Market Making, Prices, and Quantity Limits

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This article develops a model of spread and depth setting under asymmetric information where the equilibrium depth is proportionally more sensitive than the spread to changes in the degree of information asymmetry. The analysis uses a one-period model in which a risk-neutral, monopolistic market maker faces a price-sensitive liquidity trader and a better informed trader who is alternatively risk neutral and risk averse. The equilibrium depth can take values ranging from 0 to infinity, depending on the information asymmetry, the asset volatility, and the strength of the liquidity demand, while the spread remains positive and finite.

In many financial markets, dealers serve as intermediaries between buyers and sellers, earning positive profits by imposing a bid-ask spread. One determinant of this spread is asymmetric information: because the dealer makes public the prices at which he commits to sell or buy, he runs the risk of being picked off by traders who have better information and value the stock more than his ask or less than his bid. The market maker loses money when trading with them but makes money off liquidity or noise traders, who transact for some often unspecified reasons and have no private information. In equilibrium, the dealer sets the spread wide enough to recoup his losses with the informed traders but narrow enough to attract sufficient liquidity demand. The basic intuition was laid out by Bagehot (1971) and formalized by Glosten and Milgrom (1985) and by Glosten (1989).

Empiricists often consider changes in the bid-ask spread to be a reflection of variations in the market information asymmetry. Studies have shown that the spread increases before earnings or takeover announcements, events likely to be preceded by a rise in information asymmetry. However, the evidence for this scenario is not always strong [see Venkatesh and Chiang (1986)].

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Indeed, studies that focus exclusively on the spread neglect another potentially important tool the dealer can use to fend off informed traders: the quantity limit (or depth), that is, the maximum amount the market maker is willing to sell at the ask or buy at the bid.

Although academic literature has traditionally given less attention to the depth than to the spread, practitioners have long recognized its importance as a decision-making tool. Mann and Seijas (1991) even note that market makers on the New York Stock Exchange (NYSE) have more control over the depth than over the spread. Moreover, recent empirical work by Lee, Mucklow, and Ready (1993) and by Jennings (1994) indicates that the depth changes proportionally more than the spread around earnings or takeover announcements, suggesting that dealers use both spread and depth to manage their asymmetric information risk.

This article incorporates both tools into a model and finds that the theoretical dealer adjusts the depth proportionally more than the bid-ask spread in response to changes in the degree of information asymmetry. The analysis uses a static framework with a risk-neutral, monopolistic market maker who faces a price-sensitive liquidity trader and a risk-neutral or, alternatively, risk-averse informed trader. The model shows that the depth is more sensitive than the spread to changes in the degree of information asymmetry. This relative sensitivity is measured by the absolute value of the elasticity of substitution between the depth and the spread with respect to the quality of the informed trader's information; it will be referred to hereafter as "elasticity of substitution" for brevity.[1] It depends on market conditions, which are characterized by three parameters: the degree of information asymmetry, the volatility of the asset value, and the volatility of the liquidity shock. In a very favorable environment, the dealer stops imposing limits on how much he commits to trade at posted prices while his market power bounds the spread away from 0; in a very unfavorable environment, the dealer squeezes the depth to 0 while maintaining a finite spread. We also find that the elasticity of substitution is greater than 1.5 even under less extreme market conditions.

We experiment with three distributions. First, the asset value, the private signal, and the liquidity shock each take two values; the liquidity demand is linear in price; and the informed trader is risk neutral. Second, the asset value is normally distributed and the liquidity demand is linear in price. Third, the asset value is lognormally distributed and the liquidity demand is linear in the logarithm of the asset price. In the last two cases, the informed trader is alternatively risk neutral and risk averse, and the liquidity trader's demand is subject to a normally distributed shock. When the informed trader is risk neutral and the asset value is normally distributed or takes discrete values, the elasticity of substitution depends on a single variable, which combines

[1] The elasticity of substitution between two variables is the ratio of their elasticities with respect to the same parameter.

the three parameters characterizing market conditions. When the asset value is lognormally distributed, the standard deviation of the asset's returns used in the model is calibrated to a wide array of market-based values. The article does not address the role that limit orders, floor brokers, or inventory control may play in shaping the market-making process. The conclusions of the article seem robust to distributional or parameter changes.

Before introducing the technical details of the model, we present some intuition for the results. We deal only with the ask side, the bid side being symmetrical.[2] The dealer runs the risk of being picked off by a better-informed trader whose estimation of the asset value lies above the ask. The probability of such an event and the associated loss increase with the precision of the informed trader's private signal. The market maker can respond to such an increase by raising the spread or lowering the depth, or both. Increasing the spread diminishes the probability that the informed trader will buy; decreasing the depth reduces the amount sold at a loss if he buys. However, because the liquidity trader is price sensitive, increasing the price drives away liquidity demand—precisely the segment of the order flow the market maker wants to preserve. Likewise, reducing the quantity limit may prevent profitable trade with the liquidity trader.

The key difference between the demand of the informed trader and that of the liquidity trader is that the absolute value of the informed trader's demand increases with the precision of the private signal, whereas the liquidity trader's demand is independent of information asymmetry. Intuitively the informed trader is more likely than the liquidity trader to place large orders.[3] Lin, Sanger, and Booth (1995) confirm this prediction using a sample of stocks traded on the NYSE; they estimate that the adverse selection component of the bid-ask spread increases uniformly with trade size. If desired order size and private information are correlated, it makes sense for the dealer to use the posted depth to hedge against better informed traders.

The trade-off governing the setting of the depth is that it limits both unprofitable trade with the informed trader and profitable trade with the liquidity trader. As the precision of the informed trader's private signal improves, the likelihood increases that the quantity limit actually rations the informed trader's demand, which should tilt the trade-off toward lowering the depth. Naturally, increasing the spread also reduces the probability of the informed trader's valuation being outside the bid and ask quotes, so that we should

[2] When the asset value is symmetrically distributed around 0, the optimal bid price equals the optimal ask price multiplied by -1. When the asset value is lognormally distributed, the optimal bid price is not a simple function of the ask price because the expected profit combines a lognormally distributed informed trader's valuation and a normally distributed liquidity shock. However, when the informed trader is risk neutral, the difference between the mean asset value and the bid price is very close to the difference between the ask price and the mean asset value.

[3] Easley and O'Hara (1987) note that "given that they wish to trade, informed traders prefer to trade larger amounts at any given price. Since uninformed traders do not share this quantity bias, the larger the trade size, the more likely it is that the market maker is trading with an informed trader" (p. 70).

expect the dealer to use this option too. When the asset value, the private signal, and the liquidity shock each take two values, the dealer clearly uses the quantity limit to discriminate against the informed trader by setting the depth equal to the positive value of the liquidity trader's demand. When the asset value is normally or lognormally distributed, the absence of closed-form solutions makes a precise linkage between the general intuition and the inner workings of the model more elusive.

When the informed trader is risk neutral, a proportional change in the volatility of the asset has the same effect on the depth and the spread as a proportional change in the degree of information asymmetry. In fact, the price posted by the dealer is like the strike price of a free option whose value to the informed trader increases with asset volatility. The other determinant of the equilibrium spread and depth is the strength of the liquidity demand. Intuitively a deepening in information asymmetry may be partially offset by an increase in the willingness of the liquidity trader to buy at the ask or sell at the bid. This willingness is characterized by the volatility of the liquidity shock.

The structure of this article is as follows. Section 1 introduces the economic setting. In Section 2, the true asset value, the informed trader's private signal, and the liquidity shock each take two values and the informed trader is risk neutral. In Section 3, these random variables are normally or lognormally distributed and the informed trader is alternatively risk neutral and risk averse. Section 4 shows how the model is solved numerically and investigates the relative sensitivities of spread and depth to changes in the degree of information asymmetry under various market conditions.

1. Economic Setting

1.1 General framework of the model

The model features a quote-driven dealer market with firm quotes, in which a risk-neutral, monopolistic market maker facing a liquidity trader and an informed trader precommits to both price and depth on the bid and ask sides. The liquidity trader's demand is price sensitive and stochastic. The informed trader, who observes a signal correlated with the true value of the asset, buys the asset if the ask is below (or sells if the bid is above) his valuation. Buy orders and sell orders are aggregated separately; orders are not crossed between informed and liquidity traders. No limit-order book or floor brokers compete with the dealer (or specialist). The model focuses on asymmetric information and excludes other factors such as misdiversification and order processing costs. The informed trader is alternatively risk neutral and risk averse. His demand is satisfied before that of the liquidity trader. This assumption simplifies the informed trader's demand but excludes some strategies potentially available to actual traders, such as conditioning orders on the volume of liquidity trade. Informed and liquidity traders' buy or sell orders are lumped together and passed on to the dealer so that the quantity limit becomes binding if the sum of all orders is greater than the posted depth.

Because the dealer's profit can be written as a sum of two functions, one depending on the bid price and depth and the other depending on the ask price and depth (see appendix), the model presented here uses only the selling side of the dealer's quotes. Let π be the dealer's profit, a the ask price, z the ask quantity limit, x the true value of the traded asset, G the signal observed by the informed trader, $v = E[x|G]$ his valuation, $I[\,]$ the indicator function, $q(a)$ his demand at price a, and $d(a)$ the liquidity trader's demand, which is the sum of a deterministic function of a and a liquidity shock, η, independent of x and G.

