

Empty Promises and Arbitrage

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Analysis of absence of arbitrage normally ignores payoffs in states to which the agent assigns zero probability. We extend the fundamental theorem of asset pricing to the case of “no empty promises” in which the agent cannot promise arbitrarily large payments in some states. There is a superpositive pricing rule that can assign positive price to claims in zero probability states important to the market as well as assigning positive prices to claims in the states of positive probability. With continuous information arrival, no empty promises can be enforced by shutting down the agent’s subsequent investments once wealth hits zero.

Dogmatic disagreements create arbitrage opportunities in competitive markets that are sufficiently complete. There may be someone in the economy who is certain (correctly or not on objective grounds) that gold prices will go up and someone else who believes there is a positive probability that gold prices will go down. The first person will have an arbitrage opportunity if at-the-money put options on gold have a positive price, while the second person will have an arbitrage opportunity if at-the-money put options on gold have a zero or negative price. Similarly, if some people are sure their favorite sport teams or horses will win but others are not so sure, any odds posted by a competitive bookmaker will imply arbitrage for one group or the other.¹ In practice, however, dogmatic differences in beliefs do not imply actual arbitrage that can be used to generate arbitrarily large profits because people have limited resources and cannot make “empty promises” of payments that exceed their own ability to pay in states the market cares about, even if those are states they personally believe to be impossible. The purpose of this article is to extend the study of the absence of arbitrage to situations in which no empty promises are permitted.

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¹ These arbitrages are generally still present in the presence of finite spreads. Our analysis assumes strict price-taking without a spread, but as can be seen from the analysis of Jouini and Kallal (1995), absence of arbitrage in the presence of a spread is the same as the existence of prices within a spread that do not admit arbitrage, and this would be true of our notion of “robust arbitrage” as well.

The fundamental theorem of asset pricing asserts the equivalence of the absence of arbitrage, the existence of a positive linear pricing rule, and the existence of an optimal demand for some agent who prefers more to less.² This result is important, for example, since it tells us that if asset price processes admit no arbitrage, then they are consistent with equilibrium (in a single-agent economy for the agent whose existence is ensured by the theorem). In other words, assuming equilibrium places no more and no less restriction on prices than assuming no arbitrage, absent additional assumptions.

The usual presentation of the fundamental theorem of asset pricing typically ignores payoffs in states to which the agent assigns zero probability. A linear pricing rule that attaches a positive price to a state that the agent believes is impossible would be inconsistent with expected utility maximization in competitive markets since selling short the corresponding Arrow–Debreu security provides consumption now with zero probability of future loss. On the other hand, a zero or negative state price would be inconsistent with utility maximization if the agent believes the state is possible since buying the Arrow–Debreu security for some state is an arbitrage, and a marginal purchase makes the agent better off.

Preventing an agent from making empty promises can rule out strategies that exploit positive prices in states to which the agent assigns zero probability. We model this as a nonnegative wealth constraint that is imposed in all states deemed to be “important” by a regulatory agency, consensus market beliefs, or some other mechanism. The shadow price of the binding constraint equals the price of the impossible state, restoring consistency between linear pricing and expected utility maximization. We state a new version of the fundamental theorem which equates absence of robust arbitrages (defined to be arbitrages not requiring empty promises), existence of a superlinear pricing rule (which may give positive price to states valued by the market even if the agent believes them impossible), and existence of an optimum for an agent who prefers more to less and cannot make empty promises.

A potential conceptual problem with the general analysis is one of enforcement: How do we ensure that an agent will choose a strategy that does not make empty promises in states of nature that the agent believes will never happen? Our main result on enforcement is that, whenever information arrives continuously, the no-empty-promises condition can be enforced by shutting down all investments for the rest of the time once the agent’s wealth hits zero. Continuous information arrival implies that the portfolio value is continuous and therefore must hit zero before going negative. This

² See Ross (1978) and Dybvig and Ross (1987) for discussions of the fundamental theorem of asset pricing in the traditional setting.

property means that if the market halts the agent's trading the first time the wealth hits zero, then the wealth cannot become negative. Being able to enforce the no-empty-promises condition using only the paths of wealth is important because it eliminates the market's need to know the agent's strategy in all important states.³ By contrast, monitoring the path of wealth would be insufficient to enforce no empty promises if information arrival were discontinuous (as it would be in a discrete-time model or a model with unpredictable jumps) because the market may be unable to anticipate whether wealth will become negative. We can compare this to the marking-to-market process in futures markets. Daily marking to market presumably approximates continuous information arrival in that it permits exchanges to stop the activity of any trader who is in danger of having negative wealth, regardless of the trader's view of the future direction of the market. Marking-to-market would be less effective if it were done monthly or if daily price variation were more volatile, both of which represent more discontinuous information arrival, requiring more margin money to complement marking to market.

In related work, Hindy (1995) studies the viability of a pricing rule when the agent must maintain a level of "risk-adjusted" equity. Hindy claims that viability is equivalent to the existence of a linear pricing rule which is the sum of two linear rules, one representing marginal utility of consumption and the other representing the shadow price of his solvency constraint. Our pricing rule has a similar decomposition, and dogmatic differences in beliefs provide a natural reason for the shadow price to be nontrivial, something absent in Hindy's analysis. In addition, our setting allows interesting option pricing and investment models (such as the Black–Scholes model) because we allow unbounded consumption and investment strategies. We also allow markets to be incomplete.

Several studies consider consumption and investment problems when agents disagree about the possible returns of risky assets. Wu (1991) and Pikovsky and Karatzas (1996) study optimal consumption in a model in which an agent has logarithmic preferences and "anticipates" future asset prices. Bergman (1996) argues that arbitrarily conditioning return processes to lie in some range may produce arbitrage opportunities; our results suggest that these arbitrages can be ruled out by a no-empty-promises condition. Loewenstein and Willard (1997) use results presented here to study equilibrium trading strategies and prices when agents face unforeseen contingencies.

³ The difference here is analogous to the difference between using expected square integrability and using a nonnegative wealth constraint to rule out doubling strategies in a continuous-time model. Determining expected square integrability requires the market to know the agent's strategy in every state of nature; by contrast, a nonnegative wealth constraint requires only knowledge of the agent's wealth along the observed path.

Section 1 contains examples that illustrate the use of the no-empty-promises constraint. Section 2 contains the fundamental theorem of asset pricing with no empty promises for a single-period model with a finite number of states. Section 3 extends this result to a continuous-time model in which returns are assumed to be general special semimartingales. In Section 4, we show that the no-empty-promises condition can be enforced by shutting down investments when wealth hits zero, provided returns are continuous. Section 5 concludes.

1. Examples

We illustrate the connection between empty promises and arbitrage in a number of discrete- and continuous-time examples. In each of our examples, the agent has initial endowment $w > 1$ and preferences $c_0 + \log c_1$, where c_0 and c_1 represent current and terminal consumption, respectively. Assuming separable linear-logarithmic utility is in no way essential for the results, but this choice does simplify computation. The agent takes prices as given and conditions on information available at the start of trade. General results are given in later sections.

1.1 Single-period examples

In our first set of examples, there are three states of nature: a high state H , a middle state M , and a low state L . Gold, the risky asset, costs 15 units of wealth, and a riskless bond costs 10 units. Prices next period are given by the matrix

$$\begin{array}{c} H \\ M \\ L \end{array} \begin{bmatrix} 30 & 10 \\ 30 & 10 \\ 1 & 10 \end{bmatrix}.$$

The first column represents price of gold in the three states, and the second represents the price of the riskless asset.

Example 1. An agent who believes all states are possible. We use this example to contrast the no-empty-promises setting. Suppose that the agent believes states H and M each occur with probability $2/5$ and state L occurs with probability $1/5$. Here is the agent's traditional choice problem.

Problem A. Choose portfolio weights (α_1, α_2) to maximize expected utility of consumption

$$\begin{aligned} w - 15\alpha_1 - 10\alpha_2 + \frac{2}{5} \log(30\alpha_1 + 10\alpha_2) + \frac{2}{5} \log(30\alpha_1 + 10\alpha_2) \\ + \frac{1}{5} \log(\alpha_1 + 10\alpha_2). \end{aligned}$$

From the first-order conditions, the solution to Problem A equals $\alpha_1 = 23/525$ and $\alpha_2 = 6/175$.

Example 2. An agent who believes a state is impossible. Here the agent is certain gold will outperform the bond: suppose the agent believes states H and M each occur with probability $1/2$. Problem B is the traditional consumption choice problem.

