proof of Result 7 that relies on arguments similar to those used in the proof of Result 6 is given in Garmaise (1999).

Result 7 describes securities that maximize differences of opinion. If, for example, a security W of the form described in the result is optimal with $e_2(W) \geq e_1(W)$, then this security will promise full payment in all the states that have higher probability under f_2 than under f_1. (This follows from the condition $a \geq 1$.) An optimal security X of the type described in Result 6 with $e_2(X) \geq e_1(X)$, in contrast, will promise full payment in all the states that have *lower* probability under f_2 than under f_1. Security W maximizes the differences between the optimistic and pessimistic valuations, while security X minimizes them. The optimal securities in Result 7 are designed to generate a diversity of views and are risky in the sense that there is no market consensus about their value. The optimal securities in Result 6 are valued similarly by all investors and may be regarded as safe in this sense.

Result 7 shows that when investors have rational beliefs and $N \geq 3$, an optimal security will concentrate payments in the states most expected by one type of investor, but it cannot be specified whether the security will be valued more by the optimist who places a greater value on the whole claim or by the pessimist. Beliefs may be such that the firm can create greater diversity of opinion by designing a security that pessimists greatly value and that optimists do not value very much. It will more typically be the case, however, that an optimal security is designed to maximize the optimistic valuation.

Results 5, 6, and 7 provide a description of optimal securities in all cases. When investors have rational expectations, or when $N = 2$, the firm's objective is to minimize differences of opinion. It does so by issuing securities that promise high payments in the states most expected by pessimists. When investors have rational beliefs and $N \geq 3$, the firm attempts to maximize differences of opinion. It does so by issuing extreme securities as detailed in Result 7, but these securities may be designed under different circumstances to appeal to either optimists or pessimists.

The next result is concerned with the class of monotone increasing securities. The security design literature often focuses on this class of securities, in part because many of the presently available securities are monotone.

[10] Set $W(p) = p$ for all p such that $f_1(p) = 0 = f_2(p)$.

Another justification for considering this class is that moral-hazard concerns may make it impossible, under certain circumstances, for a firm to issue a nonmonotone security. For example, suppose that payments are not monotone in cash flows and that the sources of cash flows are not verifiable. Then firms can overreport their cash flow by adding outside monies to their earned income. Firms will then be able to pay out less to investors. Investors will anticipate this and regard all securities as monotone.[11]

In the result below, we characterize the optimal monotone security under a condition on beliefs. We define a *debt claim* B_a to be a security X such that for some $a \geq 0$, $X(p) = p$ for all $p \leq a$ and $X(p) = a$ for all $p \geq a$.

Definition. The density function f_2 dominates the density function f_1 in the sense of the monotone likelihood ratio property (MLRP) if $\frac{f_2(a)}{f_1(a)} \geq \frac{f_2(b)}{f_1(b)}$ for all $a \geq b$.

Result 8. Assume that investors have rational expectations, or that $N = 2$ and investors have rational beliefs. Consider the class of monotone increasing securities. Suppose that one density dominates the other in the sense of the monotone likelihood ratio property. Then either debt or full equity is optimal.

A proof is given in the appendix.

It can be quickly verified that one density dominates the other in the sense of the MLRP if and only if one belief dominates the other in the sense of the MLRP. The intuition for Result 8 is as follows. When densities are ordered by the MLRP, one density (say f_1) is pessimistic relative to the other (f_2). The pessimistic density accords a lower expected value to any monotone security, relative to the optimistic density. So for any given μ_x, the problem is to maximize $e_1(X)$, the pessimistic expectation, and to minimize $e_2(X)$, the optimistic expectation. Under MLRP the ratio of the densities is increasing, so cash flows in the very lowest states of P do the most to increase $e_1(X)$ without much increasing $e_2(X)$. The monotonicity of the security guarantees that any cash flows promised in low states must also be promised in high states, but by concentrating payoffs in low states, at least $e_1(X)$ will "benefit" from a whole range of outcomes in which $f_1 > f_2$. The MLRP property makes it the case that it is never better to have a cash flow in a higher state than in a lower state. Of course, the feasibility condition restricts the extent to which cash flows can be focused in low states. This means that optimal securities have no slack up to some point and are then flat. Payoffs in high states are less useful, but monotonicity of the security restricts them from being reduced. The form described is exactly that of a bond or, in the limit as bond face value approaches infinity, a stock.

[11] DeMarzo and Duffie (1999) make this argument precise. Suppose that $a > b$ and $X(a) < X(b)$. If outcome b is realized, the firm will add $a-b$ to its cash flows. Its total payoff in this case is $a-X(a)-(a-b) = b-X(a)$, which is better than the $b - X(b)$ the firm would have kept without adding to the cash flows. So, in fact, $X(b)$ will never be realized.

It is noteworthy that Result 8 guarantees the existence of an optimal security, despite the fact that the cash flow P is unbounded. This is achieved by means of a technical argument [given in Garmaise (1999)] invoking the compaction of the real line. A proof of Result 8 is given in the appendix.

Definition. The density function f_2 dominates the density function f_1 in the sense of *first-order stochastic dominance (FOSD)* if $F_2(a) \leq F_1(a)$ for all $a \geq 0$.

It is straightforward to verify that if f_2 dominates f_1 in the sense of MLRP, then f_2 dominates f_1 in the sense of FOSD. In the next result we restrict attention to claims X such that both $X(p)$ is monotone and $p - X(p)$ is monotone. The second restriction may be rationalized by arguing that the firm has free disposal of cash flows and thus its residual claim must be monotone (otherwise the firm would strategically dissipate cash flows). The assumption that $p - X(p)$ is monotone (i.e., that X has a monotone residual) is frequently adopted in the security design literature [e.g., Nachman and Noe (1994)]. Essentially all well-known securities are monotone and have monotone residuals.

Result 9. Assume that $N \geq 3$ and that investors have rational beliefs. Consider the class of securities that are monotone and have a monotone residual. Suppose that one density dominates the other in the sense of first-order stochastic dominance. Then full equity is optimal.