When the informed trader is risk averse, his demand remains finite; when he is risk neutral, his demand is unboundedly positive if $v > a$. Using the fact that $v = E[x|G]$ and that η is independent of x and G, one simplifies the expected profit to Equation (1) when the informed trader is risk neutral, and to Equation (2) when he is risk averse.

$$\begin{aligned} E[\pi] = {} & E\big[\, I[v \leq a]\ (a - v)\big]\ E\big[I[d(a) \geq 0]\ \min(d(a), z)\,\big] \\ & + E\big[\, I[v > a]\ (a - v)\,\big]\, z. \end{aligned} \tag{1}$$

$$\begin{aligned} E[\pi] = {} & E\big[I[v \leq a]\ (a - v)\big]\ E\big[I[d(a) \geq 0]\ \min(d(a), z)\big] \\ & + p(d(a) < 0)\ E\big[\, I[v > a]\ (a - v)\ \min(q(a), z)\,\big] \\ & + E\big[\, I[d(a) \geq 0]\ I[v > a]\ (a - v)\ \min(q(a) + d(a), z)\,\big]. \end{aligned} \tag{2}$$

First, whether the informed trader is risk neutral or risk averse, if his valuation is lower than the ask, he does not trade. The trading volume is then determined by the liquidity trader, and the unit expected profit for the dealer is $E\big[\, I[v \leq a]\ (a - v)\big]$. Second, when the informed trader is risk neutral and his valuation exceeds the ask price, the trading volume automatically equals the quantity limit, and the unit expected loss for the dealer is $E\big[\, I[v > a]\ (a - v)\big]$. Third, when he is risk averse and $v > a$, the trading volume depends both on his demand and that of the liquidity trader; sales to the liquidity trader also accrue to the dealer's loss.

When no trader is informed, the market maker does not impose any quantity limit; the spread is not 0 because of the specialist's market power. When the informed trader is risk neutral, his demand is infinite if $v > a$, and, consequently, for all finite spreads, the specialist's expected profit is unboundedly negative if no quantity limits are imposed. When the informed trader is risk averse, his demand is finite and the dealer's expected profit is well defined even if no quantity limit is imposed. We compare the dealer's maximized expected profit with and without quantity limits. Numerical solutions show that the dealer's maximized profit with quantity limits either dominates that without quantity limits or is numerically indistinguishable from it when the informed trader's private signal is poorly correlated with the true asset value.

1.2 Distributional assumptions

In the model, we experiment with three distributions:

1. x, G, and η take respective values $(-\bar{x}, \bar{x})$, $(-1, 1)$, and $(-\bar{\eta}, \bar{\eta})$, each with probability $1/2$, and $d(a) = -a + \eta$,
2. $x \sim N(0, \sigma_x^2)$, $G \sim N(0, 1)$, $\eta \sim N(0, \sigma_\eta^2)$, and $d(a) = -a + \eta$,
3. $\log(x) \sim N(-\sigma_y^2/2, \sigma_y^2)$, $G \sim N(0, 1)$, $\eta \sim N(-\sigma_\eta^2/2, \sigma_\eta^2)$, and $d(a) = -\log(a) + \eta$.

The liquidity shock is always independent of the asset value and of the informed trader's private signal. The model can accommodate more general means for the asset value and different slopes for the liquidity demand by adequately rescaling all variables, provided some assumptions are made linking liquidity demand and the distribution of the asset value (see the appendix).

The spread can be conveniently interpreted as the dollar spread when the distribution of the asset value is discrete or normal, since in this case $E[x] = 0$. It can be interpreted as the percentage spread when this distribution is lognormal, since in this case $E[x] = 1$. In the following, we use the ask half-spread—defined as the difference between the ask price and the mean asset value—instead of the spread.

2. Equilibrium with Discrete Distributions

The random variables x, G, and η take respective values $(-\bar{x}, \bar{x})$, $(-1, 1)$, and $(-\bar{\eta}, \bar{\eta})$, each with probability $1/2$, $\bar{x} > 0$ and $\bar{\eta} > 0$, and $d(a) = -a+\eta$. The informed trader's valuation v takes values $\bar{v} = E[x|G = 1]$ and $\underline{v} = E[x|G = -1]$ with probability $1/2$. The law of iterated expectations implies that $E[v] = E[x] = 0$; hence $\underline{v} = -\bar{v}$, with $\bar{v} = \rho\,\sigma_x$.[4] Naturally, $a > 0$ for the expected profit would be negative otherwise. When there is no informed trader, the dealer imposes no quantity limit, and his profit is

$$E[\pi] = E\big[\, I[d(a) \geq 0]\, d(a)\, \big]\, a = E\big[\, I[\eta \geq a]\, (\eta - a)\, \big]\, a = \frac{1}{2}\, (\bar{\eta} - a)\, a.$$

The optimal price is $\bar{\eta}/2$ and the maximum value for the profit is $\bar{\eta}^2/8$.

When there is an informed trader (which we assume to be risk neutral), the form of the profit function depends on the magnitude of a relative to $\bar{v}$,

[4] Writing $q = P(x = \bar{x}, G = 1)$ and using the distributional assumptions about G and x, we get $P(x = -\bar{x}, G = -1) = q$ and $P(x = \bar{x}, G = -1) = P(x = -\bar{x}, G = 1) = 1/2 - q$; hence $\bar{v} = \bar{x}\, P(x = \bar{x}|G = 1) - \bar{x}\, P(x = -\bar{x}|G = 1) = \bar{x}\,(4q - 1)$ and $\rho\,\sigma_x = \mathrm{cov}(x, G) = \bar{x}\,(4q - 1)$.

Table 1
Equilibrium price and quantity limit with risk-neutral informed trader and discrete distributions

$\bar{v}$	$\bar{v} \leq \frac{1}{2}\bar{\eta}$	$\frac{1}{2}\bar{\eta} < \bar{v} \leq \frac{3}{5}\bar{\eta}$	$\frac{3}{5}\bar{\eta} < \bar{v} \leq \bar{\eta}$	$\bar{\eta} < \bar{v} \leq 3\bar{\eta}$	$\bar{v} > 3\bar{\eta}$
$\max(E[\pi])$ with $a \geq \bar{v}$	$\frac{1}{8}\,\bar{\eta}^2$	$\frac{1}{2}\,\bar{v}\,(\bar{\eta} - \bar{v})$	$\frac{1}{2}\,\bar{v}\,(\bar{\eta} - \bar{v})$	0	0
$\max(E[\pi])$ with $a < \bar{v}$	n.a.	n.a.	$\frac{1}{48}\,(3\bar{\eta} - \bar{v})^2$	$\frac{1}{48}\,(3\bar{\eta} - \bar{v})^2$	0
optimal a	$\frac{1}{2}\,\bar{\eta}$	$\bar{v}$	$\frac{1}{2}\,\bar{\eta} + \frac{1}{6}\,\bar{v}$	$\frac{1}{2}\,\bar{\eta} + \frac{1}{6}\,\bar{v}$	n.a.
optimal z	n.a.	n.a.	$\frac{1}{2}\,\bar{\eta} - \frac{1}{6}\,\bar{v}$	$\frac{1}{2}\,\bar{\eta} - \frac{1}{6}\,\bar{v}$	0
$\max(E[\pi])$	$\frac{1}{8}\,\bar{\eta}^2$	$\frac{1}{2}\,\bar{v}\,(\bar{\eta} - \bar{v})$	$\frac{1}{48}\,(3\bar{\eta} - \bar{v})^2$	$\frac{1}{48}\,(3\bar{\eta} - \bar{v})^2$	0

The first two rows of this table present the dealer's maximum expected profit alternatively imposing $a \geq \bar{v}$ and $a < \bar{v}$, where $\bar{v}$ is the upper value of the informed trader's valuation of the asset. The maximum profit in each case depends on the value of $\bar{v}$ relative to $\bar{\eta}$, the upper value of the liquidity shock. When $\bar{v} \leq 3\,\bar{\eta}/5$, no value of $a < \bar{v}$ maximizes $E[\pi]$ (in the limit, the dealer wishes to set $a = \bar{v}$) and "n.a." is entered in the corresponding cells. The last three rows of the table present the optimal price, quantity limit, and the global maximum of the dealer's expected profit, obtained by comparing the maximum expected profit under the constraints $a \geq \bar{v}$ and $a < \bar{v}$. When $a \geq \bar{v}$, imposing a quantity limit is not necessary; the optimal z is undefined and "n.a." is entered in the corresponding cells. When $\bar{v} > 3\bar{\eta}$, the expected profit is never positive and the dealer exits the market by setting z to 0; the optimal a is undefined and "n.a." is entered in the corresponding cell.

$\bar{v}$ relative to $\bar{\eta}$, and z relative to a and $\bar{\eta}$. Table 1 summarizes the results. Figure 1 plots the equilibrium half-spread and depth, and Figure 2 plots the elasticity of substitution as functions of $\bar{v}/\bar{\eta} = \rho\,\sigma_x/\sigma_\eta$.