Problem B. Choose portfolio weights (α_1, α_2) to maximize expected utility of consumption

$$w - 15\alpha_1 - 10\alpha_2 + \frac{1}{2} \log(30\alpha_1 + 10\alpha_2) + \frac{1}{2} \log(30\alpha_1 + 10\alpha_2).$$

Problem B does not have a solution because there is an arbitrage. For example, lending $3/2$ units of the bond and buying 1 unit of the gold costs nothing but returns 15 units for sure under the agent's beliefs. The agent can reach any level of expected utility by undertaking enough of these net trades. However, for each trade the agent promises to pay 14 units of wealth in the event that state L occurs. In using this trade to construct the arbitrage, the agent promises to pay successively larger amounts of wealth conditional on state L , thus making an “empty promise” because potential losses will eventually exceed any given initial endowment.

The following choice problem has a solution because a no-empty-promises constraint rules out strategies that make these empty promises.

Problem C. Choose portfolio weights (α_1, α_2) to maximize expected utility of consumption

$$w - 15\alpha_1 - 10\alpha_2 + \frac{1}{2} \log(30\alpha_1 + 10\alpha_2) + \frac{1}{2} \log(30\alpha_1 + 10\alpha_2)$$

subject to the no-empty-promises constraint given by $30\alpha_1 + 10\alpha_2 \geq 0$ and $\alpha_1 + 10\alpha_2 \geq 0$.

The solution to Problem C is $\alpha_1 = 1/14$ and $\alpha_2 = -1/140$. The shadow price on the binding no-empty-promises constraint ($\alpha_1 + 10\alpha_2 \geq 0$) equals $15/29$, the market price of the Arrow–Debreu security for the low state. The shadow price takes up the slack between zero marginal utility and a positive state price. Notice that the no-empty-promises constraint in this example is weaker than a no-short-sales constraint since α_1 can be arbitrarily negative if α_2 is positive enough to cover the position.

Here is an example to show that the no-empty-promises constraint does not always eliminate arbitrage opportunities.

Example 3. A robust arbitrage opportunity. Suppose that the spot prices

of gold and the riskless bond are $[10, 10]$, and next period's prices are given by

$$\begin{array}{c} H \\ M \\ L \end{array} \begin{bmatrix} 10 & 10 \\ 10 & 10 \\ 0 & 10 \end{bmatrix}.$$

There is a solution to the choice problem when the agent believes state L is impossible, but there is no solution whenever the agent believes the gold price will fall with positive probability. This is true even under the no-empty-promises constraint because an at-the-money put option on gold has zero cost, has nonnegative payoffs, and pays a positive amount with positive probability.

In these examples we see that (i) there may be no arbitrage opportunities if the agent assigns positive probabilities to all states in which portfolios of the assets can have positive payoffs, even in the absence of a no-empty-promises constraint; (ii) if the agent assigns some of these states zero probability, then the resulting arbitrages may be ruled out by preventing the agent from making empty promises; and (iii) there may be "robust" arbitrages that will be available whether or not empty promises are permitted.

1.2 Continuous-time examples

We now demonstrate that similar conclusions hold in examples in which the agent trades continuously over a time interval $[0, T]$. In this section the exposition is informal with a few details in footnotes; formal definitions and proofs are given in a later section.

Uncertainty is generated by the Wiener process Z , and the risky asset has instantaneous return $\mu dt + \sigma dZ_t$, where μ and σ are positive constants. There is a riskless asset that has constant continuously compounded return represented by the positive constant r . We assume that the local risk premium $\mu - r$ is positive, so there is a unique state price density process ρ given by

$$\rho_t = \exp \left(- \left(r + \frac{1}{2} \eta^2 \right) t - \eta Z_t \right),$$

where $\eta = (\mu - r)/\sigma$. The current price of any random terminal payoff c_1 is given by $E[\rho_T c_1]$.

Example 4. No advance information. This is a special case of Merton (1971). We use the fact that markets are complete to state the traditional consumption choice problem.⁴

Problem D. Choose consumption (c_0, c_1) to maximize $c_0 + E[\log(c_1)]$ subject to the budget constraint $w = c_0 + E[\rho_T c_1]$.

⁴ See Duffie (1992, Chapter 8) for an elementary discussion.

The solution to Problem D is $c_0 = w - 1$ and $c_1 = \rho_T^{-1}$. The trading strategy that finances this consumption is

$$\theta_t = W_t \left(\frac{\mu - r}{\sigma^2} \right),$$

which invests a constant proportion of wealth in the risky asset.⁵

Our second example considers a case of advance information.

Example 5. Advance information I. Suppose the agent conditions prior to the start of trade on the belief that the terminal risky return will exceed the riskless return; that is, $\mu T + \sigma Z_T > rT$ or equivalently $Z_T > -\eta T$. Problem E states the traditional choice problem.

Problem E. Choose (c_0, c_1) to maximize $c_0 + E[\log(c_1) \mid Z_T > -\eta T]$ subject to the budget constraint $w = c_0 + E[\rho_T c_1]$.

This problem has no solution because the agent can write put options paying only in states for which Z_T is less than or equal to $-\eta T$. This adds current consumption without violating the budget constraint and without decreasing expected utility. (The puts are never exercised under the agent's beliefs.) A problem is that states of nature to which the agent assigns zero probability have positive market prices. As in the finite-state example, a no-empty-promises constraint can restore the existence of a solution.

Problem F. Choose consumption (c_0, c_1) to maximize $c_0 + E[\log(c_1) \mid Z_T > -\eta T]$ subject to the budget constraint $w = c_0 + E[\rho_T c_1]$ and the no-empty-promises constraint $c_1 \geq 0$ almost surely (in the unconditional probabilities).

Problem F has a solution given by $c_0 = w - 1$ and

$$c_1 = \begin{cases} \frac{1}{N(-\eta\sqrt{T})} \rho_T^{-1} & Z_T > \eta T \\ 0 & Z_T \leq \eta T, \end{cases}$$

where $N(\cdot)$ is the cumulative distribution function of a standard normal random variable. The trading strategy that finances this consumption is

$$\theta_t = W_t \left(\frac{\mu - r}{\sigma^2} + \phi(t, Z_t) \right),$$

⁵ Our class of admissible trading strategies includes those for which each strategy θ is predictable and expected square integrable:

$$E \left[\int_0^T \rho_t^2 \theta_t^2 dt \right] < +\infty.$$

This guarantees that discounted wealth is a martingale under the risk-neutral probability measure.

which is the strategy of the original Merton problem (Example 4) plus a term ϕ which represents selling consumption when the risky asset over the whole period falls below the riskless return. (This additional amount is approximately zero when the risky asset's prior returns are much larger than the riskless return.) This term is defined by

$$\phi(t, y) \equiv \frac{n\left(\frac{y-\eta T}{\sqrt{T-t}}\right) \frac{1}{\sqrt{T-t}}}{\sigma N\left(\frac{y-\eta T}{\sqrt{T-t}}\right)} \quad \text{for all } t \in [0, T)$$

and $\phi(T, y) = 0$, where n is the density function for a standard normal random variable.⁶ As in the finite-state model, the shadow price of the no-empty-promises constraint takes up the slack between the positive state price density and the zero marginal utility of consumption in impossible states.

Example 6. Advance information II. In our final example, we suppose that the agent knows exactly the terminal price of the risky asset, say $S_T = 120$. Here is the traditional choice problem.

Problem G. Choose consumption (c_0, c_1) to maximize $c_0 + E[\log(c_1) \mid S_T = 120]$ subject to the budget constraint $w = c_0 + E[\rho_T c_1]$.

Problem H includes the no-empty-promises constraint.

Problem H. Choose consumption (c_0, c_1) to maximize $c_0 + E[\log(c_1) \mid S_T = 120]$ subject to the budget constraint $w = c_0 + E[\rho_T c_1]$ and subject to the no-empty-promises constraint $c_1 \geq 0$ almost surely.

No solution exists to either problem. We demonstrate this by constructing a “robust free lunch,” which is an arbitrage payoff in the limit that does not

⁶ Here is a proof that this strategy is optimal for Problem F. Define the function

$$h(t, y) \equiv \frac{P\left(Z_T > \eta \sqrt{T} \mid Z_t = y\right)}{N(-\eta \sqrt{T})} = \frac{N\left(\frac{y-\eta T}{\sqrt{T-t}}\right)}{N(-\eta \sqrt{T})}.$$

The process $M_t \equiv h(t, Z_t)$ is a martingale under the agent's prior beliefs, and Ito's lemma implies that $dM_t = M_t \sigma \phi(t, Z_t) dZ_t$. Define $W_t \equiv \rho_t^{-1} M_t$, and note that $W_0 = 1$ and $W_T = c_1$, where c_1 is given in the solution to Problem F. To show that W is the wealth process that finances the optimal consumption, we need to solve for the optimal trading strategy. Ito's lemma implies

$$\begin{aligned} dW_t &= M_t d\rho_t^{-1} + \rho^{-1} dM_t + \rho^{-1} M_t \eta \sigma \phi dt \\ &= \rho^{-1} M_t (r + \eta^2) dt + \rho^{-1} M_t \eta dZ_t + \rho^{-1} M_t \sigma \phi dZ_t + \rho^{-1} M_t \eta \sigma \phi dt \\ &= r W_t dt + W_t \left(\frac{\eta}{\sigma} + \phi \right) \{(\mu - r) dt + \sigma dZ_t\}. \end{aligned}$$

