Information Revelation Through Option Exercise

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In many real-world situations, agents must formulate option exercise strategies with imperfect information. In such a setting, agents may infer the private signals of other agents through their observed exercise strategies. The building of an office building, the drilling of an exploratory oil well, and the commitment of a pharmaceutical company toward the research of a new drug all convey private information to other market participants. This article develops an equilibrium framework for option exercise games with asymmetric private information. Many interesting aspects of the patterns of equilibrium exercise are analyzed. In particular, informational cascades, where agents ignore their private information and jump on the exercise bandwagon, may arise endogenously.

In standard option models the timing of exercise is simultaneous and uninformative. Agents are assumed to be perfectly informed about the parameters of the option, and the optimal exercise policy is a purely deterministic function of the current state variables. However, there are likely to be many realistic situations in which agents are imperfectly and differentially informed. In such cases, agents may impute the private information of others by observing their exercise (or lack of exercise) decisions. The important questions in such a framework are how do agents determine their exercise strategies, and how is information conveyed?[1] Examples abound in both the financial and real option domains. For example, the exercise of an executive stock option may allow the market to impute insider information on the volatility of the company's stock. However, the vast majority of applications are likely to occur in real options markets. In order to aid in the intuition in the model, consider the following two examples of exercise games with private information externalities.

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[1] In a full information setting, the strategic exercise of options has been the subject of several articles. Emanuel (1983), Constantinides (1984), and Spatt and Sterbenz (1988) analyze the strategic exercise of warrants and convertible securities. Grenadier (1996) provides a game-theoretic equilibrium to a strategic option exercise game and applies the model toward analyzing real estate developer decisions.

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Example 1. Oil Exploration
A well-known case of a real option is oil exploration. Paddock, Siegel, and Smith (1988) model the optionlike characteristics of offshore oil leases. In addition to possessing the standard features of options (ongoing oil price uncertainty, large lump-sum exercise costs), it also possesses important private information externalities. Due to the leasing practices of the federal government, it is frequently the case that two or more firms lease adjacent tracts of land that may contain an oil deposit [Hendricks and Kovenock (1989)]. Prior to obtaining the lease, each firm has obtained private information about the probability of finding a deposit. Each firm obtains a noisy signal of the likelihood of striking oil by combining seismic surveys with in-house expertise. However, the existence and magnitude of the deposit can only be known through drilling. Firms then face a trade-off between the benefits of drilling and potentially obtaining oil earlier and the benefits of waiting for other firms to drill first and reveal information about the size of the deposit. Therefore, in equilibrium, firms must determine their optimal drilling exercise times in a context of private information and ongoing uncertainty.

Example 2. Real Estate Development
Another classic case of a real option is real estate investment. As in Titman (1985), Williams (1993), and Grenadier (1996), the development of real estate is analogous to an American call option on a building, where the exercise price is equal to the construction cost. However, in real estate markets, there is a great deal of private information held by various market participants. Local developers may have particular knowledge of the suitability of particular locations, while large national developers may have greater expertise on marketwide factors and trends. The timing of construction will convey potentially valuable information about the value of a particular developer's private signal. Thus the early entry of a local developer may convey positive information about the potential value of a particular location (just as the failure to enter would convey less positive information).

This article develops a model in which private information is conveyed through the revealed exercise strategies of market participants. A set of n agents holds options on a continuous-time stochastic price process, and each must determine the optimal moment at which to exercise. However, agents possess private information about a parameter that affects the option payoff. The private signals are informative, but imperfect. Agents form exercise strategies that are contingent on not only their own signal, but on the revealed signals of other agents through their observed exercise policies. In equilibrium, as the potential payoff from exercise becomes greater, agents will trade off the benefits of early exercise with the benefits of waiting for information to leak out through the actions of others. Equilibrium exercise will be sequential, with the most informed agents allowing the least informed agents to free ride on the information conveyed by their exercise, or failure to exercise.[2]

The model demonstrates that while the observed exercise decisions of other agents may convey valuable private information, there are also times when no information can be inferred. This will be the case when an informational cascade arises. As in the literature on information herding [i.e., Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992)], an informational cascade occurs when agents ignore their own private information and instead emulate the behavior of those that preceded them.[3] Specifically, in the present model, an informational cascade can arise endogenously in which all agents, regardless of their private information, exercise immediately. Agents will find their private information overwhelmed by the information conveyed by others and simply jump on the bandwagon.

Such option-exercise cascades may help explain the occurrence of "overbuilding" in real estate markets: pronounced bursts of development in the face of declining rents and property values. Such overbuilding has not been limited to any region, property type, or even country. For example, when oil patch cities such as Denver and Houston experienced unprecedented waves of office construction during the early 1980s, local office vacancy rates were already above 27% [Grenadier (1996)]. It seems unreasonable to assume that the developers (and their lenders) that had jumped on the bandwagon all had positive private signals about local market conditions. It is possible that a rational informational cascade was formed where agents found their private signals overwhelmed by the signals conveyed by previous developers.

The dynamic equilibrium nature of this model differs significantly from that of the current literature on informational cascades. The results obtained here serve to both generalize and add greater realism to the standard results on informational cascades. In the standard models of informational cascades the ordering of actions is predetermined. That is, the timing of agents' decisions is typically assumed to be exogenous and not determined by any sort of optimization decision.[4] For example, this preordering assumption would imply that even if an investor had an extremely positive private signal about the prospects of investment, if he was preassigned to be the 1,000th investor in line, he must wait for the first 999 investors to act before he may choose to invest. In this model, however, timing is the

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[2] Back (1993) provides an extension of the Kyle (1985) model of continuous insider trading that includes asymmetric information and options. While there is no strategic exercise feature in the model, information is conveyed by trading in the options. This information transmission eliminates the typical redundancy of the option. Childs, Ott, and Riddiough (1996) analyze option exercise in a context in which the underlying asset value is imperfectly observable.

[3] Herding behavior may also result from agency-related reputational problems. See Scharfstein and Stein (1990) and Zwiebel (1995).

[4] Bikhchandani, Hirshleifer, and Welch (1992) conjectured that with endogenous timing those agents with the most precise information will move first.

essence of the game; agents exercise strategically and the revealed ordering of exercise is entirely endogenous.[5] Thus this model can be applied to real investment decisions where firms are free to choose the timing of their investment outlays. In addition to providing greater realism, the endogenous nature of the timing allows for a greater depth of analysis. Since there is an explicit time dimension underlying the present model, the model can be used to analyze not simply "if" an informational cascade occurs, but also "when." In an investment context, a cascade will develop in both the decision to invest (exercise) and its timing. An additional distinction in the present model is that informational cascades may occur in an environment of ongoing uncertainty. Thus agents not only use the actions of others to inform their behavior, but also monitor the continuous stochastic movements in an underlying state variable (i.e., oil prices, building rents).

An analysis of the equilibrium exercise strategies reveals several interesting features. With diffuse and imperfect information, equilibrium exercise timing will almost always deviate from the full-information optimum. Those with the most informative signals will exercise the earliest, and will typically exercise earlier than optimal. Market observers may impute some form of "overbuilding" as is often implied in commercial real estate markets, using ex post information. In addition, the patterns of exercise across agents will be seen to differ under markets with varying structures. Markets with large information asymmetries will experience smooth exercise patterns over time, while markets with mild information asymmetries will experience quick bursts of exercise. The patterns of exercise will also vary according to the number of participants. Markets with large numbers of participants will be prone to longer lags between early exercisers but more prone to bursts of exercise for later exercisers.

The model is applied to an analysis of information markets. In markets with diffuse and imperfect information, mechanisms may develop that facilitate the spread of information. Brokers, consultants, and newsletters may emerge offering information to market participants. The model is used to analyze the value of such information services, as well as the most likely purchasers of the information.

While in the basic model the exercise of an agent impacts the information set of other agents, there is no direct impact on the actual payoff from exercise. This simplifying assumption facilitates an examination of the information revelation aspects of the option exercise game.[6] However,

[5] In Zhang (1997), agents also choose when to act. In his model, however, informational cascades always occur at the moment the first agent acts, after an initial lag. Chamley and Gale (1994) consider a discrete-time investment model under incomplete information where agents choose when to invest. In their model, agents' signals are of equal precision.

[6] Grenadier (1996) models an option exercise game where exercise results in an increase in the number of firms in an industry, and thus lowers the cash flows available to competing firms.

in many real-world situations, such exercise games will also have direct payoff externalities. In Section 6, the model is extended to take into account cases in which there are payoff externalities from exercise. For example, certain technology markets exhibit network externalities in which the benefits from adoption are increasing in the total number of adopters. A classic instance of this was in the battle between VHS and beta in videocassette recorder technology. It is found that such direct payoff externalities can have a powerful impact on the nature of equilibrium. For example, if there are significant benefits from agglomeration, the likelihood of informational cascades increases substantially. Conversely, if there are negative externalities associated with conformity, the likelihood of informational cascades can fall to zero.

The article proceeds as follows. Section 1 presents the framework and assumptions of the model of option exercise with imperfect and asymmetric information. Section 2 derives the equilibrium exercise strategies. Section 3 presents a discussion and analysis of exercise cascades. Section 4 analyzes the impact of the information environment on the patterns of exercise. Section 5 analyzes the sale of information. Section 6 generalizes the model to the case of payoff externalities. Section 7 concludes.

1. Assumptions and Model Formulation

Consider a model with n agents, with $n \geq 2$. Each agent holds an identical call option, where agents are free to exercise at any time of their choosing (i.e., the option is a perpetual American call option, and exercise decisions are made in continuous time). However, the precise payoff upon exercise is not fully known to any of the agents. In particular, each agent holds an independent signal of the true realized payoff from exercise. Each agent's optimal option exercise strategy will be contingent on not only his own signal, but also on the observed actions of other agents.

In a standard call option setting, exercise yields the difference between the observable value of the underlying asset and the exercise price. Denoting the underlying asset value process by $X(t)$ and the exercise price by I, the payoff from exercise is typically $X(t) - I$. However, in the present model, the payoff from exercise also includes an unobserved random variable, θ, whose realization directly impacts the option payoff.[7] In particular, let the net payoff from exercise be $\theta \cdot X(t) - I$. For example, $X(t)$ may represent the

[7] It is assumed that exercise does not reveal the true value of θ to those that have not yet exercised. As long as θ remains even partially unobservable, the model's results are unaffected. This assumption is more likely to hold in private, segmented markets such as real estate development where institutional frictions such as construction lags and slow information leakage make observability difficult. Relaxation of this assumption is simple in this model's equilibrium framework. The general n -person equilibrium is transformed into the two-person equilibrium (defined in Section 2.1) upon the moment of first exercise. The remaining agents act simultaneously as the "less informed agent" in the two-person equilibrium.

value of a "typical" office building, θ the uncertain quality of the particular location, and I the cost of construction. Similarly, $X(t)$ may represent the price of oil, θ the grade or quantity of the reserve, and I the cost of extraction.

