On the Recoverability of Preferences and Beliefs

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We examine the extent to which an investor's tastes and beliefs can be jointly recovered from knowledge of his/her consumption choice. More precisely, we assume that the investor's preferences admit an expected utility representation, but with subjective (unknown) probabilities, and investigate what joint restrictions can be placed on utility functions and beliefs. If the investor draws utility from intertemporal consumption, we show that the set of utility functions and beliefs that are consistent with a given consumption choice can be characterized by a martingale condition. In the Markovian case, this characterization can be restated in terms of a Riccati differential equation that must be satisfied by the investor's relative risk aversion function. To each solution of this differential equation is associated a unique utility function and a unique set of beliefs supporting the given consumption choice. Moreover, we show that the differential equation has at most one solution in the class of utility functions displaying infinite absolute risk aversion at the origin. Thus, preferences (and associated beliefs) can be uniquely recovered within this class.

Following the work of Mehra and Prescott (1985), there has been considerable empirical evidence that stock risk premia are too high relative to what would be implied by the Lucas (1978) model with a single representative agent having a constant relative risk aversion utility function, at least for "reasonable" levels of risk aversion. In attempting to solve this "puzzle," various extensions of the basic model have been proposed, based on different preference assumptions for the representative agent or on the introduction of market frictions.[1] Recently, several attempts have been made to explain the size of the equity premium puzzle by assuming that investors are heterogeneous, either in terms of their endowments [Constantinides and Duffie (1996)], or in terms of their information/beliefs [e.g., Abel (1990), Detemple and Murthy (1994)], or in terms of their risk aversion [Dumas (1989)]. One

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[1] Hansen and Jagannathan (1991) empirically demonstrate that the conclusions of Mehra and Prescott (1985) are robust and do not depend on the particular parameterization they choose for the aggregate consumption process.

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of the motivations of this article is to study under what circumstances heterogeneity in beliefs and risk aversion might be observationally equivalent.

The relevance and the apparent difficulty of this task were summarized by Kraus and Sick (1980, p. 335):

> Since restrictions on beliefs and restrictions on preferences have both shown themselves to be productive routes to theories of security market equilibrium, the question naturally arises as to whether one can distinguish the effects of beliefs and preferences in observed prices. It is certainly not obvious that this can ever be done. Since individual agent optimality conditions involve the product of probability and marginal utility, it may be that any set of equilibrium prices that are consistent with some combination of beliefs and preferences could also have resulted from different beliefs combined with different preferences.

Kraus and Sick approached this issue by considering a highly parameterized single-period model with a riskless asset, a single risky asset with a normally distributed payoff, and a representative agent having constant absolute risk aversion and derived conditions under which both the mean of the risky asset's payoff and the representative agent's risk aversion coefficient can be uniquely inferred from the two assets' prices.

In this article we consider a general dynamic model with von Neumann–Morgenstern utility functions and possibly non-Markovian price processes. By making use of martingale techniques and Girsanov's theorem, beliefs are represented in full generality as probability measures on the set of observable events. Because of this generality, it is easy to see that if the investor has a utility function for wealth or consumption at a single future date, then the problem is indeed indeterminate, in the sense that, subject to minor technical conditions, for any given utility function and consumption/investment policy, it is possible to construct a set of beliefs that would make the given policy optimal. On the other hand, if the investor draws utility from intertemporal consumption, we show in Section 2 that the set of preferences and beliefs that are consistent with a given consumption process can be characterized by a martingale condition. We formulate our dynamic model in continuous time for ease of notation, but our conclusions also apply to a discrete-time economy.

As an application of our characterization result, we consider a more general version of the "integrability problem" discussed by He and Leland (1993) and Wang (1993a,b) and relating to the possibility of recovering the representative agent's preferences from knowledge of the aggregate consumption process and the equilibrium prices.[2] In Section 3, we specialize our setting to a Markovian environment similar to that considered in the above articles and

[2] Dybvig and Rogers (1997) have recently shown that, subject to some additional assumptions on the conditional distribution of the terminal state-price density (in the case of preferences for terminal wealth) or on the utility function (in the case of preferences for intertemporal consumption), then recoverability of an investor's preferences is possible even from observation of the consumption choice along a *single* sample path.

show that our martingale condition characterizing the set of preferences and beliefs consistent with a given consumption choice can be restated in terms of a Riccati differential equation. The problem of recovering *both* preferences and beliefs amounts to solving such a differential equation. We provide sufficient conditions for this differential equation to admit a unique solution in the class of utility functions displaying infinite absolute risk aversion at the origin. Thus, under the stated conditions, preferences (and associated beliefs) can be uniquely recovered within this class.