Comparing this to the budget equation [Equation (2)] in the continuous-time section yields the claimed optimal portfolio strategy. This strategy satisfies the expected square integrability condition because of the L^2 -isometry of stochastic integrals.

require empty promises. This robust free lunch will exploit the agent's belief that $S_T = 120$ for sure and the zero market price of this event. We construct the robust free lunch using a sequence of butterfly spreads:⁷ Let $0 < \varepsilon < 120$ be given, and consider the payoff at maturity equal to $\max\{0, \varepsilon - |S_T - 120|\}$, which can be constructed using a long position in two T -maturity European calls with an exercise price of 120 and a short position in two T -maturity European calls, one having an exercise price $120 - \varepsilon$ and the other $120 + \varepsilon$. The value of the long position equals

$$-\varepsilon^2 \left\{ \frac{BS(120 + \varepsilon) - 2BS(120) + BS(120 - \varepsilon)}{\varepsilon^2} \right\},$$

where $BS(K)$ gives the Black–Scholes price of a European call with exercise price K and maturity T . As ε decreases to 0, the bracketed term converges to the (finite) second derivative of the Black–Scholes price with respect to the exercise price evaluated at $K = 120$. The price of the spread is $O(\varepsilon^2)$, and an agent with a fixed endowment can purchase $O(1/\varepsilon^2)$ spreads.⁸ Each spread pays ε conditional on $S_T = 120$, so the portfolio's payoff is $O(1/\varepsilon)$. As ε decreases to 0, the terminal payoff increases without bound conditional on S_T being equal to 120, so there is no optimum, even under the no-empty-promises constraint.

We draw the similar conclusions from the continuous-time examples as from the finite-state examples: (i) There are no arbitrage opportunities if the agents beliefs are positive on any event for which the state prices are positive. (ii) A no-empty-promises constraint can rule out arbitrage if the agent attaches zero probability to some states which have positive implicit prices. (iii) A “robust arbitrage” exists if the agent believes that a state is possible and it has a zero price.

2. The Single-Period Analysis

Having considered several examples which illustrate our no-empty-promises setting, we now turn to our general results. The starting point of our analysis is the standard neoclassical choice problem with finitely many states, no taxes or transaction costs, and possibly incomplete markets. Consistent with the partial equilibrium spirit of arbitrage arguments, we will consider the choice problem faced by an individual agent and will condition our analysis on the agent's (possibly endogenous) information at the start of the period. An agent's endowment includes a nonrandom nonnegative initial

⁷ Recall that a butterfly spread is a portfolio consisting of a short position in two call options at a given exercise price X and a long position in two call options, one of which has an exercise price higher than X and one of which has an exercise price lower than X . (This is a bet that the stock price at maturity will be near X .)

⁸ A function $f(x)$ is $O(x^k)$ if $f(x)/x^k$ is bounded as x decreases to 0.

endowment ω_0 and random nonnegative terminal endowment represented as a vector $(\omega_{11}, \dots, \omega_{1\Theta})$ of payoffs across states of nature $1, \dots, \Theta$. (Of course, this includes as a special case the assumption that all $\omega_{1\theta}$'s are zero and all endowment is received initially.) Investment opportunities are represented by an asset price vector P and a $\Theta \times N$ matrix X of terminal asset payoffs. The typical entry $X_{\theta n}$ of X is the payoff of security n in state θ . The agent's beliefs are given by a vector π of state probabilities, nonnegative and summing to one, and the agent's preferences are represented by a von Neumann–Morgenstern utility function $u(c_0, c_1) = u_0(c_0) + u_1(c_1)$, additively separable over time and strictly increasing and continuous in both arguments, which are initial and terminal consumption. We further require that the domain of u allows increases in consumption: if (c_0, c_1) is in the domain of u , then so must be (c_0', c_1') whenever $c_0' \geq c_0$ and $c_1' \geq c_1$. While we have assumed that u is additively separable over time, our results would be the same, with the same proofs, for various classes of preferences, with or without additive separability, continuity, differentiability, concavity, or state independence; what really matters is that we have a sufficiently rich class in which more is preferred to less and the agent cares only about states that happen with positive probability. The results do not get “stronger” or “weaker” as we vary our assumptions on the class of utility functions: some get stronger while others get weaker as we restrict the class.

The agent's choice variable is the portfolio weight vector α measured in units of shares purchased. Given these assumptions, the traditional choice problem is Problem 1.

Problem 1. Choose a vector α of portfolio weights to maximize expected utility of consumption $\sum_{\theta|\pi_{\theta}>0} \pi_{\theta} u(\omega_0 - P\alpha, \omega_{1\theta} + (X\alpha)_{\theta})$.

We will be concerned with a choice problem with the additional *no-empty-promises* constraint that it is not feasible to make promises that cannot be met in some set $\Theta^* \subseteq \{1, 2, \dots, \Theta\}$ of states. This is motivated by a requirement that trading partners will not permit the agent to risk insolvency. In some contexts, we could interpret Θ^* to be the set of possible states given consensus beliefs, or a superset of those possible states: leaving this issue vague permits application of the analysis to circumstances in which it is not obvious what is meant by consensus beliefs. With the no-empty-promises condition, we have Problem 2.⁹

Problem 2. Choose a vector α of portfolio weights to maximize expected utility of consumption $\sum_{\theta|\pi_{\theta}>0} \pi_{\theta} u(\omega_0 - P\alpha, \omega_{1\theta} + (X\alpha)_{\theta})$ subject to $(\forall \theta \in \Theta^*)(\omega_{1\theta} + (X\alpha)_{\theta} \geq 0)$.

⁹ For symmetry, it might be reasonable to impose a no-empty-promises constraint on initial consumption as well; this would change nothing in the analysis. Similarly, putting a lower bound different from zero on consumption in states the market cares about would not change anything.

The traditional definition of arbitrage, given by Definition 1, depends only on payoffs and the individual's beliefs.

Definition 1. An *arbitrage* is a net trade η in securities that pays off sometime with positive probability (either $(X\eta)_\theta > 0$ for some θ with $\pi_\theta > 0$ or $-P\eta > 0$) and never has a positive probability of a loss (both $-P\eta \geq 0$ and $(X\eta)_\theta \geq 0$ whenever $\pi_\theta > 0$).

For our current purposes, we want a new definition of arbitrage that is subject to a no-empty-promises condition. We cannot tell from looking at a net trade whether adding it to some proposed investment portfolio would violate the no-empty-promises condition, but we can tell whether it would if undertaken at large enough scale. The spirit of arbitrage is that it is inconsistent with optimization since it is a net trade that would continue to be feasible and improve utility at all scales and given any starting value. To remain within this spirit, we have “robust arbitrages,” given by Definition 2.

Definition 2. A *robust arbitrage* is an arbitrage η satisfying the no-empty-promises constraint $(\theta \in \Theta^*) \Rightarrow ((X\eta)_\theta \geq 0)$.

Any robust arbitrage is obviously an arbitrage, but an arbitrage need not be a robust arbitrage if there are states the market cares about but are assigned zero probability by the agent. The arbitrage is “robust” because it is feasible even if the agent cannot make empty promises.

A *positive linear pricing rule* assigns positive price to all states with positive probability and zero price to all other states.

Definition 3. A *positive linear pricing rule* is a vector p of state prices that correctly prices all assets ($P' = pX$), assigns positive price to those states with positive probability $((\pi_\theta > 0) \Rightarrow (p_\theta > 0))$, and assigns zero price to all other states $((\pi_\theta = 0) \Rightarrow (p_\theta = 0))$.

We need to consider a *superpositive linear pricing rule* that assigns positive price to all states with positive probability but may assign positive price to other states if the market cares about them.

Definition 4. A *superpositive linear pricing rule* is a vector p of state prices that correctly prices all assets ($P' = pX$), assigns positive price to those states with positive probability $((\pi_\theta > 0) \Rightarrow (p_\theta > 0))$, assigns nonnegative price to all states in which empty promises are not permitted $((\theta \in \Theta^*) \Rightarrow (p_\theta \geq 0))$, and assigns zero price to all other states $((\pi_\theta = 0) \text{ and } (\theta \notin \Theta^*)) \Rightarrow (p_\theta = 0)$.