Let the underlying asset value, $X(t)$, evolve as a geometric Brownian motion process:

$$dX = \alpha X dt + \sigma X dz, \tag{1}$$

where α is the instantaneous conditional expected percentage change in X per unit time, σ is the instantaneous conditional standard deviation per unit time, and dz is the increment of a standard Wiener process. α can be decomposed into the difference between the equilibrium expected return on the asset and its dividend rate.

At time 0, the value of θ is unknown. However, each agent holds an independent signal as to its true underlying value. The random variable θ is assumed to equal a constant plus the sum of n independent signals.[8] Specifically,

$$\theta = \mu + S_1 + S_2 + \cdots + S_n, \tag{2}$$

where μ is the expected value of θ and the S_i are independent, mean-zero random variables.[9] Each agent i possesses private information about the true value of θ by observing the true value of signal S_i.

I assume that the information available to an agent is his own private signal, plus the observed actions (exercise decisions) of other agents. The assumption that information transmission is accomplished solely through actions, and not words, is quite common in the literature [i.e., Banerjee (1992), Bikhchandani, Hirshleifer, and Welch (1992), Welch (1992)]. Embodied in this assumption is the belief that only actions are truly believable, and that communication mechanisms such as press releases and information disclosures are not credible. In many situations, agents may have a strong incentive to mislead their opponents, making truthful revelation unlikely. There could be many justifications for the incentive to mislead other agents. Agents may be compensated based on relative performance, they may wish to harm competitors in order to drive out future competition, or they may wish to elicit early exercise by others if there are benefits to agglomeration.

However, this assumption may be too strong in some cases and will be examined at several points in the article. If it is possible for agents to collude in their information or write contracts such that information sale is credible, then the information transmission mechanism may be more

[8] The assumption of independent signals is made to simplify the discussion. In essence, agents' differing signals represent idiosyncratic components of the true state variable. The model could be modified to allow for correlated signals by adding a random common component to θ, and making the signals equal to the sum of the common components plus noise.

[9] If one wishes to visualize the problem as an option on a random number of shares of X, then $\mu = 1$ would signify that the expected number of underlying shares is one.

broad. One possible assumption would be that agents know the signals of all agents that previously exercised, rather than just their actions. The model is easily modified to account for this different assumption, and this is examined in Section 3. An outline for the implications on the equilibrium is as follows. At many times under the "only actions revealed" assumption, exercise decisions are fully revealing of agents' signals. During those periods, the equilibrium is identical to the "signals revealed" equilibrium. However, when only actions are observable, there is also the potential for informational cascades to occur, upon which actions become uninformative. If signals were revealed, informational cascades would never occur, and the equilibrium would avoid the information loss due to cascades. With full revelation of signals, equilibrium exercise will always converge to the "correct" decision. While the elimination (or at least mitigation) of informational cascades may be realistic for some markets, there is also considerable anecdotal evidence that agents do sometimes follow the crowd, even when the crowd is incorrect [see Hirshleifer (1995)].

Agents will differ in the quality of their private information, as their signals will differ in the precision with which they forecast the true value of θ. That is, assume that $\text{var}(S_1) > \text{var}(S_2) > \cdots > \text{var}(S_n)$. Therefore, for all $j < k$, agent j has a more precise signal than agent k because he knows the true value of a more dispersed signal. Stated similarly, agent j's conditional expected value of θ has a lower standard error than agent k's.

Assume that the signals, S_i, have simple two point distributions:

$$S_i = \begin{cases} -\epsilon_i & \text{with probability } \frac{1}{2} \\ +\epsilon_i & \text{with probability } \frac{1}{2} \end{cases}, \qquad \text{for } i = 1, 2, \ldots, n, \tag{3}$$

where $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$ and $\mu > \sum_{i=1}^{n} \epsilon_i$.[10] At the outset of the game, agents are assumed to know the distributions of competitors' signals, but not their realizations. This common knowledge assumption only requires that agents know one value, ϵ_i, that corresponds to each competitor. This mapping might be based on such observable factors as past reputation, locational proximity, or experience within the industry. The assumption that the ϵ_i's are decreasing ensures that the signal variances are decreasing, and the assumption that the sum of the signals is less than μ ensures that

[10] The assumption of simple, discrete signal distributions is common in the literature on investment cascades. Allowing for continuous signal distributions would greatly complicate the equilibrium derivation, as well as imply very different asset price dynamics. With continuous signal distributions, information would evolve continuously, as agents refine their posterior distribution of θ through observing the actions of other agents. This implies that asset prices will evolve continuously, as both public information on $X(t)$ and private information on θ evolve continuously over time. In the current setup with discrete signals, asset prices evolve continuously during periods in which no private information is revealed, but jump by discrete increments when private information is revealed. Such mixed asset price dynamics may be particularly descriptive of real-options investments discussed in this article.

even if all of the signals are negative, for high enough value of $X(t)$, the options will eventually be exercised.

In traditional financial option pricing models, the approach to valuation is based on arbitrage arguments, where one uses the fact that one can instantaneously trade the underlying asset and a riskless asset so that the option is precisely replicated. However, such an approach requires assumptions about the liquidity of the underlying asset. Since the majority of applications of this model involve assets such as office buildings, plant, and equipment, and technological innovations that are subject to substantial transactions costs, indivisibility, and the inability to be sold short, such arbitrage arguments are particularly questionable. An equilibrium approach relaxes the tradability assumptions needed for arbitrage pricing, although an appropriate equilibrium model must be chosen.[11] For simplicity, I assume risk neutrality so that all assets are priced so as to yield an expected rate of return equal to the risk-free rate, r. This seemingly restrictive assumption can easily be relaxed by adjusting the drift rate to account for a risk premium in the manner of Cox and Ross (1976).

It shall prove useful to briefly consider the optimal exercise strategy in a world of perfect information, where θ is known to all agents. The details of the derivation are provided in the Appendix. In a world of perfect information, all agents will choose to (simultaneously) exercise the first moment that $X(t)$ equals or exceeds an optimal trigger level. Denote the full-information exercise trigger as $X^*(\theta)$, where a closed-form solution appears as Equation (A.4) in the Appendix. Thus all agents will exercise at $T^*(\theta)$, the first passage time to $X^*(\theta)$, where $T^*(\theta) = \inf[t \geq 0 : X(t) \geq X^*(\theta)]$.

In the model, however, agents will often have less than perfect information about the true value of θ. At each point in time, their information set will contain not only their own signal, S_i, but also the information conveyed by the exercise (or lack of exercise) of other agents. Agents will update their information over time by observing the actions of others in an effort to impute the signals of other agents. This attempt to learn from the actions of others will directly shape the equilibrium exercise strategies. In the standard, full-information exercise strategy, the optimal trigger strategy will equate the immediate benefits of exercise with the marginal value of waiting. However, with partial information, agents will also take into account the benefits of waiting for others to reveal information. For example, if you know that if another agent has a high signal he will exercise at a particular stopping time, then it may pay for you to wait for him to reveal

[11] In addition, asymmetric information can itself lead to the failure to price options by arbitrage. As demonstrated by Back (1993) in a model of continuous insider trading, trading in the option affects the flow of information, making a seemingly redundant asset not capable of being priced by arbitrage. As such, an equilibrium pricing approach must be used.

his information before you yourself exercise. In the equilibrium solution in the next section, these trade-offs are calculated explicitly.

2. Equilibrium with Information Revelation

In this section I derive an equilibrium for the n-person option exercise game. In this equilibrium, agents will take into account both their own signal as well as infer the signals of other players through their revealed exercise policies. Given all others' strategies, each agent's equilibrium exercise strategy will be optimal.

Before deriving the equilibrium set of exercise strategies, I will provide a bit of intuition. In equilibrium, the most informed investor will reveal his information first through his observed exercise strategy. Thus, at the beginning, all agents will know that if player 1 has good news (i.e., $S_1 = +\epsilon_1$), he will exercise early at a relatively low trigger value of $X(t)$. This will be true because S_1 contains the most dramatic information; if the signal is positive, it will pay for agent 1 to exercise even before collecting any information from the other agents' behavior. Therefore, if investor 1 exercises early, all agents learn that his trigger was positive. If investor 1 doesn't exercise early, then all agents learn that his trigger was negative. Next, agent 2 holds the most dramatic information. Similarly, all will observe his exercise strategy in order to infer his news. Thus information is revealed over time through the observed exercise behavior of agents 1 through n. Each agent optimally trades off the benefits of immediate exercise with the benefits of waiting to observe the signals revealed by other agents.

I will present the equilibrium in two stages. I will first state the equilibrium for the simplest case where $n = 2$. I then define the equilibrium for the general case where $n \geq 2$.

2.1 Equilibrium with two agents

In this equilibrium, agent 1's actions will fully reveal his signal to agent 2. Agent 2 will use this information to inform his exercise strategy. The equilibrium will now be stated and then proven below.

Equilibrium exercise strategies with $n = 2$. Each agent i's ($i = 1, 2$) exercise strategy will be contingent on the current level of $X(t)$, its signal S_i, and the revealed exercise policy of the other agent. Let $X(0) < X^*(\mu+\varepsilon_1)$.[12] A pair of Bayesian–Nash equilibria exercise strategies can be described as follows:

Agent 1 (the more informed agent): *If $S_1 = +\epsilon_1$, then exercise the first*

[12] This assumption, while not essential, ensures that equilibrium exercise does not occur in a discrete jump at an arbitrary initial starting point of the game.

moment that $X(t)$ equals the trigger $X^*(\mu + \epsilon_1)$. If $S_1 = -\epsilon_1$, then copy the exercise strategy of agent 2 by exercising immediately after him.

Agent 2 (the less informed agent): *Wait until the trigger $X^*(\mu + \epsilon_1)$ is reached. If agent 1 exercises, then infer that agent 1's signal is $+\epsilon_1$ and exercise the first moment that $X(t)$ equals or exceeds the trigger $X^*(\mu + \epsilon_1 + S_2)$. This will imply immediate exercise if $S_2 = +\varepsilon_2$. If instead agent 1 does not exercise, then infer that agent 1's signal is $-\epsilon_1$ and exercise the first moment that $X(t)$ equals the trigger $X^*(\mu - \epsilon_1 + S_2)$.*[13]

To prove that the above strategies constitute an equilibrium, I will demonstrate that given each player's strategy, the other player's strategy is optimal. First, take agent 1's strategy as given. It is clear that by observing whether agent 1 exercises at the trigger $X^*(\mu + \epsilon_1)$, agent 2 will correctly infer agent 1's signal and will then know the true value of θ. The only scenario in which agent 2 does worse than the full-information optimum is when both agents have positive signals: agent 2 exercises at the trigger $X^*(\mu+\epsilon_1)$, when with full information he would have exercised earlier at $X^*(\mu + \epsilon_1 + \varepsilon_2)$. However, if agent 2 exercises before agent 1, his expectation of θ would be $\mu+\epsilon_2$. With this expectation, the optimal exercise strategy would be to wait until the trigger $X^*(\mu + \epsilon_2)$ is reached. Since this implies waiting beyond the trigger $X^*(\mu + \epsilon_1)$, it will not be optimal to preempt the revelation of agent 1's signal.