1. If $a \geq \bar{v}$, the informed trader is shut out of the market. If $\bar{v} > \bar{\eta}$, the liquidity demand is also reduced to 0. If $\bar{v} \leq \bar{\eta}$, the optimal price is $\max[\bar{v}, \bar{\eta}/2]$ and the maximum profit is $\bar{\eta}^2/8$ if $\bar{v} \leq \bar{\eta}/2$ and $\bar{v}\,(\bar{\eta} - \bar{v})/2$ if $\bar{v} > \bar{\eta}/2$. Intuitively, since $\bar{\eta}/2$ is the optimal price with no informed trader, when $\bar{v} \leq \bar{\eta}/2$, the dealer need not raise his price any higher to exclude the informed trader.
2. If $0 < a < \bar{v}$, then $I[v \leq a] = I[v = -\bar{v}]$, $I[v > a] = I[v = \bar{v}]$, and the profit is

$$E[\pi] = \frac{1}{2}\,(a + \bar{v})\left\{E\big[\,I[a \leq \eta \leq a + z]\,(\eta - a)\,\big] + p(\eta > a + z)\,z\right\} + \frac{1}{2}\,(a - \bar{v})\,z.$$

The dealer does not set the price above $\bar{\eta}$ because the liquidity trader would not trade in that case. We can hence assume that $a < \bar{\eta}$. First, suppose that $z < \bar{\eta} - a$. The profit becomes

$$E[\pi] = z\left(\frac{1}{4}\,(a + \bar{v}) + \frac{1}{2}\,(a - \bar{v})\right) = \frac{3}{4}\,z\left(a - \frac{1}{3}\,\bar{v}\right). \qquad (3)$$

If $a < \bar{v}/3$, then $E[\pi] \leq 0$ and the optimal z is 0; if $a = \bar{v}/3$, then $E[\pi] = 0$ and z can be set arbitrarily; if $a > \bar{v}/3$, then $E[\pi] > 0$ and there is no interior solution in the interval $(0, \bar{\eta} - a)$. In the limit, the dealer would like to set z equal to $\bar{\eta} - a$. Assume now that $z \geq \bar{\eta} - a$. The profit becomes

$$E[\pi] = \frac{1}{4}\,(a + \bar{v})\,(\bar{\eta} - a) + \frac{1}{2}\,(a - \bar{v})\,z.$$

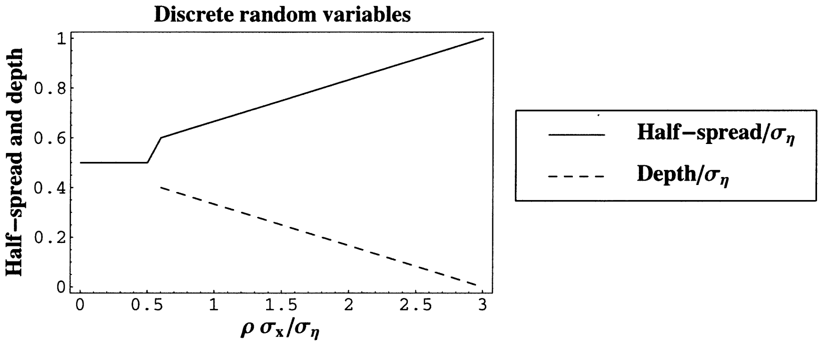


Figure 1
Half-spread and depth with risk-neutral informed trader and discrete distributions
This figure plots the half-spread and the depth, both divided by σ_η, against $\rho\,\sigma_x/\sigma_\eta$, which captures the market conditions facing the dealer. To relate this figure to Table 1, note that $\bar{v} = \rho\,\sigma_x$ and $\bar{\eta} = \sigma_\eta$. The informed trader is risk neutral. His signal (G), the asset value (x), and the liquidity shock (η) each take two values with equal probabilities. The parameters σ_x, σ_η, and ρ are respectively the standard deviations of x and of η and the coefficient of correlation between x and G. For $\rho\,\sigma_x/\sigma_\eta \le 3/5$, no quantity limit is necessary, for $3/5 < \rho\,\sigma_x/\sigma_\eta \le 3$, $z = \bar{\eta} - a$, and for $\rho\,\sigma_x/\sigma_\eta > 3$, $z = 0$.

Since $a < \bar{v}$, z should be set as low as possible, that is, to $\bar{\eta} - a$. The profit is then

$$E[\pi] = (\bar{\eta} - a)\left(\frac{1}{4}(a + \bar{v}) + \frac{1}{2}(a - \bar{v})\right) = \frac{3}{4}(\bar{\eta} - a)\,(a - \frac{1}{3}\,\bar{v}). \tag{4}$$

The dealer maximizes the function defined in Equation (4) under the constraint that $0 < a < \bar{v}$. If $\bar{v} > 3\,\bar{\eta}$, this function is negative for $a < \bar{\eta}$ and the dealer exits the market. If $\bar{v} \le 3\,\bar{\eta}$, the price that maximizes $E[\pi]$ without the constraint $a < \bar{v}$, $\bar{\eta}/2 + \bar{v}/6$, is smaller than $\bar{v}$ if $\bar{v} > 3\,\bar{\eta}/5$. Hence, if $3\,\bar{\eta}/5 < \bar{v} \le 3\,\bar{\eta}$, the optimal price is $\bar{\eta}/2 + \bar{v}/6$ and the maximum profit is $(3\bar{\eta} - \bar{v})^2/48$; if $\bar{v} \le 3\,\bar{\eta}/5$, the constraint $a < \bar{v}$ is binding; in the limit, the dealer wants to set $a = \bar{v}$.

When $3\,\bar{\eta}/5 < \bar{v} < 3\,\bar{\eta}$, imposing $a \ge \bar{v}$ results in a suboptimal profit because then $(3\bar{\eta} - \bar{v})^2/48 > \bar{v}\,(\bar{\eta} - \bar{v})/2$. The optimal quantity limit is well defined only for $3\,\bar{\eta}/5 < \bar{v} \le 3\bar{\eta}$. Imposing a quantity limit is not necessary when $\bar{v} \le 3\,\bar{\eta}/5$ because the optimal price is greater than or equal to $\bar{v}$ and the optimal quantity limit is zero when $\bar{v} > 3\,\bar{\eta}$. When $3\,\bar{\eta}/5 < \bar{v} < 3\bar{\eta}$, the elasticity of substitution is an increasing function of the ratio $\bar{v}/\bar{\eta}$, starting at 3/2 when this ratio equals 3/5 and growing without bound when it approaches 3 as the quantity limit converges to 0 while the spread remains finite.

To summarize, when x and η take only two values, it is easy to define the optimal quantity limit as a function of a because the liquidity trader's demand takes only two values: one negative, $-\bar{\eta} - a$, the other positive, $\bar{\eta} - a$

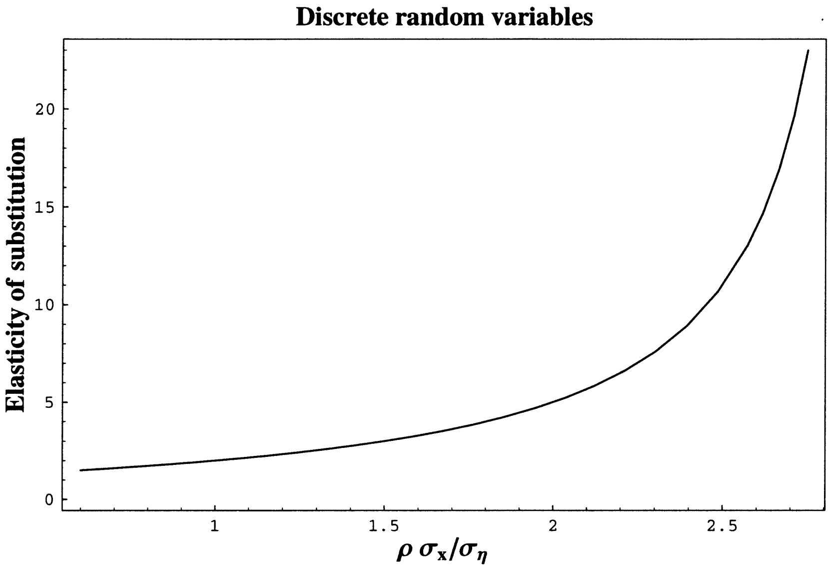


Figure 2
Elasticity of substitution with risk-neutral informed trader and discrete distributions
This figure plots the elasticity of substitution between depth and spread with respect to ρ against $\rho\,\sigma_x/\sigma_\eta$ when x, G, and η each take two values. When $\rho\,\sigma_x/\sigma_\eta$ converges to 3, the depth converges to 0 while the spread remains finite, and the elasticity of substitution converges to infinity. This elasticity is also always greater than $3/2$.

(for $a < \bar{\eta}$). Since $\bar{\eta} - a$ is the maximum liquidity demand, setting z above this value would increase potential losses (when $v > a$) without increasing potential profits (when $v \leq a$). Since $\bar{\eta} - a$ is also the minimum positive value of the liquidity demand, a quantity limit set below it would always be binding. Consequently the expected profit would be proportional to z, which would be optimally either 0 if the constant of proportionality is negative or $\bar{\eta} - a$ if this constant is positive. It follows that the optimal quantity limit, when it is positive and finite, equals $\bar{\eta} - a$. The quantity limit is used to discriminate against the informed trader, whose demand can exceed $\bar{\eta} - a$, and in favor of the liquidity trader, whose demand does not exceed $\bar{\eta} - a$. Since z equals $\bar{\eta} - a$ in equilibrium, it falls as a rises in response, for example, to an increase in the quality of the informed trader's information.

3. Equilibrium with Continuous Distributions

In the following discussion, x is either normally or lognormally distributed, and G and η are normally distributed.

3.1 Marginal conditions with a risk-neutral informed trader

When x is normally or lognormally distributed, although no equilibrium closed-form solution exists, the marginal conditions for the price and the quantity limit shed light on the trade-offs governing the dealer's profit maximization.