A proof is given in the appendix.

It may be checked that one density dominates the other in the sense of FOSD if and only if one belief dominates the other in the sense of FOSD. It is interesting to note that the optimal security design does not promise payments only in the states particularly valued by optimists. Instead, it promises the maximal payment in all states. When securities are monotone and have monotone residuals, the most efficient way to maximize the optimist's valuation is to issue full equity.

If investors have rational expectations, as N increases the expected price received by the firm when it issues a security X converges to μ_x. Thus for values of λ less than one, the firm will sell a full-equity claim when N is very large and the price discount almost disappears.

Result 10. Assume that investors have rational expectations and that $\lambda < 1$. For N sufficiently large, full equity is optimal.

A proof is given in the appendix.

Results 8, 9, and 10 describe the optimal securities under reasonable assumptions. If investor beliefs can be characterized as "optimistic" and "pessimistic" in the sense of the MLRP, then the results above indicate that optimal security designs can be quite different when the number of investors is small. The investors in this model are each capable of purchasing the entire claim, so it is appropriate to view them as large firms competing to purchase a division from a competitor, or investment banks competing to purchase a large asset-backed security. In either of these examples it is likely that N would be small. Results 8 and 9 suggest that if investors have rational expectations then debt is likely to be optimal,[12] while if investors have rational beliefs and $N \geq 3$ then full equity is optimal. This arises from the fundamental difference that minimizing diversity of opinion is optimal under rational expectations, while maximizing diversity of opinion is optimal under rational beliefs (for $N \geq 3$). For larger values of N, Result 10 shows that equity is optimal under rational expectations. The next section discusses some further important differences in optimal security design under rational beliefs and under rational expectations.

4. Multiple Securities

We now consider a model in which the firm can simultaneously offer multiple securities to the market. We presume that each security is admissible. We also assume that the markets for the different securities are independent and that the distribution of investors' beliefs is fixed and identical across markets.[13]

Definition. A finite set $\{X_i\}_{i=1}^c$ is said to be a *partition* of an admissible security X if for all $i \in \{1, \ldots, c\}$, X_i is an admissible security and $\sum_{i=1}^c X_i(p) = X(p)$ for all $p \geq 0$. Each element of a partition is referred to as a *component*.

We let $\{X_i\}_{i=1}^c$ be a partition of an admissible security X, and we denote the value realized by the firm from issuing the $\{X_i\}_{i=1}^c$ in c separate markets by $V^F(\{X_i\}_{i=1}^c, P_t, N)$. If investors have rational beliefs

$$\begin{aligned} &V_{RB}^F(\{X_i\}_{i=1}^c, P_t, N) \\ &\quad = \lambda(\mu - \mu_x) + (1 - \beta(\gamma, N)) \sum_{i=1}^c \min\{e_1(X_i), e_2(X_i)\} \\ &\qquad + \beta(\gamma, N) \sum_{i=1}^c \max\{e_1(X_i), e_2(X_i)\}, \end{aligned} \tag{11}$$

[12] If λ is sufficiently large (or N is sufficiently small), it can be shown that the condition $\lim_{p\to\infty} \frac{f_2(p)}{f_1(p)} = \infty$ (satisfied, e.g., by two normal densities with differing means and identical variances) guarantees that debt, and not equity, is optimal.

[13] It might also be specified that the securities are offered simultaneously in multiple markets to a single group of investors. The equilibrium outcomes of such a game, however, are identical to those of the game described in the text. As shown in Garmaise (1999), the firm's utility is unaffected by this change in the specification and the results below continue to apply.

and if investors have rational expectations

$$V_{RE}^F(\{X_i\}_{i=1}^c, P_t, N) = \lambda(\mu - \mu_x) + (1 - \kappa(\gamma, N)) \sum_{i=1}^{c} \min\{e_1(X_i), e_2(X_i)\} + \kappa(\gamma, N) \sum_{i=1}^{c} \max\{e_1(X_i), e_2(X_i)\}. \quad (12)$$

We note that the residual left to the firm is independent of the number of components into which the security is divided.

Result 11 establishes that selling multiple securities is optimal when investors have rational beliefs and $N \geq 3$, but is not optimal when investors have rational expectations.

Result 11. If investors have rational expectations, or if $N = 2$ and investors have rational beliefs, then selling an admissible claim X yields a weakly higher valuation to the firm than selling any partition of X. If $N \geq 3$ and investors have rational beliefs, then for any partition $\{X_i\}_{i=1}^c$ of X, there exists a partition $\{X_j\}_{j=1}^2$ of X into two components that yields the firm a weakly higher valuation.

A proof is given in the appendix.

If investors have rational expectations, then the firm will attempt to minimize differences of opinion. Selling multiple claims has the opposite effect. In essence, the firm receives a price closer to the minimum investor valuation than to the maximum investor valuation. When the firm partitions its claim, it receives a price close to the minimal valuation on each component, which is clearly worse than receiving a price close to the minimal valuation of the whole claim. When investors have rational beliefs and $N \geq 3$, the firm receives a price closer to the maximal valuation of each component, and partition is optimal.[14] A partition raises the firm's valuation in the rational beliefs ($N \geq 3$) case only if one component is valued more by one investor type, while the other is valued more by the second investor type. If both components are valued more by the same investor type, the firm does not benefit from partitioning the claim; in this case, the maximum of the whole is equal to the sum of the maxima of the components. Since there are only two investor belief types, no partition into more than two components will be strictly better than the optimal two-component partition. The firm could just as well, for each investor type, combine all the securities valued more by that type into one claim and then issue these two claims. In general, under rational beliefs (for $N \geq 3$) it is optimal for the firm to issue as many claims as there are investor beliefs.

Definition. A partition $\{X_i\}_{i=1}^c$ of a monotone security X is said to be a monotone partition if each X_i is monotone.

[14] Other arguments for issuing multiple securities may be found in Allen and Gale (1988), Axelson (1999), Boot and Thakor (1993), DeMarzo (1999), and Dewatripont and Tirole (1994).