Alternatively, our results can be interpreted as providing sufficient conditions for given dividend, stock price, and interest rate processes to be supported in equilibrium by a representative agent with time-additive preferences. Differently from the existing literature [Bick (1987, 1990), He and Leland (1993)] that treats the representative agent's beliefs as exogenous and his utility function as endogenous, we allow both the representative agent's beliefs and his utility function to be endogenously determined when trying to "rationalize" the given processes. Finally, if the beliefs are exogenously fixed, our differential equation reduces to a condition on the equity risk premium that must be satisfied for the assumed price dynamics to be compatible with the existence of a representative agent having a von Neumann–Morgenstern utility function. This extends the result of He and Leland (1993) to a setting with intertemporal consumption and a general interest rate process.

1. The Economy

We consider a standard continuous-time infinite-horizon economy with a dynamically complete financial market.[3]

1.1 Information structure

The uncertainty is represented by the canonical probability space for n-dimensional Brownian motion, defined as follows. Let $\Omega = C([0,\infty))^n$ be the space of continuous functions $\omega : [0,\infty) \to \mathbb{R}^n$. On this space, define the coordinate mapping process $w(t,\omega) = \omega(t)$. Finally, let $\mathcal{F}_t^w = \sigma\{w_s : 0 \leq s \leq t\}$ denote the σ-field generated by the sample path of w on $[0,t]$ and let P^w denote the Wiener measure on $\mathcal{F}_\infty^w$, that is, the probability measure under which w is an n-dimensional Brownian motion. All the stochastic processes to appear in the sequel are assumed to be adapted to the filtration $\{\mathcal{F}_t^w\}$.[4] We remark that the probability measure P^w does not need to be interpreted as

[3] The result in Theorem 1 would be unchanged in a finite-horizon economy. However, the ODE in Theorem 2 would in general become a PDE in a finite-horizon economy if the equilibrium price coefficients depend on calendar time.

[4] We work with the incomplete filtration $\{\mathcal{F}_t^w\}$ in order to avoid the difficulty illustrated by Example 1.7.6 in Karatzas and Shreve (1998). Completion of the filtration is not needed for our subsequent application of Girsanov's theorem and the Martingale Representation theorem.

representing "objective" probabilities or investors' beliefs: as it will become clear in the sequel, the only role of P^w is to identify the set of events to which agents assign positive probability.

1.2 Securities market

The investment opportunities are represented by $n+1$ long-lived securities. The first security is a locally riskless bond whose price is given by

$$B(t) = \exp\left(\int_0^t r(\tau)\, d\tau\right) \tag{1}$$

for some interest rate process r with

$$\int_0^t |r(\tau)|\, d\tau < \infty \quad \text{a.s.}$$

for all $t \in [0, \infty)$. The remaining n assets are risky. Letting $S = (S_1, \ldots, S_n)$ denote their price process and $D = (D_1, \ldots, D_n)$ their cumulative dividend process, we assume that $S + D$ is an Itô process:

$$S(t) + D(t) = S(0) + \int_0^t I_S(\tau)\mu(\tau)\, d\tau + \int_0^t I_S(\tau)\sigma(\tau)\, dw(\tau), \tag{2}$$

where $I_S(t)$ denotes the $n \times n$ diagonal matrix with elements $S(t)$ and

$$\int_0^t |I_S(\tau)\mu(\tau)|\, d\tau + \int_0^t |I_S(\tau)\sigma(\tau)|^2\, d\tau < \infty \quad \text{a.s.}$$

for all $t \in [0, \infty)$.

The following two assumptions on the security price processes will be maintained throughout the article.

Assumption 1. *The diffusion matrix σ satisfies the nondegeneracy condition*

$$x^\top \sigma(t)\sigma(t)^\top x \geq \epsilon |x|^2 \quad \textit{a.s.} \tag{3}$$

for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ and some $\epsilon > 0$.

Assumption 2. *The process*

$$\kappa(t) = \sigma(t)^{-1}(\mu(t) - r(t)\bar{1}),$$

where $\bar{1} = (1, \ldots, 1)^\top \in \mathbb{R}^n$, satisfies

$$\int_0^t |\kappa(\tau)|^2\, d\tau < \infty \quad \textit{a.s.}$$

for all $t \in [0, \infty)$. Moreover, the positive local martingale

$$\xi(t) = \exp\left(-\int_0^t \kappa(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\kappa(\tau)|^2 d\tau\right) \tag{4}$$

is in fact a martingale, that is, $\mathrm{E}_{P^w}[\xi(t)] = 1$ for all $t \in [0, \infty)$, where E_{P^w} denotes the expectation operator under P^w.