Now I demonstrate that agent 1's strategy is optimal, taking agent 2's strategy as given. First, note that if $S_1 = -\epsilon_1$, by copying agent 2 he achieves the full-information optimum. Thus he could do no better. Now, if $S_1 = +\epsilon_1$, agent 1 exercises at the trigger $X^*(\mu + \varepsilon_1)$, without knowledge of agent 2's signal, and thus does not achieve the full-information optimum. However, the only possible reason for agent 1 to deviate from this strategy would be to wait for agent 2 to reveal his signal. Thus, suppose agent 1 waits for agent 2 to reveal his signal. This will take place at the trigger $X^*(\mu - \epsilon_1 + \epsilon_2)$: if agent 2 exercises at this trigger his signal is positive, otherwise it is negative. However, regardless of whether or not agent 2 exercises, the full-information strategy will always dictate immediate exercise. This is because for any realized value of S_2 , the full-information trigger $X^*(\mu + \varepsilon_1 + S_2)$ is less than the trigger $X^*(\mu - \epsilon_1 + \epsilon_2)$. Since this delaying strategy yields the same expected payoff as the equilibrium strategy, $(\mu + \varepsilon_1)X - I$, but at a suboptimal exercise time, the optimal exercise trigger remains at $X^*(\mu+\epsilon_1)$.

The intuition for the equilibrium is quite simple. If agent 1 has a positive signal, the expected value of exercising will become so great as to overwhelm any benefit from waiting to learn from agent 2's actions. If instead

[13] Off-equilibrium actions play a very limited role in the basic model. Since an agent's exercise payoff does not depend on what later agents do, there is no incentive to make an out-of-equilibrium move to try to influence a later player. Thus there is a multiplicity of off-equilibrium conjectures associated with this equilibrium.

agent 1's signal is low, agent 2 will infer this by noting that agent 1 did not exercise early. Then, once agent 1's information has leaked out, agent 1 knows that agent 2 is fully informed and can therefore copy agent 2's fully informed exercise strategy. In almost all cases, agent 2 achieves the full-information optimum by waiting for agent 1 to reveal his information, and then proceeding to exercise optimally. Only in the case in which both signals are positive does agent 2 regret not exercising earlier. However, agent 2 knows that if agent 1 also has a positive signal, agent 1's optimism will be greater and will always choose to exercise at an earlier moment.

2.2 Equilibrium with n $\geq$ 2 agents

When there are more than two players, much of the intuition from the two-player game continues to hold. Information will stream from the agent with the most informative signal (agent 1) to the agent with the least informative signal (agent n). If agent 1 has a positive signal, he will find it optimal to exercise first, without waiting for more information. If his signal is negative, he will not exercise early, but will simply copy the next agent who exercises. Recognizing this, agent 1's signal will be fully revealed by whether or not he exercises early. After agent 1 reveals his signal through his actions, the remaining agents play a game with $(n - 1)$ players, using the information conveyed by agent 1. Now, it is agent 2 who is the most informed agent. The remaining agents will wait for him to reveal his signal before playing an $(n-2)$ player game. This sequence ends with a two-player game as before.

Therefore, the definition of equilibrium is relatively straightforward. The general n-person game can be defined recursively by the $(n - 1)$-person game. The only complication in the general n-player game is the potential for informational cascades to occur. Informational cascades will be discussed in greater detail in the following section. In the two-player game, an agent is able to perfectly impute the signal of the other player through his revealed exercise policy. With many players (in fact with $n > 3$), it may be possible that agents' actions will be utterly uninformative of their private signals. Agents will form a herd, each copying the other by exercising immediately regardless of their private signal.[14]

An informational cascade will occur whenever two consecutive agents reveal positive signals. Suppose agents j and $j + 1$ each receive positive signals. Agent j will reveal his positive signal by his early exercise. Since agent $j+1$ will be even more optimistic (agent j did not know that $j+1$ also had a positive signal), agent $j + 1$ will exercise immediately after agent j. Now, what will agent $j + 2$ do? If he has a positive signal, he will clearly

[14] As discussed in footnote (7), if the true value of θ is revealed when the first agent exercises, the equilibrium is "truncated." That is, once the option is first exercised (by the agent with the greatest signal), all of the remaining agents are fully informed and exercise the first moment $X(t)$ equals or exceeds the trigger $X^*(\theta)$.

exercise immediately since he is more optimistic than both previous agents. But even if he has a negative signal, he will exercise immediately because he is still more optimistic than agent j, who found it optimal to exercise at this moment. This is because when agent j found it optimal to exercise, his expectation was that $E(S_{j+1}) = E(S_{j+2}) = 0$. Even if agent $j+2$ knows that $S_{j+1} = +\varepsilon_{j+1}$ and $S_{j+2} = -\varepsilon_{j+2}$, he is still more optimistic than agent j because $\varepsilon_{j+1} > \varepsilon_{j+2}$. Thus it will be optimal for agent $j+2$ to exercise regardless of his private signal. Since agent $j+3$ will find himself in the same position as agent $j+2$, he too will find it optimal to exercise immediately regardless of his private signal. This will cascade to all remaining agents; once an informational cascade forms following two consecutive positive signals, each remaining agent will exercise immediately and the game is over.[15]

The discussion above can now be summarized in the statement of equilibrium for the general case of n players.

Equilibrium exercise strategies with $n \geq 2$. Each agent i's ($i = 1, 2, \ldots, n$) exercise strategy will be contingent on the current level of $X(t)$, its signal S_i, and the revealed exercise policies of other agents. Let $X(0) < X^*(\mu+\varepsilon_1)$. Let $\gamma_0 = \mu$, and let $\gamma_j = \gamma_{j-1}+S_j$ for $j = 1, 2, \ldots, n$. If there exists some j such that both $S_j > 0$ and $S_{j+1} > 0$, an informational cascade is said to begin with agent i^c where $i^c = \min[1 \leq j \leq n-2 : S_j > 0, S_{j+1} > 0]$. Otherwise there will be no informational cascades and we set $i^c = n+1$. The n agents will follow a set of Bayesian–Nash equilibrium exercise strategies, where the equilibrium strategy for player $i = 1, 2, \ldots, n$ is as follows:

(i) *If $i \geq i^c$, then exercise at the trigger $X^*(\gamma_{i^c})$.*
(ii) *If $i < i^c$ and $S_i > 0$, then exercise at the trigger $X^*(\gamma_i)$.*
(iii) *If $i < i^c$, $i < n$, and $S_i < 0$, then copy the next agent that exercises by exercising immediately after him.*
(iv) *If $i < i^c$, $i = n$, and $S_i < 0$, then exercise the first moment that $X(t)$ equals or exceeds the trigger $X^*(\gamma_n)$.*

Figure 1 provides an example of an equilibrium realization for a 10-player option exercise game. A hypothetical sample path of $X(t)$ is plotted, along with arrows denoting the equilibrium exercise times of the players. In a

[15] It is interesting to briefly consider the impact of allowing agents to possess options with differing payoff functions. For example, agents may have different costs of exercise. While much of the spirit of the model will remain (agents trade off the benefits of exercise with the benefits of waiting to learn from the actions of others), there will be one important difference. When agents have different payoff functions, it will not necessarily be the case that information is transmitted from the most informed agents to the least informed agents. If an agent has a low exercise price, he may exercise earlier than another agent with a more precise signal, as his lower exercise price overcomes his information disadvantage. This may lead to a lessening of the likelihood of informational cascades, as the information inferred from prior exercise may no longer overwhelm the remaining private information.

table below the graph are the signals of the players. Since player 1 receives a positive signal, he exercises when the trigger $X^*(\mu + S_1) = X^*(5.4)$ is reached at time $t = 0.13$. Agent 1's exercise reveals his signal to the remaining players. Since agent 2 has a negative signal, he delays his exercise until the next player exercises. This inaction reveals his negative signal to the market. Agent 3 has a positive signal and exercises at time $t = 0.195$ when the trigger $X^*(\mu + S_1 + S_2 + S_3) = X^*(5.36)$ is reached. The market infers agent 3's positive signal, and agent 2 immediately copies agent 3. Agent 4 has a negative signal and delays his exercise until the next player exercises. This inaction reveals his negative signal to the market. Since agents 5 and 6 each have positive signals, a cascade forms beginning with agent 5. All agents that have yet to exercise at time $t = 0.84$ when the trigger $X^*(\mu + S_1 + S_2 + S_3 + S_4 + S_5) = X^*(5.32)$ is reached.

2.3 Equilibrium option values

Given the equilibrium exercise strategies of all n players, it is straightforward to value their underlying options. The value of each player's option will simply be the discounted expected value of the payoff, where the payoff is defined by the equilibrium exercise strategies.

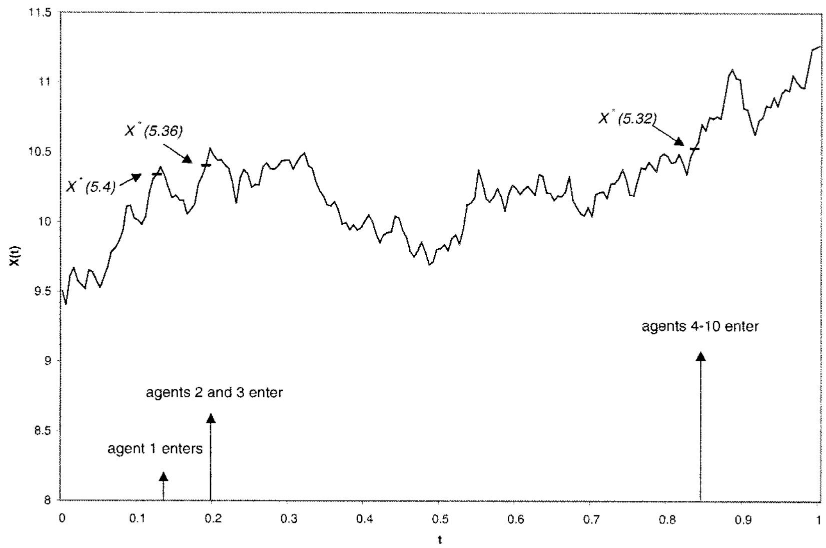
The equilibrium defines the exercise trigger for each agent's option, as well as the information set on which it conditions its expected payoff from exercise. In general, the realized exercise trigger for agent j will be a function of all agents' signals. Let $\bar{X}_j(S_1, \ldots, S_n)$ denote agent j's equilibrium exercise trigger, and let $\bar{T}_j(S_1, \ldots, S_n)$ denote the first passage time to the trigger $\bar{X}_j$. For example, for any set of n realized signals, $s_1, \ldots, s_n$, agent j will exercise the first moment $X(t)$ rises to the trigger $\bar{X}_j(s_1, \ldots, s_n)$ at time $\bar{T}_j(s_1, \ldots, s_n)$.

At the beginning of the game, each agent knows only his own signal, S_j. Define the equilibrium value of agent j's option (as of time zero) as $V_j(X; S_j)$. Due to the updating of information, at the moment of exercise each agent forms a conditional expected payoff based on the information learned up to that point. Thus the conditional expectation of θ at the exercise time $\bar{T}_j(s_1, \ldots, s_n)$ will contain all of the information learned by observing the actions of all agents up through time $\bar{T}_j(s_1, \ldots, s_n)$.