Result 12. Assume that $N \geq 3$ and that investors have rational beliefs. If one density dominates the other in the sense of first-order stochastic dominance, then selling a whole claim yields the same valuation to the firm as selling any monotone partition of that claim.

A proof may be found in the appendix.

If one belief dominates the other in the sense of first-order stochastic dominance, then all securities are more valued by optimistic investors. As argued above, in this case partitioning a claim does not raise the revenue expected to be realized by the firm.

Definition. The density function f_1 dominates the density function f_2 in the sense of the *single-intersection property (SIP)* if there exists $a > 0$ such that $F_1(a) \in (0, 1)$, $F_1(p) \leq F_2(p)$ for all $p \leq a$ and $F_2(p) \leq F_1(p)$ for all $p \geq a$.

It can be shown that if $e_1(P) = e_2(P)$ then SIP implies second-order stochastic dominance [Garmaise (1999)]. Several examples of common densities that satisfy the SIP are given below. We define a *junior equity claim* J_a to be a security X such that for some $a \geq 0$, $X(p) = 0$ for all $p \leq a$ and $X(p) = p - a$ for all $p \geq a$.

Result 13. Assume that $N \geq 3$ and that investors have rational beliefs. Suppose that f_1 dominates the density function f_2 in the sense of the SIP. Consider a security X that is monotone and has a monotone residual. A partition of the form $\{B_a, J_a\}$ for some $a > 0$ dominates any monotone partition of X.

A proof is given in the appendix. It may be verified that one density dominates the other in the sense of the SIP if and only if one belief dominates the other in the sense of the SIP. The proof shows that investors with belief A (if present) purchase the debt claim, while investors with belief B (if present) purchase the junior equity claim. Result 13 has two implications. The first implication is that if the firm is considering issuing multiple securities and is restricted to partitioning a claim that is monotone and that has a monotone residual, then it is optimal for the firm to issue a full-equity claim. Given that most observed claims are monotone and have monotone residuals, this is a strong implication. When investor beliefs satisfy the SIP, the firm does best to issue the largest security possible. The second implication is that the full-equity claim should be partitioned into a senior debt claim and a junior equity claim.

Result 13 provides an explanation for the very common practice of dividing asset-backed securities into multiple prioritized tranches.[15] In this security design, the holders of the highest priority tranche receive all cash flows until

[15] See, for example, the July 2000 issue of *Asset Finance International* (vol. 274), which includes many descriptions of securitizations organized into tranches.

their claim is paid in full. The highest priority tranche is thus identical to a debt claim. If there are two tranches, the lower priority tranche receives all residual cash flows and is thus identical to a junior equity claim. Result 11 shows that if investors have rational expectations, then structuring a security into tranches can only harm the issuing firm. The practice of tranching is therefore difficult to explain in a rational expectations framework. Result 13 demonstrates that if investors have rational beliefs, N is greater than 3 and beliefs are ordered by the SIP, then the prioritized tranche structure is the optimal security design.

The following are examples of densities on $\Re^+$ for which f_1 dominates f_2 in the sense of the SIP.

Example 1. Density f_1 is uniform (b, c), f_2 is uniform (a, d) for $0 < a < b < c < d$.
Example 2. Density f_1 is Weibull (a_1, b_1), f_2 is Weibull (a_2, b_2) for $b_1 > b_2$.
Example 3. Density f_1 is log normal (μ_1, σ_1), f_2 is log normal (μ_2, σ_2) for $\sigma_1 < \sigma_2$.
Example 4. Density f_1 is gamma (α_1, β_1), f_2 is gamma (α_2, β_2) for $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$.

A proof that these examples satisfy the SIP is given in Garmaise (1999). The four examples provide some insight into the types of beliefs that cause a prioritized two tranche partition to be optimal. There are two seemingly obvious intuitions about which investors will purchase the debt tranche and which will invest in the junior equity tranche. The first is that optimistic investors will purchase the junior equity claim. The second is that investors with high-variance beliefs will purchase the option-like junior equity tranche. The examples show that neither intuition holds as a general rule. In Example 1, for a sufficiently small and b and c sufficiently large, it will be that $e_1(P) > e_2(P)$. Despite this fact, however, the first (high-priority) tranche is designed to be sold to investors who receive signal A (and is valued more by these investors). That is, the first (debt) tranche is sold to optimists, not to pessimists. It is the case, however, that investors who receive the signal A assign a lower variance to P than investors who receive the signal B. (Straightforward calculations show that the variance of belief A is lower than the variance of belief B if and only if the variance of f_1 is less than the variance of f_2.) That is, in Example 1 investors with low-variance beliefs are assigned the debt tranche, even if they have high assessments of the mean of P. Similar examples can be constructed using the log normal densities of Example 3. Example 4, however, provides instances in which investors who are both more optimistic and who have a higher assessment of the variance of P are assigned the debt tranche. The mean and variance of a gamma (α, β) random variable are $\alpha\beta$ and $\alpha\beta^2$, respectively. If α_1 is much greater than α_2, but β_1 is not much smaller than β_2 (e.g., $\alpha_1 = 5, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2$), then investors receiving the signal A are allocated the debt tranche, despite having higher assessments of both the mean and variance of P. The critical condition is that belief B is more optimistic about the highest states. This is more important than the relative means or variances of P under the two beliefs.

5. Empirical Implications

A final important topic is the empirical implications of the model presented in this article. We begin by considering two predictions for which evidence is available. First, Results 1 and 9 show that if investors have rational beliefs then greater investor belief variability will encourage firms in many-investor markets to issue riskier securities like equity. Results 2, 5, and 8 show that if investors have rational expectations, then greater investor belief variability will encourage firms to issue safer securities like debt. Best and Best (1995) provide some supportive evidence for the contention that investors have rational beliefs. They show that analyst earnings forecast errors for public firms that issue equity are much higher than for public firms that issue debt. Analyst forecast errors are highly correlated with the dispersion of analyst forecasts and can be regarded as a proxy for the diversity of investor beliefs. This suggests that investors have rational beliefs and not rational expectations.[16] As Result 10 makes clear, however, the prediction that the firm will issue debt under rational expectations holds only for a smaller number of investors. This prediction of the article may therefore be better tested by considering the private placement market.

