Valuing American Put Options Using Gaussian Quadrature

Michael A. Sullivan
Office of the Comptroller of the Currency

This article develops an efficient and accurate method for numerical evaluation of the integral equation which defines the American put option value function. Numerical integration using Gaussian quadrature and function approximation using Chebyshev polynomials are combined to evaluate recursive expectations and produce an approximation of the option value function in two dimensions, across stock prices and over time to maturity. A set of such solutions results in a multidimensional approximation that is extremely accurate and very quick to compute. The method is an effective alternative to finite difference methods, the binomial model, and various analytic approximations.

American-style options allow early exercise at the discretion of the option holder. This simple feature leads to major complications for valuation theory and defeats attempts to find closed-form solutions for American option values. In response, a variety of techniques have been offered over the years, each building in some way on the insights used to price European options. This article develops an efficient and accurate technique for approximating the option value function using a combination of numerical integration and function approximation.

The binomial method of Cox, Ross, and Rubinstein (1979) is the benchmark technique for pricing American options because it is easy to implement and delivers accurate results. However, Geske and Shastri (1985) show that the finite difference method can have an advantage, producing the same level of accuracy with less computation time, when many options need to be valued. Both approaches use a discrete time and discrete stock price process to approximate the underlying continuous process, but the finite difference method produces option values on a grid of stock prices while the binomial tabulates a single option value at a time. Several other routines developed recently are based on analytical results that avoid discretization. They also produce explicit solutions for hedge ratios which would require multiple applications of the binomial method in most circumstances.

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The technique we propose builds on the risk-neutral pricing relation introduced by the binomial method and follows the same strategy of recursively solving for values at discrete exercise dates. However, we do not make the stock price process discrete. We use Gaussian quadrature to evaluate risk-neutral expectations, keeping the log normal density rather than replacing it with a binomial. And we use Chebyshev polynomial approximation of the option value function, rather than restricting it to discrete points.[1] The routine produces a complete description of the option value function in the stock price and maturity dimensions. Based on a set of such solutions we show how to implement multidimensional polynomial approximation that greatly reduces marginal computation time while preserving accuracy. Hedge ratios can be directly evaluated from this approximating function.

The following sections describe the application of Gaussian quadrature and Chebyshev polynomial approximation, explore the convergence properties of the resulting method with comparisons to the finite difference and binomial methods, develop the multidimensional approximation, and compare the routine to several analytical approximations. Results are developed for American put options on a nondividend paying stock when stock prices vary while volatility and interest rates are fixed over time. Extensions of the method are laid out in the conclusion.

1. Valuing Discretely Exercisable Options

We use the Black and Scholes (1973) setting of perfect capital markets with continuous trading to value an option which can only be exercised on discrete dates. Consider a put option written at an initial time 0 with a strike price K and maturity date M on a nondividend paying stock. Assume the stock price follows geometric Brownian motion with constant variance parameter σ^2 and the risk-free rate r is constant. Suppose the option may be exercised early but only at n dates separated by time intervals of length $\Delta = M/n$. The value of the option is the greater of the value on immediate exercise and that from holding the option. The latter equals the risk-neutral expectation of its value at the next possible exercise date discounted at r. The expectation is taken for changes in the log stock price, z, which are normally distributed with mean $u_z = (r - \sigma^2/2)\Delta$ and variance $\sigma_z^2 = \sigma^2\Delta$. For an option with remaining maturity m

$$V(S, m) = \max\left(K - S, e^{-r\Delta} E\left[V(Se^z, m - \Delta)\right]\right) \qquad m = \Delta, 2\Delta, \ldots, M \quad (1)$$

[1] Judd (1992) develops a similar approach, explains its advantages, and applies it to a stochastic growth problem.

and at maturity $V(S,0) = \max(K - S, 0)$. The largest stock price at which the put option value equals its exercise value is the exercise boundary, $B(m)$, which satisfies $K - B(m) = V(B(m), m)$. At current stock prices above the exercise boundary the option holder will choose not to exercise.

The integral representation of Equation (1) forms the basis of our approximation routine. Let $f(z)$ be the probability density function of log stock price changes. Breaking up the integration interval of the risk-neutral expectation at $B(m - \Delta)$ gives two components of the option value when $S > B(m)$:

$$\begin{aligned} V(S,m) &= e^{-r\Delta}\int_{-\infty}^{\ln(B(m-\Delta)/S)} (K - Se^{z}) f(z)\,dz \\ &\qquad + e^{-r\Delta}\int_{\ln(B(m-\Delta)/S)}^{\infty} V(Se^{z}, m - \Delta) f(z)\,dz \\ &\equiv U(S,m) + W(S,m). \end{aligned} \tag{2}$$

The first component has an explicit solution similar to the standard Black−Scholes option value

$$\begin{aligned} U(S,m) &= Ke^{-r\Delta}N\big(d_2(S, B(m-\Delta))\big) - SN\big(d_1(S, B(m-\Delta))\big) \\ d_2(x,y) &= \big(\ln(y/x) - u_z\big)/\sigma_z \\ d_1(x,y) &= d_2(x,y) + \sigma_z, \end{aligned} \tag{3}$$

where $N(d)$ is the cumulative standard normal distribution evaluated at d. The value differs from that of a European put in that here the time to possible exercise is Δ and the exercise boundary is less than the exercise price except at maturity. The second component, $W(S,m)$, is defined recursively and we will use numerical integration to evaluate it.

1.1 Approximate integration

In general terms, the problem is to estimate the integral of a given function, $g(x)$, over an interval $[\beta, \alpha]$. Quadrature rules approximate the integral with a linear combination of function values in the interval. A particular rule specifies a set of abscissas x_i and associated weights w_i for estimating the value of the integral by a weighted sum:

$$\int_{\beta}^{\alpha} g(x)\,dx \cong \sum_i g(x_i) w_i. \tag{4}$$

Routines can be devised to give exact results when $g(x)$ is a polynomial function of a given degree. The highest degree polynomial which is exactly evaluated is called the order of the quadrature rule. The classic techniques for approximate integration, the trapezoidal rule and Simpson's rule, use a fixed number of equally spaced abscissas and choose the weights to achieve maximum order. They are commonly used in compound form: the interval of integration is divided into q subintervals of equal size, the rule is applied to each subinterval, and the sum of the results is used as an estimate of the integral. A binomial lattice with q time steps has similar structure, with $q+1$ terminal stock prices at which the option payoff is evaluated, but the weights are binomial probabilities.