The value of the option, $V_j(X; S_j)$, can be derived as follows. Let $G_j(X; s_1, \ldots, s_n)$ denote the "conditional" value of agent j's option, conditional on the realized n signals, $s_1, \ldots, s_n$. The value of the option is then the expected value of G_j, conditional on only the value of agent j's signal.

Just as in the derivation of the full-information option value in the Appendix, the equilibrium conditional option value must satisfy the following differential equation:

$$0 = \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 G_j}{\partial X^2} + \alpha X \frac{\partial G_j}{\partial X} - r G_j. \tag{4}$$



Realized Signals	
s_1	+0.40
s_2	-0.36
s_3	+0.32
s_4	-0.28
s_5	+0.24
s_6	+0.20
s_7	-0.16
s_8	-0.12
s_9	+0.08
s_{10}	-0.04

Figure 1
Illustration of equilibrium exercise times
This figure depicts the equilibrium exercise times of 10 agents, for a given sample path of $X(t)$ and a given set of realized signals. The curve depicts a hypothetical sample path of $X(t)$, and the arrows depict the equilibrium exercise times of agents. The exercise strategies follow the definition of equilibrium provided in Section 2.2. Agent 1 has a positive signal and exercises at the trigger $X*(\mu + s_1)$. The remaining agents update their private information after inferring s_1. Agent 2 has a negative signal and will copy the next agent who exercises. The remaining agents infer s_2 through agent 2's inaction. Agent 3 has a positive signal and exercises at the trigger $X*(\mu + s_1 + s_2 + s_3)$, revealing s_3 to the market. Agent 2 immediately copies agent 3. Agent 4 has a negative signal and will copy the next agent who exercises, revealing s_4 through his inaction. Since agents 5 and 6 each have positive signals, a cascade forms and all remaining agents exercise at the trigger $X*(\mu + s_1 + s_2 + s_3 + s_4 + s_5)$. The assumed parameter values are $\alpha = 0.04$, $\sigma = 0.1$, $r = 0.05$, $I = 10$, $X(0) = 9.5$, and $\mu = 5$.

$G_j(X; s_1, \ldots, s_n)$ must also satisfy the boundary condition:

$$G_j(\bar{X}_j(s_1, \ldots, s_n); s_1, \ldots, s_n) = \bar{X}_j(s_1, \ldots, s_n) \cdot E\left[\theta \,\middle|\, \bar{T}_j(s_1, \ldots, s_n)\right] - I, \tag{5}$$

where the expectation implies that when agent j exercises at the equilibrium trigger, he conditions his expectation of θ on all information revealed up to time $\bar{T}_j$. In addition, the boundary condition $G_j(0; s_1, \ldots, s_n) = 0$ must hold.

Finally, to get the *unconditional* value of agent j's option, one must take the expected value of $G_j(X; S_1, \ldots, S_n)$, conditional on agent j's knowledge of S_j. Therefore the equilibrium value of agent j's option, $V_j(X; S_j)$, can be written as

$$V_j(X; S_j) = E\left[G_j(X; S_1, \ldots, S_n)\,\middle|\,S_j\right]. \tag{6}$$

In the context of real estate development options, the value of undeveloped land would equal the option value to the owner. Thus the value of agent j 's land equals $V_j(X; S_j)$. At all times at which there is no exercise (development), the value of land evolves in a continuous fashion, reflecting only the innovations in the fundamental state variable, $X(t)$. However, at any moment when any other agent begins development (provided that there is no cascade), there is a discontinuous jump in the value of land reflecting the discrete release of private information. Therefore, in the setting of this model, land prices evolve in a continuous fashion corresponding to movements in market fundamentals and jump by discrete amounts following the release of private information.

While closed-form solutions are easily obtained, for large values of n the solutions become quite lengthy. In Section 5 a closed-form solution for $V_1(X; S_1)$ is presented and analyzed. In addition, the solution to the system of Equations (4), (5), and (6) may be obtained through numerical methods.

3. Endogenous Exercise Cascades

An informational cascade is said to occur when agents ignore their own private information and instead emulate the behavior of previous actors. It is important to note that an informational cascade is not simply the act of simultaneous exercise; for example, simultaneous exercise always occurs in a full-information model. What is distinct about an informational cascade is that agents will ignore their private information and mimic the actions of previous decision makers. As is apparent from the equilibrium in the general model with more than two agents, an informational cascade will occur whenever two consecutive agents reveal positive signals. As agents witness the exercise of better informed agents, they will ignore their own private information and exercise simultaneously. Despite the typically negative connotation associated with herdlike behavior, this behavior is entirely rational. While from a social welfare criteria it is suboptimal in that it wastes information, it is individually rational to defer to apparently overwhelmingly positive information.

Many have attributed the pronounced waves of construction in the face of declining rents and property values to irrational overbuilding.[16] In terms of national office construction, Birch (1990) estimates that 43% of all office space ever built in the United States was built during the 1980s. Much of this construction was initiated when the national office vacancy rate was rising rapidly and nearing historically high levels. In local markets, the same phenomenon was occurring. Consider the cases of such oilpatch cities as Denver and Houston. Over the 30-year period from 1960 through 1990, over half of all office construction was completed during the four-year interval 1982–1985. During the time of their concentrated burst of construction in 1982–1985, each had vacancy rates near 30%. In fact, by 1983, vacancy rates were already above 27% in each of these cities.

However, some of the development patterns that emerge in commercial real estate markets may be due to rational, informational cascades. As stated in the introduction, real estate development represents the exercise of an option, where agents possess noisy private signals of building values. Much of the real estate market is private, and such information is regarded as valuable and proprietary. While information is revealed in many ways, observing the actions of developers is clearly an important medium of information conveyance. As in the model, if a few developers have optimistic forecasts of future building values, then it may be entirely rational for other developers to ignore their own information and form a herd. A particular example would be if local developers were optimistic and began development. Then outside developers may also enter the market and begin construction, even if they possess negative signals about the state of the national real estate market.

Although the informational aspects of real estate markets may make them prone to informational cascades, this propensity may be mitigated by the direct payoff externalities associated with development. As discussed in Section 6, when there are negative payoff externalities associated with option exercise (i.e., the exercise of one agent's option lowers the prospective payoffs on other agents' options), the likelihood of the occurrence of informational cascades can be significantly diminished. Intuitively, if the exercise of an option lowers the payoffs for future option exercise, the remaining agents will delay exercise until building values rise substantially, thus lowering or eliminating the likelihood of a cascade. Even if the private information revealed by previous exercise is significantly positive, agents may still defer exercise because the negative direct payoff externality overwhelms the importance of the private information revelation. In the case of real estate development, there are likely to be first-mover advantages. The

[16] Grenadier (1996) provides an alternative rational explanation for such seemingly irrational developer behavior.

first developer to build in a local area can act as a monopolist (perhaps only temporarily) and lock in the most desirable tenants into his building. When future development options are exercised, the rents that can be charged may be lower, as the demand curve for space is likely to be downward-sloping. Therefore the true susceptibility of a real estate market to the occurrence of informational cascades will depend on the interaction between the informational and direct payoff externalities.

As mentioned in the introduction, the informational cascade phenomenon in the present model differs significantly from that in the standard models of information-oriented herding found in the literature [e.g., Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992)]. First, in this model informational cascades occur in a setting where the ordering of actions (exercise) is determined endogenously. Agents choose the timing of their actions strategically. Second, agents face a continuous rather than binary choice set. That is, they may exercise at any stopping time. Their decision to exercise at a point in time is as informative as their decision not to exercise at a point in time. The timing of decisions is an essential element of the model. Thus herding will develop in both the decision to exercise and the timing of exercise. Finally, agents must make their decision in an environment in which there is ongoing uncertainty, $X(t)$, in addition to the uncertainty revealed through observed actions.

It is straightforward to derive the probability that an informational cascade will occur in a market with n agents. It will be seen that the probability of an informational cascade will approach one as n goes to infinity. Even for moderate sizes of n, the likelihood of an informational cascade will be very high. However, it must be kept in mind that even though there may be a high likelihood of an informational cascade, the timing of such an event could be very far off in the future. This is because even though an informational cascade will occur if two consecutive agents have positive signals, it may take a long time before $X(t)$ rises to the necessary trigger value to entice these agents to exercise and reveal their triggers. I will assume that the stochastic process of $X(t)$ possesses the property that any finite trigger will eventually be reached with probability one. Thus I shall assume $\alpha - 1/2\sigma^2 > 0$.

The probability that an informational cascade forms in a market of size n, p_n, is derived in the Appendix. p_n can be written as

$$p_n = 1 - \frac{4}{\sqrt{5}}\left[\left(\frac{1}{\sqrt{5}-1}\right)^{1+n} + \left(\frac{1}{1+\sqrt{5}}\right)\left(\frac{-1}{1+\sqrt{5}}\right)^{n}\right], \quad \forall n \geq 0. \tag{7}$$

The probability of an informational cascade is increasing and concave. Even for reasonably small values of n, the likelihood of an eventual informational cascade can be very large. With only 10 agents, the probability of an eventual informational cascade is almost 83%. However, as stated previously, the time at which such an event may occur could be far off in the future.

Informational cascades are harmful from a social welfare standpoint because information is wasted. When there are no informational cascades, the observed exercise decisions are fully revealing, as they convey the true private signals of previous agents. Thus if there is no cascade prior to agent j's exercise decision, agent j will know the values of signals 1 through j. However, if there is a cascade beginning with agent $i < j$, agent j will have no information about the values of signals $i + 2$ through $j - 1$. This lost information leads the remaining agents to, on average, make less accurate exercise decisions.

If the assumption that only previous actions are observable is replaced by the assumption that previous signals are observable, this information loss could be avoided. Recall that the only time that actions are not fully revealing is during a cascade. Thus suppose that two consecutive agents receive positive signals and exercise simultaneously. As in the model, the next agent will find it optimal to exercise immediately, regardless of his signal. Now, suppose that this agent also publicizes his signal. If this pronouncement is credible, then all remaining agents update their information and an informational cascade will never form. The equilibrium defined in Section 2.2 can easily be modified to account for this altered assumption. Specifically, the term i^c is always set equal to $n + 1$, and the remaining definition holds.

The assumption that only actions are credible (i.e., that actions speak louder than words) is standard in the information herding literature. Whether this assumption is accurate or not will depend on the type of market structure being considered. There are several factors that can erode the credibility of words. If agents are compensated based on relative performance, rather than absolute performance, then agents will have an incentive to misreport their private signals. This is in keeping with Keynes's observation that "it is better for reputation to fail conventionally than to succeed unconventionally." Another justification is that agents may wish to eliminate future competition. If agents must compete with each other in other markets, this will provide a disincentive from truthfully revealing private information. Similarly, if there are payoff externalities, agents will have an incentive to reveal information in a self-serving fashion.

Under certain conditions it may be possible to overcome the factors making actions the only credible information source. In particular, since the information lost during a cascade is valuable to remaining agents, a market for information sale may develop. However, if it is not possible to write contracts which are ex post verifiable, then the information sold will not be credible and will be worthless. Even if the information is credible, a market for information sales may still not develop. As discussed in Section 5, if agents can free ride on the purchase of information, then the market price of the information may only represent only a small fraction of the public benefits of information. Thus the market price of the information may not be large enough to compensate for the costs of information acquisition.

