Market Efficiency and Natural Selection in a Commodity Futures Market

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While the literature usually justifies informational efficiency in the context of rationality, this article shows informational efficiency by applying the evolutionary idea of natural selection. In a dynamic futures market, speculators are assumed to merely act upon their predetermined trading types (buyer or seller), their predetermined fractions of wealth allocated for speculation, and their inherent abilities to predict the spot price, reflected in their distributions of prediction errors with respect to the spot price. This article shows that the proportion of time that the futures price equals the spot price converges to one with probability 1.

Much financial economic theory has been developed to examine the informational efficiency of markets. If traders are rational, in the sense that they maximize expected utility and form rational expectations, then informational efficiency can be achieved, as shown by Grossman (1976, 1978), Radner (1979), Hellwig (1980), Allen (1981), and Bray (1981). Instead of relying on rationality to justify market efficiency, this article takes an evolutionary approach. An evolutionary model of natural selection is offered. It produces a unique long-run outcome: market efficiency. The evolutionary approach abandons any

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rationality assumption by adopting a framework where it is not possible for agents to learn about other traders or the market environment; nor is it possible for agents to maximize particular objectives. Agents' behavior are exogenously preprogrammed.[1] [Economic justifications for moving away from assuming rationality on the part of agents (speculators) can be found in Alchian (1950) and Tversky and Kahneman (1974)]. By adopting this extreme framework with such mechanical rules, we can emphasize the market's ability to promote market efficiency through the selection process. The market rewards the speculator whose behavior happens to fit the market environment the most, and takes wealth away from the speculator whose behavior happens to not fit well with the market. Economic natural selection drives the market to a state where informational efficiency arises.

The following economic modeling approach is analogous to the biological approach where a variety of animals compete under a common environment for food (or resources) and where natural selection selects only those who are most fit. Here, there are speculators, with a variety of abilities to predict the future spot price, who compete in the futures and spot market for wealth and where market selection rewards only those who are most fit.

Consider an economy with a commodity spot and a futures market. The futures market consists of both speculators and producers. Producers sell contracts in the futures market to hedge against the risk in the spot market. Speculators enter the futures market sequentially with an initial endowment of wealth V_0. Speculators either buy futures contracts with the hope of selling at a higher spot market price (acting as a buyer) or short sell futures contracts with the hope of buying at a lower spot market price (acting as a seller). In addition, the fraction of wealth allocated for speculative activity differs across speculators. Furthermore, the ability to predict the future spot price also differs across speculators. A speculator's ability to predict the future spot price is modeled as inherent and is characterized by the distribution of his or her prediction error with respect to the future spot price. A speculator's prediction error represents the amount by which this speculator overpredicts or underpredicts the spot price at that time period. Each speculator is labeled according to his or her trading type (buyer or seller), his or her fraction of wealth allocated for speculative activities, and his or her inherent probability distribution of prediction error with respect to the spot price, which are all randomly determined upon his or her entry period and are fixed thereafter.[2] In this article the speculators are modeled as

[1] In the traditional rationality models agents make choices to maximize certain objectives. In this article, the evolutionary approach is deployed for the purpose of abandoning the rationality of making choices. Instead the choice variables are exogenously prespecified.

[2] Analogous to the biological term of genotype, the economic agent's type is characterized by his or her choice variable. The choice variable could be interpreted as arising from an agent maximizing his or her own objective within his or her resource constraints, information constraints, and ability constraints.

unsophisticated and they merely act upon their predetermined trading type (buyer or seller), and a predetermined fraction of wealth that they allocate on speculative activities, and their predetermined abilities to predict the future spot price. All speculators are noise traders in the sense that the distribution of each speculator's prediction error generates a strictly positive probability of overpredicting or underpredicting the spot price. In other words, each trader has a strictly positive probability of acting upon noise as if it were information. Nevertheless, some speculators have a higher probability of predicting the future spot price than others. That is, some speculators are less noisy than others.