Second, Result 11 implies that firms selling an asset-backed security to a public market will partition the claim if investors have rational beliefs and will issue a single whole claim if investors have rational expectations. As discussed earlier, the use of prioritized tranches is very common in this market.

The model also makes several untested predictions. First, if we assume that there is some cost to approaching the public market, Result 1 implies that if investors have rational beliefs then firms about which there is great diversity of opinion will be more willing to bear this cost. As a result, we would expect firms (or industries) with greater dispersion of analyst forecasts to make more use of public markets for finance through the issuance of debt or equity. Result 2 implies that if investors have rational expectations then public issuance is less attractive when investor beliefs are diverse. Second, Result 11 suggests that if investors have rational beliefs, then claim splitting should occur more often in many-investor markets. Consequently the model

[16] It is also difficult to explain this result from within an asymmetric information framework such as Myers (1984); in that model the firms with the greatest information asymmetries should issue debt, not equity.

implies that conglomerates divesting a division in a given industry will tend to sell the division whole if there are two potential buyers in the industry and separate the division into multiple entities in a many-buyer industry.

Third, Result 11 also indicates that industries with many firms should experience relatively more piecemeal liquidations, while outright acquisitions should be relatively more common in highly concentrated industries.

Fourth, the model in this article applies quite directly to the privatization of state-owned enterprises. It follows from Results 8 and 9 that firm (government, in this case) retention of the riskiest part of the cash flows will occur more often in few-investor games, if investors have rational beliefs. Consequently the model predicts that government retention will be greater in private sales of assets than in public offerings of shares.

Fifth, the model makes several predictions about the design of asset-backed securities. Result 3 indicates that asset-backed claims should pay out all the underlying cash flows in at least some states of the world, irrespective of investor beliefs. Result 11 implies that if investors have rational beliefs, then claim splitting will occur more often in many-investor markets. Results 12 and 13 have the following implication if investors have rational beliefs and the firm is selling to a public market. Generally speaking, if investors disagree primarily about the mean of an asset's payoff (i.e., FOSD dominance of one belief by the other), then the firm should issue a single claim. If investors disagree primarily about the spread of the asset's payoff (i.e., SIP dominance of one belief by the other), then the firm should issue two prioritized tranches.

6. Conclusion

This article studies the impact of investors' beliefs on a firm's choice of a security design. We show that optimal designs are similar under rational beliefs and rational expectations when there are two investors. When there are more than two investors, however, the optimal securities are rather different. Under rational beliefs a firm may issue securities even if it is not cash constrained, while this is not true under rational expectations. Safe securities such as debt are optimal under rational expectations, while riskier securities are optimal under rational beliefs. Claim splitting benefits firms under rational beliefs, but is never optimal under rational expectations. The issuance of multiple securities backed by the same assets is quite common and is difficult to explain in a rational expectations framework. Moreover, for the rational beliefs case we present conditions under which it is optimal for the firm to sell prioritized tranches, which is a structure that is often observed in practice.

The results in this article also contrast rather starkly with those generated by models in which the firm possesses an informational advantage over investors. The emergence of full equity as an optimal claim is very unusual in that literature. More generally, this article rationalizes the widespread presence of highly risky securities in the market, a phenomenon not well explained by either the rational expectations case in this article or the literature on firm asymmetric information.

Our model makes use of a strategic variation on rational beliefs theory. We have proposed a new motivation for security design, and we have suggested that investor rational beliefs can explain part of the diversity of extant securities. We argue that firms can benefit from analyzing the state of market beliefs before issuing securities. Investment bankers may undertake this activity on behalf of their client firms.

The model presented here has the interesting implication that uncertainty surrounding the firm's assets is costly to issuers in a two-investor market, but advantageous for issuers facing many investors. Firms in many-investor markets may thus benefit from some mystique about their prospects.

Appendix

In this section we provide proofs for several of the results given in the main body of the article.

Proof of Result 1.

Case 1. $e_2(X) > e_1(X)$: We propose the following symmetric equilibrium. Investors receiving the A signal bid their valuations $b_A = e_1(X)\gamma + e_2(X)(1-\gamma)$. Investors receiving the B signal bid b_B, which is a random variable with cumulative distribution function (CDF) F, where $F\colon [b_A, \hat{s}] \to [0, 1]$ is implicitly defined by

$$\begin{aligned}(e_1(X)(1-\gamma)+e_2(X)\gamma-s)\sum_{j=0}^{N-1}\left(\binom{N-1}{j}(1/2)^{N-1}F(s)^{N-1-j}\right)\\ = \frac{(2\gamma-1)(e_2(X)-e_1(X))}{2^{N-1}},\end{aligned} \tag{13}$$

and $\hat{s}$ is such that $F(\hat{s}) = 1$. The unique solution to Equation (13) is

$$F(s) = -1 + \left(\frac{e_1(X)(1-\gamma)+e_2(X)\gamma-s}{(2\gamma-1)(e_2(X)-e_1(X))}\right)^{\frac{1}{1-N}}. \tag{14}$$

F is strictly increasing, and $F(e_1(X)(1-\gamma)+e_2(X)\gamma-2^{1-N}((2\gamma-1)(e_2(X)-e_1(X)))) = 1$. A proof that this is the unique symmetric equilibrium is given in Garmaise (1999). The expected seller revenue is equal to the difference between the total social surplus from the sale and the sum of the expected revenues of the investors. Expected seller revenue is therefore given by

$$\begin{aligned}\left(\frac{1}{2}\right)^N(e_1(X)\gamma+e_2(X)(1-\gamma))+\left(1-\left(\frac{1}{2}\right)^N\right)(e_1(X)(1-\gamma)+e_2(X)\gamma)\\ -N\left(\left(\frac{1}{2}\right)0+\left(\frac{1}{2}\right)\frac{(2\gamma-1)(e_2(X)-e_1(X))}{2^{N-1}}\right).\end{aligned} \tag{15}$$

Simplifying this equation gives the result.