The first derivative of the specialist's expected profit with respect to the quantity limit is given in Equation (5); the one with respect to the ask is given in Equation (6), where $d'(a)$ denotes the slope of the liquidity demand at price a. Equations (5) and (6) are based on Proposition 2 in the appendix.

$$\frac{\partial}{\partial z} E[\pi] = p(d(a) > z)\ E\big[I[v \leq a]\ (a - v)\big] + E\big[I[v > a]\ (a - v)\big], \quad (5)$$

$$\frac{\partial}{\partial a} E[\pi] = p(v > a)\ z + p(v \leq a)\ E\big[\ I[d(a) \geq 0]\ \min(d(a), z)\big] \\ + d'(a)\ p(0 \leq d(a) \leq z)\ E\big[\ I[v \leq a]\ (a - v)\big]. \quad (6)$$

Equation (5) has an intuitive interpretation. Increasing the quantity limit by an infinitesimal amount brings the market maker a profit equal to $E\big[\ I[v \leq a]\ (a - v)\big]$ if the limit is hit by a liquidity trader, or a loss equal to $E\big[\ I[v > a]\ (a - v)\big]$ if it is hit by an informed trader. Equation (6) is similar to the standard monopolistic case in which increasing the price by \$1 makes the monopolist earn another \$1 on the total volume but reduces the overall demand. The first two terms on the right-hand side of Equation (6) represent the expected sales when the informed trader wants to trade, $p(v > a)\ z$, and when he does not, $p(v \leq a)\ E\big[\ I[d(a) \geq 0]\ \min(d(a), z)\big]$. The last term represents the impact on profits resulting from the reduction in overall demand. A price increase depresses demand if the quantity limit is not binding, that is, if the liquidity trader's demand is positive but lower than z and the informed trader's evaluation is below a. In this event, the effect on overall demand is proportional to the slope of the liquidity trader's demand.

3.2 Marginal conditions with a risk-averse informed trader when the asset value is normally distributed

When the asset value is normally distributed, the informed trader is endowed with a negative exponential utility function with risk aversion γ. Equivalently the informed trader maximizes $E[\pi^i|G] - \gamma\, \text{var}(\pi^i|G)/2$, where π^i is the informed trader's profit and $\text{var}(\pi^i|G)$ is the conditional variance of his profit given G. We assume that his initial wealth is conditionally independent of the value of the asset and the liquidity shock. The informed trader chooses the quantity q to maximize

$$q\ (E[x|G] - a) - \frac{1}{2}\, q^2\ \gamma\ \text{var}(x|G), \quad (7)$$

where $\text{var}(x|G)$ is the conditional variance of x given G. As the informed trader's orders are filled before those of the liquidity trader, the informed trader's demand function is

$$q(a) = \max\left[0, \left(\gamma \ \mathrm{var}(x|G)\right)^{-1} (v - a)\right], \tag{8}$$

with $\mathrm{var}(x|G) = \sigma_x^2(1-\rho^2)$. The first derivatives of the informed trader's demand are given by Equation (9) for the quantity limit and by Equation (10) for the price. These equations are obtained using Proposition 2 in the appendix.

$$\begin{aligned}\frac{\partial}{\partial z}E[\pi] = \ & p(d(a) > z)\ a \\ & + p(d(a) < 0)\ E\left[\ I[q(a) > z]\ (a - v)\ \right] \\ & + E\left[\ I[0 \le d(a) \le z]\ I[d(a) + q(a) > z]\ (a - v)\ \right]. \end{aligned} \tag{9}$$

There are three terms on the right-hand side of Equation (9). In each, the quantity limit is hit since varying the limit by an infinitesimal amount affects profits only if this limit is binding. In the first term, the liquidity trader's demand exceeds the limit. In the second term, the liquidity agent does not trade, the quantity limit is saturated by the informed trader. In the third term, the liquidity demand does not hit the limit, but aggregate demand is greater than the limit because of orders coming from the informed trader.

The derivative with respect to the price is also intuitive albeit more complex.

$$\frac{\partial}{\partial a}E[\pi] = f_1(a, z) + f_2(a, z) + f_3(a, z), \tag{10}$$

with

$$\begin{aligned} f_1(a, z) = \ & \left\{p(d(a) > z) + p(d(a) < 0)\ p(q(a) > z)\right. \\ & \left. + p(0 \le d(a) \le z, d(a) + q(a) > z)\right\} z, \\ f_2(a, z) = \ & p(d(a) < 0)\ E\left[I[q(a) \le z]\ I[v > a]\ q(a)\right] \\ & + p(v \le a)\ E\left[I[0 \le d(a) \le z]\ d(a)\right] \\ & + E\left[I[0 \le d(a) \le z]\ I[v > a]\ I[d(a) + q(a) \le z]\right. \\ & \quad \left.(d(a) + q(a))\right], \\ f_3(a, z) = \ & q'(a)\ p(d(a) < 0)\ E\left[I[q(a) \le z]\ I[v > a]\ (a - v)\right] \\ & + d'(a)\ p(0 \le d(a) \le z)\ E\left[I[v \le a]\ (a - v)\right] \\ & + (q'(a) + d'(a))\ E\left[I[0 \le d(a) \le z]\right. \\ & \quad \left. I[d(a) + q(a) \le z]\ I[v > a]\ (a - v)\right], \end{aligned} \tag{11}$$

where $q'(a)$ denotes the slope of the informed trader's demand at price a. These conditions are similar to the standard monopolist case. Increasing the price drives up the unit profit on the quantities sold but reduces demand for the good. The reduction depends on the slope of the demand function. Equation (11) consists of three blocks. In the first block, $f_1(a, z)$, the quantity limit is hit. This occurs in three cases: first, when the liquidity demand exceeds z; second, when the liquidity agent does not trade but the informed trader's demand exceeds z; third, when the liquidity demand is less than z but the total demand (liquidity plus informed traders' order flow) is greater than the limit. The second block, $f_2(a, z)$, describes the volume sold when the quantity limit is not hit. This occurs in three cases: first, when the liquidity agent does not trade but the informed agent does; second, when only the liquidity agent trades; third, when both agents trade. The last block, $f_3(a, z)$, shows the decrease in overall sales that results from the downward sloping demands. A price increase depresses sales only if the quantity limit is not hit. Then, as $q'(a)$ is the slope of the informed demand and $d'(a)$ is that of the liquidity demand, the aggregate demand decreases by $q'(a)$ if the informed agent is the sole trader (the first term of the equation), by $d'(a)$ if the liquidity agent is the only one to trade (the second term), and by $q'(a) + d'(a)$ if both agents trade (the last term).

3.3 Expected dealer's profit when the asset value is lognormally distributed

When x is lognormally distributed, we assume that the informed trader maximizes the function defined in Equation (7); his demand is still defined by Equation (8) but is not linear in v because the conditional variance of x given G depends on the realization of G: $\mathrm{var}(x|G) = \left[\exp\left((1-\rho^2)\sigma_y^2\right) - 1\right] v^2$, where σ_y is the standard deviation of $y = \log(x)$ and $\rho = \mathrm{corr}(G, y)$.[5] The informed trader's demand is

$$q(a, v) = \frac{1}{\theta} \max\left[0, \frac{1}{v} - \frac{a}{v^2}\right],$$

where $\theta = \gamma\left[\exp\left((1-\rho^2)\sigma_y^2\right) - 1\right]$. When x is lognormally distributed, as opposed to normally distributed, the informed trader's demand decreases with v after some threshold. Intuitively, two effects are at play when v increases. First, the difference between v and a becomes larger, inducing the informed trader to purchase more of the asset. Second, the conditional variance $\mathrm{var}(x|G)$ increases too, reducing the attractiveness of the asset to the informed trader since he is risk averse.

[5] To compute the conditional variance in the lognormal case, use $\mathrm{var}(x|G) = E[x^2|G] - E[x|G]^2$ and Proposition 3 in the appendix, together with the fact that $\log(x)$, $\log(x^2)$, and G are jointly normally distributed.

4. Numerical Solutions

When x is normally or lognormally distributed, we search for the optimal price (or spread) and for the optimal depth numerically. The dealer's profit maximization simplifies to a one-dimensional problem when the informed trader is risk neutral, because the optimal depth in this case can be written as a function of the spread.

When the informed trader is risk averse and x is normally distributed, we divide the spread and the depth by σ_x, multiply γ by σ_x^2 (the value of the product is set at 0.5), and search for the optimal spread and depth simultaneously by using a procedure based on the Newton method; we choose as initial values the optimal spread and depth when the informed trader is risk neutral. We also maximize the expected profit function without quantity limit and compare it with the maximum achieved when a quantity limit is imposed. When ρ is small enough, the optimized dealer's profits with or without quantity limit are numerically indistinguishable, indicating that imposing a finite quantity limit may not always be optimal for the dealer when the risk-averse informed trader is poorly informed. This property, which holds for discrete distributions, may stem from the dealer's monopolistic power. His spread may be large enough to make up for losses incurred when selling to the informed trader without imposing a finite quantity limit. In contrast, for higher values of ρ, imposing a quantity limit is always better for the dealer.

When the informed trader is risk averse and x is lognormally distributed, we search for the optimal spread and quantity limit by directly maximizing the dealer's expected profit function, assuming the risk-aversion coefficient is 10. Likewise, we compare the optimum with and without quantity limits; profits with limits are distinctively greater than those without limits. Note that risk-aversion coefficients cannot be compared across distributions since one is scaled by σ_x^2 and the other is not.