The classic rules converge to the actual integral as q increases when the integrand is a continuous function, as discussed in Davis and Rabinowitz (1984). But convergence can be slow so that an accurate estimate may require large q and many function evaluations. Gaussian rules choose the q abscissas and q weights to produce a $2q-1$ order rule, the highest possible with q points. The Gauss–Legendre quadrature rule over the interval $[-1, 1]$ is the solution to

$$x_1^j w_1 + x_2^j w_2 + \ldots + x_q^j w_q = \int_{-1}^{1} x^j \quad j = 0, 1, \ldots, 2q-1. \tag{5}$$

Abscissas and weights which solve Equation (5) for various q are tabulated in Abramowitz and Stegun (1964). To apply the results to an arbitrary finite interval the tabulated solutions are rescaled: the abscissas for an interval $[\beta, \alpha]$ are $z_i = (x_i(\alpha - \beta) + \alpha + \beta)/2$, and the weights are $y_i = w_i(\alpha - \beta)/2$. Gaussian rules converge to the actual integral as q goes to infinity and often deliver high accuracy with small q conserving on the number of function evaluations.

To compare the various quadrature rules, consider a European put option for which the Black–Scholes formula gives a benchmark solution. The option's value is defined by an integral taken over log stock price changes ranging from $-\infty$ to $\ln(K/S)$. To estimate the value using quadrature rules, we truncate the interval at a log stock price change 6 standard deviations below its mean, $\beta = u_z - 6\sigma_z$, and use an upper boundary of $\alpha = \ln(K/S)$. Table 1 shows that Gaussian quadrature is orders of magnitude more efficient than the other methods for a particular set of parameters and the same holds for other choices of S, K, M, σ, and r.

We will apply Gaussian quadrature to evaluate $W(S, m)$. In that case the interval of integration has a finite lower boundary but goes to $+\infty$. We truncate it to a finite interval whose boundaries depend on the current stock price as follows: $\beta = \max[\ln(B(m - \Delta)/S), u_z - 6\sigma_z]$ and $\alpha = u_z + 6\sigma_z$. For lower stock prices there is a significant chance of being exercised at the next possible date and β corresponds to a stock price that just reaches the exercise boundary. For higher stock prices there may be only a remote possibility of exercise and β is set at 6 standard deviations below the mean of z.

Table 1
Comparing methods for approximating a European put option price

q	Binomial	Trapezoidal	Simpson's	Gauss—Legendre
4	2.6567	1.8361	2.3783	2.2858
8	2.5248	2.2965	2.4500	2.4276
16	2.4726	2.3955	2.4284	2.4275
32	2.4492	2.4195	2.4275	2.4275
64	2.4382	2.4255	2.4275	2.4275
128	2.4328	2.4270	2.4275	2.4275
256	2.4301	2.4273	2.4275	2.4275
Exact solution 2.4275				
		Percent approximation error		
4	9.4452	− 24.3624	− 2.0235	− 5.8376
8	4.0112	− 5.3939	0.9289	0.0050
16	1.8595	− 1.3187	0.0397	− 1E-06
32	0.8965	− 0.3279	0.0023	− 1E-06
64	0.4403	− 0.0819	0.0001	− 1E-06
128	0.2182	− 0.0205	1E-05	− 1E-06
256	0.1086	− 0.0051	− 1E-07	− 1E-06

Four different methods for approximating the integral representation of the expectation term defining the value of a European put option, $e^{-rM}\int_{-\infty}^{\ln(K/S)}(K - Se^{z})f(z)dz$, by a weighted sum of integrand evaluations are compared. Option parameters are strike price $K = 40$, years to maturity $M = 0.3333$, annualized volatility $\sigma = 0.30$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$; z is the random log change in stock price over $M, f(z)$ is the pdf of the normal distribution with mean $(r - \sigma^2/2)M$ and standard deviation of $\sigma M^{1/2}$. The binomial method uses a q-step lattice evolving from S that effectively divides the range of Se^z into q subintervals, defines the $q + 1$ endpoints for evaluating the option payoff, and sets the associated weights as binomial probabilities. For the other methods the integral is approximated over the interval $[\beta,\alpha]$, where β is 6 standard deviations below the mean of $\ln(Se^z)$ and $\alpha = \ln(K/S)$. The trapezoidal method and Simpson's rule evaluate the option payoff at the $q + 1$ endpoints of q equal-size subintervals of $[\beta,\alpha]$. The trapezoidal method uses weights that exactly integrate linear functions over each subinterval. Simpson's rule uses weights that exactly integrate cubic functions over pairs of adjacent subintervals. Gauss–Legendre quadrature evaluates the option payoff at q interior points of $[\beta,\alpha]$ that correspond to the zeroes of a q-order Legendre polynomial and uses weights that exactly integrate polynomials up to order $2q - 1$ over the integration interval. The exact solution is given by the Black–Scholes formula.

To value an option that has n exercise dates we work backwards from the maturity date. At each step we determine how to value an option assuming it is not immediately exercised and then locate the exercise boundary above which it is optimal not to exercise. At $m = \Delta$ an option that is not exercised has the same value as a European option and the Black–Scholes formula gives its price. The exercise boundary is found by trial and error as the stock price $B(\Delta)$ which satisfies $K - B(\Delta) = U(B(\Delta), \Delta)$. At $m = 2\Delta$ $U(S, 2\Delta)$ is valued using Equation (3) above with the previously calculated $B(\Delta)$. For $W(S, 2\Delta)$ Gauss–Legendre quadrature is applied. Abscissas, z_i, and weights, y_i, are scaled to the truncated integration interval defined above to get the approximation

$$W(S,2\Delta) \cong \sum_{i=1}^{q} U(Se^{z_i},\Delta)f(z_i)y_i. \tag{6}$$

The exercise boundary is the solution to $K - B(2\Delta) = U(B(2\Delta), 2\Delta) + W(B(2\Delta), 2\Delta)$.

Unfortunately, as we move farther back from the maturity date there is an exponential increase in the effort required to calculate option prices. Direct application of quadrature leads to nested sums to approximate the multidimensional integrals that arise from the recursive structure of the problem, for example, evaluating $W(S, 3\Delta)$ requires $q + q^2$ calculations. Huang, Subrahmanyam, and Yu (1996) discuss some possible modifications but conclude that even for moderate n they will be slow. Richardson extrapolation could be used to estimate a value based on solutions with small n. An alternative is to summarize the information gained at each step about the option value in an approximating function. A compact and accurate representation of $V(S, m)$ would substitute for the recursive application of quadrature and produce a routine whose computational effort would grow only linearly with n.

1.2 Function approximation

Function approximation is used in situations where a function of interest, say $h(y)$, is difficult to calculate. The goal is to find another function which closely fits $h(y)$ but is easier to compute. The Weierstrass theorem states that if $h(y)$ is continuous in a finite interval, then there is a polynomial of sufficiently high degree that reduces the maximum approximation error in the interval to an arbitrarily small value. The question in a given situation is whether a good enough fit is possible with a moderate-order polynomial. Young and Gregory (1972) show how to construct the best polynomial approximation in the minimax sense, but the routine can be time consuming to apply.