Case 2. $e_2(X) = e_1(X) = e(X)$: In this case, in the only symmetric equilibrium, all investors bid $e(X)$. Seller revenue is equal to $e(X)$.

Proof of Result 2.

Case 1. $e_2(X) > e_1(X)$: We propose the following symmetric equilibrium. Investors receiving the A signal bid $b_A = \frac{e_1(X)\gamma^N + e_2(X)(1-\gamma)^N}{\gamma^N + (1-\gamma)^N}$. We define $e_1 = e_1(X)$ and $e_2 = e_2(X)$. Investors receiving the B signal bid b_B, which is a random variable with CDF F, where F: $[b_A, \hat{s}] \to [0, 1]$ is implicitly defined by

$$\begin{aligned}(1-\gamma)(e_1 - s)\sum_{j=0}^{N-1}\binom{N-1}{j}\gamma^j(1-\gamma)^{N-1-j}F^{N-1-j}\\ +(\gamma)(e_2 - s)\sum_{j=0}^{N-1}\binom{N-1}{j}(1-\gamma)^j(\gamma)^{N-1-j}F^{N-1-j}\\ = \frac{(e_2-e_1)((1-\gamma)^{N-1}\gamma^{N-1}(2\gamma-1))}{\gamma^N+(1-\gamma)^N}\end{aligned} \tag{16}$$

and $\hat{s}$ is such that $F(\hat{s}) = 1$. We solve Equation (16) for $s(F)$ and find that the function $s(F)$ is continuous and strictly increasing on [0, 1]. The inverse function $F(s)$ is thus well defined, continuous, and strictly increasing. A proof that this is the unique symmetric equilibrium is given in Garmaise (1999). Expected seller revenue is given by

$$\frac{e_1+e_2}{2} - N\left(\left(\frac{1}{2}\right)0 + \left(\frac{1}{2}\right)\frac{(e_2-e_1)((1-\gamma)^{N-1}\gamma^{N-1}(2\gamma-1))}{\gamma^N+(1-\gamma)^N}\right). \tag{17}$$

Simplifying this equation gives the result.

Case 2. $e_2(X) = e_1(X) = e(X)$: In this case, in the only symmetric equilibrium, all investors bid $e(X)$. Seller revenue is equal to $e(X)$.

Proof of Result 5. We have $\mu = e_1(P) = e_2(P)$. The firm's valuation from issuing full equity is μ. Let any admissible security X be given. We assume, without loss of generality, that $e_1(X) \le e_2(X)$. Under the premise of the result, the firm's valuation from issuing X can be written as $e_1(X)(d - \frac{\lambda}{2}) + e_2(X)((1-d) - \frac{\lambda}{2}) + \lambda\mu$, for some $d > \frac{1}{2}$. We have

$$(e_1(X) - e_2(X))\left(d - \frac{\lambda}{2}\right) + e_2(X)(1-\lambda) + \lambda\mu \le 0 + e_2(P)(1-\lambda) + \lambda\mu = \mu. \tag{18}$$

Proof of Result 6. We note that f_1 and f_2 are measurable since they are densities. This implies that X is measurable. The design X is feasible for all a, b, so the designs described in the statement of the result are certainly admissible.

Throughout this proof we will make use of three quickly verified properties of $X_{a,b}$ securities. Together these properties are roughly equivalent to continuity in the dictionary ordering.

(i) For $i = 1, 2$ $e_i(X_{a+\frac{1}{n},1}) \downarrow e_i(X_{a,1})$ as $n \to \infty$.
(ii) For $i = 1, 2$ $e_i(X_{a-\frac{1}{n},1}) \uparrow e_i(X_{a,0})$ as $n \to \infty$.
(iii) For $i = 1, 2$ $e_i(X_{n,1}) \uparrow e_i(X_{\infty,0})$ as $n \to \infty$.

Claim 1. There is a design $X_{a,b}$ such that $e_1(X_{a,b}) = e_2(X_{a,b})$, for some $a \ge 1$.

Proof of claim. We define $g(a) = e_1(X_{a,1}) - e_2(X_{a,1})$. We note that $g(a) \ge 0$ for all $a \in [0, 1]$ and that g is monotonically decreasing in a for $a \ge 1$. We set $g(\infty) = e_1(P) - e_2(P) < 0$.

If there is an $a \in [1, \infty]$ such that $g(a) = 0$, then $e_1(X_{a,1}) = e_2(X_{a,1})$ and the claim is proved. Suppose no such a exists. First we define $J = \{a\colon g(a) \ge 0\}$ and then set $a^* = \sup_a(J)$.

We note that it must be that $a^* \ge 1$. Given our assumptions, there are two cases:

(i) $g(a^*) > 0$.
(ii) $g(a^*) < 0$.

Using property (i) we can show that $g(a^*) > 0$ cannot occur, so $g(a^*) < 0$. First we note that $a^* > 1$. It must be that $g(a) > 0$ for all $a < a^*$ since a^* is a least upper bound. Now we define $h(a) = e_1(X_{a,0}) - e_2(X_{a,0})$. Using properties (ii) and (iii) we can show that $h(a^*) \geq 0 > g(a^*)$. Define

$$q(b) = \int_{\{p:\frac{f_2(p)}{f_1(p)}<a^*\}} (p)(f_1(p) - f_2(p))\,dp + b\int_{\{p:\frac{f_2(p)}{f_1(p)}=a^*\}} (p)(f_1(p) - f_2(p))\,dp. \tag{19}$$

We note that $q(0) = h(a^*) \geq 0$ and that $q(1) = g(a^*) < 0$ so $q(b^*) = 0$ for some $b^* \in [0, 1]$. We then have that $e_1(X_{a^*,b^*}) = e_2(X_{a^*,b^*})$ for $a^* > 1$ and the claim is proved. As a final remark, we note that since $g(a^*) \leq 0$ and $a^* \geq 1$, it must be the case that $e_2(X_{a,b}) \geq e_1(X_{a,b})$ for any pair (a, b) such that either $a > a^*$ or $a = a^*, b \geq b^*$.