As more and more speculators enter the economy, at any point in time, whoever acts upon better predictions makes profit at the expense of his or her trading counterparts who act on less reliable predictions. For those buyers (sellers), who predict exactly right the spot price with a low probability and overpredict (underpredict) the spot price with a high probability, their wealth will be frequently reduced. Consequently, their ability to influence the futures price will be overshadowed by buyers (sellers) with a high probability of predicting exactly right the spot price and a low probability of overpredicting (underpredicting) the spot price. Therefore, as time goes by, with this natural selection of information caused by the reallocation of wealth away from more noisy traders to less noisy traders and with an ongoing flow of entering speculators, some of whom have better predictive ability than others, the convergence of the futures price to the spot price eventually occurs.

This article is not alone in applying natural selection to examine market behavior. Early articles which use the idea of natural selection to examine market efficiency are Figlewski (1978, 1982). In Figlewski (1978), in a pure speculation market with two representative traders with different quality of information, traders with better information make profits at the expense of those with less information. Figlewski (1982) extends the discussion to N traders. However, efficiency cannot be obtained in the long run.[3] A fundamental reason for the difference in results is that, unlike Figlewski (1978,1982), this article allows for sequential entry of traders with a wide diversity of predictive abilities. This ongoing arrival of new agents, some of whom have better predictive ability, prevents the economy from being stuck in an inefficient market equilibrium.

Among the few recent articles that have analytically explored this issue are Blume and Easley (1992), Biais and Shadur (1993), and Luo (1995a). These articles illustrate that market selection will not necessarily lead to the elimination of noise or inefficiency. For example, Blume and Easley (1992),

[3] In Figlewski (1982), informational efficiency is obtained in the long run only in the special case when all traders have independent information. In contrast, this is not a required assumption in the evolutionary model of this article.

in a dynamic market with wealth flows between traders, find that economic natural selection does not necessarily lead to efficient markets. Similarly, Biais and Shadur (1993) arrive at the same conclusions using Darwinian dynamics in a nonoverlapping generations model of a risky asset. In the above two articles the reason for the market's departure from efficiency is that, aside from not allowing traders to enter with better predictive abilities than existing traders, traders' behavior rules are modeled as being linked to their utility functions. Since utility maximizing rules do not guarantee that traders have accurate predictions, it follows that wealth may not be maximized. Therefore, traders with rational rules do not necessarily dominate the market in the long run. On the contrary, traders with irrational rules may come to dominate the market. On the other hand, Luo's (1995a) examination of competition among firms shows that with sequential entry of various types of firms over time, including those that happen to produce at or near the minimum efficient scale, the market selects the most efficient firms.

This article is organized into four sections. The framework of the model is described in Section 1. Section 2 provides the results of the model. Section 4 concludes the article.

1. The Model

Consider a dynamic economy with an industry producing one commodity. Discrete time is indexed by $t, t = 1, 2, \ldots$. It is assumed that the commodity is nonstorable and cannot be carried to the next time period. There is a commodity futures market and there is a spot market. The futures market opens at the beginning of each time period and the spot market opens at the end of each time period. The participants in the futures market consist of speculators and producers. It is assumed that the spot price is determined at the beginning of each time period.[4] Nevertheless, the realized spot price is unknown to each producer and to each speculator at the beginning of each time period, and the spot price is only revealed to all producers and all speculators at the end of each time period when the output is delivered to the spot market. Therefore, the spot price is independent of the futures price in the futures market.

At the beginning of each time period, each producer contracts a percentage of their production in the futures market to hedge against the risk associated with an unknown spot price. Producers' aggregate contracts brought

[4] The justification for this assumption is as follows. For each of a large number of independent producers the output level is randomly determined at the beginning of each time period. Since the commodity is nonstorable, the total output of all producers must be delivered to the spot market at the end of each time period and sold at a market clearing spot price. Since the individual output levels of producers are determined in the beginning of each time period, the industry's aggregate output is also determined at the beginning of each time period. Given any exogenously specified continuous demand schedule in the spot market, the spot price is determined at the beginning of each time period.

to the futures market at the beginning of each time period are assumed to be drawn from an interval $U = (0, \overline{S}]$ according to an arbitrarily given distribution, where $0 < \overline{S} < \infty$.[5] Denote producers' aggregate contracts brought to the futures market at the beginning of time period t as S_t^f.