To understand the connection between pricing and absence of robust arbitrages, it is useful to examine the choice problem. Under appropriate regularity assumptions, first-order necessary and sufficient conditions to

Problem 2 are

$$\begin{aligned} u'_0(\omega_0 - P\alpha)P &= \sum_{\theta | \pi_\theta > 0} \pi_\theta u'_1(\omega_{1\theta} + (X\alpha)_\theta)X_\theta + \sum_{\theta \in \Theta^*} \gamma_\theta X_\theta \\ (\forall \theta \in \Theta^*) \quad \gamma_\theta(\omega_{1\theta} + (X\alpha)_\theta) &= 0 \\ \gamma_\theta &\geq 0, \quad \omega_{1\theta} + (X\alpha)_\theta \geq 0. \end{aligned}$$

for some Lagrange multipliers $\gamma_\theta, \theta \in \Theta^*$. If there is a solution to Problem 2, then the vector

$$p_\theta = \frac{\pi_\theta u'_1(\omega_{1\theta} + (X\alpha)_\theta) + \gamma_\theta}{u'_0(\omega_0 - P\alpha)}, \quad \theta \in \Theta, \quad (1)$$

forms a superpositive linear pricing rule. In an important state with positive shadow price but zero probability for the agent, the shadow price $\gamma_\theta > 0$ on the binding nonnegative wealth constraint takes up the slack between the zero marginal utility of consumption and the positive state price.

By now we have accumulated the definitions needed to state the fundamental theorem of asset pricing in the single-period world with finitely many states, both in its original version and in its new version without empty promises.

Theorem 1. (the fundamental theorem of asset pricing). *The following are equivalent: (i) absence of arbitrage, (ii) existence of a positive linear pricing rule, and (iii) existence of an optimum in the traditional problem (Problem 1) for some hypothetical agent who prefers more to less.*

Proof. See Dybvig and Ross (1987). ■

Here is the new version of the fundamental theorem of asset pricing. While the proof follows the same broad outline as the proof of Dybvig and Ross (1987), some details are more subtle.

Theorem 2. (the fundamental theorem of asset pricing with no empty promises). *The following are equivalent: (i) absence of robust arbitrages, (ii) existence of a superpositive linear pricing rule, and (iii) existence of an optimum in the problem without empty promises (Problem 2) for some hypothetical agent who prefers more to less for some endowment.*

Proof. (iii) $\Rightarrow$ (i): We want to show that existence of an optimum implies absence of robust arbitrage. Suppose to the contrary that there is an optimum but that there is also available a robust arbitrage. Since $u(\cdot, \cdot)$ is strictly increasing in both arguments, it follows that adding the robust arbitrage to the claimed optimum would not decrease value in any state, would increase value in the positive-probability state (or time 0) when consumption is in-

creased, and would not violate the no-empty-promises constraint. Therefore it would dominate the claimed optimum, which is a contradiction.

(ii) $\Rightarrow$ (iii): Given the existence of a superpositive linear pricing rule p , we need to show that some hypothetical agent has a maximum. We will show that a hypothetical agent with time-separable exponential von Neumann–Morgenstern utility function $u(c_0, c_1) = -\exp(-c_0) + \log(c_1)$ and a carefully chosen endowment has an optimum. The endowment we select has

$$\omega_{1\theta} = \begin{cases} \pi_\theta / p_\theta & \text{when } p_\theta > 0 \text{ and } \pi_\theta > 0 \\ 0 & \text{when } p_\theta = 0 \text{ or } \pi_\theta = 0 \end{cases}$$

and $\omega_0 = 0$. Then it is easy to verify that $\alpha = 0$ is an optimal portfolio choice given this endowment using the first-order condition [Equation (1)] previously developed, with

$$\gamma_\theta = \begin{cases} p_\theta & \text{when } \pi_\theta = 0 \text{ and } \theta \in \Theta^* \\ 0 & \text{otherwise.} \end{cases}$$

(i) $\Rightarrow$ (ii): The consumption space in our problem can be represented by $\mathfrak{N}^{1+\Theta}$, with the first component representing the amount of consumption at time 0 and the remaining components representing the consumptions across states at time 1. In consumption space, let A be the set of net trades that are candidate robust arbitrages, that is, $c \equiv (c_0, c_{11}, \dots, c_{1\Theta})$ is in A if (a) both $c_0 \geq 0$ and $c_{1\theta} \geq 0$ for all θ with $\pi_\theta > 0$, (b) either $c_0 > 0$ or $c_{1\theta} > 0$ for some θ with $\pi_\theta > 0$, and (c) $c_{1\theta} \geq 0$ if $\theta \in \Theta^*$. Also in consumption space, let M be the set of marketed net trades (ignoring the empty promise and nonnegative consumption constraint), that is, $c \equiv (c_0, c_{11}, \dots, c_{1\Theta})$ is in M if there exists a portfolio α such that $c_0 = -P\alpha$ and $(c_{11}, \dots, c_{1\Theta}) = X\alpha$. We are given that $A \cap M = \emptyset$, and we want to show that there exists a state price vector $p \in \mathfrak{N}_+^\Theta$ such that (1) $P' = pX$, (2) $p_\theta > 0$ whenever $\pi_\theta > 0$, and (3) $p_\theta = 0$ whenever $\pi_\theta = 0$ and $\theta \notin \Theta^*$.

Since A and M are nonempty disjoint convex sets, there exists a dual (price) vector $\phi \in \mathfrak{N}^{1+\Theta}$, $\phi \neq 0$, such that $\phi a \geq \phi m$ for all $a \in A$ and $m \in M$. We will show shortly that such a ϕ can be chosen to satisfy $\phi_0 > 0$ and $\phi_\theta > 0$ whenever $\pi_\theta > 0$. In that case, $p \equiv \phi_0^{-1}(\phi_{11}, \dots, \phi_{1\Theta})$ will be the required state price vector. Property (1), $P' = pX$, follows from the fact that ϕ separates M , since M includes the result of investment of +1 or -1 unit of each asset individually. Property (2), $p_\theta > 0$ whenever $\pi_\theta > 0$, follows from the selection of ϕ with like properties. Property (3), $p_\theta = 0$ whenever $\pi_\theta = 0$ and $\theta \notin \Theta^*$, follows from the fact that ϕ separates A , since $\pi_\theta = 0$ implies that A contains $(1, 0, \dots, 0)$ plus K times the unit vector in the direction of the 1θ coordinate, for all positive and negative K .

It remains to show that the vector ϕ separating A and M can be chosen with $\phi_0 > 0$ and $\phi_{1\theta} > 0$ whenever $\pi_\theta > 0$. It suffices to show that we

can choose separate ϕ 's to make each corresponding element (ϕ_0 or $\phi_{1\theta}$) positive, since the sum of such ϕ 's will still separate A and M but will make them all positive. Fix θ with $\pi_\theta > 0$, and suppose there are no robust arbitrages. (The argument for ϕ_0 is identical; only the notation is slightly different.) Arguing by contradiction, assume no ϕ separating A and M has $\phi_{1\theta} > 0$. Recall that the dual to a set $X \in \Re^N$ is defined to be the convex set $X^+ \in \Re^N$ defined by $X^+ \equiv \{y \mid (\forall x \in X) yx \geq 0\}$. The set of ϕ 's separating A and M is given by the nonzero elements of $A^+ \cap (-M)^+$.¹⁰ Since (by our assumption in the argument by contradiction) no ϕ separating M and A has $\phi_{1\theta} > 0$, and since no ϕ separating A and M can have $\phi_{1\theta} < 0$ (or it would not separate A , since $\pi_\theta > 0$), it follows that $A^+ \cap M^+ \subseteq \{\phi \mid \phi_{1\theta} = 0\}$, or equivalently, $\{\phi \mid \phi_{1\theta} = 0\}^+ \subset (A^+ \cap M^+)^+$ [by property (i) of convex cones given before Karlin (1959) Theorem 5.3.1]. By the duality theorem for closed convex cones [Karlin (1959), Theorem B.3.1, parts I and II], it follows, using the usual overline notation for set closure, that $\{\phi \mid \phi_{1\theta} = 0\}^+ \subseteq (A^+ \cap M^+)^+ \subseteq \overline{M} + \overline{A} = \overline{M} + \overline{A} = M + \overline{A}$, where the first equality follows from the fact that $\overline{A}$ is a polyhedral cone and $\overline{M}$ is a subspace [Rockafellar (1970), Theorem 20.3], and the second equality comes from the fact that M is closed. However, this contradicts absence of robust arbitrage, since the set $\{\phi \mid \phi_{1\theta} = 0\}^+$ includes the vector with all zeros except for -1 in the component 1θ , and therefore M includes an element of $\overline{A}$ plus the vector of all zeros except for 1 in the component 1θ , which is itself an element of A . This completes the proof that (i) $\Rightarrow$ (ii). ■

3. The Continuous-Time Analysis

We now turn to the continuous-time version of the fundamental theorem of asset pricing without empty promises. The intuition of the finite-state results holds once we add structure to accommodate infinite-dimensional state spaces typically used in investments and option pricing.