4.1 Calibration

When the informed trader is risk neutral, the elasticity of substitution depends only on $\rho\,\sigma_x/\sigma_\eta$ when the asset value is normally distributed, or on $\rho\,\sigma_y/\sigma_\eta$ and σ_η when it is lognormally distributed. When the informed trader is risk averse, the dependence of the elasticity of substitution on ρ, σ_x (or σ_y), and σ_η is more complex and we need to calibrate the model to observable variables, such as asset return volatilities.

Return volatilities show considerable heterogeneity. The standard deviation of monthly returns ranges from 8.79% for the top-capitalization quintile to 21.12% for the bottom one on a sample of firms quoted on the NYSE and AMEX between 1985 and 1995 (Duffee 1996). Historical or implied volatilities display similar variations (Franks and Schwartz 1991; McInish and Wood 1991). Moreover, calibrations need to take into account the time for which the dealer's quotes hold.[6] The interval between trades—a proxy for this parameter—varies greatly across stocks. Dimson (1979) estimates that the mean price age of continuously listed shares in the London Share Price Database between 1955 and 1974 ranges from less than 1/10 of a day for the most-traded decile to more than 12 days for the least-traded one. The great variety in asset return volatilities is reflected in the bid and ask spreads. Aiyagari and Gertler (1991) note that the spread on actively traded assets (about half the market) averages 0.52%, while the spread on the rest of the assets averages about 1%, peaking at 6.55% for firms with assets of less than $10 million.

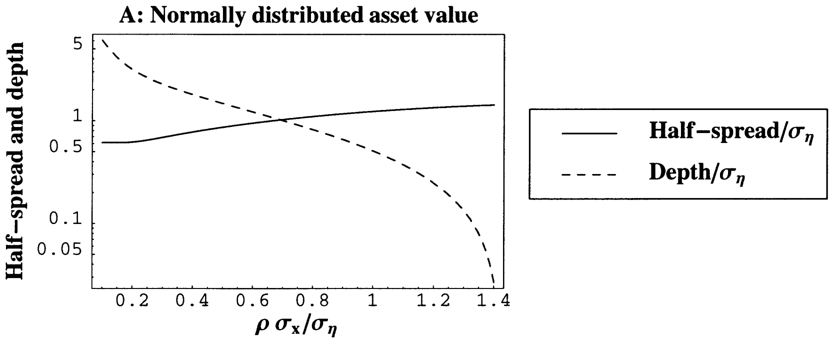
To match the wide array of return volatilities observed in the market, we vary σ_η from 1 to 35% when the informed trader is risk neutral, and we set σ_y and σ_η first to 1 and 2% and then to 10 and 20% when the informed trader is risk averse.

4.2 Results

Figure 3 displays the half-spread and depth, both divided by σ_η, against $\rho\,\sigma_x/\sigma_\eta$ when x is normally distributed (panel A) and against $\rho\,\sigma_y/\sigma_\eta$ when x is lognormally distributed and $\sigma_\eta = 0.1$ (panel B). In both cases, the informed trader is risk neutral. The graphs use a logarithmic scale for the vertical axis. The spread increases and the depth decreases with $\rho\,\sigma_x/\sigma_\eta$ or $\rho\,\sigma_y/\sigma_\eta$. When market conditions are very favorable—for example, the informed trader is poorly informed—the depth grows without bound while the spread remains finite because of the dealer's market power. When market conditions become very unfavorable, the dealer exits the market by setting the depth to 0 while maintaining a finite spread. When $\rho\,\sigma_x/\sigma_\eta$ is held constant, half-spread and depth are proportional to σ_η if x is normally distributed. If x is lognormally distributed, half-spread and depth are approximately proportional to σ_η. For example, for $\rho\,\sigma_y/\sigma_\eta = .5$, the half-spread equals about .8% when $\sigma_\eta = .01$ and 9% when $\sigma_\eta = .1$. Making the informed trader risk averse and choosing different volatilities for the asset and the liquidity shock do not qualitatively affect the results.

When the informed trader is risk neutral, the elasticity of substitution traces a U-shaped curve with a steeper slope at the low end (Figures 4 and 5). This elasticity increases to infinity when $\rho\,\sigma_x/\sigma_\eta$ or $\rho\,\sigma_y/\sigma_\eta$ reach their upper or lower bounds, because the depth either converges to 0 or to infinity while the spread remains positive and finite. Figures 4 and 5 show two regimes for the relative sensitivity of spread and depth. When market conditions are either very favorable (low $\rho\,\sigma_x/\sigma_\eta$ or $\rho\,\sigma_y/\sigma_\eta$) or very unfavorable (high $\rho\,\sigma_x/\sigma_\eta$ or $\rho\,\sigma_y/\sigma_\eta$), the depth is proportionally far more sensitive than the spread

[6] If x_t, the asset value, follows a geometric Brownian motion ($dx_t = \mu\, x_t\, dt + \sigma\, x_t\, dW_t$, where μ and σ are the instantaneous mean and standard deviation of the asset returns), x_t is lognormally distributed, and the standard deviation of $\log(x_t) - \log(x_0)$ (the logarithmic returns between times 0 and t) equals $\sigma\sqrt{t}$. In the model, t can be benchmarked on the time for which the dealer's quotes hold.



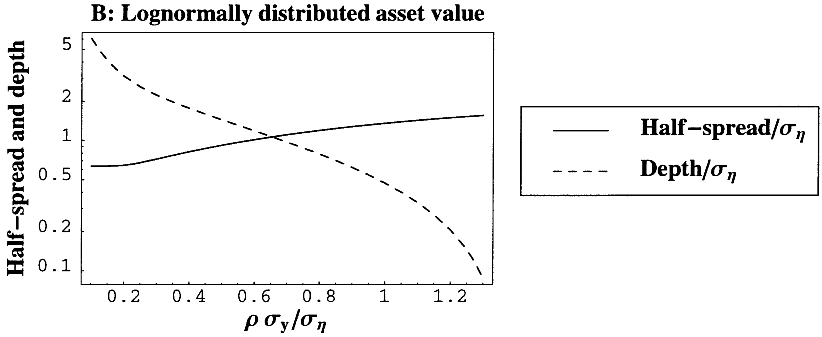


Figure 3
Half-spread and depth with risk-neutral informed trader and normally or lognormally distributed asset value
The graph in panel A plots the half-spread and the depth, both divided by σ_η, against $\rho\,\sigma_x/\sigma_\eta$ when the informed trader is risk neutral and x is normally distributed; ρ measures the precision of the informed trader's private signal. The graph in panel B plots the half-spread and the depth, both divided by σ_η, against $\rho\,\sigma_y/\sigma_\eta$ when the informed trader is risk neutral, x is lognormally distributed, and $\sigma_\eta = 0.1$; $y = \log(x)$. In both panels, the variable on the horizontal axis captures the market conditions facing the dealer; the vertical axis uses a logarithmic scale.

to changes in the degree of information asymmetry; when market conditions are average, the depth is still about 1.5–4 times as sensitive as the spread, increasing rapidly as market conditions worsen.

Figures 6 and 7 display the elasticity of substitution when the informed trader is risk averse. We set σ_η equal to σ_x, $2\,\sigma_x$, and $\sigma_x/2$ when x is normally distributed and (σ_η, σ_y) equal to (.1, .1), (.2, .1), and (.1, .2) when x is lognormally distributed. (When σ_y and σ_η equal .01 or .02, optimal spreads and depths are very close to their risk-neutral levels.) For both distributions, the elasticity of substitution appears higher when the informed trader is risk averse. However, the elasticity of substitution is decreasing in γ for $\gamma > 0.5$ and $\rho\,\sigma_y/\sigma_\eta = 0.5$.

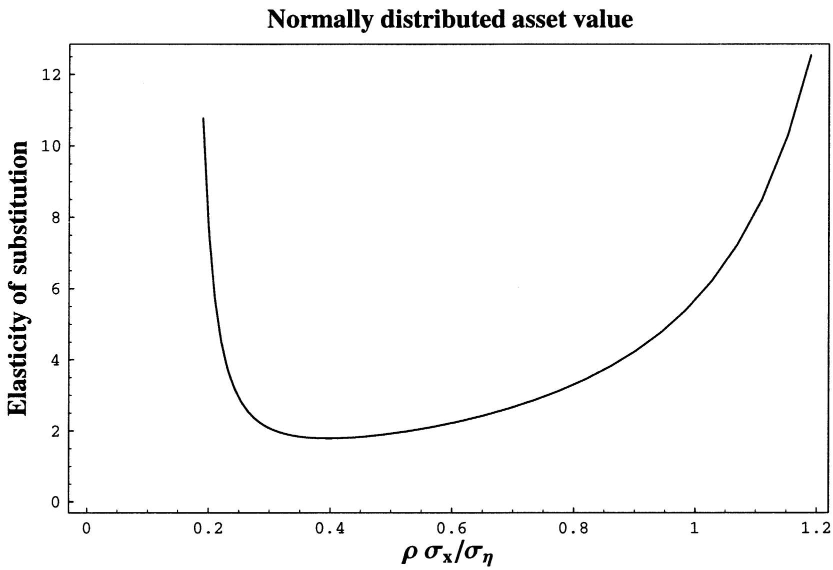


Figure 4
Elasticity of substitution with risk-neutral informed trader and normally distributed asset value
This figure plots the elasticity of substitution between depth and spread with respect to ρ as a function of $\rho\,\sigma_x/\sigma_\eta$ when the informed trader is risk neutral and x is normally distributed.