Since the commodity is nonstorable and cannot be carried to the next period, the futures contract is a one-period contract. All payments among all futures market participants are settled at the end of each time period after both the futures and the spot markets close.[6] Each speculator's wealth at the end of a time period is defined as accumulated profits up to the end of that time period.

The futures contract size is perfectly divisible. The price of the commodity in the spot market p_t and the price of a futures contract p_t^f at time t (where $t = 1, 2, \cdots$) are always quoted as nonnegative multiples (m) of a unit size d (> 0). The spot price p_t is randomly drawn at the beginning of time period t (before the futures market opens) from the set $\{d, d+1, \ldots, (\overline{m}-1)d, \overline{m}d\}$, where $\overline{m} \geq 1$ is a positive integer, according to a distribution function. The futures price p_t^f is merely the market clearing price in the commodity futures market and for any t, p_t^f is constrained to be greater than or equal to d. The determination of p_t^f is described in Section 1.3.

The following subsections provide further detailed descriptions of the model.

1.1 Speculators' types

Speculators enter the commodity futures and spot markets sequentially over time. At the beginning of the time period t, where $t = 1, 2, \ldots$, only one speculator (called speculator t) is allowed to enter the futures and spot markets. The spot price at time t is determined at the beginning of time period t, but it is unknown to all speculators and all producers. At time s, where $s \geq t$, speculator $t's$ prediction or belief about the spot price, denoted as b_s^t, consists of the summation of the spot price p_s and a noise term ν_s^t which characterizes the prediction error of speculator t at time s. That is,

$$b_s^t = p_s + \nu_s^t, \tag{1}$$

[5] A more elaborate production model, which characterizes the costs of individual producers, could be specified. This adds very little to the issues examined in this article, which focuses on the effect of various speculators' behavior on the futures market.

[6] The producers honor their futures contracts sold at the beginning of each time period, by transferring, at the end of the time period, to the buyers of the futures contracts the corresponding revenue received in the spot market and in return the producers receive from the buyers the value of these same contracts at the futures price previously agreed upon. The short sellers deliver to the buyers their short sales valued at the spot price and receive from the buyers the value of the same contracts at the previously agreed upon futures price.

as in Grossman (1976, 1978) and Hellwig (1980). It is assumed that $\nu_s^t \in \{-\underline{N}d, -(\underline{N}-1)d, \ldots, -2d, -d, 0, d, 2d, \ldots, (\overline{N}-1)d, \overline{N}d\}$,[7] where $\underline{N}$ and $\overline{N}$ are both positive whole numbers. The smaller a prediction error is, the more accurate is the information reflected in the prediction. It is assumed that ν_s^t is independent of p_s at time s. For a fixed t, the ν_s^t are assumed to be independently and identically distributed across time $s = t, t+1, \ldots$.[8] Therefore denote

$$\theta_1^t = \Pr(\nu_s^t > 0), \quad \theta_2^t = \Pr(\nu_s^t = 0), \text{ and } \theta_3^t = \Pr(\nu_s^t < 0),$$

where superscript t indicates speculator t. It is assumed that the vector $(\theta_1^t, \theta_2^t, \theta_3^t)$ is randomly taken in the beginning of time period t from a set $\Theta = \{(\theta_1, \theta_2, \theta_3) \in (0,1)^3 : \theta_1 + \theta_2 + \theta_3 = 1\}$ according to an arbitrarily given distribution with its support being Θ. And furthermore, the vector $(\theta_1^t, \theta_2^t, \theta_3^t)$ is independent of p_t. The vector $(\theta_1^t, \theta_2^t, \theta_3^t)$ describes the probability distribution of speculator t's overprediction, exact prediction, and underprediction. This probability distribution characterizes speculator t's predictive ability. Once speculator t's predictive ability $(\theta_1^t, \theta_2^t, \theta_3^t)$ is determined in the beginning of his or her entry time period t, it is fixed thereafter. One justification for modeling a speculator's probability distribution of prediction error being fixed through time is that speculators display systematic biases due to cognitive errors [e.g., Tversky and Kahneman (1974)].