Assumption 1 implies in particular that $\sigma(t)$ has full rank a.s. for all $t \in [0, \infty)$, so that the financial market is dynamically complete. Assumption 2 guarantees the existence of a martingale measure [cf. Karatzas and Shreve (1998), Proposition 1.7.4] and rules out arbitrage opportunities.[5]

1.3 Trading strategies

Trading takes place continuously and there are no market frictions. An admissible trading strategy is an $(n+1)$-dimensional vector process (α, θ)—where $\alpha(t)$ and $\theta_k(t)$ denote, respectively, the dollar amount invested at time t in the bond and the kth risky asset—satisfying

$$\int_0^t |\alpha(\tau) r(\tau)|\, d\tau + \int_0^t |\theta(\tau)^\top \mu(\tau)|\, d\tau + \int_0^t |\theta(\tau)^\top \sigma(\tau)|^2\, d\tau < \infty \quad \text{a.s.} \tag{5}$$

and the nonnegative wealth constraint $\alpha(t) + \theta(t)^\top \bar{1} \geq 0$ a.s. for all $t \in [0, \infty)$.

1.4 Consumption set

There is a single consumption good (the numeraire). A *consumption plan* is a strictly positive consumption rate process c with $\int_0^t c(\tau)\, d\tau < \infty$ a.s. for all $t \in [0, \infty)$.

1.5 Preferences

Investors' preferences are represented by an additive expected utility function U for intertemporal consumption,

$$U(c) = \mathrm{E}_P\left[\int_0^\infty e^{-\rho t} u(c(t))\, dt\right],$$

where P is a probability measure on $\mathcal{F}_\infty^w$, $\rho \in \mathbb{R}$ and $u : (0, \infty) \to \mathbb{R}$ is an increasing, strictly concave and three times continuously differentiable felicity function. We will denote by $\mathcal{U}$ the set of felicity functions u satisfying the stated assumptions and by $\mathcal{U}'$ the subset of felicity functions displaying infinite absolute risk aversion at the origin:

$$\mathcal{U}' = \left\{ u \in \mathcal{U} : \lim_{c \downarrow 0} u''(c)/u'(c) = \infty \right\}.$$

The only a priori restriction placed on P is that it be *locally equivalent* to P^w, that is, that for all $t \in [0, \infty)$ the restriction P_t of P to $\mathcal{F}_t^w$ be equivalent to the restriction P_t^w of P^w to $\mathcal{F}_t^w$.[6] We will denote by $\mathcal{P}$ the set of possible "beliefs", that is, the set of probability measures locally equivalent to P^w.

[5] We remark that the martingale measure whose existence is guaranteed by Assumption 2 is not necessarily equivalent to P^w, although it is locally equivalent. The definition of local equivalence will be introduced momentarily.

[6] Two probability measures are *equivalent* if they attribute zero probability to the same events.

Letting

$$\eta_P(t) = \frac{dP_t}{dP_t^w} \tag{6}$$

denote the density process of a probability measure $P \in \mathcal{P}$ with respect to P^w, Theorem III.3.4 in Jacod and Shiryaev (1987) and Proposition 7.11(b) in Jacod (1979) show that η_P can be taken to be a strictly positive P^w-martingale. It then follows from the Martingale representation theorem [Jacod and Shiryaev (1987), Theorem III.4.33] and Girsanov's theorem [Jacod and Shiryaev (1987), Theorem III.3.11] that η_P has the stochastic exponential representation

$$\eta_P(t) = \exp\left(-\int_0^t \gamma_P(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\gamma_P(\tau)|^2\, d\tau\right)$$

for some n-dimensional process γ_P with $\int_0^t |\gamma_P(\tau)|^2\, d\tau < \infty$ a.s. for all $t \in [0, \infty)$ and that the process

$$w_P(t) = w(t) + \int_0^t \gamma_P(\tau)\, d\tau$$

is a standard Brownian motion under P. We remark that under P, the gain process $S + D$ is still an Itô process, with representation

$$S(t) + D(t) = S(0) + \int_0^t I_S(\tau)\mu_P(\tau)\, d\tau + \int_0^t I_S(\tau)\sigma(\tau)\, dw_P(\tau),$$

where $\mu_P = \mu - \sigma\gamma_P$.