We know that there exists an $a^* \geq 1$ and a $b^* \in [0, 1]$ such that $e_1(X_{a^*,b^*}) = e_2(X_{a^*,b^*}) =: \mu_{x^*}$. Let any design Y be given. We will show that Y is dominated by a security $X_{a,b}$.

First suppose that $0 \leq \mu_y \leq \mu_{x^*}$. It is clear from Equation (8) that X_{a^*,b^*} dominates Y. We now suppose that $\mu_{x^*} < \mu_y \leq \mu$.

Claim 2. There exists a security $X_{a,b}$ such that its stationary mean μ_x is equal to μ_y.

Proof of claim. The stationary mean is increasing in both a and b, so if we find the $X_{a,b}$ specified in the statement of the claim, it must be that either $a > a^*$ or $a = a^*, b > b^*$. Define $u(a) = \frac{1}{2}(e_1(X_{a,1}) + e_2(X_{a,1}))$. We note that $u(a)$ is increasing. We also see that $u(a^* - \epsilon) \leq \mu_{x^*} \leq u(a^*)$ for all $\epsilon > 0$. We define $J' = \{a\colon u(a) \leq \mu_y\}$ and set $a' = \sup_a(J')$. It must be that $a' \geq a^*$. If $u(a) = \mu_y$ for any a then we are done. We suppose otherwise. Using property (i) we can show that it must be that $u(a') > \mu_y$. We note that for all $\hat{a} < a'$, $u(\hat{a}) < \mu_y$. We define $v(a) = \frac{1}{2}(e_1(X_{a,0}) + e_2(X_{a,0}))$. Using property (ii) we can show that $v(a') = u(a' - \frac{1}{n}) \uparrow \leq \mu_y$.

The stationary mean of $X_{a',0}$ is below μ_y and the stationary mean of $X_{a',1}$ is above μ_y. The stationary mean is continuous in b for a fixed a, so there is a $b' \in [0, 1]$ such that the stationary mean of $X_{a',b'}$ is precisely equal to μ_y.

Claim 3. The security $X_{a',b'}$ with stationary mean equal to μ_y dominates the security Y.

Proof of claim. To ease notation we will denote $X_{a',b'}$ by X. Recall that $a' \geq 1$. We define $A = \{p\colon X(p) < Y(p)\}$, $B = \{p\colon X(p) = Y(p)\}$, and $C = \{p\colon X(p) > Y(p)\}$. We have that $A \subseteq \{p\colon \frac{f_2(p)}{f_1(p)} \geq a'\}$ and $C \subseteq \{p : \frac{f_2(p)}{f_1(p)} \leq a'\}$.

We define $c = \int_B Y(p)f_2(p)\,dp + \int_A X(p)f_2(p)\,dp + \int_C Y(p)f_2(p)\,dp$, $d = \int_B Y(p)f_1(p)\,dp + \int_A X(p)f_1(p)\,dp + \int_C Y(p)f_1(p)\,dp$, $e = \int_C(X(p) - Y(p))f_2(p)\,dp$, $f = \int_C(X(p) - Y(p))f_1(p)\,dp$, $g = \int_A(Y(p) - X(p))f_2(p)\,dp$, and $h = \int_A(Y(p) - X(p))f_1(p)\,dp$. With this formulation we can write $\frac{e_2(X)}{e_1(X)} = \frac{e+c}{f+d}$ and $\frac{e_2(Y)}{e_1(Y)} = \frac{g+c}{h+d}$.

We first presume that $a' < \infty$. It is the case that

$$0 \leq a' \int_A (Y(p) - X(p))f_1(p)\,dp \leq \int_A (Y(p) - X(p))f_2(p)\,dp. \tag{20}$$

If the right-most term is zero, then X and Y have the same means under both measures and hence yield the same value to the firm. Otherwise

$$\frac{\int_A(Y(p) - X(p))f_1(p)\,dp}{\int_A(Y(p) - X(p))f_2(p)\,dp} \leq \frac{1}{a'}. \tag{21}$$

A symmetric argument gives

$$\frac{\int_C(X(p) - Y(p))f_1(p)\,dp}{\int_C(X(p) - Y(p))f_2(p)\,dp} \geq \frac{1}{a'}. \tag{22}$$

We note that all these integrals are nonnegative. From the above we can then infer that $\frac{g}{h} \geq \frac{e}{f}$.

Since $\mu_x = \mu_y$ we know that $g + h = e + f$. If $h > f$ then $e > g$, which would contradict the fact that $\frac{g}{h} \geq \frac{e}{f}$. So it must be $h \leq f$ and $e \leq g$. This means that $e_2(X) \leq e_2(Y)$ and $e_1(X) \geq e_1(Y)$. We recall from the remark following the proof of the first claim that $e_2(X) \geq e_1(X)$. So by Equation (8), X dominates Y. If $a' = \infty$ then $f_1(p) = 0$ for all $p \in A$, so $h = 0$ and thus $e_1(X) \geq e_1(Y)$. The same argument now applies. Thus if there is an optimal security, it will be dominated by a security of the form $X_{a,b}$.

Proof of Result 8.

Claim. Let X be a monotone security. Then $e_2(X) \geq e_1(X)$.

Proof of claim. We first note that if f_2 dominates f_1 in the sense of the MLRP then it must also be the case that f_2 first-order stochastically dominates f_1. The claim then follows from the usual ranking property of the expectations of a monotone function of stochastically ordered distributions.

We may now restrict attention to monotone securities X such that $e_2(X) \geq e_1(X) \geq 0$ and $\mu_x \in [0, \mu]$. Consider a bond B_a. We define $g(a) = \frac{1}{2}(e_1(B_a) + e_2(B_a))$, and we set $g(\infty) = \mu$. Note that g is increasing. The dominated convergence theorem shows that g is continuous and that $g(n) \uparrow g(\infty)$.