An alternative which nearly meets the minimax criteria is Chebyshev approximation: candidate polynomial approximations come from the set of all possible linear combinations of the p Chebyshev polynomials, $T_j(x) = \cos(j \arccos(x)), j = 0, 1, \ldots, p - 1$, where $-1 \le x \le 1$, with the best approximation being the one which equals $h(y)$ at the set of p zeros of $T_p(x)$, $x_j = \cos(\pi(j - 1/2)/p), j = 1, 2, \ldots, p$. Chebyshev polynomials follow a recurrence, $T_{p+1}(x) = 2xT_p(x) - T_{p-1}(x)$, with $T_0(x) = 1$ and $T_1(x) = x$, which relates them to regular polynomials, for example, $T_3(x) = 4x^3 - 3x$. Therefore this is just a particular type of polynomial approximation, distinguished by the set of basis functions used to construct the approximating function and the set of points used to determine its coefficients. The fact that Chebyshev polynomials are defined over the interval $[-1, 1]$ is not restrictive, we have already seen how to rescale to an arbitrary finite interval in applying Gaussian quadrature.

To find the coefficients in the approximating function, denoted c_j, pick a finite interval $[b, a]$ in which to approximate $h(y)$, evaluate the function at p points, $y_j = (x_j(a - b) + a + b)/2$, scaled from the interval $[-1, 1]$ to match the x_j defined above, and solve the following system of equations:

$$c_0 T_0(x_j) + c_1 T_1(x_j) + \ldots + c_{p-1} T_{p-1}(x_j) = h(y_j) \quad \text{for } j = 1, 2, \ldots, p. \tag{7}$$

Then $h(y)$ is approximated at arbitrary y by an initial rescaling to $x = (2y - (a + b))/(a - b)$ followed by evaluation of the p-term polynomial

$$h(y) \cong \sum_{j=0}^{p-1} c_j T_j(x). \tag{8}$$

Routines for calculating the coefficients and evaluating the Chebyshev approximation are in Press et al. (1992), with a more extensive discussion in Fox and Parker (1968).

To reduce the size of percent errors across the estimation interval it is preferable to approximate the option value function in logs, fitting log option prices to log stock prices. The lower boundary of the interval is naturally the exercise boundary. For the upper boundary we pick a stock price at which the option value is very small: $A(m)$ that satisfies $V(A(m), m) = V^*$, where V^* is an arbitrary small value and we used $V^* = 0.005$. In logs the estimation interval is $[b(m), a(m)]$, where $b(m) = \ln(B(m))$ and $a(m) = \ln(A(m))$. In this interval we locate the p points that correspond to zeros of $T_p(x)$, find the corresponding option prices using quadrature, convert to logs, and calculate coefficients in an approximating polynomial. Errors introduced by using function approximation can be directly measured by comparison to prices given by direct application of quadrature. For a wide range of parameters we found $p = 8$ reduced the maximum error in the estimation interval to an absolute value of about 0.001% while producing no noticeable extrapolation errors for moderately large stock prices.

With the approximating function, the Gaussian rule need only be employed for one-dimensional integrals. No matter the number of future exercise dates, only a double sum is needed to approximate the value of an option with time to maturity m at a given stock price $S > B(m)$:

$$V(S, m) \cong U(S, m) + e^{-r\Delta} \sum_{i=1}^{q} \sum_{j=0}^{p-1} \exp\big(c_j T_j(x_i)\big) f(z_i) y_i, \tag{9}$$

where the x_i is a rescaling of log stock prices, $\ln(S) + z_i$, from the interval $[b(m - \Delta), a(m - \Delta)]$ to $[-1, 1]$, and z_i and y_i are the quadrature abscissas and weights for the interval $[\beta, \alpha]$. The coefficients in the approximating polynomial will vary with time to maturity, a dependence suppressed in the notation. With this routine for pricing the option the exercise boundary can be found. Then a new estimation interval is set, the option price is evaluated at p stock prices, and new coefficients are found. This sequence produces the necessary inputs for taking another step back: the exercise boundary and the approximate option value function for time to maturity m.

2. Convergence, Accuracy, and Speed

As the number of exercise dates increases for a given maturity the terms of the option more closely match those of a continuously exercisable option. If the pricing for discrete exercise is accurate, then the method should converge to the true value as n increases. The individual components of the method, Gaussian quadrature and Chebyshev approximation, converge for continuous functions as q and p increase. However, their combination to approximate a multidimensional integral produces a nonlinear recursion whose theoretical convergence properties are difficult to analyze. On the other hand, empirical convergence and computational efficiency are properties that can be easily checked to indicate whether more extensive analysis would be useful.

The important parameters controlling the method are the number of quadrature points, the number of terms in the Chebyshev approximation, and the number of exercise dates. Two limitations appeared in preliminary testing. First, approximating polynomials had substantial extrapolation errors for $p > 8$. Second, a larger number of exercise dates required a higher degree of quadrature. The latter is demonstrated in Table 2 which lists solutions produced by various combinations of q and n, all using $p = 8$. The solutions for various degrees of quadrature match for $n = 16$, but at the lower $q = 16$ they diverge at higher n by an increasing amount compared to solutions with $q > 16$. In addition, the option value eventually falls with increasing n for $q = 16$, counter to its theoretical properties, as adding exercise dates should not cause a decline in value. An increase in the degree of quadrature addresses the problem: for $n \leq 1{,}024$ solutions with $q = 24$ and 32 match exactly to 5 digits and differ from $q = 20$ at $n = 1{,}024$ by only 0.0001. Additional testing revealed that $n = 16{,}384$ would require $q = 40$ to avoid declines in value with increasing n and that the resulting solution matched the $n = 1{,}024$ solution up to 0.0001.

These are robust results. The method allows a very fine division of time to maturity that will be a close substitute for continuous exercise

Table 2
Convergence and extrapolation of Gaussian quadrature method

	q-Point quadrature			
n Exercise dates	$q = 16$	$q = 20$	$q = 24$	$q = 32$
Extensive solutions				
$n = 16$	2.4775	2.4775	2.4775	2.4775
$n = 64$	2.4809	2.4812	2.4812	2.4812
$n = 256$	2.4807	2.4822	2.4822	2.4822
$n = 1024$	2.4745	2.4825	2.4826	2.4826
Extrapolated solutions				
$n = 1,2,3,4$	2.4839	2.4839	2.4839	2.4839
$n = 1,2,4,8$	2.4828	2.4828	2.4828	2.4828
$n = 1,3,9,27$	2.4825	2.4825	2.4825	2.4825

Option parameters are strike price $K = 40$, years to maturity $M = 0.3333$, annualized volatility $\sigma = 0.30$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$. All solutions use Gauss–Legendre quadrature with q points and $p = 8$ term Chebyshev polynomial approximation. Extensive solutions are for a put option that can only be exercised at n dates. Extrapolated solutions are linear combinations of four extensive solutions using weights determined from Richardson extrapolation for the indicated spacing of n.

possibilities. To be practical it must also be fast. For $q = 20$ and $p = 8$ on a 166-MHz Pentium computer, the routine does about 100 iterations per second. Based on Table 2, 4-digit accuracy would take at most 3 seconds for one option price. Additional option values for given parameters and different stock prices can be calculated with little additional effort using the coefficients in the approximating function. Derivatives with respect to stock price can similarly be evaluated directly from that polynomial.