3.1 Definitions

The agent trades finitely many risky securities in a frictionless, competitive, and possibly incomplete market over a trading interval $[0, T]$. Security returns are defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω contains the set of states of nature and the σ -algebra $\mathcal{F}$ contains the events distinguishable at time T . The probability measure P is a reference measure and is used only to define returns in states to which the agent may or may not assign positive probability. We have no need to specify separately the price and

¹⁰ Since the set M of marketed net trades is a subspace, $-M = M$ and therefore $(-M)^+$ is the same as M^+ , but this fact is not needed for the proof.

dividend processes, so we simply assume that returns are generated by some special semimartingale process G , as in Back (1991).¹¹ We also assume that there is an asset with a locally riskless instantaneous return represented by $r_t dt$. Return processes and their coefficients are assumed to be adapted to a given right-continuous complete filtration $\mathbf{F} = \{\mathcal{F}_t: t \in [0, T]\}$ that satisfies $\mathcal{F}_T = \mathcal{F}$. Corresponding to a given trading strategy θ for the risky assets, there is a right-continuous wealth process that satisfies

$$(\forall t \in [0, T]) \quad W_t = W_0 + \int_0^t r_u W_u du + \int_0^t \theta_u dG_u, \quad (2)$$

P almost everywhere. To ensure that the process in Equation (2) is well defined, we assume that each trading strategy is $\mathbf{F}$ predictable and satisfies integrability conditions used to define stochastic integrals. Let Θ denote the linear space of these strategies for which Equation (2) has a unique solution.¹²

The agent's beliefs are represented by a probability measure P^I . We assume that P^I is absolutely continuous relative to the reference measure P so that, after the fact, the agent and the market agree on trading profits.¹³ Preferences for consumption plans (c_0, c_1) are represented by the expected utility function

$$U(c_0, c_1) \equiv u_0(c_0) + \int_{\Omega} u_1(\omega, c_1(\omega)) P^I(d\omega),$$

where $u_0(\cdot)$ and $u_1(\omega, \cdot)$ are continuous, increasing, and defined on $\mathfrak{R}_+$. The examples in Section 1 are special cases of this setting when P is interpreted as a weighted average of the agent's prior and conditional beliefs.

We now define the commodity space for the agent. In infinite-state models such as ours, the topology of the commodity space is important because it influences the definitions of arbitrage and linear pricing. In our case, the topology must also allow convergence in important states of nature, even if the agent assigns them zero probability. A commodity space that is suitable for our purposes is $\mathcal{C} \equiv \mathfrak{R} \times L^p(\Omega, \mathcal{F}, P)$, for some $1 \leq p < \infty$, where $L^p(\Omega, \mathcal{F}, P)$ is the space of P -equivalent random variables that have finite

¹¹ A special semimartingale is a process that is uniquely representable as the sum of a predictable right-continuous finite-variation process and a right-continuous local martingale [Dellacherie and Meyer (1982, VII.23)]. A special semimartingale may be discontinuous.

¹² Predictability requires strategies at time t to depend on information available only strictly before time t . See Dellacherie and Meyer (1982, Chapter VIII) for conditions sufficient to define stochastic integrals and Protter (1990, Chapter 5.3) for conditions sufficient to ensure unique solutions.

¹³ In discrete models, trading profits are defined sample path by sample path, and this would not be an issue, but stochastic integrals defining trading profits in continuous time are not. Agreeing on trading profits after the fact does seem to be a feature of the actual economy, and absolute continuity of P^I in P implies that two agents agree on trading profits almost surely in both P and P^I . This seems like a very minimal sort of rationality assumption for us to make.

pth moments.¹⁴ We will take convergence and continuity to be in the product norm topology on $\mathcal{C}$.

The traditional choice problem of the agent is given in Problem 3.

Problem 3. Choose consumption $(c_0, c_1) \in \mathcal{C}$ to maximize expected utility $U(c_0, c_1)$ subject to the following conditions:

(a) There is a trading strategy $\theta \in \Theta$ for which wealth satisfies Equation (2) and the conditions $W_0 \leq w - c_0$ and $c_1 \leq W_T$, P almost surely.

(b) c_1 and W are nonnegative in states important to the agent ($P^I(c_1 \geq 0) = 1$) and $((\forall t \in [0, T]), W_t \geq 0)$, P^I almost surely).¹⁵

In the no-empty-promises setting, the agent faces the additional constraint of being unable to make “empty promises” in some subset Ω^* of important states: the agent must maintain nonnegative wealth along P almost all paths $t \mapsto W_t(\omega)$, $\omega \in \Omega^*$. We assume that Ω^* belongs to the σ -algebra $\mathcal{F}$. As in the finite-state model, the no-empty-promises constraint may be motivated by a requirement that trading partners will not permit the agent to risk insolvency. Notice that if $P(\Omega^*) = 0$, then the no-empty-promises restriction is vacuous, and the problem would be essentially the same with or without empty promises.

We include the no-empty-promises constraint in Problem 4.

Problem 4. Choose consumption $(c_0, c_1) \in \mathcal{C}$ to maximize expected utility $U(c_0, c_1)$ subject to the following conditions:

(a) There is a trading strategy $\theta \in \Theta$ for which wealth satisfies Equation (2) and the conditions $W_0 \leq w - c_0$ and $c_1 \leq W_T$, P almost surely,

(b) c_1 and W are nonnegative in states important to the agent ($P^I(c_1 \geq 0) = 1$) and $((\forall t \in [0, T]), W_t \geq 0)$, P^I almost surely).

(c) Consumption and wealth satisfy the no-empty-promises condition ($c_1 \geq 0$ and $((\forall t \in [0, T]), W_t \geq 0)$, for P almost all $\omega \in \Omega^*$).

Unlike Problem 3, Problem 4 includes the no-empty-promises constraint (c), which requires consumption and the paths of wealth to be nonnegative in almost all states important to the market.

We now define a traditional arbitrage. The intuition is similar to the finite-state setting except that a nonnegative wealth constraint is imposed only along paths the agent cares about.

¹⁴ An additional assumption that the Radon–Nikodym derivative $\frac{dP^I}{dP}$ is essentially bounded would ensure that the commodity space is consistent for all agents and any consumption plan is p integrable against any agent’s beliefs. This assumption is satisfied, for example, in our finite-state analysis and Examples 4 and 5 of Section 1; however, we do not use this assumption in our analysis.

¹⁵ The agent can choose arbitrarily negative consumption in states occurring with zero P^I probability because these states do not enter into the expected utility calculation.

Definition 5. An *arbitrage opportunity* is a consumption plan $(c_0, c_1) \in \mathcal{C}$ financed by a trading strategy $\theta \in \Theta$ such that

- (a) Wealth satisfies Equation (2) with no initial investment ($c_0 = -W_0$).
- (b) Wealth is nonnegative on paths the agent believes are possible; that is, $((\forall t \in [0, T]) W_t \geq 0)$, P^I almost surely.
- (c) The agent believes that (c_0, c_1) is nonnegative and provides positive consumption with positive probability; that is, $[P^I(c_1 \geq 0) = 1 \text{ and } c_0 \geq 0]$ and [either $P^I(c_1 > 0) > 0$ or $c_0 > 0$].

A robust arbitrage additionally enforces the no-empty-promises condition along paths the market cares about. It is “robust” because it will be feasible even if no empty promises are permitted.

Definition 6. A *robust arbitrage* is an arbitrage opportunity which satisfies the no-empty-promises constraint: $((\forall t \in [0, T]) W_t \geq 0)$ for P almost all $\omega \in \Omega^*$.

In general, we cannot tell whether adding a given net trade to some unknown portfolio violates the nonnegative wealth or the no-empty-promises constraint of an investor; however, we can tell if it will when undertaken at an arbitrarily large scale. The spirit of arbitrage is that it is inconsistent with optimization since it is a net trade which would continue to be feasible and improve utility at all scales and given any initial feasible plan. Both definitions of arbitrage here remain within this spirit.

In contrast to the finite-state case, the set of marketed net trades generally is not the intersection of an affine subspace with the positive orthant due to free disposal implicit in the financing condition $c_1 \leq W_T$, and more subtly, due to suicidal strategies even if $c_1 = W_T$ holds.¹⁶ However, this set does form a convex cone, which we denote by M . Formally, M is the convex cone of marketed net trades ignoring nonnegative consumption constraints: the set of $(c_0, c_1) \in \mathcal{C}$ such that (a) there is a trading strategy $\theta \in \Theta$ with W satisfying Equation (2) and the conditions $c_0 \leq -W_0$ and $c_1 \leq W_T$, and (b) the condition $((\forall t \in [0, T]) W_t \geq 0)$ holds P^I almost surely. Similarly define $\hat{M}$ for the no-empty-promises setting by adding to (b) the condition that $((\forall t \in [0, T]) W_t \geq 0)$ for P almost all $\omega \in \Omega^*$.

Here is the definition of a positive linear pricing rule. In this and the following definitions, we use L^P to denote the linear space $L^P(\Omega, \mathcal{F}, P)$.

Definition 7. A continuous linear functional $\psi: L^P \rightarrow \Re$ defines a positive linear pricing rule if it satisfies the following conditions:

- (a) Each net trade $(c_0, c_1) \in M$ has a nonpositive price equal to $c_0 + \psi(c_1) \leq 0$.