2. Viable Consumption Plans

Under the stated assumptions, it is well known that the investor's problem of maximizing the lifetime expected utility over the consumption plans that are financed by an admissible trading strategy can be written in the following static form [cf. Huang and Pagès (1992) or Karatzas and Shreve (1998)]:

$$\max_{c \geq 0} \mathrm{E}_P\left[\int_0^\infty e^{-\rho t} u(c(t))\, dt\right]$$

$$\text{s.t. } \mathrm{E}_P\left[\int_0^\infty \pi_P(t) c(t)\, dt\right] = W_0,$$

where W_0 denotes the time 0 value of the investor's lifetime endowment,

$$\pi_P(t) = B(t)^{-1} \exp\left(-\int_0^t \kappa_P(\tau)^\top dw_P(\tau) - \frac{1}{2}\int_0^t |\kappa_P(\tau)|^2\, d\tau\right), \tag{7}$$

and $\kappa_P = \sigma^{-1}(\mu_P - r\bar{1})$. The process π_P identifies the state-price density with respect to the investor's beliefs P.

The above immediately implies that, if a consumption plan $c^* \in \mathcal{C}$ is optimal for a pair $(u, P) \in \mathcal{U} \times \mathcal{P}$ of preferences and beliefs, then it must satisfy the first-order condition

$$e^{-\rho t} u_c(c^*(t)) = \psi \pi_P(t) \tag{8}$$

for some Lagrangian multiplier $\psi > 0$. Defining the state-price density with respect to the Wiener measure P^w by

$$\pi(t) = B(t)^{-1} \exp\left(-\int_0^t \kappa(\tau)^\top dw(\tau) - \frac{1}{2}\int_0^t |\kappa(\tau)|^2 \, d\tau\right),$$

it can be easily verified that $\pi_P = \eta_P^{-1}\pi$, where η_P is the process in Equation (6). Thus we can rewrite Equation (8) as

$$e^{\rho t}\pi(t)/u_c(c^*(t)) = \eta_P(t)/\psi, \tag{9}$$

and we have the following characterization of the consumption plans which are optimal for a given utility function $u \in \mathcal{U}$ and some set of beliefs $P \in \mathcal{P}$.

Theorem 1. *Let $c^* \in \mathcal{C}$ be a consumption plan and let $\rho \in \mathbb{R}$ and $u \in \mathcal{U}$ be given. Then there exists a $P \in \mathcal{P}$ such that c^* is optimal for an investor with felicity function $u \in \mathcal{U}$, beliefs $P \in \mathcal{P}$, and endowment*

$$W_0 = \mathrm{E}_{P^w}\left[\int_0^\infty \pi(t) c^*(t)\, dt\right]$$

if and only if the process $e^{\rho t}\pi(t)/u_c(c^*(t))$ is a P^w-martingale. If the above condition is satisfied, then the beliefs P are uniquely determined by

$$\frac{dP_t}{dP_t^w} = \frac{u_c(c^*(0))e^{\rho t}\pi(t)}{u_c(c^*(t))} \tag{10}$$

for all $t \in [0, \infty]$.

Proof. Necessity is immediate from Equation (9), the fact that η_P is a P^w-martingale, and the fact that

$$\begin{aligned}\mathrm{E}_{P^w}\left[\int_0^\infty \pi(t)c^*(t)\,dt\right] &= \mathrm{E}_{P^w}\left[\int_0^\infty \eta_P(t)\pi_P(t)c^*(t)\,dt\right] \\ &= \mathrm{E}_P\left[\int_0^\infty \pi_P(t)c^*(t)\,dt\right] = W_0.\end{aligned}$$

Conversely, if $e^{\rho t}\pi(t)/u_c(c^*(t))$ is a P^w-martingale, then Theorem VIII.1.13 in Revuz and Yor (1991) implies that there exists a unique probability measure P on $\mathcal{F}_\infty^w$ such that

$$\eta_P(t) = \frac{dP_t}{dP_t^w} = \frac{u_c(c^*(0))e^{\rho t}\pi(t)}{u_c(c^*(t))},$$

so that Equation (9) is satisfied and c^* is optimal for (u, P). The fact that $P \in \mathcal{P}$ (i.e., that P is locally equivalent to P^w) follows from Propositions 7.14(a) and 7.11(b) in Jacod (1979). ■

While in the case of preferences for terminal wealth only, it is easy to see that, given any consumption plan $c^* \in \mathcal{C}$ and any felicity function $u \in \mathcal{U}$ satisfying minor technical conditions, it is always possible to construct beliefs $P \in \mathcal{P}$ that would make the given policy optimal for that particular felicity function. Theorem 1 shows that a similar result does not apply to the case of preferences for intertemporal consumption. A given consumption plan is not necessarily optimal for an arbitrary felicity function given an appropriate choice of beliefs, as the above martingale condition is not necessarily satisfied for arbitrary $u \in \mathcal{U}$.