Richardson extrapolation, introduced for option price computations by Geske and Johnson (1984), boosts efficiency substantially. A common implementation is based on solutions with $n = 1,2,3$, and 4. Omberg (1987) makes the case for geometric spacing of n, impractical for Geske and Johnson's routine but easily implemented here. Table 2 includes extrapolated solutions with various spacing. Since all of them are based on small n, the value of q has no impact. With $n = 1,3,9$, and 27 it is possible to virtually duplicate the extensive solutions while dramatically reducing the number of iterations and the computation time.

A comparison to the finite difference and binomial methods will put the properties of the approximation technique in perspective. Table 3 lists approximation errors and computation times for the set of parameters in Cox, Ross, and Rubinstein (1979), using the binomial method with $n = 10{,}000$ as a benchmark solution. Each of the three methods easily achieves "penny accuracy" while allowing an order of magnitude reduction in errors with additional work. The extensive Gaussian quadrature solutions display convergence to the benchmark solution from below, consistent with being accurate values for n-date exercisable options. However, they are slow. For similar levels of accuracy the finite

Table 3
Comparison of finite difference, binomial, and Gaussian quadrature solutions

Option parameters			Binomial	Approximation errors relative to binomial n = 10,000 solution							
				Finite difference		Binomial		Gaussian quadrature			
K	σ	M	n = 10,000	$n=256$	$n=1024$	$n=150$	$n=500$	$n=64$	$n=256$	n = 1,2,3,4	n = 1,3,9,27
35	0.2	0.0833	0.0062	0.0000	0.0000	−0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
35	0.2	0.3333	0.2004	0.0006	0.0001	−0.0009	−0.0003	−0.0003	−0.0001	−0.0004	−0.0001
35	0.2	0.5833	0.4328	0.0022	0.0005	0.0012	0.0003	−0.0006	−0.0002	−0.0007	0.0000
40	0.2	0.0833	0.8522	0.0012	0.0003	−0.0010	−0.0003	−0.0004	−0.0001	0.0002	0.0000
40	0.2	0.3333	1.5798	−0.0005	0.0006	−0.0015	−0.0004	−0.0013	−0.0003	0.0009	0.0000
40	0.2	0.5833	1.9904	0.0036	0.0009	−0.0018	−0.0005	−0.0022	−0.0005	0.0000	0.0001
45	0.2	0.0833	5.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	−0.0015	−0.0001
45	0.2	0.3333	5.0883	0.0005	0.0000	0.0003	0.0002	−0.0030	−0.0006	0.0067	−0.0009
45	0.2	0.5833	5.2670	0.0025	0.0004	0.0007	0.0003	−0.0048	−0.0011	0.0049	−0.0024
35	0.3	0.0833	0.0774	0.0005	0.0001	0.0001	−0.0001	0.0000	0.0000	−0.0001	0.0000
35	0.3	0.3333	0.6975	0.0030	0.0007	0.0018	0.0007	−0.0005	−0.0001	−0.0007	0.0000
35	0.3	0.5833	1.2198	0.0012	0.0009	0.0041	0.0013	−0.0011	−0.0003	−0.0008	0.0000
40	0.3	0.0833	1.3099	0.0024	0.0001	−0.0016	−0.0005	−0.0003	0.0000	0.0000	0.0000
40	0.3	0.3333	2.4825	0.0043	0.0011	−0.0026	−0.0007	−0.0013	−0.0003	0.0014	0.0000
40	0.3	0.5833	3.1696	−0.0019	0.0013	−0.0031	−0.0009	−0.0024	−0.0006	0.0021	0.0000
45	0.3	0.0833	5.0597	0.0000	−0.0001	0.0003	0.0000	−0.0006	0.0001	0.0001	−0.0004
45	0.3	0.3333	5.7056	0.0022	0.0002	0.0009	0.0009	−0.0025	−0.0005	−0.0046	−0.0003
45	0.3	0.5833	6.2436	0.0010	0.0004	0.0012	0.0008	−0.0041	−0.0009	−0.0072	−0.0004
35	0.4	0.0833	0.2466	0.0011	0.0002	−0.0012	−0.0004	−0.0001	−0.0001	−0.0001	−0.0001
35	0.4	0.3333	1.3460	0.0040	0.0003	0.0045	0.0013	−0.0007	−0.0001	−0.0009	0.0000
35	0.4	0.5833	2.1549	−0.0032	0.0014	0.0053	0.0000	−0.0015	−0.0004	−0.0009	−0.0001
40	0.4	0.0833	1.7681	0.0031	0.0001	−0.0023	−0.0007	−0.0003	0.0001	−0.0002	0.0000
40	0.4	0.3333	3.3874	0.0051	0.0013	−0.0039	−0.0011	−0.0014	−0.0002	0.0012	0.0000
40	0.4	0.5833	4.3526	−0.0036	0.0017	−0.0046	−0.0013	−0.0025	−0.0005	0.0027	0.0000
45	0.4	0.0833	5.2868	0.0009	0.0001	0.0007	0.0004	−0.0007	0.0001	−0.0010	−0.0002
45	0.4	0.3333	6.5099	0.0032	0.0001	0.0004	0.0010	−0.0024	−0.0005	−0.0007	−0.0003
45	0.4	0.5833	7.3830	0.0004	0.0013	0.0067	0.0018	−0.0038	−0.0008	−0.0019	−0.0002
		RMSE		0.0024	0.0007	0.0026	0.0008	0.0020	0.0004	0.0025	0.0005
		Computation time		1.37	7.14	0.66	3.46	5.22	17.90	0.55	2.99

Fixed option parameters are current stock price S = 40 and annual continuously compounded interest rate r = 0.0488. Table entries show results for combinations of strike price K, years to maturity M, and annualized volatility σ. Finite difference is an explicit method with a grid of n time steps and p stock prices, p chosen to ensure convergence [Hull and White (1990)], and linear interpolation for option values off the grid. Gaussian quadrature uses q = 20 point Gauss–Legendre quadrature and p = 8 term Chebyshev approximation for extensive solutions (n = 64,256) for a put option that can only be exercised at n dates and q = 16, p = 8 for extrapolated solutions that are linear combinations of four extensive solutions using weights determined from Richardson extrapolation for arithmetic ($n = 1, 2, 3, 4$) and geometric ($n = 1, 3, 9, 27$) spacing. RMSE is root mean-squared error relative to the benchmark binomial model with n = 10,000 time steps. Computation time measures in seconds the time for calculating all 27 option prices. All routines were programmed using the GAUSS language and run on a 166-MHz Pentium computer.

difference method takes less time, and the binomial method does much better than either. But the extrapolated quadrature method has the best accuracy-speed combination for this set of option parameters, performing 15% faster than the binomial method while producing lower errors on average.