¹⁶ An example of a “suicidal strategy” is a doubling strategy run in reverse. A suicidal strategy is a net trade that permits an agent to throw away wealth; see Dybvig and Huang (1989) for an analysis of doubling and suicidal strategies in the presence of a nonnegative wealth constraint.

(b) Any terminal consumption plan c_1 which the investor believes is positive has a positive price (i.e., together $P^I(c_1 > 0) > 0$ and $P^I(c_1 \geq 0) = 1$ imply that $\psi(c_1) > 0$).

(c) For any terminal consumption plan c_1 for which there is a feasible trading strategy $\theta \in \Theta$ such that wealth satisfies Equation (2) and the condition $c_1 \leq W_T$, we have $\psi(c_1) \leq W_0$.

(d) Terminal consumption plans positive only on events to which the agent assigns zero probability are costless (i.e., $(\forall E \in \mathcal{F})(P^I(E) = 0 \Rightarrow \psi(1_E) = 0)$).¹⁷

Here is the definition of a “superpositive” linear pricing rule which may assign positive prices to important states, even to those impossible from the agent’s perspective.

Definition 8. A continuous linear functional $\psi: L^P \rightarrow \Re$ defines a superpositive linear pricing rule if it satisfies the following conditions:

(a) Any net trade $(c_0, c_1) \in M$ has a nonpositive price equal to $c_0 + \psi(c_1) \leq 0$.

(b) Any terminal consumption plan c_1 that satisfies the no-empty-promises condition and which the agent believes is positive has a positive terminal price (i.e., the conditions $P^I(c_1 > 0) > 0$, $P^I(c_1 \geq 0) = 1$, and $c_1 \geq 0$ for P almost all $\omega \in \Omega^*$ imply that $\psi(c_1) > 0$).

(c) For any c_1 for which there is a feasible strategy $\theta \in \Theta$ such that wealth satisfies Equation (2) and $c_1 \leq W_T$, we have $\psi(c_1) \leq W_0$.

(d) Consumption plans that are positive only on events that are important to neither the agent nor the market are costless (i.e., $(\forall E \in \mathcal{F})([P^I(E) = 0 \& P(E \cap \Omega^*) = 0] \Rightarrow \psi(1_E) = 0)$).

We now formalize a notion of arbitrage in this continuous-time setting. We will use the concept of a “free lunch” to extend the notion of arbitrage to include continuity and free disposal.¹⁸ A free lunch differs from an arbitrage in that a free lunch relies on the topology of the commodity space. The trouble with a free lunch is that each consumption plan in the sequence may be infeasible, yet the limit is called a free lunch. A free lunch will be attractive if there exists an agent in the economy who is willing to absorb a deviation that is “small” in the topology.¹⁹

We now define the notation that we use to define a robust free lunch. Let

¹⁷ The indicator function 1_E equals 1 if $\omega \in E$ and 0 otherwise.

¹⁸ Absence of arbitrage is generally insufficient to guarantee the existence of a continuous linear pricing rule. A problem is that the interior of the nonnegative orthant is empty in most interesting infinite-dimensional spaces, thus invalidating most separating hyperplane theorems. See Ross (1978), Kreps (1981), and Back and Pliska (1991) for more details.

¹⁹ This agent cannot be our agent because adding the topologically small deviation may cause our agent’s consumption to lie outside the consumption set.

A be candidate arbitrage payoffs in the traditional setting,

$$A \equiv \{(c_0, c_1) \in \mathcal{C}: P^I(c_1 \geq 0) = 1 \text{ and } [P^I(c_1 > 0) > 0 \text{ or } c_0 > 0]\},$$

and similarly define $\hat{A}$ in the no-empty-promises setting to be

$$\hat{A} \equiv A \cap \{(c_0, c_1) \in \mathcal{C}: c_1 \geq 0 \text{ for } P\text{-almost all } \omega \in \Omega^*\}.$$

Note that A and $\hat{A}$ are convex cones that exclude the origin. Let $\overline{X}$ denote closure in the product norm topology for a subset X of $\mathcal{C}$. Here is a version of arbitrage which suits our purpose.

Definition 9. A *free lunch* is a candidate arbitrage which is the limit of a sequence of net trades less free disposal taken in states important to the agent; that is, it is an element of $A \cap (\overline{M} - A)$. A *robust free lunch* is a candidate robust arbitrage which is the limit of a similar sequence with free disposal taken additionally in states important to the market; that is, it is an element of $\hat{A} \cap (\overline{\hat{M}} - \hat{A})$.

Note that a robust arbitrage is a robust free lunch. The limit of the sequence of portfolios of butterfly spreads in Example 6 of Section 1 is an example of a robust free lunch. Note, however, that none of the portfolios in the sequence is a robust arbitrage because of the (arbitrarily small but fixed) initial wealth required to purchase the portfolio.

Free disposal in this setting allows us to consider sequences of net trades which might otherwise become “too good” to remain in the commodity space by freely disposing of consumption in states important to either the agent or the market. This is needed for technical reasons: a consumption sequence becoming larger and larger in important states may not converge and would fail to be an arbitrage opportunity. The definition of a free lunch captures our intuition that something dominating an arbitrage payoff in important states also should be an arbitrage opportunity.

We now turn to our definition of optimality. Back and Pliska (1991) suggest that something stronger than optimality is required for the existence of an optimum for some agent to imply either continuous linear pricing or absence of free lunches. Their suggestion applies here.²⁰ Our final definition provides a notion of optimality sufficient for our purpose.

Definition 10. An optimal consumption demand (c_0, c_1) is called a *strong optimum* if it is also optimal in the closure of the set that consists of the elements in the budget set plus net trades less free disposal.

²⁰ This is again a problem of the empty interior of the nonnegative orthant of the commodity space. Previous studies of the fundamental theorem often assume that preferences are continuous and defined over negative consumption. [See, for example, Kreps (1981) and Back (1991)]. Our model requires something different because we rule out negative consumption.

The budget set, the set of marketed net trades, and the set of potential arbitrage opportunities depend on whether or not the agent faces the no-empty-promises constraint. Let B denote the budget set of consumption satisfying (a) and (b) in the traditional choice problem (Problem 3), and let $\hat{B}$ denote the analogous budget set for Problem 4. Strong optimality in the no-empty-promises problem, for example, requires the agent to be unable to approximate a consumption plan providing higher expected utility than (c_0, c_1) using a sequence from the set $\hat{B} + \hat{M} - \hat{A}$. The need to take closure arises because $\hat{B} + \hat{M} - \hat{A}$ may not be closed, even if the individual sets $\hat{B}$, $\hat{M}$, and $\hat{A}$ are.²¹

3.2 Results

We now have enough to state a continuous-time version of the traditional fundamental theorem of asset pricing.

Theorem 3. (the fundamental theorem of asset pricing). *The following statements are true. (i) The absence of free lunches is equivalent to the existence of a positive linear pricing rule. (ii) If there exists a positive linear pricing rule, then there exists a solution to Problem 3 for some hypothetical agent who prefers more to less. (iii) If there exists a solution to Problem 3 for an agent who prefers more to less, then there are no arbitrage opportunities. If, in addition, there is a strong optimum for Problem 3, then there are no free lunches.*

Theorem 3 can be proven by adapting the proof of our next theorem, so we postpone the proof of Theorem 3. Here is the version of the fundamental theorem of asset pricing that has the no-empty-promises constraint.

Theorem 4. (the fundamental theorem of asset pricing without empty promises). *The following statements are true. (i) The absence of robust free lunches is equivalent to the existence of a superpositive linear pricing rule. (ii) If there exists a superpositive linear pricing rule, then there exists a solution to Problem 4 for some hypothetical agent who prefers more to less. (iii) If there exists a solution to Problem 4 for an agent who prefers more to less, then there are no robust arbitrage opportunities. If, in addition, there is a strong optimum for Problem 4, then there are no robust free lunches.*

Proof of Theorem 4. (i) Suppose there are no robust free lunches; by definition, this means that $\hat{A} \cap (\hat{M} - \hat{A}) = \emptyset$. The set of marketed net trades $\hat{M}$ and the set of potential robust arbitrage payoffs $\hat{A}$ are nonempty disjoint convex cones, so we can use the separation theorem of Clark (1993, Theorem 5)

²¹ The appendix of Dybvig and Huang (1989) presents an example of this in a portfolio problem, but errors in the typesetting make the example unreadable. The original correct version is available from Dybvig.

to obtain a continuous linear functional $\phi: \mathcal{C} \rightarrow \Re$ that strictly separates $\hat{M}$ from $\hat{A}$; that is, ϕ satisfies the condition $(\forall a \in \hat{A})(\forall m \in \hat{M}) \quad \phi(m) \leq 0 < \phi(a)$.²² Let E be an arbitrary event that occurs with positive P^I probability. Notice that the consumption plan $(1, 0)$, which provides one unit of consumption today and none next period, and the consumption plan $(0, 1_E)$, which provides no consumption today and one unit next period if event E occurs, are potential robust arbitrage opportunities. Because ϕ separates $\hat{A}$ and $\hat{M}$, the inequalities $\phi(1, 0) > 0$ and $\phi(0, 1_E) > 0$ must hold.