It is also worthwhile remarking that the condition of Theorem 1 does not depend on the particular choice of the probability measure P^w: given any $P' \in \mathcal{P}$, it follows immediately from Proposition III.3.8(a) in Jacod and Shiryaev (1987) that the process $e^{\rho t}\pi(t)/u_c(c^*(t))$ is a P^w-martingale if and only if the process $e^{\rho t}\pi_{P'}(t)/u_c(c^*(t))$ is a P'-martingale, in which case the beliefs P of Theorem 1 also satisfy

$$\frac{dP_t}{dP'_t} = \frac{u_c(c^*(0))e^{\rho t}\pi_{P'}(t)}{u_c(c^*(t))}.$$

3. The Recoverability Problem

One application of the above result relates to the literature on the *recoverability* (or *integrability*) problem. This line of research is concerned with the possibility of recovering preferences of the investor by observing consumption/investment decisions. Preferences are assumed to have an expected utility representation, and beliefs are treated as objective and hence assumed to be known. Wang (1993a) studied the recoverability problem in a time-homogeneous model with intertemporal consumption and showed that the representative agent's felicity function can be recovered (although not uniquely) under mild technical conditions. On the other hand, this recoverability result was based on the assumption that the only traded asset was a claim to aggregate consumption and thus ignored the information contained in the interest rate process. In a discrete-time setting with a Markov output growth rate process and two traded assets (a bond and a claim to aggregate consumption), Wang (1993b) showed that recoverability is *generically* possible within the larger class of preferences introduced by Kreps and Porteus (1978).

Finally, He and Leland (1993) considered a setting with a constant interest rate, a single risky asset which pays no dividends and follows a diffusion, and a representative agent that maximizes utility from terminal wealth. They showed that the representative agent's utility function can be recovered from knowledge of the equity risk premium at the terminal date and obtained a partial differential equation that must be satisfied by the relative risk premium for it to be consistent with the representative agent optimally holding the risky asset [see also Bick (1987, 1990)]. Again, they ignored the information in the interest rate process, since a model with no intertemporal consumption places no restriction on the equilibrium interest rate.

We investigate next to what extent simultaneous recovery of tastes and beliefs is possible. As in Wang (1993a), we consider a simple continuous-time time-homogeneous Lucas economy with a single representative agent. However, we assume that two assets are traded: a locally riskless bond and a risky asset. The supply of the risky asset is normalized to one share, while the riskless asset is taken to be in zero net supply. The representative agent is endowed with the share of the risky asset at time 0. The risky asset pays a strictly positive dividend rate process y, where

$$y(t) = y(0) + \int_0^t m(y(\tau))y(\tau)\,d\tau + \int_0^t s(y(\tau))y(\tau)\,dw(\tau) \tag{11}$$

for some continuous functions $m, s : (0, \infty) \to \mathbb{R}$ with $s(y) \neq 0$ for all $y \in (0, \infty)$. We assume that for all $t \in (0, \infty)$ the random variable $y(t)$ has support $(0, \infty)$ and focus on Markovian equilibria in which

$$B(t) = \exp\left(\int_0^t \hat{r}(y(\tau))\,d\tau\right) \tag{12}$$

and

$$S(t) + \int_0^t y(\tau)\,d\tau = S(0) + \int_0^t \hat{\mu}(y(\tau))S(\tau)\,d\tau + \int_0^t \hat{\sigma}(y(\tau))S(\tau)\,dw(\tau) \tag{13}$$

for some continuous functions $\hat{r}, \hat{\mu}, \hat{\sigma} : (0, \infty) \to \mathbb{R}$ with $\hat{\sigma}(y) \neq 0$ for all $y \in (0, \infty)$. Clearly, Equations (12) and (13) are special cases of Equations (1) and (2), respectively. The equilibrium condition for this economy is that the representative agent optimally holds the share of the risky asset, consuming the dividend y. The following theorem completely characterizes the set of tastes and beliefs $(u, P) \in \mathcal{U} \times \mathcal{P}$ for the representative agent that are consistent with an equilibrium.