3. Multidimensional Approximation

So far we have focused on approximating the option's value in a single dimension, that is, as a function of the stock price alone. This is consistent with the pricing model because the other factors which affect the option value are assumed fixed. However, the effect of changes in those factors on the option value are also of interest when calculating implied volatilities or hedge ratios. A given method can be applied repeatedly for different parameters to get the desired solutions. Computation speed becomes critical to meeting practical demands for timely evaluation of the option price function in this situation. Chebyshev approximation has already been shown to be an effective substitute to extensive tabulation and it can also be used to extend the pricing function to multiple dimensions.

A by-product of the extensive solution using Gaussian quadrature is a tabulation of the coefficients in the approximating function at equally spaced times to maturity from M to 0. Therefore the method produces a two-dimensional approximation across stock prices and maturity, although in the maturity dimension the approximation is only at discrete points. To construct an approximating function, set upper and lower maturities, m_{a} and m_{b}, to define the approximation interval and then choose the number of terms in the approximating polynomial, p_{m}. These determine the location of the zeros of a p_{m}-degree Chebyshev polynomial in the interval $[m_b, m_a]$, denoted $m_i, i = 1, \ldots, p_m$, at which option prices need to be evaluated. A single run of the extensive Gaussian quadrature procedure, augmented by another p_m applications of one-dimensional quadrature, produces those prices. For each maturity needed, locate the equally spaced maturity value closest to and less than m_i and calculate the discounted expected option value with Gaussian quadrature, adjusting the risk-neutral distribution of log stock price changes to match the time interval between m_i and the already tabulated maturity.

This two-dimensional approximation is defined by the option values on a grid of $p + 2$ by p_{m} points in the stock price-maturity space, corresponding to the two boundaries of the approximation interval and the p interior points for each of the p_m times to maturity. They are the basis for determining the coefficients in a two-dimensional polynomial in S and m which matches the option value exactly at those points. The only complication is that the grid is not square. The exercise boundary changes with time to maturity and so does the upper boundary for the stock price interval. To get an accurate option value for a target pair (m, S) it is important to first evaluate the approximating polynomial at the target maturity, producing a one-dimensional approximation of the option value as a function of stock price, and then evaluate it for the target stock price. The position of the target stock price relative to the exercise boundary depends on time to maturity. If $S < B$, then the option will be exercised and its value is not given by the approximating function but by its exercise value. Therefore the exercise boundary should be determined first for the target maturity to get an accurate price.

A similar procedure is used to extend the approximation in other directions. To include the volatility dimension, we set boundaries for the approximation interval in that dimension and the number of terms in the approximating polynomial. Several extensive solutions need to be produced, one for each of the target volatilities in the interval, to get the coefficients in the volatility dimension. Given these, an option price for target parameters (σ, m, S) is found by evaluating the three-dimensional polynomial.

To explore the potential of multidimensional approximation we started with the parameter set $K = 40, M = 0.3333, S = 40, \sigma = 0.30$, and $r = 0.0488$ and constructed one-dimensional approximations separately in the maturity and volatility dimensions. We used the binomial method with $n = 10{,}000$ to get option prices at 10 points in the maturity interval [0.10, 5.00] and then calculated polynomial coefficients. To check on accuracy, the binomial method with $n = 10{,}000$ was also used to get prices at a different set of points, midway (in logs) between the 10 previously tabulated prices and at the endpoints of the interval, chosen to reflect the highest errors. Polynomial approximations of various degrees were tested for accuracy. The same procedure was used to check whether it is possible to accurately estimate option values in the volatility dimension.

Table 4 indicates that multidimensional approximation is feasible. Polynomials with four terms were enough to reduce approximation

Table 4
Approximate pricing in maturity and volatility dimensions

Terms in the approximating polynomial	Maximum absolute approximation error: Maturity interval $M = [0.10, 5.00]$ — Difference	Percent	Volatility interval $\sigma = [0.10, 0.50]$ — Difference	Percent
3	0.0587	0.8474	0.0085	0.2421
4	0.0062	0.0900	0.0009	0.0221
5	0.0002	0.0049	< 0.0001	0.0005
6	0.0001	0.0018	< 0.0001	0.0001
7	< 0.0001	0.0005	< 0.0001	< 0.0001
8	< 0.0001	0.0001	< 0.0001	< 0.0001

Benchmark prices are from the binomial model with $n = 10{,}000$ time steps. Approximations were produced by fitting coefficients of Chebyshev polynomials of various degrees using benchmark prices at a set of maturities and volatilities that correspond to the zeroes of various order Chebyshev polynomials. A second set of benchmark option prices were calculated at a different set of maturities and volatilities, including the endpoints of the intervals and points midway in logs between the ones used to produce the coefficients in the approximating polynomials. Errors are measured by comparing that second set of benchmark prices to option prices estimated by polynomial approximations. Table entries are maximum absolute errors (Difference) and percent errors (Percent). Base option parameters are strike price $K = 40$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$. Errors are measured for maturities in the interval $M = [0.10, 5.00]$ for annualized volatility $\sigma = 0.30$, and for annualized volatilities in the interval $\sigma = [0.10, 0.50]$ for years to maturity $M = 0.3333$.

errors below 0.10%. The errors rapidly decrease and are eliminated for all practical purposes with an eight-term polynomial approximation. The maturity dimension is slightly more difficult to fit because the exercise boundary becomes very sensitive to declines in maturity at small m. Polynomials are less successful in approximating functions with sharp curves and an accurate estimate of the exercise boundary is required. Later, the full 10-term approximating polynomial will be used to generate benchmark prices to assess the accuracy of various approximations. For now, we take the results as evidence that multidimensional approximation can succeed.

We constructed a four-dimensional approximation to the American put option value function for interest rates in the interval [0.02, 0.10], volatilities in [0.10, 0.50], maturities in [0.10, 5.00], and stock prices going from the corresponding exercise boundaries to upper boundaries at which the option value was 0.01. We treated the maturity dimension differently from the others, breaking it into two sections, [0.10, 1.50] and [1.50, 5.00], and calculating approximations over each section separately. This accommodates the variation in the exercise boundary with respect to M and allows use of a lower-order approximation in each section. We used Chebyshev approximation with $p = 8$ terms in the stock price dimension and 6 terms in each of the others. Extensive solutions with $n = 1{,}024$ were the basis for tabulating option values at the required points. Including the varying boundaries on the stock price interval and the two maturity intervals, a total of 4,320 points were tabulated using 72 runs of the extensive Gaussian quadrature method.