We will now use ϕ to construct our continuous superpositive linear pricing rule. Define a new continuous linear functional $\hat{\phi}$ by

$$\hat{\phi}(c_0, c_1) = \frac{\phi(c_0, c_1)}{\phi(1, 0)} = \frac{c_0\phi(1, 0) + \phi(0, c_1)}{\phi(1, 0)} \equiv c_0 + \psi(c_1).$$

(Recall that c_0 is nonrandom and ϕ is linear.) It is easy to check that $\hat{\phi}$ is also a continuous linear functional that separates $\hat{M}$ from $\hat{A}$, so condition (a) of Definition 8 holds. Condition (b) is true for ϕ , and it is true for $\hat{\phi}$ for the same reasons. To verify condition (c), note that if c_1 is financed by W_0 , then the consumption plan $(-W_0, c_1)$ is a net trade in $\hat{M}$. Thus condition (a) implies that $\psi(c_1) \leq W_0$. We now need to show that condition (d) holds. Let $E \in \mathcal{F}$ be an arbitrary event. For any given whole number n , the consumption plan $(0, \frac{1}{n}1_\Omega + 1_E)$ belongs to $\hat{A}$ and must have a positive price; therefore, $\psi(1_E) \geq 0$ by the continuity of ϕ . Furthermore, note that for any $E \in \mathcal{F}$ with $P^I(E) = 0$ and $P(E \cap \Omega^*) = 0$, the consumption plan $(0, 1_\Omega + \lambda 1_E)$ belongs to $\hat{A}$ for all real λ . Strict separation requires the condition

$$\psi(1_\Omega + \lambda 1_E) = \psi(1_\Omega) + \lambda \psi(1_E) > 0 \quad \text{for all real } \lambda$$

to hold, which would be true only if $\psi(1_E) = 0$. Thus ψ defines a continuous superpositive linear pricing rule.

Conversely, suppose that we are given some ψ that defines a superpositive linear pricing rule; we want to show that there are no robust free lunches. Suppose to the contrary that one exists; we will obtain a contradiction. For $m = (c_0, c_1) \in \mathcal{C}$, define the continuous linear functional $\phi(m) = c_0 + \psi(c_1)$. Our contrary hypothesis is that there exists a sequence $\{m_n - a_n\} \subset \hat{M} - \hat{A}$ converging to some $a \in \hat{A}$. This is impossible because ϕ is continuous and by definition must satisfy the condition $(\forall n) \quad \phi(m_n - a_n) < \phi(m_n) \leq 0 < \phi(a)$. Therefore, there cannot be a robust arbitrage.

(ii) We now want to show that there exists an optimum to the consumption choice problem of a hypothetical agent who faces a no-empty-promises constraint, but we must develop some notation first. Let ξ^I define the Radon–

²² Specifically, Clark shows that two nonempty convex cones J and K (with vertices at the origin) in a separable Banach space (like our consumption space) can be strictly separated if and only if $J \cap (\overline{K - J}) = \emptyset$.

Nikodym derivative dP^I/dP , and let ψ be the linear functional that defines the given superpositive linear pricing rule. The Riesz representation theorem implies that there exists $v \in L^q(\Omega, \mathcal{F}, P)$, $1/q + 1/p = 1$, such that

$$\psi(c_1) = E[v c_1] \quad \text{for all } c_1 \in L^p(\Omega, \mathcal{F}, P). \quad (3)$$

Because ψ defines a superpositive linear pricing rule, $v(\omega) \geq 0$, P^I almost surely and for P almost all $\omega \in \Omega^*$, and $v(\omega) > 0$ if $\xi^I(\omega) > 0$.

We now demonstrate that there is a solution to Problem 4 for an agent with initial endowment $w = 0$ and state-dependent preferences

$$U(c_0, c_1) = -\exp(-c_0) - \int_{\Omega} \exp(-(y(\omega) + c_1(\omega))) P^I(d\omega),$$

where

$$y(\omega) = \begin{cases} \log \frac{\xi^I(\omega)}{v(\omega)} & \text{if } \xi^I(\omega) > 0 \\ 0 & \text{if } \xi^I(\omega) = 0. \end{cases}$$

Consider the following new choice problem: choose $(c_0, c_1) \in \mathcal{C}$ to maximize $U(c_0, c_1)$ subject to the budget constraint $c_0 + \psi(c_1) = c_0 + E[v c_1] \leq 0$ and the nonnegativity condition $c_1 \geq 0$, that holds P^I almost surely and for P almost all $\omega \in \Omega^*$. A solution to this new problem is optimal for Problem 4 if there is a trading strategy that finances it. First-order sufficient conditions for a solution to the new problem are

$$\begin{aligned} \xi^I(\omega) \exp(-(y(\omega) + c_1(\omega))) + \lambda_1(\omega) &= \gamma v(\omega) \\ \exp(-c_0) + \lambda_0 &= \gamma \\ \gamma(c_0 + E[v c_1]) &= 0 \\ \lambda_1(\omega) c_1(\omega) &= 0, \text{ and } \lambda_0 c_0 = 0 \quad \text{almost surely} \end{aligned}$$

for some Lagrange multipliers $\lambda_1 \in L_+^q$ and $\lambda_0, \gamma \in \Re_+$ [see, e.g., Duffie (1988, p. 77)]. The Lagrange multipliers λ_0 and λ_1 are the shadow prices for the nonnegative constraints on c_0 and c_1 , respectively, and γ is the marginal utility of wealth. A solution to the new problem is $c_0 = 0, c_1 \equiv 0, \gamma = 1$, and $\lambda(\omega) = v(\omega)$ if $\xi^I(\omega) = 0$ holds and $\lambda(\omega) = 0$ otherwise. The multiplier λ belongs to L_+^q since it satisfies $0 \leq \lambda \leq v$ and $v \in L_+^q$, so c_1, λ , and γ satisfy the first-order and the complementary slackness conditions. This is a solution to Problem 4 since $\theta \equiv 0$ is trivially feasible and finances c_1 .

(iii) The existence of an optimal demand rules out robust arbitrage opportunities, for the net trade could be added to any candidate optimum, increasing expected utility without violating the no-empty-promises constraint. If (c_0, c_1) is a strong optimum, then the intersection of $(c_0, c_1) + \hat{A}$ with $(\hat{B} + \hat{M} - \hat{A})$ must be empty since preferences are monotonically in-

creasing. Because $(c_0, c_1) \in \hat{B}$, we have $\hat{A} \cap \overline{(\hat{M} - \hat{A})} = \emptyset$, which is the condition for absence of robust free lunches. This finishes our proof. ■

Remark: Hindy (1995) writes his pricing rule as the sum of the direct contribution to expected utility and the shadow price of his solvency constraint. In our notation, the price in Equation (3) can be written as

$$\psi(c_1) = E[1_{\{\xi^I > 0\}} \nu c_1] + E[1_{\{\xi^I = 0\}} \nu c_1] \equiv \psi^I(c_1) + \psi^*(c_1).$$

where ψ^I and ψ^* are continuous linear functionals. The latter is nontrivial whenever the no-empty-promises constraint is binding and nonvacuous (i.e., $P(\{\xi^I = 0\} \cap \Omega^*) > 0$). Thus ψ^* can be interpreted as the shadow price of the no-empty-promises constraint, analogous to Hindy's "solvency price." For the examples in Section 1, the state price density in the original problem by Merton could be used to define a superpositive linear pricing rule for an agent who has advance information. In these examples, the linear functional ψ^* would be nontrivial.

We now prove Theorem 3.

Proof of Theorem 3. We make use of our work in the proof of Theorem 4. (i) The proof of parts (a), (b), and (c) of Definition 7 is exactly the same as the corresponding parts of Definition 8 given in the proof of Theorem 4, where $\hat{A}$ and $\hat{M}$ are replaced by A and M , respectively. To prove part (d), note that for any $E \in \mathcal{F}$ that satisfies the condition $P^I(E) = 0$, the corresponding consumption plan $(0, 1_\Omega + \lambda 1_E)$ belongs to A for all real λ . Strict separation implies that

$$\psi(1_\Omega + \lambda 1_E) = \psi(1_\Omega) + \lambda \psi(1_E) > 0 \quad \text{for all real } \lambda,$$

which holds only if the condition $\psi(1_E) = 0$ is satisfied.

(ii) This part is nearly the same, except for the absence of the no-empty-promises constraint. However, the sets $\{\nu(\omega) > 0\}$ and $\{\xi^I(\omega) > 0\}$ differ only by a nullset, so analogous first-order conditions hold at an optimum.