Let any admissible monotone security X be given. If $\mu_x = 0$, then $e_1(X) = 0 = e_2(X)$ and X is equivalent to B_0. If $\mu_x = \mu$, it must be that $e_j(X) = e_j(P)$ for $j \in \{1, 2\}$, since by feasibility $e_j(X) \leq e_j(P)$. Then issuing X gives the firm the same utility as issuing a full-equity claim. Now suppose $\mu_x \in (0, \mu)$. By the continuity of g there exists an $a \in [0, n]$ such that $g(a) = \mu_x$. We denote the a bond by Y.

We set $\zeta = \{p\colon X(p) > Y(p)\}$ and define $H = \inf(\zeta)$. First we note that if $\zeta = \emptyset$, then it must be that $e_j(Y) = e_j(X)$ for $j \in \{1, 2\}$, since $\mu_x = \mu_y$. In this case issuing X yields the same utility for the firm as issuing Y. Let us suppose that $\zeta \neq \emptyset$. We note that a is a lower bound for ζ since $Y(p) = p \geq X(p)$ for all $p \leq a$. This means that $H \geq a$ so $Y(H) = a$.

We define $A = \{p\colon Y(p) \geq X(p)\}$. Straightforward arguments show that $A \subseteq [0, H]$ and $\zeta \subseteq [H, +\infty)$. For convenience, we denote $\frac{f_2(H)}{f_1(H)}$ by α. By the MLRP, for all $x \in A$, $\frac{f_2(x)}{f_1(x)} \leq \alpha$, and for all $x \in \zeta$, $\alpha \leq \frac{f_2(x)}{f_1(x)}$. This implies that

$$\alpha \int_A (Y(p) - X(p)) f_1(p)\, dp \geq \int_A (Y(p) - X(p)) f_2(p)\, dp. \tag{23}$$

If $\int_A (Y(p) - X(p)) f_1(p)\, dp = 0$, then $e_j(X) \geq e_j(Y)$ for $j \in \{1, 2\}$. However, $\mu_x = \mu_y$ so these two inequalities must be equalities and Y and X would therefore yield the firm identical utilities. Otherwise

$$\alpha \geq \frac{\int_A (Y(p) - X(p)) f_2(p)\, dp}{\int_A (Y(p) - X(p)) f_1(p)\, dp}. \tag{24}$$

By similar reasoning,

$$\alpha \int_\zeta (X(p) - Y(p)) f_1(p)\, dp \leq \int_\zeta (X(p) - Y(p)) f_2(p)\, dp \tag{25}$$

and unless Y and X are equivalent,

$$\alpha \leq \frac{\int_\zeta (X(p) - Y(p)) f_2(p)\, dp}{\int_\zeta (X(p) - Y(p)) f_1(p)\, dp}. \tag{26}$$

An argument similar to the one given in the proof of Result 6 shows that $e_2(X) \geq e_2(Y) \geq e_1(Y) \geq e_1(X) \geq 0$. By Equation (8), it must be that Y dominates X. A proof of the existence of an optimal debt or full equity contract is given in Garmaise (1999).

Proof of Result 9. Let an admissible security X that is monotone and has a monotone residual be given. We define Y to be full equity. Equation (5) shows that for some $\beta(\gamma, N) \geq \frac{1}{2} \geq \frac{\lambda}{2}$,

$$V^F_{RB}(Y, P_t, N) - V^F_{RB}(X, P_t, N) = (e_1(Y) - e_1(X))\left(1 - \beta(\gamma, N) - \frac{\lambda}{2}\right) + (e_2(Y) - e_2(X))\left(\beta(\gamma, N) - \frac{\lambda}{2}\right). \tag{27}$$

The security $Y - X$ is monotone by assumption, so (FOSD) implies that $e_2(Y - X) \geq e_1(Y - X)$. We thus have

$$V^F_{RB}(Y, P_t, N) - V^F_{RB}(X, P_t, N) \geq e_1(Y - X)(1 - \lambda) \geq 0. \tag{28}$$

Proof of Result 10. Let an admissible security X be given. We define Y to be full equity. Straightforward calculations show that $\kappa(\gamma, N) \leq \frac{1}{2}$ for all N and $\lim_{N\to\infty} \kappa(\gamma, N) = \frac{1}{2}$. There exists N' such that for all $N \geq N'$ $\kappa(\gamma, N) > \frac{\lambda}{2}$. For such N, Equation (7) shows that

$$V^F_{RE}(Y, P_t, N) - V^F_{RE}(X, P_t, N) = (e_1(Y) - e_1(X))\left(1 - \kappa(\gamma, N) - \frac{\lambda}{2}\right) + (e_2(Y) - e_2(X))\left(\kappa(\gamma, N) - \frac{\lambda}{2}\right) \geq 0. \tag{29}$$

The final inequality follows from the fact that $e_i(Y - X) \geq 0$ for $i \in \{1, 2\}$.

Proof of Result 11. Let an admissible claim X be given, and let $\{X_i\}_{i=1}^c$ be a partition of X. It is clear that $\sum_{i=1}^c \min\{e_1(X_i),\ e_2(X_i)\} \leq \min\{\sum_{i=1}^c e_1(X_i),\ \sum_{i=1}^c e_2(X_i)\}$. If Equation (12) holds, we have

$$V^F_{RE}(\{X_i\}_{i=1}^c, P_t, N) - V^F_{RE}(X, P_t, N) \tag{30}$$

$$= (1 - 2k(\gamma, N))\left(\sum_{i=1}^c \min\{e_1(X_i), e_2(X_i)\} - \min\left\{\sum_{i=1}^c e_1(X_i), \sum_{i=1}^c e_2(X_i)\right\}\right) \leq 0. \tag{31}$$