4. Comparison to Analytical Methods

For further evidence on the efficiency of our technique we compared the various quadrature-based solutions to other approximation routines: MacMillan (1986), Broadie and Detemple (1996), Huang, Subrahmanyam, and Yu (1996), and Carr (1988). All of them incorporate analytical results in the solution technique that improves computation speed, particularly in calculating hedge ratios. Maintaining accuracy is a challenge met in different ways. Huang, Subrahmanyam, and Yu (1996) and Carr (1998) use discrete exercise opportunities and so can produce different solutions depending on the choice of n. The complexity of the models in such that increasing the number of exercise dates causes a very rapid rise in computation time. Therefore each are generally used in combination with extrapolation routines to improve accuracy. MacMillan (1986) and Broadie and Detemple (1996) assume continuously exercisable options and so produce a single solution. Broadie and Detemple use adjustment factors to improve accuracy, determined by regression using prices from the binomial method with $n = 15{,}000$. In contrast to this calibration against an external benchmark, our four-dimensional approximation is based only on prices produced by the Gaussian quadrature method.

Approximation errors for the various methods were measured relative to benchmark prices from the one-dimensional approximations described above, using a 10-term approximating polynomial based on values from the binomial method with $n = 10{,}000$ in each of the stock price, maturity, and volatility dimensions. Graphical comparisons reveal the accuracy of quadrature-based approximations. Figure 1 shows errors for several methods as stock prices vary. Among the analytical routines, Carr's method maintains penny accuracy throughout the interval, while Broadie and Detemple have higher errors near the exercise boundary and Huang, Subrahmanyam, and Yu (1996) show substantial variation. Geometric extrapolation using quadrature solutions for $n = 1, 3, 9$, and

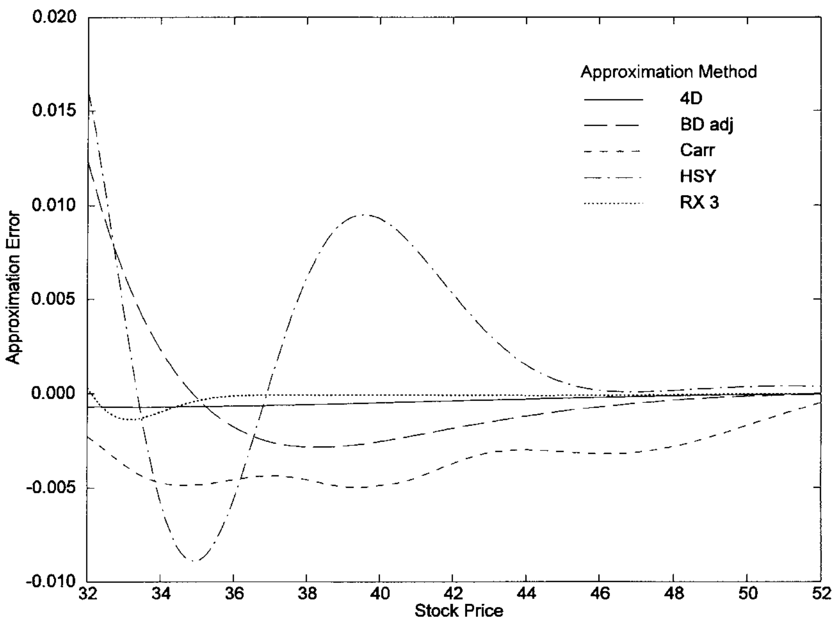

Figure 1
Approximation errors in the stock price dimension
Errors are measured by differences from benchmark prices based on the binomial method with $n = 10{,}000$ time steps. Option parameters are strike price $K = 40$, years to maturity $M = 0.3333$, annualized volatility $\sigma = 0.30$, and annual continuously compounded interest rate $r = 0.0488$. Errors are shown for a range of current stock prices for the Broadie and Detemple (1996) lower-bound approximation adjusted by regression coefficients (BD adj), the Carr (1998) four-point extrapolation (Carr), the Huang, Subrahmanyam, and Yu (1996) four-point extrapolation (HSY), extrapolated quadrature with $n = 1{,}3{,}9{,}27$ (RX 3), and quadrature-based four-dimensional approximation (4D).

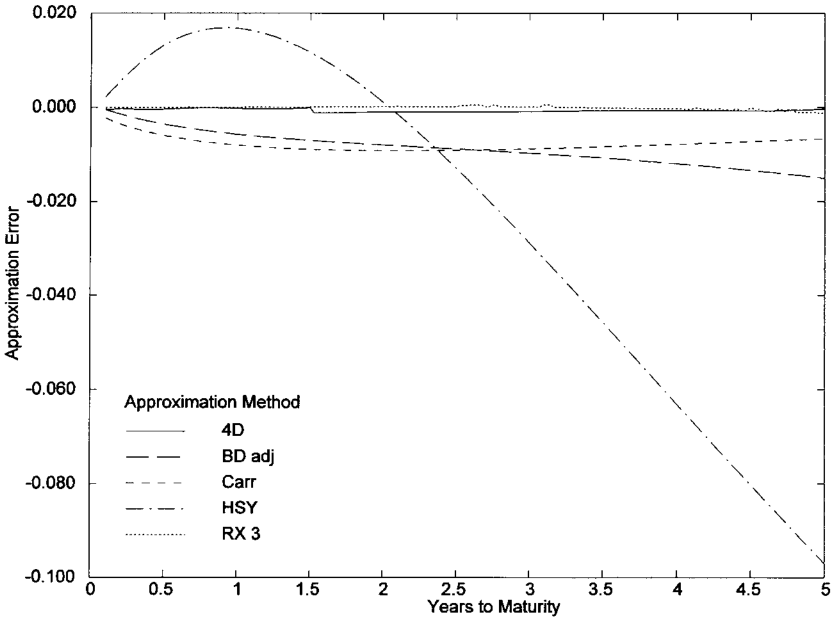

Figure 2
Approximation errors in the maturity dimension
Errors are measured by differences from benchmark prices based on the binomial method with $n = 10{,}000$ time steps. Option parameters are strike price $K = 40$, annualized volatility $\sigma = 0.30$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$. Errors are shown for a range of years to maturity for the Broadie and Detemple (1996) lower-bound approximation adjusted by regression coefficients (BD adj), the Carr (1998) four-point extrapolation (Carr), the Huang, Subrahmanyam, and Yu (1996) four-point extrapolation (HSY), extrapolated quadrature with $n = 1,3,9,27$ (RX 3), and quadrature-based four-dimensional approximation (4D).

27 has essentially no error except near the exercise boundary, and the quadrature-based four-dimensional approximation performs nearly as well. Figure 2 demonstrates that the same results carry over to variation in time to maturity. The analytical routines have errors which generally increase with M, and only Carr stays within a penny of the benchmark, while the quadrature approximations remain extremely accurate. For the range of volatility displayed in Figure 3, the analytical routines more closely match the benchmark, though again not as well as the extremely accurate geometric extrapolation and the four-dimensional approximations.