Theorem 2. *Let $\hat{\kappa}(y) = (\hat{\mu}(y) - \hat{r}(y))/\hat{\sigma}(y)$ and let $\rho \in \mathbb{R}$ be given. There exists a pair $(u, P) \in \mathcal{U} \times \mathcal{P}$ of tastes and beliefs for the representative agent that is consistent with an equilibrium for the above economy if and only if there exists a strictly positive solution T to the Riccati differential equation*

$$\begin{aligned}\frac{1}{2}s(y)^2y^2[T'(y) - 1] - [m(y) - \hat{\kappa}(y)s(y)]yT(y)\\ + [\hat{r}(y) - \rho]T(y)^2 = 0\end{aligned} \tag{14}$$

that satisfies the integrability condition

$$\begin{aligned}\mathrm{E}_{P^w}\Bigg[\exp\Bigg(-\int_0^t \left(\hat{\kappa}(y(\tau)) - \frac{s(y(\tau))y(\tau)}{T(y(\tau))}\right)dw(\tau)\\ - \frac{1}{2}\int_0^t \left(\hat{\kappa}(y(\tau)) - \frac{s(y(\tau))y(\tau)}{T(y(\tau))}\right)^2 d\tau\Bigg)\Bigg] = 1\end{aligned}$$

for all $t \in [0, \infty)$. *If the above condition is satisfied, then*

$$u(c) = \psi_1 + \psi_2 \int_{y_0}^{c} \exp\left(- \int_{y_0}^{x} \frac{dy}{T(y)} \right) dx \tag{15}$$

and

$$\frac{dP_t}{dP_t^w} = \exp\left(- \int_0^t \hat{\gamma}_P(y(\tau))\, dw(\tau) - \frac{1}{2} \int_0^t \hat{\gamma}_P(y(\tau))^2\, d\tau \right) \tag{16}$$

for all $t \in [0, \infty)$, *where* $\psi_1 \in \mathbb{R}$ *and* $\psi_2 > 0$ *are arbitrary constants that do not affect preferences and*

$$\hat{\gamma}_P(y) = \hat{\kappa}(y) - \frac{s(y)y}{T(y)}. \tag{17}$$

Proof. Suppose first that $(u, P) \in \mathcal{U} \times \mathcal{P}$ is a pair of preferences and beliefs for the representative agent that is consistent with an equilibrium. Let $\psi_1 = u(y_0)$, $\psi_2 = u_c(y_0)$, $v(c) = \log(u_c(c)/\psi_2)$, and

$$T(y) = -\frac{u_c(y)}{u_{cc}(y)} = -\frac{1}{v'(y)}.$$

Then

$$v(c) \;=\; - \int_{y_0}^{c} \frac{dy}{T(y)}$$

and

$$u_c(c) = \psi_2 \exp(v(c)) = \psi_2 \exp\left(- \int_{y_0}^{c} \frac{dy}{T(y)} \right). \tag{18}$$

In analogy with Equation (7), let

$$\pi(t) = B(t)^{-1} \exp\left(- \int_0^t \hat{\kappa}(y(\tau))^\top dw(\tau) - \frac{1}{2} \int_0^t |\hat{\kappa}(y(\tau))|^2\, d\tau \right). \tag{19}$$

By Theorem 1, the pair (u, P) is consistent with an equilibrium if and only if the process $e^{\rho t}\pi(t)/u_c(y(t))$ is a P^w-martingale, in which case

$$\frac{dP_t}{dP_t^w} = \frac{\psi_2 e^{\rho t}\pi(t)}{u_c(y(t))}.$$

By the martingale representation theorem, the above condition amounts to the existence of a process γ_P such that

$$\frac{\psi_2 e^{\rho t}\pi(t)}{u_c(y(t))} = \exp\left(- \int_0^t \gamma_P(\tau)\, dw(\tau) - \frac{1}{2} \int_0^t \gamma_P(\tau)^2\, d\tau \right), \tag{20}$$

or, equivalently [using Equations (18) and (19) and taking logs],

$$- \int_0^t \hat{\kappa}(y(\tau))\, dw(\tau) - \int_0^t \left(\hat{r}(y(\tau)) - \rho + \frac{1}{2}\hat{\kappa}(y(\tau))^2 \right) d\tau - v(y(t))$$
$$= - \int_0^t \gamma_P(\tau)\, dw(\tau) - \frac{1}{2} \int_0^t \gamma_P(\tau)^2\, d\tau.$$

Also, by Itô's lemma,

$$
\begin{aligned}
v(y(t)) &= \int_0^t s(y(\tau))y(\tau)v'(y(\tau))\,dw(\tau) \\
&\quad + \int_0^t \left(\frac{s(y(\tau))^2}{2}\, y(\tau)^2 v''(y(\tau)) + m(y(\tau))y(\tau)v'(y(\tau)) \right) d\tau \\
&= -\int_0^t \frac{s(y(\tau))y(\tau)}{T(y(\tau))}\, dw(\tau) \\
&\quad + \int_0^t \left(\frac{s(y(\tau))^2 y(\tau)^2 T'(y(\tau))}{2T(y(\tau))^2} - \frac{m(y(\tau))y(\tau)}{T(y(\tau))} \right) d\tau.
\end{aligned}
$$