(iii) Apply the same argument to A and M . ■

4. Enforcement of No Empty Promises

A potential conceptual problem with the general analysis is one of enforcement: How do we ensure that an agent chooses a strategy with nonnegative wealth in states that the agent believes will never happen? We now show that if information arrival is continuous, then shutting down the agent's investments for the duration once wealth hits zero is essentially equivalent to ruling out empty promises. With continuous information arrival, the market knows the agent's wealth cannot go below zero without first hitting zero. By contrast, the wealth could jump from positive to negative without warn-

ing in a model with discrete trading or unpredictable jumps in the return distribution.

For simplicity, we assume that the risky assets' instantaneous returns are represented by

$$\mu_t dt + \sigma_t dZ_t,$$

where Z is a standard multidimensional Brownian motion. We follow the usual practice of defining the filtration $\mathbf{F}$, which represents information arrival, to be the filtration generated by Z and augmented with P nullsets; this filtration is continuous [Karatzas and Shreve (1988, Section 2.2.7)].

Trading strategies must satisfy the conditions assumed in Section 4, including those needed to define stochastic integrals. In particular, trading strategies must be predictable. Modifying a strategy to stop investment when wealth first hits zero will not violate any of these conditions, since the time when wealth first hits zero is a stopping time.

For implementation, we want to think of two different types of stopping times. A private version of stopping time available to the agent is the usual definition of stopping time. In this definition it is assumed that the stochastic process is known (in this case from knowledge of the agent's strategy). A public type of stopping time is one that depends on the wealth history and not on knowledge of the agent's particular strategy. An example of a public stopping time is stopping when the agent's wealth hits zero, which is something the market can implement without knowing the agent's strategy. An example of a private stopping time is the last time before wealth first goes negative (which is a stopping time because of the continuous information arrival). A private stopping time is available to the agent (to whom the strategy is known) for construction of an arbitrage, but not to the market's regulatory mechanism. Our proof of enforceability of no-empty-promises uses stopping when wealth hits zero, which is our example of a public stopping time.

We state a new consumption choice problem which includes a constraint that shuts down the agent's investment when wealth first hits zero.

Problem 5. Choose consumption $(c_0, c_1) \in \mathcal{C}$ to maximize expected utility $U(c_0, c_1)$ subject to the following conditions:

- (a) There is a trading strategy $\theta \in \Theta$ for which wealth satisfies Equation (2) and the conditions $W_0 \leq w - c_0$ and $c_1 \leq W_T$, P almost surely.
- (b) c_1 and W are nonnegative in states important to the agent ($P^I(c_1 \geq 0) = 1$) and $((\forall t \in [0, T]), W_t \geq 0)$, P^I almost surely).
- (c) The agent's trades are shut down at the first instant that wealth becomes zero; that is, $((\forall t)(W_t = 0 \Rightarrow (\forall s > t)\theta_s = W_s = 0))$, P^I almost surely.

Problem 5 differs from Problem 4 only in constraint (c): the no-empty-promises condition in Problem 4 is replaced by the shutdown constraint in

Problem 5. In the shutdown constraint, we use the agent's information set since optimization is from the agent's perspective. Nothing essential would be changed if we required this to hold in states important to the market as well.

We now state our main result on enforcement. It says that from the perspective of optimal choice, the no-empty-promises constraint can be replaced by a constraint that shuts down the agent's investment when wealth first hits zero.

Theorem 5. *Assume that information arrival is continuous and that Problem 4 or Problem 5 has a solution. Further assume that there exists a trading strategy giving a payoff bounded below away from zero and a positive wealth process (as there would if there exists a locally riskless asset with a nonnegative interest rate process). Then the problems are equivalent in the following sense:*

(i) Suppose (θ, W, C) is feasible in one of Problem 4 and Problem 5. Then define the new strategy, $(\hat{\theta}, \hat{W}, \hat{C})$, by closing out the agent's investments when wealth hits zero. Let

$$\tau = \begin{cases} \min\{t \mid W_t \leq 0\} & \text{if such a } t \text{ exists} \\ T & \text{otherwise.} \end{cases}$$

Letting $\wedge$ indicate the minimum, we define $\hat{\theta}_t \equiv \theta_t 1_{\{t < \tau\}}$, $\hat{W}_t \equiv W_{t \wedge \tau}$, and $\hat{C} \equiv C 1_{\{\tau < T\}}$. This new strategy is just as desirable to the agent as (θ, W, C) and is feasible for both problems.

(ii) There is a strategy (θ, W, C) that is a solution to both Problem 4 and Problem 5.

We now prove Theorem 5.

Proof of Theorem 5. (i) Feasibility requires $\hat{\theta}$ to be predictable and to satisfy the integrability conditions needed to define stochastic integrals. It will be predictable because it is adapted and because adapted strategies coincide with predictable ones whenever information arrival is continuous. It will be integrable because, by construction, integrals of the modified strategy $\hat{\theta}$ equal integrals of the original strategy θ up to the stopping time τ .²³ Thus the new strategies are integrable if the original strategies are.

The statement that the two strategies are equally desirable follows from a claim that $\hat{W}_T = W_T, P^I$ almost surely, which we now show. If (θ, W, C) is feasible for Problem 5, the claim is immediate because stopping wealth

²³ The fact that τ is a stopping time follows from the fact that optional and predictable times coincide whenever informational arrival is continuous [see Karatzas and Shreve (1988, Problem 1.2.7)].

is redundant on paths of P^I -positive probability given the shutdown constraint. To prove the claim when (θ, W, C) is feasible only in Problem 4, we show that if it is false then there is a robust arbitrage available to the agent, which will be a contradiction to the assumption that Problem 4 or Problem 5 has a solution.

Suppose that the claim is false. The P^I -nonnegative wealth constraint implies that there is a P^I -positive probability set E that contains paths along which wealth hits zero and then eventually becomes positive by time T ; note that $0 = \hat{W}_T < W_T$ on E . Define a new stopping time τ' by $\inf\{t | W_t < 0\}$ if such a t exists or T otherwise. The trading strategy $\theta_t 1_{\{t \in [\tau, \tau')\}}$, which starts investment at τ and shuts it down when wealth first becomes negative, pays a P^I -positive amount on E and ensures that wealth satisfies no-empty-promises (i.e., wealth is pathwise nonnegative P almost surely). Thus we have constructed a robust arbitrage.

The existence of the robust arbitrage immediately rules out an optimum for Problem 4. Adding the arbitrage to an arbitrary feasible strategy in Problem 5 may be infeasible, however, because the shutdown constraint may stop investment before the arbitrage is complete. Thus we will rule out a solution to Problem 5 by showing that the agent can shift a small amount of initial wealth from the strategy to make the arbitrage available, and for a small enough transfer the arbitrage will make it worthwhile at large enough scale. Together these will imply a contradiction to our assumption that Problem 4 or Problem 5 has a solution.

Let B be the payoff of a trading strategy giving a payoff bounded below away from zero and a positive wealth process, scaled to cost w_0 initially. Furthermore, let $X > 0$, $X \neq 0$ be the payoff to the arbitrage strategy as constructed above. Using free disposal if necessary, we may assume without loss of generality that X is a bounded random variable. Monotonicity implies that any terminal consumption plan C in a candidate optimum will be strictly less desirable than $C + X$; however, the latter may be infeasible, since if C involves wealth hitting zero, then the agent may be shut down before the arbitrage generating X has been played out. Let ε and n be positive numbers. The consumption plan $(1 - \varepsilon)C + \varepsilon B + nX$, formed by shifting some initial wealth from C to B , is feasible because it requires initial investment w_0 and because the investment in B prevents wealth from hitting zero. If utility U is bounded, expected utility is continuous in ε and n by dominated convergence, and for $n = 1$ and small enough ε , $(1 - \varepsilon)C + \varepsilon B + X$ is strictly preferred to C . If U is unbounded, then we can choose n to be large and ε (which may depend on n and can be uniformly bounded below by an arbitrary positive number) to obtain unbounded expected utility on the P^I -positive probability set of states with $X > 0$. In either case, Problem 5 cannot have a solution, and our claim is proven.

(ii) By (i), just use τ to construct a new solution to the problem that is given to have a solution. ■

5. Conclusion

We have demonstrated a version of the fundamental theorem of asset pricing when agents have dogmatic differences of opinion but are prevented from making empty promises. The primary difference is that linear pricing will be superpositive since a state assigned zero probability by the agent may have a positive price if it is important to the market. Superpositive linear pricing is reconciled with expected utility maximization by ruling out empty promises in important states. The shadow price of the no-empty-promises constraint takes up slack between positive state prices and zero marginal utility. These results are proven for finite-state and continuous-time models.

With continuous information arrival, the no-empty-promises constraint can be enforced by permanently shutting down the agent's investments from the time wealth first hits zero.

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