The other important factor is computation speed. Several of the routines base their solutions on an approximate exercise boundary. Since this is constant for a given strike price, maturity, volatility, and interest rate, and involves a search process which can use a significant

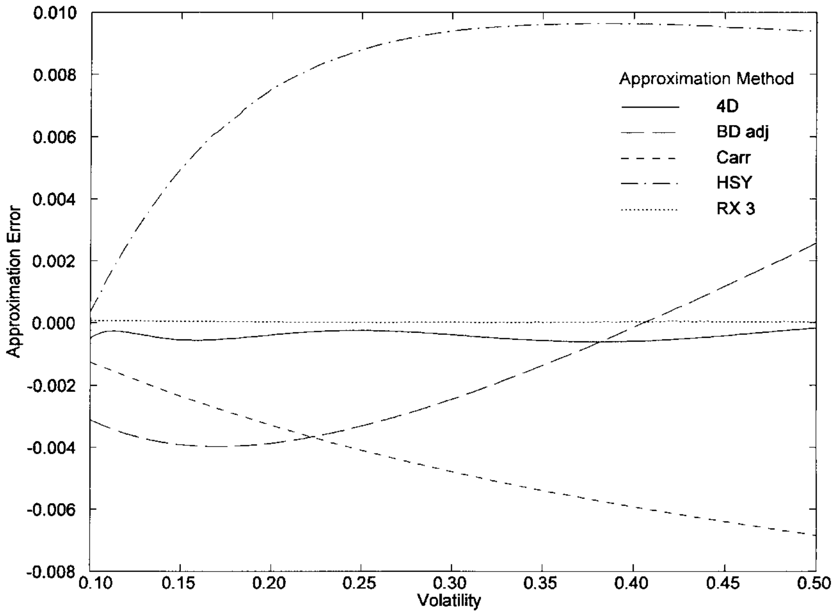

Figure 3
Approximation errors in the volatility dimension
Errors are measured by differences from benchmark prices based on the binomial method with $n = 10{,}000$ time steps. Option parameters are strike price $K = 40$, years to maturity $M = 0.3333$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$. Errors are shown for a range of annualized volatility for the Broadie and Detemple (1996) lower bound approximation adjusted by regression coefficients (BD adj), the Carr (1998) four-point extrapolation (Carr), the Huang, Subrahmanyam, and Yu (1996) four-point extrapolation (HSY), extrapolated quadrature with $n = 1,3,9,27$ (RX 3), and quadrature-based four-dimensional approximation (4D).

fraction of total computation time, these methods will have markedly faster times in the stock price dimension. Similarly, a single run of the quadrature routine produces a polynomial approximation that can be evaluated for many stock prices with little additional effort. As volatility or maturity varies, however, there will be no favorable spillovers from previously calculated values and computation times will revert to the average time required to produce an approximation from scratch. If speed is essential, then it makes sense to store information about the option value function for reuse. This is the idea behind the multidimensional approximation, which stores in the coefficients of the approximating polynomial information gained from a set of quadrature solutions. And it also applies to the adjustments used by Broadie and Detemple.

The effort to get the coefficients in the four-dimensional approximation and to get the adjustment factors in a calibrated method can be substantial. To run the Gaussian quadrature method with $n = 1{,}024$,

$q = 20$, and $p = 8$ the necessary 72 times and calculate the coefficients took about 9 minutes. Broadie and Detemple (1996) do not report the time needed to solve for their regression coefficients, but we found that a single solution for the binomial method with $n = 15{,}000$ takes about 200 seconds, so that the 2,500 solutions on which they base their adjustments would be done in about 5 days. The use of preliminary calculations gives these two methods an advantage in marginal computation time which should be weighed against the initial effort. To highlight this distinction, we report average errors and computation times for various methods, indicating ones that rely on preliminary calculations not counted in the table entries.

Table 5 shows that in the stock price dimension, the most accurate by far is geometric extrapolation based on Gaussian quadrature. Nearly as accurate but also fastest among all the methods is the four-dimensional quadrature-based approximation. The binomial method is the slowest overall, though with $n = 150$ it does better than Carr's routine. The finite difference method is faster than either at similar levels of accuracy. MacMillan's technique is very quick, but also has the highest average errors. Broadie and Detemple's lower bound approximation is nearly as inaccurate, but is much improved by the regression coefficient adjustment. The procedure of Huang, Subrahmanyam, and Yu produces average errors of less than 0.2% at one of the fastest times. Lower still are the errors of the quadrature-based routines, which have the lowest percent errors nearly across the board and among the best times for given accuracies.

In the maturity dimension, the ranking in terms of speed changes significantly. Only the binomial and Broadie and Detemple times are essentially unaffected, bringing them up in accuracy-speed rankings. Over a broad maturity interval, Broadie and Detemple are fastest, after the highly inaccurate MacMillan method, and achieve penny accuracy on average. The four-dimensional approximation is 20% slower but 10 times more accurate. Next in terms of speed is Huang, Subrahmanyam, and Yu, but its average errors are high. The binomial method achieves much better accuracy while taking a bit more time, and it turns in a better performance than Carr's four-point extrapolation and the finite difference method. When tested over a range of volatilities there are some changes in the size of average errors, with MacMillan's technique showing a noticeable drop. The general pattern is the same. Geometric extrapolation based on Gaussian quadrature produces the most accurate results of all, albeit slowly. The four-dimensional approximation captures that accuracy while also being one of the fastest routines.

Analytical methods have the advantage that option hedge ratios can be evaluated using the exercise boundary estimated when approximating prices. With Gaussian quadrature it is sometimes that simple. A

Table 5
Approximation errors and computation time

Approximation method	Stock price interval $S = [32, 52]$			Maturity interval $M = [0.10, 5.00]$			Volatility interval $\sigma = [0.10, 0.50]$		
	RMSE	RMS%E	Time	RMSE	RMS%E	Time	RMSE	RMS%E	Time
Finite difference									
$n = 256$	0.0021	0.2790	0.19	0.0049	0.0942	26.48	0.0021	0.0963	28.06
$n = 512$	0.0009	0.1160	0.36	0.0025	0.0474	57.23	0.0017	0.0649	62.23
$n = 1024$	0.0004	0.0554	0.79	0.0013	0.0242	128.14	0.0008	0.0312	143.68
Binomial									
$n = 150$	0.0021	0.2354	4.78	0.0053	0.0972	4.67	0.0030	0.1053	4.89
$n = 300$	0.0011	0.1161	11.98	0.0026	0.0472	11.98	0.0015	0.0516	12.19
$n = 500$	0.0006	0.0663	25.70	0.0015	0.0275	25.98	0.0009	0.0302	25.98
MacMillan	0.0133	0.7870	0.0076	0.0600	0.9374	0.58	0.0041	0.1666	0.64
Broadie and Detemple									
Lower bound	0.0123	0.4932	1.60	0.0405	0.7281	1.42	0.0128	0.5795	1.54
Adjusted lower bound*	0.0028	0.0822	1.69	0.0094	0.1624	1.51	0.0027	0.1993	1.63
Carr	0.0037	0.3313	6.43	0.0080	0.1609	14.78	0.0048	0.1893	14.99
Huang, Subrahmanyam, and Yu	0.0054	0.1882	0.07	0.0436	0.6851	2.95	0.0084	0.3598	2.96
Gaussian quadrature									
Extensive $n = 128$	0.0007	0.0332	1.04	0.0063	0.1020	210.86	0.0007	0.0380	203.00
Extrapolation, $n = 1,2,3,4$	0.0021	0.0704	0.37	0.0218	0.3402	9.98	0.0011	0.0561	10.33
Extrapolation, $n = 1,3,9,27$	0.0004	0.0093	0.65	0.0003	0.0052	66.49	0.0000	0.0028	65.91
Four-dimensional*	0.0004	0.0196	0.0060	0.0007	0.0132	1.83	0.0004	0.0245	1.76