4. An Analysis of the Patterns of Exercise

In this section I analyze the patterns that may emerge from equilibrium exercise strategies. For example, in some markets we may observe a period of rapid exercise by all agents. This may not necessarily be caused by informational cascades. Agents may simply find it optimal to act quickly after observing another agent's exercise. Caplin and Leahy (1993) provide an example of a sudden burst of exercise behavior. Manhattan's Lower Sixth Avenue was experiencing an attenuated retail depression when Bed Bath & Beyond opened the first superstore in the area. Other retailers then quickly exercised their options to locate on Lower Sixth Avenue. Bed Bath & Beyond's entry conveyed powerful information about its confidence that consumers would be willing to travel to such an inconvenient location. In addition, the fact that there was such a long period of inactivity in the area signifies that retailers were also learning from the reticence of others. Conversely, we also find environments in which exercise occurs fairly evenly over time. Agents may only follow up a previous agent's exercise after a cautious delay. Hendricks, Porter, and Boudreau (1987) find an even pattern of exercise in oil drilling. They find in a sample of 1,056 wildcat tracts, a pattern of drilling that is fairly evenly distributed across the different lease years.

While with perfect information all agents would exercise simultaneously, with imperfect information agents will typically exercise sequentially over time. Agents will find it optimal to balance the benefits of immediate exercise with the benefits of waiting to learn from the actions of others. The agents with the most informative positive signals will exercise early, and those with the least informative signals will exercise later. It will now be demonstrated that a consequence of this trade-off will be that the most informed agents will typically exercise "too early" while the least informed agents will exercise "too late."

Recall that the full-information exercise trigger is $X^*(\theta)$. Thus if agents have perfect information they will exercise the first moment that $X(t)$ rises to the trigger $X^*(\theta)$, denoted as $T^*(\theta)$. Without perfect information, the equilibrium exercise trigger of agent j is $\bar{X}_j(S_1, \ldots, S_n)$. As detailed in the equilibrium, $\bar{X}_j(S_1, \ldots, S_n)$ will in general depend on the signals of all agents. Agent j will exercise the first moment that $X(t)$ rises to the trigger $\bar{X}_j(S_1, \ldots, S_n)$, denoted by $\bar{T}_j(S_1, \ldots, S_n)$.

The following two propositions demonstrate that the most informed agent will typically exercise "too early" and the least informed agent will typically exercise "too late."

Proposition 1. $E[\bar{T}_1(S_1, \ldots, S_n) - T^*(\theta)] < 0$. *Thus, on average, the agent with the most precise signal will exercise earlier than the full-information optimum.*

Proof. See Appendix.

Proposition 1 is simply a result of Jensen's inequality. From the definition of equilibrium in Section 2.2, agent 1 exercises at the trigger $X^*[E_1(\theta)]$, where E_1 denotes agent 1 's conditional expectation of θ at the moment of exercise. Since $X^*(\theta)$ is convex, a quick application of Jensen's inequality (and the law of iterated expectations) implies that $E[\bar{X}_1(S_1, \ldots, S_n)] < E[X^*(\theta)]$. Since agent 1 is expected to exercise at a trigger less than the full-information trigger, he is expected to exercise too early. The intuition for Proposition 1 is as follows. Agent 1 is the most informed agent, and in equilibrium is always the first to reveal his private signal. While other agents face the trade-off between the benefits of immediate exercise and the benefits of waiting for more informed agents to reveal their private signals, agent 1 faces no such trade-off. Agent 1 has no incentive to delay and wait beyond his partial information exercise strategy.

Proposition 2. $E[T_n(S_1, \ldots, S_n) - T^*(\theta)] > 0$. *Thus, on average, the agent with the least precise signal will exercise later than the full-information optimum.*

Proof. See Appendix.

The intuition for Proposition 2 is as follows. If no cascade occurs, then agent n always has full information and will never exercise before the full-information trigger. If a cascade occurs, then all agents that participate in the cascade (which always includes agent n) will rush to exercise given the overwhelmingly positive information of those that began the cascade. On average, agents that follow a cascade exercise later than they would have had they possessed full information. Therefore, whether or not a cascade occurs, agent n is expected to exercise at or after the full-information trigger.

Propositions 1 and 2 demonstrate that there will typically be a lag between the exercise of several agents, as agents trade off the benefits of immediate exercise with the benefits of waiting for information to be revealed. I now demonstrate that the degree to which agents wait will depend critically upon the structure of the information environment.

Consider a market with n agents. The expected lag between the exercise of agent j and that of agent $j+1$ is $E[\bar{T}_{j+1}(S_1, \ldots, S_n) - \bar{T}_j(S_1, \ldots, S_n)]$, denoted as $L_j(S_1, \ldots, S_n)$ for $j = 1, \ldots, n-1$. This can be expressed as

$$L_j(S_1, \ldots, S_n) = \frac{1}{\left(\alpha - 1/2\sigma^2\right)} E\left\{\ln\left[\bar{X}_{j+1}(S_1, \ldots, S_n)/\bar{X}_j(S_1, \ldots, S_n)\right]\right\}, \tag{8}$$

using a slight transformation of a result in Harrison (1985). The expected lag between the exercise of agent j and that of agent $j+1$ will be a function of the information environment in the market. This is evident from its dependence on the distribution of the n signals, $S_1, \ldots, S_n$.

I now analyze how different information environments affect the speed of exercise. For example, do agents in markets with significant asymmetries in private information act more cautiously and wait longer between exercise? Are markets with large numbers of players more prone to bursts of exercise? To answer these sorts of questions, I analyze the impact of two important characteristics of the market structure: the distribution of private information and the number of players.

In order to analyze alterations in the structure of the market, I parameterize the information structure in a simple manner. In a market of size n, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be evenly spaced such that $\varepsilon_n = \delta$, $\varepsilon_{n-1} = 2\delta, \ldots,$ $\varepsilon_1 = n\delta$. Therefore $\varepsilon_j = (n-j+1)\cdot\delta$, for $j = 1, \ldots, n$. Thus δ represents the magnitude of asymmetries in private information. The greater δ is, the greater the degree to which agents can benefit by learning the signals of others. The effect of changes in δ and n on the clustering of equilibrium exercise are derived through simulation. While the results are presented for a base-case set of parameter values, the qualitative results apply for a wide range of reasonable parameter values.

First, consider the effect of increasing δ, the magnitude of differences in private information. For a fixed number of players, the observed actions of other players reveal more information about the true state of the market. The key will be to focus on the effect of bad news. This is true because, regardless of the amount of information, there will always be simultaneous exercise between consecutive agents with good news. There will be time between exercise only in the case of good news, followed by bad news, followed by good news. Thus consider the timing of exercise when an agent exercises (due to good news) and the next agent does not exercise immediately (due to bad news). Now, in a market with great distinctions in private information, when an agent reveals (through failing to exercise) that his signal is negative, the next agent to exercise will be more cautious before exercising due to the severity of the revelation. Conversely, if there were only small distinctions in private information, bad news should have little impact on the timing of the next exercise; the next agent with good news will exercise very soon thereafter.

Table 1 depicts the effect of differing levels of private information asymmetry on the clustering of exercise. The table displays the average number of days between exercise in a 10-player game for three levels of δ. There are two important results. First, for any level of δ , the average delay will be greatest for the earliest exercisers. Thus the lag between players 1 and 2 will be the greatest, and the lag between players $n-1$ and n will be the smallest. This is because the earliest agents reveal the most informative signals. A

Table 1
The effect of private information on the time between exercise

	$\delta = 0.01$	$\delta = 0.04$	$\delta = 0.07$
L_1	45.99	186.52	327.77
L_2	21.17	86.51	172.28
L_3	19.71	75.56	160.97
L_4	10.95	52.56	93.81
L_5	8.40	36.50	73.73
L_6	5.48	24.09	52.20
L_7	3.36	14.97	32.49
L_8	1.83	7.67	18.62
L_9	0.67	3.42	6.94

This table presents the expected number of days between the exercise of agents, as the underlying amount of private information varies. This table demonstrates that as the amount of private information increases, the expected lag between the exercise times of any two agents becomes greater. In the model, each of n agents may exercise a call option paying $\theta X(t)$, at any time t, by paying an exercise price of I. $X(t)$ follows a geometric Brownian motion process with drift α and volatility σ. θ is an unknown parameter, whose value is represented by the sum $\mu + S_1 + S_2 + \cdots + S_n$. Agent j knows the realized value of the mean zero signal S_j, whose random value is $+(n - j + 1)\delta$ or $-(n - j + 1)\delta$, with equal probability. As δ increases, the degree of private information that is held by other agents increases. The expected time lag between the exercise of agent j and agent $j + 1$ is denoted L_j. The expected lag is estimated through performing 10,000 Monte Carlo simulation runs. The assumed parameter values are $\alpha = 0.04$, $\sigma = 0.1$, $r = 0.05$, $I = 10$, $X(0) = 9.5$, $n = 10$, and $\mu = 5$.

negative early signal will inspire the greatest amount of caution. Second, as expected, the greater the magnitude of private information asymmetries, the greater the average lag between exercise. For example, if $\delta = 0.01$, then the expected lag between the exercise of agents 2 and 3 is only about 21 days. However, when $\delta = 0.07$, the expected lag is almost half a year.

Now, consider the effect of increasing the number of players in the market. There will be two effects. First, for any given player, the greater the number of remaining players in a market, the greater the amount of private information that is unknown to the player. This leads to a greater desire to wait for others to reveal their information. In equilibrium, this will lead to a greater lag between the exercise of any given two consecutive players. Second, the greater the number of players, the greater the likelihood of an informational cascade. Once a cascade ensues, the time between exercise is zero for all remaining players. Therefore intuition suggests that the average time between the exercise of later players will fall to zero as the number of agents increases. As the number of agents becomes very large, the equilibrium approaches that of the model of Zhang (1997). In Zhang, all agents form an informational cascade with probability one. In the current model, as the number of agents increases to infinity, all but a finite number of players form a cascade with probability one.

Table 2
The effect of market size on the time between exercise

	$n = 5$	$n = 10$	$n = 15$
L_1	83.37	186.52	321.20
L_2	31.31	86.51	161.33
L_3	21.33	75.55	147.02
L_4	7.72	52.56	110.49
L_5	—	36.50	85.03
L_6	—	24.09	71.46
L_7	—	14.96	51.57
L_8	—	7.66	35.41
L_9	—	3.42	27.74
L_{10}	—	—	19.06
L_{11}	—	—	13.33
L_{12}	—	—	7.51
L_{13}	—	—	4.27
L_{14}	—	—	1.56

This table presents the expected number of days between the exercise of agents, as the number of agents varies. This table demonstrates that as the number of agents increases, the expected lag between the exercise times of earlier agents increases, while the expected lag between the exercise times of later agents decreases. In the model, each of n agents may exercise a call option paying $\theta X(t)$, at any time t, by paying an exercise price of I. $X(t)$ follows a geometric Brownian motion process with drift α and volatility σ. θ is an unknown parameter, whose value is represented by the sum $\mu + S_1 + S_2 + \cdots + S_n$. Agent j knows the realized value of the mean zero signal S_j, whose random value is $+(n - j + 1)\delta$ or $-(n - j + 1)\delta$, with equal probability. The expected time lag between the exercise of agent j and agent $j + 1$ is denoted L_j. The expected time lag between the exercise of agent j and agent $j + 1$ is denoted as L_j. The expected lag is estimated through performing 10,000 Monte Carlo simulation runs. The assumed parameter values are $\alpha = 0.04$, $\sigma = 0.1$, $r = 0.05$, $I = 10$, $X(0) = 9.5$, $\delta = 0.04$, and $\mu = 5$.