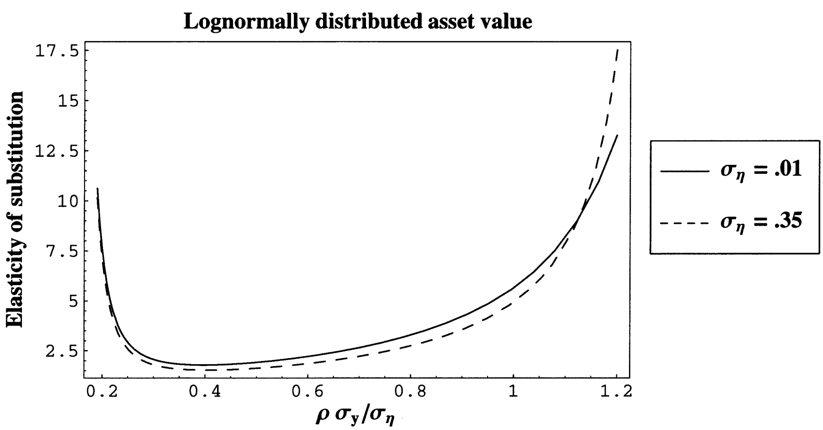


Figure 5
Elasticity of substitution with risk-neutral informed trader and lognormally distributed asset value
This figure plots the elasticity of substitution between depth and spread with respect to ρ as a function of $\rho\,\sigma_y/\sigma_\eta$ when the informed trader is risk neutral and x is lognormally distributed, $y = \log(x)$, $\sigma_\eta = .01$, and $\sigma_\eta = .35$.

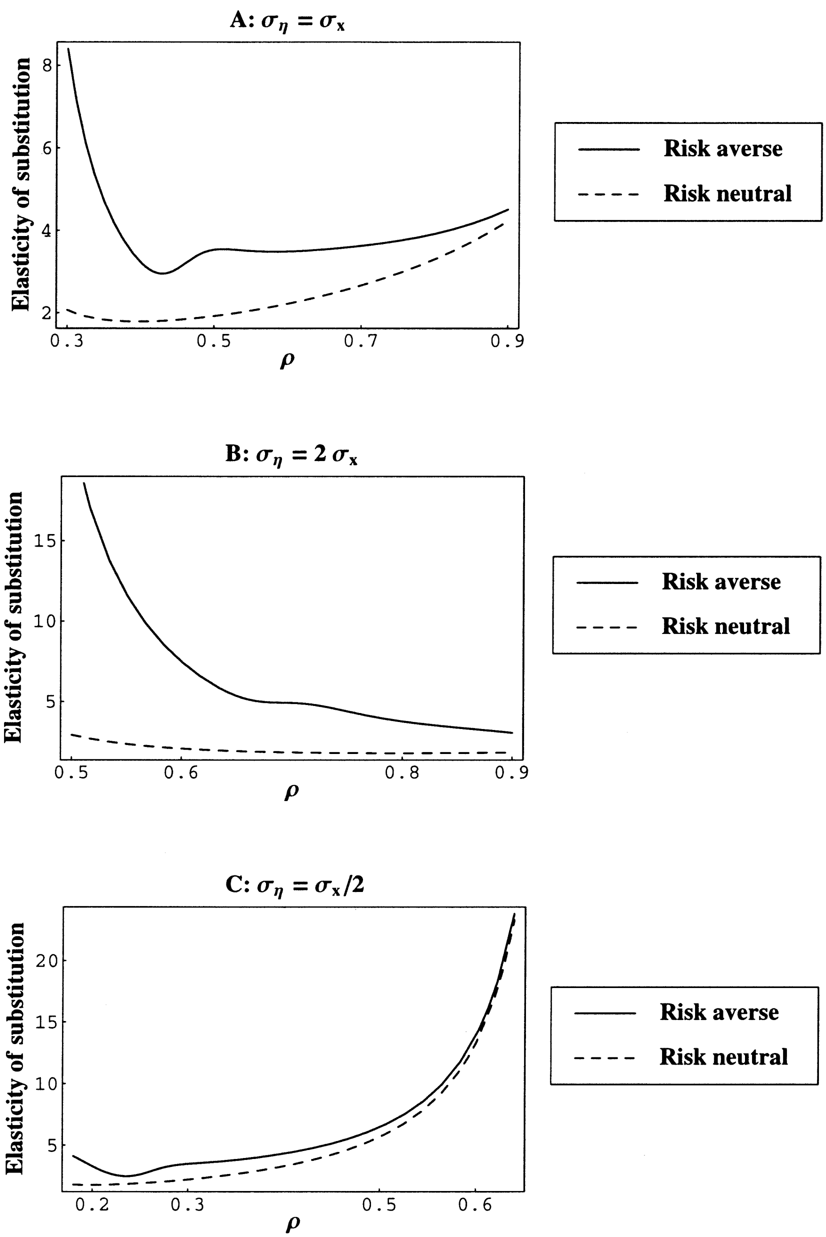


Figure 6
Elasticity of substitution with risk-averse informed trader and normally distributed asset value
This figure plots the elasticity of substitution between depth and spread with respect to ρ as a function of ρ when the informed trader is risk averse, x is normally distributed, $\gamma\sigma_x^2 = .5$, and $\sigma_\eta = \sigma_x$ (panel A), $\sigma_\eta = 2\sigma_x$ (panel B), $\sigma_\eta = \sigma_x/2$ (panel C), where ρ measures the precision of the informed trader's private signal and γ is the informed trader's risk-aversion coefficient. The dashed line represents the elasticity of substitution when the informed trader is risk neutral.

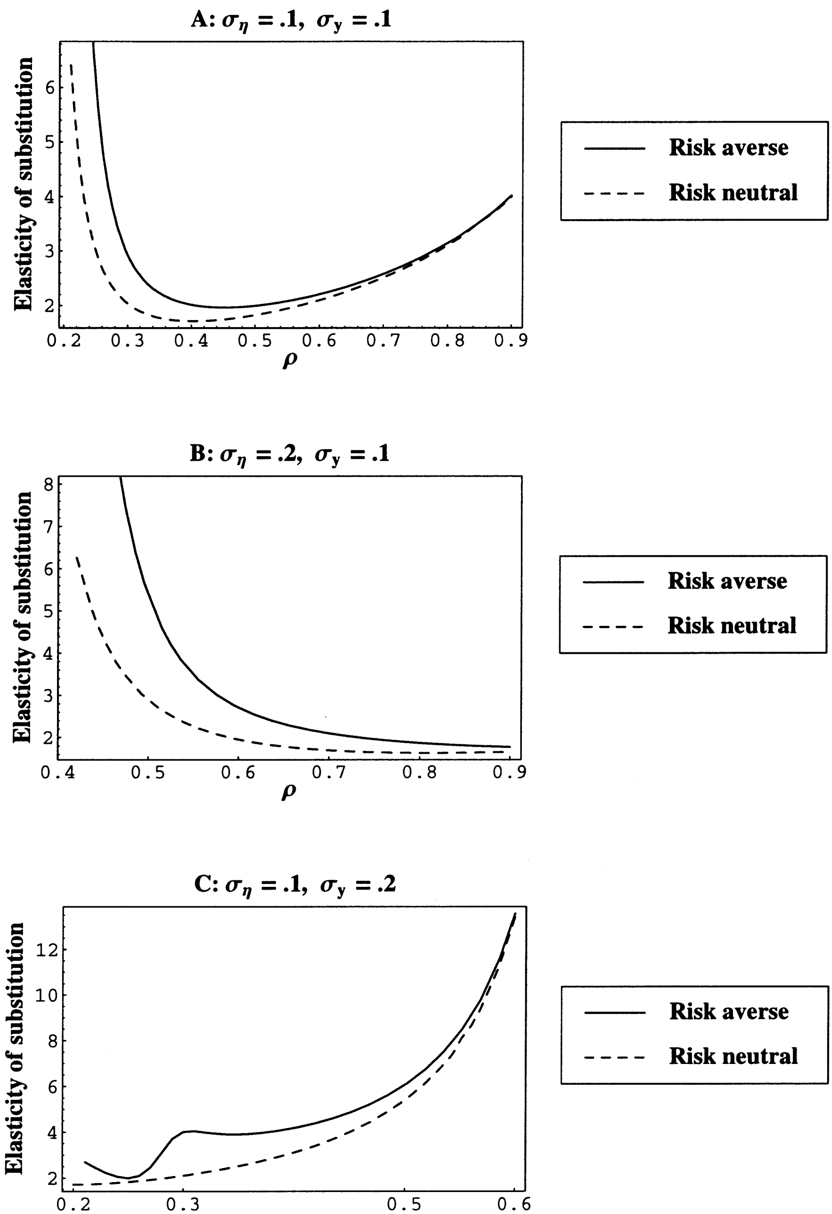


Figure 7
Elasticity of substitution with risk-averse informed trader and lognormally distributed asset value
This figure plots the elasticity of substitution between depth and spread with respect to ρ as a function of ρ when the informed trader is risk averse, x is lognormally distributed, $y = \log(x)$, $\gamma = 10$, $\sigma_\eta = \sigma_y = .1$ (panel A), $\sigma_\eta = .2$ and $\sigma_y = .1$ (panel B), $\sigma_\eta = .1$ and $\sigma_y = .2$ (panel C), where ρ measures the precision of the informed trader's private signal and γ is the informed trader's risk-aversion coefficient. The dashed line represents the elasticity of substitution when the informed trader is risk neutral.