If the speculator is buying (short selling) at time s, b_s^t indicates the highest (lowest) price that a speculator is willing to pay for (supply) a contract at time s. Essentially b_s^t is speculator t's reservation price at time s. The terms prediction, belief, bid, and reservation bid are used interchangeably in the remainder of this article. This way of modeling the speculator's predictions reflects diverse information among the speculators, diverse abilities to process the same information among the speculators, or diverse bidding strategies.[9]

There are two further characterizations of speculators. First, speculator t on entry is endowed with one of the following two trading types:

(i) Speculator t participates only as a buyer at all times $s \geq t$. That is, speculator t only buys contracts at a price no higher than his or her

[7] To prevent unrealistic negative predictions, the range of predictions sometimes may have to be truncated if $p_s + \nu_s^t < d$. More precisely, $b_s^t = \max[p_s + \nu_s^t, d]$.

[8] This reflects the impossibility of speculators learning from other traders and learning from their past experiences.

[9] One can interpret the b_s^t as a signal that speculator t receives at time s. The prediction error ν_s^t is the noise that speculator t brings into the market at time s. As long as b_s^t is not equal to p_s speculator t is acting upon noise. Of course, given the differences in the distribution of speculators' prediction errors, different speculators have different probabilities of acting upon noise. In this article no assumptions are made with respect to the type of the distribution of the prediction error. (This stands in contrast to most ones including recent ones by Kyle and Wang (1997) and Fischer and Verrecchia (1997).]

reservation bid b_s^t at all times $s \geq t$. Denote this type of speculator as $z_t = 1$, where subscript t indicates speculator t.

(ii) Speculator t participates only as a short seller at all times $s \geq t$. That is, speculator t only sells contracts at a price no lower than his or her reservation bid b_s^t at all times $s \geq t$. Denote this type of speculator as $z_t = -1$, where subscript t indicates speculator t.

Therefore, speculator t's trading type, represented by a random variable z_t, is randomly taken from a set $\{-1, 1\}$ in the beginning of time period t according to an arbitrarily given discrete probability distribution with its support being $\{-1, 1\}$. And furthermore, z_t is independent of p_t for $t = 1, 2, \ldots$.

The second further characterization is that each speculator spends a fraction of his or her total wealth on speculative activity. This fraction could be viewed as a reflection of a speculator's inherent attitude toward risk. A smaller fraction indicates more risk aversion. The remaining fraction of total wealth can be viewed as risk-free and earns no return. This fraction is randomly determined in the beginning of his or her entry period and fixed thereafter. Denote f_t as speculator t's fraction of wealth allocated for speculative activities. f_t is randomly taken from an interval $(0, 1]$ according to a distribution with its support being $(0, 1]$ in the beginning of time period t. And furthermore, f_t is independent of p_t for $t = 1, 2, \ldots$.

It is assumed that the random vectors $(\theta_1^t, \theta_2^t, \theta_3^t)$, (z_t), and (f_t) are mutually independent for a fixed t and the random vector $(\theta_1^t, \theta_2^t, \theta_3^t, z_t, f_t)$ is independently and identically distributed across speculators $t = 1, 2, \ldots$.

To summarize, on entry speculator t is characterized by random draws, represented by the ordered vector $(\theta_1^t, \theta_2^t, \theta_3^t, z_t, f_t)$, where the first three elements characterize the distribution of speculator t's prediction error, the fourth element describes speculator t's trading type, and the last element describes the fraction of wealth spent by speculator t on speculative activity. Figure 1 shows the timing of the above events. Since there is an ongoing entry of speculators with characteristics $(\theta_1^t, \theta_2^t, \theta_3^t, z_t, f_t)$ randomly drawn from the supports of their distributions, collectively the speculators can be described as having a continuous spectrum of characteristics. Since θ_1^t, θ_2^t, and θ_3^t for all t are strictly between 0 and 1, all speculators are noise traders. Nevertheless, some speculators with a higher probability of predicting the spot price are less noisy than others with a lower probability of predicting the spot price.