Table 2 depicts the effect of increasing the number of players on the clustering of exercise. The previous intuition is confirmed. First, as the number of players increases, the average lag between any two given players increases. Agents find it in their interest to increase the time between exercise because of the greater amount of private information remaining in markets of greater size. For example, with five players the average lag between the exercise of players 3 and 4 is about 21 days, while with 15 players the average lag is about 147 days. Second, as the size of the market increases, the average lag between the exercise of later players becomes very small, due to the increased likelihood of a cascade. With five players, where there is a 50% probability of a cascade, the average lag between the exercise of the final two players is 7.7 days. However, with 15 players, where there is a 94% probability of a cascade, the average lag between the exercise of the final two players is only about 1.5 days.

The model's empirical implications with respect to the patterns of exercise can be summarized as follows. First, on average, the earliest entrant will exercise earlier than the full-information optimum, while the last entrant will exercise later than the full-information optimum. Second, the average lag between exercise will be greatest for the earliest exercisers. Thus markets will typically grow slowly at first, as the early entrants exercise cautiously. Then markets will pick up steam with a rapid succession of later exercise. Third, markets with the greatest amount of information asymmetries will typically have the greatest lag between exercise. Fourth, markets with the greatest number of participants will have the longest average lags between the agents who exercise first, and the shortest average lags between the agents who exercise last.

5. The Sale of Information

It is clear that even in an equilibrium in which there is some information revealed through exercise, most agents would be better off having more information. In markets with diffuse and imperfect information, mechanisms may develop that facilitate the spread of information. In such markets, brokers, consultants, newsletters, and other services may provide information and advice to those who wish to take advantage of their greater market knowledge.

In the context of the present model, the purchase of additional information by any agent will have a positive externality on all other market participants. The increase in market information will push the industry closer to a fully informed equilibrium. Information will leak out through its use, even to those who did not buy it, as the market will extract the information from the purchaser's observed exercise policy.

Even though the individual agents may not truthfully reveal their information, I assume that firms that sell information will communicate truthfully. They will offer to sell their information, and will charge a price so as to maximize their profits. While many forms of information sale can be envisioned, I shall concentrate on a simple formulation: the sale of perfect information. More detailed treatments of the economics of information sale can be found in Admati and Pfleiderer (1986, 1988).

Suppose an information service offers to sell knowledge about the true value of θ. Any agent who buys this information will then exercise at the full-information optimal trigger, $X^*(\theta)$. What should the information service charge? Who will purchase this information?

If any firm purchases the information, then all other participants will simply emulate the exercise strategy of the perfectly informed agent.[17] Thus all

[17] I am assuming that the identity of any agent that purchases information becomes common knowledge. It may be possible in certain industry environments for this information to be kept secret. However, the

other agents can free ride off the purchase of information by a single agent. Therefore, at most one agent will purchase the information. The full welfare gains from the provision of information will not be captured by the one agent who pays for the information. If the cost of extracting the information for the information providing firm is larger than the private benefits to the purchaser, the information market will shut down. This may lead to a potential argument for government intervention. Similarly, groups of agents could benefit by forming coalitions that share information. The ability to form such coalitions will depend on such factors as antitrust enforcement and the ability to create mechanisms that ensure truthful information revelation.

The firm that sells the information will sell the information to the agent that will pay the highest price for the information. This will always be firm 1. Firm 1, the firm with the most informative signal, will always pay the most for information, because all other firms have access to his information, but he does not necessarily have access to their information. Agent 1 always reveals his information first and all other agents are at least as well off as agent 1, since they never have less information at their disposal than agent 1.

The value of this information to agent 1 would be the difference between the expected value of exercising his option at the full-information exercise trigger and the value of exercising his option according to the partial-information equilibrium of Section 2. The full-information value of agent 1's option is $W(X;\theta)$, as derived in Equation (A.3) in the Appendix. Prior to obtaining the information, the expected value of the full-information option to agent 1 is $E[W(X;\theta)|S_1]$, where agent 1's expectation of θ is conditioned on knowledge of his own signal S_1. This expectation can be written as

$$E\left[W(X;\theta)\,|S_1\right] = \sum_{m_2=1}^{2}\sum_{m_3=1}^{2}\cdots\sum_{m_n=1}^{2}\left(\frac{1}{2}\right)^{n-1} W\left[X;\mu+S_1+\sum_{j=2}^{n}\varepsilon_j\cdot(-1)^{m_j}\right]. \quad (9)$$

The value of agent 1's option without the purchase of information, $V_1(X;S_1)$, is determined by the equilibrium exercise strategy detailed in Section 2.3. This value is conditional on S_1. If $S_1=+\varepsilon_1$ and agent 1 has the highest signal, he will exercise first at the trigger $X^*(\mu+\epsilon_1)$ and will obtain a value of $W(X;\mu+\epsilon_1)$. If $S_1=-\varepsilon_1$, then he will exercise when the first agent with a positive signal exercises. For example, agent j will be the first to reveal a positive signal if $S_j=+\epsilon_j$, but $S_i=-\epsilon_i$ for $i<j$. The probability of this occurring is $(\frac{1}{2})^{j-1}$. In that case, agent 1 would exercise at the trigger $X^*(\mu-\epsilon_1-\cdots-\epsilon_{j-1}+\epsilon_j)$ and will obtain a value of $W(X;\mu-\sum_{k=1}^{j-1}\epsilon_k+\epsilon_j)$. If no agent has a positive signal, all will exercise at

purchaser might still be identified by his exercise strategy.

the trigger $X^*(\mu - \sum_{k=1}^{n} \epsilon_k)$ and will obtain a value of $W(X; \mu - \sum_{k=1}^{n} \epsilon_k)$. The value of agent j's option, $V_1(X; S_1)$, can be written in closed form as

$$V_1(X; S_1) = \begin{cases} W(X; \mu + \epsilon_1) & \text{if } S_1 = +\varepsilon_1 \\ \sum_{j=2}^{n} \left(\frac{1}{2}\right)^{j-1} W(X; \mu - \sum_{k=1}^{j-1} \epsilon_k + \epsilon_j) \\ \quad + \left(\frac{1}{2}\right)^{n-1} W(X; \mu - \sum_{k=1}^{n} \epsilon_k) & \text{if } S_1 = -\varepsilon_1. \end{cases} \tag{10}$$

Therefore the value of the information to agent 1 will equal the expected net benefit of achieving the full-information option value over the equilibrium partial-information option value. This will be a function of the current level of X and the signal of agent 1, S_1. Denote the value of this information as $N(X, S_1)$. The value of the information can be written as

$$N(X, S_1) = E\left[W(X; \theta)\,|S_1\right] - V_1(X; S_1). \tag{11}$$

If the information provider is a monopolist, then it will sell the information for the maximum price, $N(X, S_1)$, provided the cost of extracting the information is less than the expected benefits of selling the information. If the information providing industry is competitive, then free entry will ensure that the information will sell for its production cost, provided the cost is less than the value of the information to agent 1.

6. Equilibrium Exercise of Options with Payoff Externalities

In the basic model, even though information is conveyed through exercise, the actual payoff from exercise is independent of the actions of others. In many real-world situations, exercise produces informational as well as direct payoff externalities. In this section I generalize the model by providing an example of a case in which exercise by one agent directly affects the payoffs on the option of another agent. It will be seen that under certain conditions, positive exercise externalities can increase the likelihood of cascades. Conversely, negative exercise externalities can entirely eliminate the occurrence of cascades. The example is simple, but conveys much of the intuition for more complicated applications. More general discussion of this phenomenon can be found in Dybvig and Spatt (1983) and Katz and Shapiro (1986).

There are numerous examples of payoff externalities from the exercise of options to invest. For example, certain technology markets exhibit network externalities in which the benefits from adoption are increasing in the total number of adopters. A classic instance of this was in the battle between VHS and beta in videocassette recorder technology. As more and more consumers adopted the VHS technology, the benefits for others who adopted VHS became greater. Another example of exercise externalities would be in the case of retail stores exercising to enter a retail area. The presence of anchor tenants entices traffic flow into the area. In addition, a diversified array of specialty stores increases traffic flow for the anchors. There are also examples in which the exercise externalities are negative. An example would be firms considering the exercise of an option to begin manufacturing a product with a downward-sloping demand curve. As more firms exercise, the payoffs decline for all.

Consider a modified version of the basic model that illustrates the equilibrium in a game with dependent payoffs from exercise. For simplicity I shall solve for the equilibrium set of exercise strategies for the case in which $n = 2$, where the payoff externality is positive. The nature of the payoff externalities are as follows: The payoff from option exercise when there is only one option exercised is $\theta \cdot X$, as in the basic model. However, when there are two options exercised, each receives an additional payoff of $\kappa \cdot X$, where $\kappa \geq 0$ is a measure of the benefits from agglomeration. Therefore the first agent to exercise receives a payoff of $\theta \cdot X - I$, and will receive an additional payoff of $\kappa \cdot X$ if and when the other agent exercises. The second agent to exercise receives an immediate payoff of $(\theta + \kappa) \cdot X - I$.

The nature of the equilibrium will depend on the magnitude of the agglomeration effect. If κ is small, then the equilibrium exercise strategies are similar to that of the basic model. Agent 1 reveals his signal by his early exercise decision, and agent 2 combines this information with his own private information to condition his exercise strategy. However, if κ is large, then the equilibrium will be quite different. Agent 2 may not always use his private information in forming his exercise strategy. In particular, if agent 1 reveals a positive signal, agent 2 will "herd" and exercise immediately regardless of his private signal. That is, informational cascades can occur even after only one positive signal is revealed.

The equilibrium set of exercise strategies for this example is now summarized.