5. Conclusion

Security market dealers have access to two important means of maximizing profits: their choice of bid and ask prices, and their choice of market depth, defined as the maximum amount a dealer is willing to sell at the ask or buy at the bid. Researchers have devoted much greater attention to the bid-ask spread than to the market depth. The stylized theoretical framework presented in this article shows that a monopolistic, risk-neutral dealer narrows his depth proportionally more than he widens his spread to respond to an increase in the degree of information asymmetry. The elasticity of substitution between the depth and the spread with respect to the quality of the informed trader's information depends on market conditions, which are characterized by the information asymmetry, the volatility of the asset, and the strength of liquidity demand. This elasticity approaches infinity when market conditions become either extremely favorable (the depth expands to infinity while the spread remains positive) or extremely unfavorable (the depth contracts to 0 while the spread remains finite). Whether the informed trader is risk neutral or risk averse does not fundamentally affect the results. However, the elasticity of substitution with respect to other parameters could behave differently, and the depth may not be more sensitive than the spread to simultaneous changes in several parameters. For example, when the asset value is normally distributed and the informed trader is risk neutral, spread and depth are both proportional to the volatility of the liquidity shock when this parameter and the volatility of the asset value (or the precision of the informed trader's signal) are multiplied by the same factor.

The conclusions drawn from the model seem to concur with the empirical results of Lee, Mucklow, and Ready (1993) and Jennings (1994), who suggest a higher sensitivity for the depth than for the spread. However, these authors do not distinguish between trades the dealer executes on his own books and those stemming from limit orders or floor brokers. A formal test of the model requires filtering out the effect of random limit order execution on the quoted depth, using, for example, the methodology of Kavajecz (1999). Alternatively, one could try to include limit orders in the model, as in Seppi (1997). Some of the model's insights could be applied to multiple-dealer markets. As noted in Dennert (1993), a risk-neutral informed trader hits every dealer posting an ask price below or a bid price above his own estimate of the asset value, whereas liquidity demand is spread among all dealers.

Appendix

Proposition 1. *The market-maker's profit can be written as the sum of two functions, each of which involves only price and quantity quotes on the ask side or the bid side.*

Proof. Let b be the bid price, a the ask price, z_b the bid quantity limit, and z_a the ask quantity limit. Assume that $b \leq a$ so that $d(b) \geq d(a)$. The specialist's profit function is determined by the actions of the informed trader and of the liquidity trader on the bid and ask sides. For the informed trader, three cases can be distinguished: $v < b$, the informed trader wants to sell to the specialist; $b \leq v \leq a$, the informed agent does not trade; $v > a$, the informed trader wants to buy from the specialist. For the liquidity trader, three cases can be distinguished on the ask side (the bid side is symmetric): $d(a) < 0$, the liquidity agent does not buy; $0 \leq d(a) \leq z_a$, the liquidity trader buys, the quantity limit might be hit depending on how much the informed agent is willing to trade; $d(a) > z_a$, the quantity limit is hit and trade is restricted. The specialist's profit, π, is defined below, where trades and quantity limits are positive on the ask side and negative on the bid side.

$$\begin{aligned}
\pi = {} & I[v < b]\ I[d(a) \geq 0]\big\{(b-x)\ \max(q(b), z_b) + (a-x)\ \min(d(a), z_a)\big\} \\
& + I[v < b]\ I[d(b) \leq 0]\ (b-x)\ \max(q(b) + d(b), z_b) \\
& + I[v < b]\ I\big[d(a) < 0 < d(b)\big]\ (b-x)\ \max(q(b), z_b) \\
& + I[b \leq v \leq a]\ \big\{I[d(b) \leq 0]\ (b-x)\ \max(d(b), z_b) \\
& \qquad\qquad + I[d(a) \geq 0]\ (a-x)\ \min(d(a), z_a)\big\} \\
& + I[v > a]\ I[d(b) \leq 0]\big\{(b-x)\ \max(d(b), z_b) + (a-x)\ \min(q(a), z_a)\big\} \\
& + I[v > a]\ I[d(a) \geq 0]\ (a-x)\ \min(d(a) + q(a), z_a) \\
& + I[v > a]\ I\big[d(a) < 0 < d(b)\big]\ (a-x)\ \min(q(a), z_a).
\end{aligned} \tag{12}$$

The profit function $\pi(a, z_a, b, z_b)$ can be written $\pi_a(a, z_a) + \pi_b(b, z_b)$, with

$$\begin{aligned}
\pi_a = {} & I[v \leq a]\ I[d(a) \geq 0]\ (a-x)\ \min(d(a), z_a) \\
& + I[v > a]\ I[d(a) < 0]\ (a-x)\ \min(q(a), z_a) \\
& + I[v > a]\ I[d(a) \geq 0]\ (a-x)\ \min(q(a) + d(a), z_a),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\pi_b = {} & I[v \geq b]\ I[d(b) \leq 0]\ (b-x)\ \max(d(b), z_b) \\
& + I[v < b]\ I[d(b) > 0]\ (b-x)\ \max(q(b), z_b) \\
& + I[v < b]\ I[d(b) \leq 0]\ (b-x)\ \max(q(b) + d(b), z_b).
\end{aligned} \tag{14}$$

The remainder of the article uses the ask side of the market only; the bid side can be incorporated by analogy. The variable z now stands for z_a, and π for π_a.

Normalizations

Capital letters denote variables that are not normalized. Let X be the asset value, $D(P)$ the demand of the liquidity trader at price P, P^* the price that would clear the market were the liquidity trader the sole participant (that is, $D(P^*) = 0$). Whether $D(P)$ is linear in P or in $\log(P)$, assuming $E[P^*] = E[X]$, we can normalize the slope of the liquidity trader's demand to unity by adequately rescaling all variables. The elasticity of substitution between spread and depth is not affected by the change of units, and the standard deviation of the normalized liquidity shock is inversely proportional to the original slope of the liquidity demand.

1. Suppose that X is symmetrically distributed around its mean and that $D(P) = F - HP + \varepsilon$, where $H > 0$ and ε is symmetrically distributed around 0. The random variables can take discrete values or be normally distributed. Subtract $E[X]$ from prices, divide quantities by H, and let $x = X - E[X]$, $\eta = \varepsilon/H$, and $d(p) = D(p + E[X])/H$. Since $E[X] = E[P^*] = F/H$, $d(p) = -p + \eta$.

(a) When the informed trader is risk neutral, further simplifications are possible. Let $\hat{x} = x/\sigma_\eta$, $\hat{v} = E[\hat{x}|G] = v/\sigma_\eta$, $\hat{a} = a/\sigma_\eta$, and $\hat{z} = z/\sigma_\eta$. $E[\pi(a, z, x, \eta, v)]$ is proportional to $E[\pi(\hat{a}, \hat{z}, \hat{x}, \hat{\eta}, \hat{v})]$, which depends on ρ, σ_x and σ_η through the combined parameter $\rho\,\sigma_x/\sigma_\eta$.

(b) When the informed trader is risk averse and $x \sim N(0, \sigma_x^2)$, let $\hat{x} = x/\sigma_x$, $\hat{v} = v/\sigma_x$, $\hat{\eta} = \eta/\sigma_x$, $\hat{a} = a/\sigma_x$, $\hat{z} = z/\sigma_x$, and $\hat{\gamma} = \gamma\,\sigma_x^2$. $E[\pi(a, z, x, \eta, v, \gamma)]$ is proportional to $E[\pi(\hat{a}, \hat{z}, \hat{x}, \hat{\eta}, \hat{v}, \hat{\gamma})]$.

2. Suppose now that X is lognormally distributed and that $D(P) = F - H\log(P) + \varepsilon$, with $H > 0$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. Divide prices by $E[X]$ and quantities by H, and let $x = X/E[X]$, $\eta = \varepsilon/H - (\sigma_\varepsilon/H)^2/2$, and $d(p) = D(E[X]\,p)/H$. Since $E[X] = E[P^*] = \exp(F/H + (\sigma_\varepsilon/H)^2/2)$, $d(p) = -\log(p) + \eta$, with $\eta \sim N(-\sigma_\eta^2/2, \sigma_\eta^2)$.

(a) When the informed trader is risk neutral, the expected profit of the dealer depends on ρ and σ_y only through their product (see Proposition 3); thus the elasticity of substitution can be written as a function of $\rho\,\sigma_y/\sigma_\eta$ and σ_η.

(b) When the informed trader is risk averse, no further normalizations are possible.