If Equation (11) holds, an analogous argument shows that $V^F_{RB}(\{X_i\}_{i=1}^c,\ P_t, N) \geq V^F_{RB}(X, P_t, N)$ for $N \geq 3$ and $V^F_{RB}(\{X_i\}_{i=1}^c, P_t, 2) \leq V^F_{RB}(X, P_t, 2)$. Lastly, we show that there is a partition of X into two components that yields the firm a valuation as least as high as that arising from the partition $\{X_i\}_{i=1}^c$ for $c \geq 3$. We relabel the X_i so that there exists k, $0 \leq k \leq c$, such that $e_2(X_j) \geq e_1(X_j)$ for all $j \leq k$ and $e_2(X_j) < e_1(X_j)$ for all $j > k$ (this may be done without loss of generality). We note that

$$\begin{aligned}\sum_{i=1}^c \max\{e_1(X_i), e_2(X_i)\} &= \sum_{i=1}^k e_2(X_i) + \sum_{i=k+1}^c e_1(X_i) \\ &= \max\left\{\sum_{i=1}^k e_1(X_i), \sum_{i=1}^k e_2(X_i)\right\} + \max\left\{\sum_{i=k+1}^c e_1(X_i), \sum_{i=k+1}^c e_2(X_i)\right\}\end{aligned} \tag{32}$$

and an analogous result holds for the minima. This proves that $V^F_{RB}(\{\sum_{i=1}^k X_i,\ \sum_{i=k+1}^c X_i\},$ $P_t, N) = V^F_{RB}(\{X_i\}_{i=1}^c, P_t, N)$.

Proof of Result 12. Let an admissible claim X be given, and let $\{X_i\}_{i=1}^c$ be a monotone partition of X. Assume, without loss of generality, that f_2 dominates f_1 in the sense of first-order stochastic dominance. We then have $e_2(X_i) \geq e_1(X_i)$ for all $i \in \{1, \ldots, c\}$. This implies that

$$\sum_{i=1}^{c} \min\{e_1(X_i), e_2(X_i)\} = \sum_{i=1}^{c} e_1(X_i) = \min\left\{\sum_{i=1}^{c} e_1(X_i), \sum_{i=1}^{c} e_2(X_i)\right\} \tag{33}$$

and similarly for the maxima. Equation (11) implies that $V_{RB}^F(\{X_i\}_{i=1}^c, P_t, N) = V_{RB}^F(X, P_t, N)$.

Proof of Result 13. By the premise, there exists $b > 0$ such that $F_1(p) \leq F_2(p)$ for all $p \leq b$ and $F_2(p) \leq F_1(p)$ for all $p \geq b$. Set $a = b$. We define two densities $g_1(x) = \frac{f_1(x)}{F_1(a)}$ and $g_2(x) = \frac{f_2(x)}{F_1(a)}$ on $[0, a]$ and note that g_1 dominates g_2 in the sense of FOSD. We also define $h_1(x) = \frac{f_1(x)}{1-F_1(a)}$ and $h_2(x) = \frac{f_2(x)}{1-F_1(a)}$ on $[a, \infty)$ and note that h_2 dominates h_1 in the sense of FOSD.

The proof of Result 11 shows that we can limit attention to two-element partitions. Let a security X_1 in the partition $\{X_1, X_2\}$ of X be given. We have $p - X_1(p) = (p - X(p)) + (X(p) - X_1(p)) = (p - X(p)) + (X_2(p))$. Both terms are monotone increasing in p by assumption, so X_1 is monotone and has a monotone residual. A similar argument applies to X_2. Let $Y \in \{X, X_1, X_2\}$ be given. We have that $B_a - Y$ is monotone on $[0, a]$ and $Y - B_a$ is monotone on $[a, \infty)$. This implies that

$$\begin{aligned}\int_0^a (B_a(p) - Y(p)) f_1(p)\, dp &= F_1(a) \int_0^a (B_a(p) - Y(p)) g_1(p)\, dp \\ &\geq F_1(a) \int_0^a (B_a(p) - Y(p)) g_2(p)\, dp \\ &= \int_0^a (B_a(p) - Y(p)) f_2(p)\, dp \end{aligned} \tag{34}$$

and

$$\begin{aligned}\int_a^\infty (Y - B_a(p)) f_2(p)\, dp &= (1 - F_1(a)) \int_a^\infty (Y - B_a(p)) h_2(p)\, dp \\ &\geq (1 - F_1(a)) \int_a^\infty (Y(p) - B_a(p)) h_1(p)\, dp \\ &= \int_a^\infty (Y(p) - B_a(p)) f_1(p)\, dp. \end{aligned} \tag{35}$$

These two inequalities imply that $e_1(B_a) - e_1(Y) \geq e_2(B_a) - e_2(Y)$. Analogous arguments show that $e_2(J_a) - e_2(Y) \geq e_1(J_a) - e_1(Y)$. It is clear that $e_1(B_a) \geq e_2(B_a)$ and $e_2(J_a) \geq e_1(J_a)$. If $e_2(X_1) \geq e_1(X_1)$ and $e_2(X_2) \geq e_1(X_2)$,

$$\begin{aligned} &V^F(\{B_a, J_a\}, P_t, N) - V^F(\{X_1, X_2\}, P_t, N) \\ &\quad = (\mu - \mu_x)(1 - \lambda) + \left(\beta(\gamma, N) - \frac{1}{2}\right)(e_2(J_a) - e_1(J_a) + e_1(B_a) \\ &\qquad - e_2(B_a) - (e_2(X) - e_1(X))) \geq 0. \end{aligned} \tag{36}$$

The inequality follows from the fact that $e_2(J_a) - e_1(J_a) \geq e_2(X) - e_1(X)$. A similar argument applies if $e_2(X_1) \leq e_1(X_1)$ and $e_2(X_2) \leq e_1(X_2)$.If (without loss of generality) $e_1(X_1) \geq e_2(X_1)$

and $e_2(X_2) \geq e_1(X_2)$, then

$$\begin{aligned}
&V^F(\{B_a, J_a\}, P_t, N) - V^F(\{X_1, X_2\}, P_t, N)\\
&\quad = (\mu - \mu_x)(1-\lambda) + \left(\beta(\gamma, N) - \frac{1}{2}\right)(e_2(J_a) - e_1(J_a) - (e_2(X_2)\\
&\qquad - e_1(X_1)) + e_1(B_a) - e_2(B_a) - (e_1(X_1) - e_1(X_1))) \geq 0. \qquad (37)
\end{aligned}$$

The inequality follows from the work above.

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