Equilibrium exercise strategies for a two-player game with positive payoff externalities

In an option exercise game in which there is a positive payoff externality of κ, each agent i's ($i = 1, 2$) exercise strategy will be contingent on the current level of $X(t)$, its signal S_i, and the revealed exercise policy of the other agent. Let $X(0) < X^*[\mu + \epsilon_1 + \max(\kappa/2, \kappa - \varepsilon_2)]$. The equilibrium exercise strategies will differ according to whether $\kappa \geq 2\varepsilon_2$. A pair of equilibrium exercise strategies for the two cases is as follows:

Case A. $\kappa \geq 2\varepsilon_2$. Agent 1 (the more informed agent): *If $S_1 = +\epsilon_1$, then exercise the first moment that $X(t)$ reaches the trigger $X^*(\mu + \epsilon_1 - \varepsilon_2 + \kappa)$. If $S_1 = -\epsilon_1$, then copy the exercise strategy of agent 2 by exercising immediately after him.*

Agent 2 (the less informed agent): *Wait until the trigger* $X^*(\mu+\epsilon_1-\varepsilon_2+\kappa)$ *is reached. If agent* 1 *exercises, then infer that agent* 1*'s signal is* $+\varepsilon_1$ *and then exercise immediately, regardless of the sign of* S_2*. If instead agent* 1 *does not exercise, then infer that agent* 1*'s signal is* $-\epsilon_1$ *and exercise the first moment that* $X(t)$ *equals or exceeds the trigger* $X^*(\mu-\varepsilon_1+S_2+\kappa)$.[18]

Case B. $\kappa < 2\varepsilon_2$. Agent 1 (the more informed agent): *If* $S_1 = +\epsilon_1$, *then exercise the first moment that* $X(t)$ *reaches the trigger* $X^*(\mu+\epsilon_1+\kappa/2)$*. If* $S_1 = -\epsilon_1$*, then copy the exercise strategy of agent* 2 *by exercising immediately after him.*

Agent 2 (the less informed agent): *Wait until the trigger* $X^*(\mu+\epsilon_1+\kappa/2)$ *is reached. If agent* 1 *exercises, then infer that agent* 1*'s signal is* $+\epsilon_1$ *and exercise the first moment that* $X(t)$ *equals or exceeds the trigger* $X^*(\mu+\varepsilon_1+S_2+\kappa)$*. This will imply immediate exercise if* $S_2 = +\varepsilon_2$*. If instead agent* 1 *does not exercise, then infer that agent* 1*'s signal is* $-\epsilon_1$ *and exercise the first moment that* $X(t)$ *equals or exceeds the trigger* $X^*(\mu-\varepsilon_1+S_2+\kappa)$.[19]

The proof that the above strategies constitute an equilibrium appears in the Appendix.

This example illustrates the fact that direct payoff externalities can overwhelm information externalities. Specifically, if the payoff from the benefits of agglomeration are large enough, the information that is revealed by exercise strategies can be totally lost. In Case A, agent 2 will respond to agent 1's early exercise by exercising immediately regardless of his private information. When extended to a market of size $n > 2$, an informational cascade will now take place after the revelation of the first positive signal. Such a market will then be highly informationally inefficient in that many agents will exercise with little or no updating of information. Agents will find the benefits of agglomeration so powerful so as to overwhelm the benefits of learning. Empirically this would imply that markets that display the greatest benefits from agglomeration would be the most prone to cascades. Such environments might include markets for technological innovations and retail development.

In a similar manner to the agglomeration example, it can be shown that for the case of negative exercise externalities, the opposite effect is true.

[18] I make the following off-equilibrium assumptions. If agent 1 exercises at any time prior to the trigger $X^*(\mu+\varepsilon_1-\varepsilon_2+\kappa)$, then agent 2 infers that agent 1 has a positive signal. Agent 2 will then exercise according to the trigger $X^*(\mu+\varepsilon_1+S_2+\kappa)$. Similarly, if agent 2 exercises before the trigger $X^*(\mu+\varepsilon_1-\varepsilon_2+\kappa)$, then agent 1 infers that agent 2 has a positive signal and exercises the first moment that $X(t)$ equals or exceeds the trigger $X^*(\mu+S_1+\varepsilon_2+\kappa)$.

[19] I make the following off-equilibrium assumptions. If agent 1 exercises prior to the trigger $X^*(\mu+\varepsilon_1+\kappa/2)$, then agent 2 infers that agent 1 has a positive signal and then exercises according to the trigger $X^*(\mu+\varepsilon_1+S_2+\kappa)$. Similarly, if agent 2 exercises prior to the trigger $X^*(\mu+\varepsilon_1+\kappa/2)$, then agent 1 infers that agent 2 has a positive signal and then exercises according to the trigger $X^*(\mu+S_1+\varepsilon_2+\kappa)$.

With negative payoffs associated with conformity, agents may never find it optimal to exercise simultaneously. If the externalities are sufficiently negative, agents will always respond to a previous agent's exercise by waiting for $X(t)$ to rise even further as to justify incurring the negative externality. Thus agents' exercise strategies will always be fully revealing, leading to the elimination of informational cascades. Empirically this would imply that markets that display the greatest negative externalities associated with agglomeration would be the least prone to cascades. Such markets might include products with steeply downward-sloping demand curves or with "snob appeal" where consumers value uniqueness over conformity.

7. Conclusion

In a world of perfect information, option exercise can be conducted in isolation. However, in markets with both public and private information, the exercise of options must be determined as part of a strategic equilibrium. This article presents a model of equilibrium option exercise policies and information revelation in markets with private signals. The act of exercise provides both a private benefit and a public benefit. The option exerciser receives an expected positive payoff, while other option holders may be able to impute private information. The model provides insights into the patterns of exercise in a variety of realistic economic settings. Cascades and boom-and-bust behavior may result from rational underlying forces. The model is used to determine the value and nature of market mechanisms that may form to provide private information. In addition, the model is extended to cases in which there exist both information and direct payoff externalities.

Appendix

A.1 Derivation of full-information exercise trigger

Let $W(X;\theta)$ denote the value of each agent's option in a world where θ is a known parameter. Using standard arguments [i.e., Dixit and Pindyck (1994), Grenadier (1995)], $W(X;\theta)$ must solve the following equilibrium differential equation:

$$0 = \frac{1}{2}\sigma^2 X^2 W_{xx} + \alpha X W_x - rW. \tag{A.1}$$

Equation (A.1) must be solved subject to appropriate boundary conditions. These boundary conditions serve to ensure that an optimal exercise strategy is chosen:

$$\begin{aligned} W\left[X^*(\theta),\theta\right] &= \theta X^*(\theta) - I \\ W_x\left[X^*(\theta),\theta\right] &= \theta. \end{aligned} \tag{A.2}$$

Here, $X^*(\theta)$ is the value of $X(t)$ that triggers entry. The first boundary condition is the value-matching condition. It simply states that at the moment the option is exercised, the expected payoff is $\theta X - I$. The second boundary condition is the smooth-pasting or high-contact condition.[20] This condition ensures that the exercise trigger is chosen so as to maximize the value of the option.

Closed-form solutions for $W(X, \theta)$ and the full-information trigger $X^*(\theta)$ are easily obtained. The value of the perfect information option strategy can be written as

$$W(X; \theta) = \begin{cases} \left(\frac{I}{\gamma-1}\right)^{1-\gamma} \left(\frac{\theta}{\gamma}\right)^{\gamma} X^{\gamma} & \text{for } X < X^*(\theta) \\ \theta X - I & \text{for } X \geq X^*(\theta), \end{cases} \tag{A.3}$$

where

$$\begin{aligned} X^*(\theta) &= \frac{\gamma}{\gamma-1} \cdot \frac{I}{\theta} \\ \gamma &= \frac{-(\alpha-\sigma^2/2)+\sqrt{(\alpha-\sigma^2/2)^2+2r\sigma^2}}{\sigma^2} > 1, \end{aligned} \tag{A.4}$$

and where $\alpha < r$ to ensure convergence. Note that $X^{*\prime}(\theta) < 0,\ X^{*\prime\prime}(\theta) > 0$.

A.2 Proofs of Propositions 1 and 2

Proposition 1. $E[\bar{T}_1(S_1, \ldots, S_n) - T^*(\theta)] < 0$. *Thus, on average, the agent with the most precise signal will exercise earlier than the full-information optimum.*

Proof. Define I_1 as the information available to agent 1 at the moment he exercises. I_1 will include the realization of S_1, plus any information [a subset of $(S_2, \ldots, S_n)$] revealed by other agents. The equilibrium describes the exercise strategy of agent 1 as follows. If agent 1 has a positive signal, he exercises at the trigger $X^*(\mu + \varepsilon_1)$. If agent 1 has a negative signal, and agent j is the first agent to reveal a positive signal through exercise, then he exercises at the trigger $X^*(\mu - \sum_{k=1}^{j-1} \varepsilon_k + \varepsilon_j)$. If no agent has a positive signal, then he exercises at the trigger $X^*(\mu - \sum_{k=1}^{n} \varepsilon_k)$. From this specification of agent 1's equilibrium exercise strategy, it follows that $\bar{X}_1(S_1, \ldots, S_n) = X^*[E_1(\theta)]$, where E_1 denotes agent 1's conditional expectation of θ at the moment of exercise.

Since $X^*(\theta)$ is convex, Jensen's inequality implies that $X^*[E_1(\theta)] < E_1[X^*(\theta)]$. Using the law of iterated expectations, $E[\bar{X}_1] = E\{X^*[E_1(\theta)]\} < E\{E_1[X^*(\theta)]\} = E[X^*(\theta)]$. Therefore $E[\bar{X}_1 - X^*(\theta)] < 0$. Thus, on average, agent 1 exercises earlier than the full-information exercise time.[21]

Proposition 2. $E[T_n(S_1, \ldots, S_n) - T^*(\theta)] > 0$. *Thus, on average, the agent with the least precise signal will exercise later than the full-information optimum.*

Proof. The proof will proceed in two parts. The first part will demonstrate that when no cascades occur, agent n always exercises at or after the full-information trigger. The second part will demonstrate that if a cascade occurs, agent n is expected to exercise after the full-information trigger.

First, assume that no cascades occur. From the definition of equilibrium in Section 2.2, agent n will know the true value of θ when he exercises. He will never exercise prior to the full-information trigger.

Second, assume that a cascade occurs, beginning with agent $i \leq n - 2$ (noting that the latest point at which a cascade can begin is with agent $n - 2$). According to the

[20] See Merton (1973) for a derivation of the high-contact condition.

[21] Recall that we assume that $\alpha > \frac{1}{2}\sigma^2$ in order for the expectation with respect to first passage times to exist.

equilibrium, $i^c = i$, agents $i+1$ through n will exercise simultaneously at the trigger $X^*[E_i(\theta)]$, where $E_i(\theta)$ denotes agent i's conditional expectation of θ at the moment of his exercise. Thus, conditional on a cascade beginning with agent i, $\bar{X}_n = X^*[E_i(\theta)]$. Given that a cascade begins with agent i, we know that agent $i+1$ exercises immediately and reveals $S_{i+1} = +\varepsilon_{i+1}$. Therefore the distribution of θ, conditional on agent $i+1$'s information, is $E_i(\theta) + \varepsilon_{i+1} + \sum_{j=i+2}^{n} S_j$.