Proposition 2. *Let ε be a random variable with continuous density function f and $u(a, z, \varepsilon)$, $v(a, z, \varepsilon)$, $x(a, z)$, $y(a, z)$, four continuously differentiable functions in all their arguments such that $u(a, z, x(a, z)) = v(a, z, x(a, z))$ and either $u(a, z, y(a, z)) = 0$ or $y(a, z)$ is independent of a and z. Let $H(a, z) = E\big[\, I[\, y(a, z) \le \varepsilon \le x(a, z)]\, u(a, z, t) + I[\, \varepsilon > x(a, z)]\, v(a, z, t)\,\big]$, then $\frac{\partial}{\partial a}H(a, z) = E\big[\, I[\, y(a, z) \le \varepsilon \le x(a, z)]\, \frac{\partial}{\partial a}u(a, z, t) + I[\, \varepsilon > x(a, z)]\, \frac{\partial}{\partial a}v(a, z, t)\,\big]$.*

Proof. $H(a, z) = \int_{y(a,z)}^{x(a,z)} u(a, z, t)\, f(t)\, dt + \int_{x(a,z)}^{+\infty} v(a, z, t)\, f(t)\, dt$. Using Leibniz's rule and the fact that $u(a, z, y(a, z)) = 0$ or $\frac{\partial}{\partial a}y(a, z) = 0$, we get

$$\begin{aligned}
\frac{\partial}{\partial a}\int_{y(a,z)}^{x(a,z)} u(a, z, t)\, f(t)\, dt &= \frac{\partial}{\partial a}x(a, z)\, u(a, z, x(a, z))\, f(x(a, z)) \\
&\quad + \int_{y(a,z)}^{x(a,z)} \frac{\partial}{\partial a}u(a, z, t)\, f(t)\, dt, \\
\frac{\partial}{\partial a}\int_{x(a,z)}^{+\infty} v(a, z, t)\, f(t)\, dt &= -\frac{\partial}{\partial a}x(a, z)\, v(a, z, x(a, z))\, f(x(a, z)) \\
&\quad + \int_{x(a,z)}^{+\infty} \frac{\partial}{\partial a}v(a, z, t)\, f(t)\, dt.
\end{aligned}$$

Using $u(a, z, x(a, z)) = v(a, z, x(a, z))$ yields the result.
Application:

$$\begin{aligned}
\frac{\partial}{\partial a}E\big[I[d(a) \ge 0]\ \min(d(a), z)\big] &= d'(a)\, p(0 \le d(a) \le z), \\
\frac{\partial}{\partial z}E\big[I[d(a) \ge 0]\ \min(d(a), z)\big] &= p(d(a) > z), \\
\frac{\partial}{\partial a}E\big[I[v \le a]\,(a - v)\big] &= p(v \le a). \qquad (15)
\end{aligned}$$

Generalization: Computation of $\frac{\partial}{\partial a}E[\pi]$ when the informed trader is risk averse and the asset value is normally distributed.

Let $\Lambda(a,z) = E[\pi] - E\big[\, I[v \le a]\ (a-v)\big]\ E\big[\, I[d(a) \ge 0]\ \min(d(a), z)\big]$,

$$\begin{aligned}\Lambda(a,z) = {} & p(d(a)<0)\ E\big[I[v>a, q(a)\le z]\ (a-v)\ q(a)\big]\\ & + p(d(a)<0)\ E\big[I[v>a, q(a)>z]\ (a-v)\big]\ z\\ & + E\big[I[0\le d(a)\le z]\ I[v>a, q(a)\le z-d(a)]\\ & \qquad \times (a-v)\ \ (q(a)+d(a))\big]\\ & + E\big[I[0\le d(a)\le z]\ I[v>a, q(a)>z-d(a)]\ (a-v)\big]\ z\\ & + p(d(a)>z)\ E\big[\ I[v>a]\,(a-v)\ \big]\ z. \end{aligned} \tag{16}$$

The liquidity demand $d(a)$ is increasing in η. Let $x_1(a,z)$ and $x_2(a,z)$ be such that $d(a)=0$ for $\eta = x_1(a,z)$, and $d(a)=z$ for $\eta = x_2(a,z)$.

$$\begin{aligned}\Lambda(a,z) = E\big[\ & I[\eta < x_1(a,z)]\ \mu(a,z,v)\\ & + I[x_1(a,z)\le \eta \le x_2(a,z)]\ \varphi(a,z,v,\eta)\\ & + I[\eta > x_2(a,z)]\ \psi(a,z,v)\big],\end{aligned}$$

with

$$\begin{aligned}\mu(a,z,v) &= E\big[\ I[v>a, q(a)\le z]\ (a-v)\ q(a)\\ &\qquad\qquad + I[v>a, q(a)>z]\ (a-v)\,z\big],\\ \varphi(a,z,v,\eta) &= E\big[\ I[v>a, q(a)\le z-d(a)]\ (a-v)\ \ (q(a)+d(a))\ |\ \eta\big]\\ &\qquad + E\big[\ I[v>a, q(a)>z-d(a)]\ (a-v)\,z\,|\,\eta\big],\\ \psi(a,z,v) &= E\big[\ I[v>a]\,(a-v)\,z\big],\end{aligned} \tag{17}$$

where $E[\ |\,\eta]$ denotes the conditional expectation conditioned on η. Since $\mu(a,z,v) = \varphi(a,z,v,x_1(a,z))$ and $\varphi(a,z,v,x_2(a,z)) = \psi(a,z,v)$, then

$$\begin{aligned}\frac{\partial}{\partial a}\Lambda(a,z) = E\Big[\ & I[\eta < x_1(a,z)]\ \frac{\partial\mu}{\partial a}\\ & + I[x_1(a,z)\le\eta\le x_2(a,z)]\ \frac{\partial\varphi}{\partial a} + I[\eta > x_2(a,z)]\ \frac{\partial\psi}{\partial a}\Big].\end{aligned}$$

Using Proposition 2 to compute $\frac{\partial}{\partial a}\big\{E\big[\ I[v\le a]\ (a-v)\big]\ E\big[\ I[d(a)\ge 0]\ \min(d(a),z)\big]\big\}$, $\frac{\partial\mu}{\partial a}$, $\frac{\partial\varphi}{\partial a}$, $\frac{\partial\psi}{\partial a}$, and recombining terms yields Equation (10). The derivation of Equation (9) is similar.

Proposition 3. *Let y and G be jointly normally distributed. G has mean 0 and variance 1. The correlation coefficient ρ between y and G is positive. Let $X = \exp(y)$, $x = X/E[X]$, $v = E[x|G]$, $\omega = E[y|G]$, and $\sigma_\omega = \rho\,\sigma_y$. Then $v = \exp(-\sigma_\omega^2/2 + \sigma_\omega G)$.*

Proof. First, $y = E[y|G]+\xi$, with $E[y|G] = E[y]+\rho\,\sigma_y\,G = E[y]+\sigma_\omega\,G$ and $\sigma_y^2 = \sigma_\omega^2+\sigma_\xi^2$. Second, $E[X|G] = E[e^y|G] = E[e^{E[y|G]+\xi}\,|G] = E[e^{E[y|G]}\,e^\xi\,|G]$. As $e^{E[y|G]}$ is measurable with respect to G, $E[X|G] = e^{E[y|G]}\,E[e^\xi|G]$. $E[\xi|G]=0$, ξ and G are jointly normally distributed; hence ξ and G are independent and $E[e^\xi|G] = E[e^\xi] = e^{\sigma_\xi^2/2}$. Hence $E[X|G] = \exp(E[y] + \sigma_\omega\,G + (\sigma_y^2 - \sigma_\omega^2)/2)$. As $E[X] = E[e^y] = \exp(E[y]+\sigma_y^2/2)$, $E[X|G] = E[X]e^{-\sigma_\omega^2/2+\sigma_\omega G}$ and $v = E[x|G] = e^{-\sigma_\omega^2/2+\sigma_\omega G}$.

References

Aiyagari, S., and M. Gertler, 1991, "Asset Returns with Transactions Costs and Uninsured Individual Risk," *Journal of Monetary Economics*, 27, 311–331.

Bagehot, W. (pseud), 1971, "The Only Game in Town," *Financial Analysts Journal*, 27, 12–14.

Dennert, J., 1993, "Price Competition between Market Makers," *Review of Economic Studies*, 60, 735–751.

Dimson, E., 1979, "Risk Measurement when Shares are Subject to Infrequent Trading," *Journal of Financial Economics*, 7, 197–226.

Duffee, G., 1996, "Counterparty Credit Exposure on Total Rate of Return Swaps," internal memorandum, Board of Governors of the Federal Reserve System.

Easley, D., and M. O'Hara, 1987, "Price, Trade Size, and Information in Securities Markets," *Journal of Financial Economics*, 19, 69–90.

Franks, J., and E. Schwartz, 1991, "The Stochastic Behaviour of Market Variance Implied in the Prices of Index Options," *Economic Journal*, 101, 1460–1475.

Glosten, L., 1989, "Insider Trading, Liquidity, and the Role of the Monopolist Specialist," *Journal of Business*, 62, 211–235.

Glosten, L., and P. Milgrom, 1985, "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14, 71–100.

Jennings, R., 1994, "Intraday Changes in Target Firms' Share Price and Bid-Ask Quotes around Takeover Announcements," *Journal of Financial Research*, 17, 255–270.

Kavajecz, K., 1999, "A Specialist's Quoted Depth and the Limit Order Book," *Journal of Finance*, 54, 747–771.

Lee, C., B. Mucklow, and M. Ready, 1993, "Spreads, Depths, and the Impact of Earnings Information: An Intraday Analysis," *Review of Financial Studies*, 6, 345–374.

Lin, J., G. Sanger, and G. Booth, 1995, "Trade Size and Components of the Bid-Ask Spread," *Review of Financial Studies*, 8, 1153–1183.

Mann, S., and R. Seijas, 1991, "Bid-Ask Spreads, NYSE Specialists, and NASD Dealers," *Journal of Portfolio Management*, 17, 54–58.

McInish, T., and R. Wood, 1991, "Hourly Returns, Volume, Trade Size, and Number of Trades," *Journal of Financial Research*, 14, 303–315.

Seppi, D., 1997, "Liquidity Provision with Limit Orders and a Strategic Specialist," *Review of Financial Studies*, 10, 103–150.

Venkatesh, P., and R. Chiang, 1986, "Information Asymmetry and the Dealer's Bid-Ask Spread: A Case Study of Earnings and Dividend Announcements," *Journal of Finance*, 41, 1089–1102.