The last two equations imply $\gamma_P(t) = \hat{\gamma}_P(y(t))$, where $\hat{\gamma}_P(y) = \hat{\kappa}(y) - \frac{s(y)y}{T(y)}$, and

$$
\begin{aligned}
\frac{s(y)^2 y^2 T'(y)}{2T(y)^2} - \frac{m(y)y}{T(y)} + \hat{r}(y) - \rho &= \frac{1}{2}\left(\hat{\gamma}_P(y)^2 - \hat{\kappa}(y)^2 \right) \\
&= \frac{s(y)^2 y^2}{2T(y)^2} - \frac{\hat{\kappa}(y)s(y)y}{T(y)}.
\end{aligned}
$$

The strict positivity of T follows from the definition and the fact that $u \in \mathcal{U}$, while the integrability condition follows from Equation (20) and the martingale property of $e^{\rho t}\pi(t)/u_c(y(t))$.

The proof of sufficiency is similar. ∎

Theorem 2 shows that recovering the set of pairs (u, P) consistent with the aggregate consumption process y in Equation (11) and the equilibrium price process (B, S) in Equations (12) and (13) amounts to determining the set of strictly positive solutions T to the ordinary differential equation (ODE) [Equation (14)], which in addition satisfy the integrability condition. Each such solution T corresponds to a unique (up to a linear transformation that does not affect preferences) utility function $u \in \mathcal{U}$ through Equation (15) and a unique set of beliefs $P \in \mathcal{P}$ through Equation (16) that are consistent with an equilibrium. The function T corresponds to the Arrow–Pratt coefficient of absolute risk tolerance associated with u.

We remark that the set of solutions T to the ODE [Equation (14)] is, as we would expect, unaffected by the particular choice of the probability measure P^w, as we have from Girsanov's theorem

$$
m_{P'} - \hat{\kappa}_{P'} s = m - \hat{\gamma}_{P'} s - \frac{\hat{\mu} - \hat{\gamma}_{P'}\sigma - \hat{r}}{\sigma} s = m - \hat{\kappa} s
$$

for all $P' \in \mathcal{P}$.

In the case of objective beliefs ($P = P^w$), we have $\hat{\gamma}_P \equiv 0$, and thus, assuming that $\hat{\kappa}(y) \neq 0$ for all $y \in (0, \infty)$, Equation (17) implies $T(y) = s(y)y/\hat{\kappa}(y)$, so that the representative agent's utility function is uniquely identified (up to a linear transformation). The ODE [Equation (14)] then becomes a condition that must be satisfied by the coefficients of the price process for them to be consistent with an equilibrium with a single representative agent with a felicity function in $\mathcal{U}$. This generalizes the results of Bick (1987, 1990) and He and Leland (1993) to allow for intertemporal consumption and a nonconstant interest rate process.

The next proposition provides sufficient conditions for the above ODE to admit a unique solution satisfying the boundary condition $\lim_{y\downarrow 0} T(y) = 0$. Thus, preferences (and associated beliefs) are in this case uniquely determined in the class $\mathcal{U}'$ of utility function displaying infinite absolute risk aversion at the origin (a class which includes the CRRA utility functions typically used in finance). This extends the recoverability result of Kraus and Sick (1980) to an intertemporal setting and to a much larger nonparametric class of preferences and beliefs.

Proposition 1. *Suppose that the functions $a(y) = \frac{m(y)-\hat{\kappa}(y)s(y)}{s(y)^2 y}$ and $b(y) = \frac{\hat{r}(y)-\rho}{s(y)^2 y^2}$ are continuous on $[0, \infty)$, with b nonnegative. Then the ODE [Equation (14)] has a unique strictly positive solution T with $\lim_{y\downarrow 0} T(y) = 0$. Thus, there exists a unique pair $(u, P) \in \mathcal{U}' \times \mathcal{P}$ (up to a linear transformation of u that does not affect preferences) of tastes and beliefs for the representative agent that is consistent with an equilibrium.*

Proof. Under the given assumptions, the ODE [Equation (14)] can be rewritten as

$$T'(y) = 1 + 2a(y)T(y) - 2b(y)T(y)^2. \tag{21}$$

The fact that the Riccati equation [Equation (21)] subject to the boundary condition $T(0) = 0$ has a nonnegative solution on $[0, \infty)$ follows from Theorem IV.5.2 in Fleming and Rishel (1975), while uniqueness follows from standard results [e.g., Corollary 2.4.3 in Miller and Michel (1982)]. The fact that the solution is strictly positive on $(0, \infty)$ follows from the fact that $T(0) = 0$ and Equation (21) implies $T'(y) = 1$ whenever $T(y) = 0$. ■

The conditions of the previous proposition do not apply to the case in which the coefficients $\hat{r}$, $\hat{\kappa}$, m, and s are constant. However, for this case we can prove a stronger result.