I shall prove that $X^*[E_i(\theta)] > E_{i+1}\{X^*[E_i(\theta) + \varepsilon_{i+1} + \sum_{j=i+2}^{n} S_j]\}$ by induction. First, let $i = n-2$. Conditional on a cascade beginning with agent $n-2$,

$$\begin{aligned} X^*\left[E_{n-2}(\theta)\right] &> X^*\left[E_{n-2}(\theta) + \varepsilon_{n-1} - \varepsilon_n\right] \\ &> \frac{1}{2}\left\{X^*\left[E_{n-2}(\theta) + \varepsilon_{n-1} + \varepsilon_n\right]\right. \\ &\qquad \left. + X^*\left[E_{n-2}(\theta) + \varepsilon_{n-1} - \varepsilon_n\right]\right\} \\ &= E_{n-1}\left\{X^*\left[E_{n-2}(\theta) + \varepsilon_{n-1} + S_n\right]\right\}, \end{aligned} \tag{A.5}$$

where the first two lines follow from the fact that $X^*(\theta)$ is decreasing and that $\varepsilon_{n-1} > \varepsilon_n$, and the third line follows from the definition of agent $n-1$'s conditional expectation.

Next, assume that $X^*[E_{i+1}(\theta)] > E_{i+2}\{X^*[E_{i+1}(\theta) + \varepsilon_{i+2} + \sum_{j=i+3}^{n} S_j]\}$. I now demonstrate that $X^*[E_i(\theta)] > E_{i+1}\{X^*[E_i(\theta) + \varepsilon_{i+1} + \sum_{j=i+2}^{n} S_j]\}$. Conditional on a cascade beginning with agent i,

$$\begin{aligned} X^*\left[E_i(\theta)\right] &= X^*\left[E_{i+1}(\theta) - \varepsilon_{i+1}\right] \\ &> E_{i+2}\left\{X^*\left[E_{i+1}(\theta) - \varepsilon_{i+1} + \varepsilon_{i+2} + \sum_{j=i+3}^{n} S_j\right]\right\} \\ &= E_{i+2}\left\{X^*\left[E_i(\theta) + \varepsilon_{i+2} + \sum_{j=i+3}^{n} S_j\right]\right\} \\ &> E_{i+2}\left\{X^*\left[E_i(\theta) + \varepsilon_{i+1} - \varepsilon_{i+2} + \sum_{j=i+3}^{n} S_j\right]\right\} \\ &= E_{i+1}\left\{X^*\left[E_i(\theta) + \varepsilon_{i+1} - \varepsilon_{i+2} + \sum_{j=i+3}^{n} S_j\right]\right\} \\ &> E_{i+1}\left\{X^*\left[E_i(\theta) + \varepsilon_{i+1} + \sum_{j=i+2}^{n} S_j\right]\right\}, \end{aligned} \tag{A.6}$$

where the first and second equalities follow because when a cascade begins with agent i, $E_{i+1}(\theta) = E_i(\theta) + \varepsilon_{i+1}$, where the third equality follows because $E_{i+2}[f(\sum_{j=i+3}^{n} S_j)] = E_{i+1}[f(\sum_{j=i+3}^{n} S_j)]$, for any function f, where the first inequality follows by assumption (in our induction proof), and where the second and third inequalities follow from the fact that $X^*(\theta)$ is decreasing.

Since we have proved that conditional on $i^c = i$, the inequality in Equation (A.6) holds, applying the law of conditional iterated expectations yields $E[\bar{X}_n | i^c = i] > E[X^*(\theta) | i^c = i]$. Therefore we have proved by induction that, conditional on the occurrence of a cascade beginning with any agent i , the exercise of agent n is expected to be at a trigger greater than the full-information trigger.

A.3 Derivation of the probability of informational cascades

Define the probability that a cascade forms in a market of size n as p_n. p_n can be derived by recursion. The probability that a cascade occurs in a market of size n is equal to the probability that a cascade occurs among the first $n-1$ agents plus the probability that a cascade does not occur among the first $n-1$ agents, but does occur with the final agent. In order for there to be no cascade among the first $n-1$ agents, but for a cascade to occur with n agents, it must be the case that there is no cascade for the first $n-3$ agents, a negative signal for agent $n-3$, and positive signals for agents $n-2$ and $n-1$. This can then be written recursively as

$$p_n = p_{n-1} + (1 - p_{n-3}) \cdot \left(\frac{1}{2}\right)^3, \qquad \text{for } n > 3 \tag{A.7}$$

and with $p_1 = 0,\ p_2 = 0,\ p_3 = 1/4$.

This third-order difference equation has the solution

$$p_n = 1 - \frac{4}{\sqrt{5}}\left[\left(\frac{1}{\sqrt{5}-1}\right)^{1+n} + \left(\frac{1}{1+\sqrt{5}}\right)\left(\frac{-1}{1+\sqrt{5}}\right)^{n}\right], \qquad \forall n \geq 0. \tag{A.8}$$

A.4 Proof of equilibrium exercise strategies for example with payoff externalities

Case A. $\kappa \geq 2\varepsilon_2$.

To prove that the strategies constitute an equilibrium, I will demonstrate that given each player's strategy, the other player's strategy is optimal. First, I demonstrate that agent 2's strategy is optimal, taking agent 1's strategy as given. Note that agent 2 achieves the full-information optimal exercise strategy (and therefore cannot be improved), with the sole exception of when $S_1 = +\varepsilon_1$ and $S_2 = +\varepsilon_2$. When both signals are positive, agent 1 might prefer to exercise earlier than the equilibrium trigger. This would mean that agent 2 would forego the benefits of learning agent 1's private signal. However, if agent 2 preempts agent 1, what will agent 1 do? If $S_1 < 0$, then agent 1 will not exercise immediately, but will wait until the trigger $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$ is reached. Thus, at best (if agent 2 exercises very early, agent 1 might still defer exercising), agent 1 will exercise immediately along with agent 2 only if $S_1 > 0$. Therefore agent 2's expected payoff from early exercise would be $(\mu + \varepsilon_2 + \kappa/2) \cdot X - I$ plus an expected payoff of $\kappa/2$ in the future at the trigger $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$. Since the future payoff is fixed, the optimal moment to receive this expected payoff is at $X^*(\mu + \varepsilon_2 + \kappa/2)$. Since $\kappa \geq 2\varepsilon_2$, $\mu + \varepsilon_2 + \kappa/2 < \mu + \varepsilon_1 - \varepsilon_2 + \kappa$, and thus $X^*(\mu + \varepsilon_2 + \kappa/2) > X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$, which implies that agent 2 is always better off waiting until the trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$ before exercising, even if his signal is positive.

Now, taking agent 2's strategy as given, I demonstrate the optimality of agent 1's strategy. If agent 1 has a negative signal, then his equilibrium strategy achieves the full-information optimum. Thus I focus on the case in which $S_1 = +\varepsilon_1$, where the equilibrium strategy is to exercise at the trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$. First, I show that it never pays to exercise earlier. If agent 1 exercises earlier, then (at best) agent 2 will exercise immediately only if $S_2 > 0$. This is because if agent 2 has a negative signal, his optimal strategy is to wait until the trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$ is reached. Thus an earlier than equilibrium exercise by agent 1 will provide an expected immediate payoff of $(\mu + \varepsilon_1 + \kappa/2) \cdot X - I$ plus an expected payoff of $\kappa/2$ in the future (when agent 2 exercises). Since the future payoff is fixed, the optimal moment to receive this payoff is at $X^*(\mu+\varepsilon_1+\kappa/2)$. Since $\kappa \geq 2\varepsilon_2$, $\mu+\varepsilon_1+\kappa/2 < \mu+\varepsilon_1-\varepsilon_2+\kappa$, which implies that it is better for agent 1 to always wait at least until the trigger $X^*(\mu+\varepsilon_1-\varepsilon_2+\kappa)$ is reached before exercising. Second, I show that it never pays to exercise later. The only reason for agent 1 (with $S_1 > 0$) to wait would be to "pretend" to have a negative signal and learn agent 2's private signal when the trigger $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$ is reached. However, this information will prove useless because agent 1 will exercise immediately regardless of the sign of S_2 . This is true because even if $S_2 = -\varepsilon_2$, the optimal exercise trigger is $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$, which implies immediate exercise. Since the "pretend" strategy implies waiting until $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$ and then always exercising immediately, this implies an expected payoff of $(\mu + \varepsilon_1 + \kappa) \cdot X - I$. The value of receiving this payoff is decreasing at any point above the trigger $X^*(\mu+\varepsilon_1+\kappa)$, which implies it is not optimal to wait beyond the equilibrium trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$.

Case B. $\kappa < 2\varepsilon_2$.

First, I demonstrate that agent 1's strategy is optimal, taking agent 2's strategy as given. If agent 1 has a negative signal, then his equilibrium strategy achieves the full-information optimum. Thus I focus on the case in which $S_1 = +\varepsilon_1$, where the equilibrium strategy is to exercise at the trigger $X^*(\mu + \varepsilon_1 + \kappa/2)$. I begin by showing that agent 1 will never enter earlier than the trigger $X^*(\mu+\varepsilon_1+\kappa/2)$. Agent 1 knows that if agent 2 exercises immediately after him if $S_2 = +\varepsilon_2$, and at the trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$ if $S_2 = -\varepsilon_2$, then his expected payoff from being the first to exercise is $(\mu+\varepsilon_1+\kappa/2)\cdot X-I$ paid immediately, plus $\kappa/2 \cdot X$ in the future when the trigger $X^*(\mu + \varepsilon_1 - \varepsilon_2 + \kappa)$ is hit. Since the future payoff is fixed, the optimal moment to receive this payoff is at $X^*(\mu + \varepsilon_1 + \kappa/2)$. Thus there is no incentive to exercise earlier. The only potential reason to wait longer would be to acquire information on agent 2's signal. In order to do so, agent 1 would have to wait until the trigger $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$ for knowledge of S_2. However, it will always prove optimal for agent 1 to exercise immediately at the trigger $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$, regardless of information about the sign of S_2. This is because the immediate payoff would be $(\mu + \varepsilon_1 + S_2 + \kappa) \cdot X - I$, and the optimal exercise strategy for such a payoff is to exercise the first moment $X(t)$ equals or exceeds the trigger $X^*(\mu + \varepsilon_1 + S_2 + \kappa)$, which is always less than $X^*(\mu - \varepsilon_1 + \varepsilon_2 + \kappa)$. Since this strategy of waiting for more information provides precisely the same expected payoff as the equilibrium strategy, but at a stopping time greater than the time at which the value maximizing trigger $X^*(\mu + \varepsilon_1 + \kappa/2)$ is reached, it never pays to exercise later.

Now, taking agent 1's strategy as given, I demonstrate the optimality of agent 2's strategy. It is clear that by observing whether agent 1 exercises the first moment that $X(t)$ equals or exceeds $X^*(\mu+\epsilon_1+\kappa/2)$, agent 2 will correctly infer agent 1's signal and will then know the true value of θ. It is never optimal to wait longer than the full-information trigger. It would also never be optimal for agent 2 to take the lead and exercise earlier. Even if agent 2 is optimistic ($S_2 = +\varepsilon_2$), prior to learning agent 1 's signal his optimal exercise strategy would be to wait until the trigger $X^*(\mu + \varepsilon_2 + \kappa/2)$, as was shown above in demonstrating agent 1's optimal strategy. Since this trigger is greater than the trigger $X^*(\mu + \epsilon_1 + \kappa/2)$, it will prove optimal to wait for S_1 to be revealed.

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