Errors are measured by differences from benchmark prices based on the binomial method with $n = 10{,}000$. Base option parameters are strike price $K = 40$, years to maturity $M = 0.3333$, annualized volatility $\sigma = 0.30$, current stock price $S = 40$, and annual continuously compounded interest rate $r = 0.0488$. A set of 201 equally spaced option prices were calculated for each of the indicated intervals by each method and compared to benchmark prices to get root mean-squared errors (RMSE) and root mean-squared percent errors (RMS%E). Finite Difference is an explicit method with a grid of n time steps and p stock prices, p chosen to ensure convergence [Hull and White (1990)], and linear interpolation for option values off the grid. The Carr and Huang, Subrahmanyam, and Yu routines use four-point extrapolation. Gaussian quadrature uses $q = 20$-point Gauss−Legendre quadrature and $p = 8$ term Chebyshev approximation for an extensive solution ($n = 128$) for an option that can only be exercised at n dates, and $q = 16$, $p = 8$ for extrapolated solutions that are linear combinations of four extensive solutions using weights determined from Richardson extrapolation for arithmetic ($n = 1,2,3,4$) and geometric ($n = 1,3,9,27$) spacing. Time measures in seconds the time used to solve for all 201 option values in a given interval. All routines were programmed using the GAUSS language and run on a 166-MHz Pentium computer. *indicates methods that use preliminary calculations which are not counted in computation time. It takes about 9 minutes to get the coefficients in the four-dimensional approximation using Gaussian quadrature. It would take about 5 days to produce the set of 2,500 benchmark prices using the binomial method with $n = 15{,}000$ time steps on a Pentium 166-MHz computer to get the regression coefficients for the Broadie and Detemple adjusted lower bound approximation.

single application of the routine produces a complete approximation as a function of stock prices given the other parameters. The delta and gamma of an option can be found by differentiating the approximating polynomial and evaluating the derivative at a given stock price. This involves little additional effort and also produces an estimate of the option's theta. On the other hand, to calculate vega and rho, several solutions would be needed to map out the function with respect to volatility or the interest rate and approximate the required derivative. That would be more time consuming.

The multidimensional approximation contains all of the information needed to evaluate the derivatives of the option value function and it delivers accurate prices quickly. For example, a set of prices for a range of volatilities can be tabulated and converted into coefficients in an approximating polynomial whose derivative provides an estimate of vega. The accuracy demonstrated here in terms of option prices should carry over to estimates of the hedge ratios and the multidimensional approximation would be competitive with the analytical methods in terms of computation speed.

5. Conclusion

Gaussian quadrature produces extremely accurate approximations of American put option prices. Chebyshev approximation captures that accuracy by converting a set of prices into an estimate of the option value function, significantly reducing computational effort. This allows use of geometric extrapolation which delivers accuracy far superior to existing analytical techniques, virtually duplicating the results of the binomial method with $n = 10{,}000$. For applications that emphasize speed because a large number of option prices need to be evaluated, the extension of Chebyshev approximation to multiple dimensions compromises accuracy only slightly while significantly reducing marginal computation time.

The quadrature technique easily extends to the case of a stock that pays continuous dividends by adjusting the trend in stock prices. The extensive discrete time methods, finite difference, binomial, and quadrature can also deal with discrete dividends. The analytic methods have more difficulty with that variation if they rely on extrapolation, since the discrete payments cannot be directly incorporated in a setup with only a few exercise dates. Deterministic changes in interest rates or volatility as a function of either time or the stock price would be reflected in the quadrature method results through the coefficients in the approximating function. These are the types of changes in the underlying variables that allow risk-neutral pricing.

Stochastic changes in interest rates and volatility complicate the derivation of option pricing formulas and their numerical evaluation. Option theories must rely on assumptions about preferences or about risk-neutral probabilities to obtain pricing equations. And the resulting functions are multidimensional. The success of the multidimensional approximation demonstrated here shows that it is possible to incorporate that type of variation. Gaussian quadrature combined with Chebyshev approximation should be an effective alternative to the binomial models commonly employed in these cases. And with the increased complexity comes a more detailed description of the option value function.

References

Abramowitz, M., and I. A. Stegun, 1964, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series 53, U.S. Government Printing Office, Washington, D.C.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–654.

Broadie, M., and J. Detemple, 1996, "American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods," *Review of Financial Studies*, 9, 1211–1250.

Carr, P., 1998, "Randomization and the American Put," *Review of Financial Studies*, 11, 597–626.

Cox, J., S. Ross, and M. Rubinstein, 1979, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229–264.

Davis, P. J., and P. Rabinowitz, 1984, *Methods of Numerical Integration* (2nd ed.), Academic Press, Orlando, Fla.

Fox, L., and I. B. Parker, 1968, *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, London.

Geske, R., and H. Johnson, 1984, "The American Put Option Valued Analytically," *Journal of Finance*, 39, 1511–1524.

Geske, R., and K. Shastri, 1985, "Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques," *Journal of Financial and Quantitative Analysis*, 20, 45–72.

Huang, J., M. Subrahmanyam, and G. Yu, 1996, "Pricing and Hedging American Options: A Recursive Integration Approach," *Review of Financial Studies*, 9, 277–300.

Hull, J., and A. White, 1990, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25, 87–100.

Judd, K., 1992, "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, 410–452.

MacMillan, L., 1986, "Analytic Approximation for the American Put Option," *Advances in Futures and Options Research*, 1, Part A, 119–139.

Omberg, E., 1987, "A Note on the Convergence of the Binomial-Pricing and Compound-Option Models," *Journal of Finance*, 42, 463–469.

Press, W., S. Teukolsky, W. Vetterling, and B. Flannery, 1992, *Numerical Recipes in C* (2nd ed.), Cambridge University Press, New York.

Young, D. M., and R. T. Gregory, 1972, *A Survey of Numerical Mathematics*, Volume 1, Addison-Wesley, Menlo Park, Calif.