Proposition 2. *Suppose that the functions $\hat{r}$, $\hat{\kappa}$, m, and s are constant on $(0, \infty)$, with $\hat{r} > \rho$ and $s > 0$. Then there exists a unique pair $(u, P) \in \mathcal{U} \times \mathcal{P}$ (up to a linear transformation of u) of tastes and beliefs for the representative agent that is consistent with an equilibrium. In particular, letting*

$$\alpha = \frac{\frac{s^2}{2} - m + \hat{\kappa}s + \sqrt{\left(\frac{s^2}{2} - m + \hat{\kappa}s\right)^2 + 2s^2(r - \rho)}}{s^2} > 0 \tag{22}$$

and

$$\hat{\gamma}_P = \hat{\kappa} - \alpha s,$$

we have

$$u(c) = \begin{cases} c^{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(c) & \text{otherwise} \end{cases} \tag{23}$$

and

$$\frac{dP_t}{dP_t^w} = \exp\left(\hat{\gamma}_P w(t) - \frac{1}{2}\hat{\gamma}_P^2 t\right). \tag{24}$$

Proof. Under the stated assumptions, the solution of the ODE [Equation (14)] with an arbitrary initial condition $T(y_0) = T_0 > 0$ is given by

$$T(y) = \frac{(\alpha s^2 y_0 + 2(r-\rho)T_0)y^\beta - \alpha s^2(y_0 - \alpha T_0)y_0^\beta}{\alpha(\alpha s^2 y_0 + 2(r-\rho)T_0)y^\beta + 2(r-\rho)(y_0 - \alpha T_0)y_0^\beta}\, y, \tag{25}$$

where α is the constant in Equation (22) and

$$\beta = 2\,\frac{\sqrt{\left(\frac{s^2}{2} - m + \hat{\kappa}s\right)^2 + 2s^2(r-\rho)}}{s^2} > 0.$$

Taking $T_0 = y_0/\alpha$, Equation (25) gives $T(y) = y/\alpha$ for all y, which is the only solution that is defined and strictly positive on $(0, \infty)$. Since $y/T(y)$ is constant in this case, the integrability condition of Theorem 1 is trivially satisfied. Thus, both preferences and beliefs can be uniquely recovered in the class $\mathcal{U} \times \mathcal{P}$ (or in $\mathcal{U}' \times \mathcal{P}$) in this case and are given by Equations (23) and (24), respectively. ■

Remark. The case of constant coefficients provides a simple illustration of the implications of treating beliefs as subjective. If beliefs are treated as objective (i.e., $P = P^w$), then $\hat{\gamma}_P = 0$ and Equation (17) immediately gives $T(y) = ys/\hat{\kappa}$, implying a constant relative risk aversion function with relative risk aversion coefficient $\hat{\kappa}/s$ (which is consistent with the CCAPM). Equation (14) then becomes a restriction on the price coefficients,

$$\hat{r} - \rho = m\frac{\hat{\kappa}}{s} - \frac{1}{2}\hat{\kappa}s\left(\frac{\hat{\kappa}}{s} + 1\right), \tag{26}$$

which restates the well-known relation between the equilibrium interest rate and the rate of change in the marginal utility of the representative agent:

$$\hat{r} - \rho = -\frac{myu_{cc}(y) + \frac{1}{2}s^2y^2u_{ccc}(y)}{u_c(y)}. \tag{27}$$

Treating beliefs as subjective, Proposition 2 shows that the representative agents' utility function can still be uniquely recovered and that it must still display constant relative risk aversion. Moreover, a given price process can be supported in equilibrium whether or not Equation (26) holds, as Equation (27) is always satisfied under the appropriate choice of beliefs (as given in Proposition 2).

4. Conclusion

We have examined the extent to which a given consumption/investment strategy can be "rationalized" by a simultaneous choice of an agent's preferences and beliefs. We have shown that when the investor derives utility from intertemporal consumption, knowledge of the consumption strategy does allow inferences about preferences and beliefs. In general, the problem becomes one of considering the possible existence of utility functions that will satisfy a martingale condition. In the Markovian case, this martingale condition can be stated as a Riccati differential equation that must be satisfied by the risk aversion function. Associated beliefs can then also be identified. Finally, we have provided a set of technical conditions guaranteeing that both preferences and beliefs can be uniquely recovered in a large nonparametric class. The extension of previous results to allow for the joint recoverability of both preferences and beliefs is made possible by properly exploiting the information in the equilibrium interest rate process.

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