1.2 Speculators in the market selection process

Each speculator is assumed to be endowed with initial wealth V_0, where $0 < V_0 < \infty$, in the entry time period. The futures market is assumed to have no speculators at time 0. After time 0 speculators enter sequentially. Denote speculator t's wealth at the end of time period s (where $s \geq t$) as V_s^t

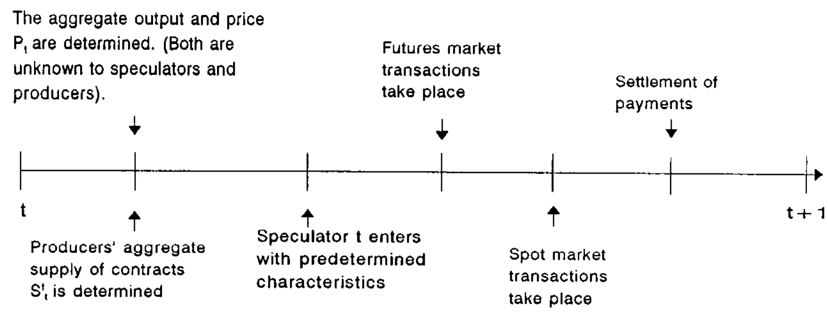

Figure 1
The timing of events

and $V_{t-1}^t = V_0$. The futures market will not open if there are only sellers in the futures market.

Speculator t's prediction about the spot price (where $t = 1, 2, \ldots$) at time s (where $s \geq t$) is characterized by b_s^t. If speculator t is a buyer, speculator t is willing to buy contracts up to what his or her speculative wealth permits at a price no higher than his or her prediction b_s^t, at time s, where $s \geq t$. Therefore, for $z_t = 1$, speculator t's demand for futures contracts in time period s, denoted as $q_s^t(p_s^f)$, is[10]

$$q_s^t(p_s^f) = \begin{cases} \left(\dfrac{f_t V_{s-1}^t}{p_s^f + d}, \dfrac{f_t V_{s-1}^t}{p_s^f} \right] & \text{if } p_s^f = b_s^t - rd, \\ & \quad \text{where } r = 1, 2, \ldots, \widehat{R}, \\ \left[0, \dfrac{f_t V_{s-1}^t}{p_s^f} \right] & \text{if } p_s^f = b_s^t, \\ 0 & \text{if } p_s^f > b_s^t. \end{cases}$$

where $\widehat{R}$ is characterized by

$$\begin{cases} b_s^t - (\widehat{R} + 1)d < 0 \\ b_s^t - \widehat{R}d \geq 0. \end{cases}$$

Figure 2 provides an example of this type of demand curve. If speculator t is a short seller, at time s, where $s \geq t$, speculator t is willing to sell contracts up to what his or her speculative wealth permits at a price no lower than his or her prediction b_s^t. To ensure that short sellers honor their

[10] Because of the discreteness of prices, when $p_s^f < b_s^t$ a buyer may not be able to spend all of his or her speculative wealth ($f_t V_{s-1}^t$) if the supply of futures contracts lies between $\frac{f_t V_{s-1}^t}{p_s^f + d}$ and $\frac{f_t V_{s-1}^t}{p_s^f}$. Thus, a steplike demand curve is used for any buyer.

contracts, short sellers are constrained to invest $\frac{f_t}{\overline{m}-1}$ of their total wealth.[11] The reason is that if short sellers are allowed to invest f_t of their wealth, and if the spot price exceeds twice the futures price then short sellers would not be able to honor their contracts. Since the maximum loss that a short seller would incur is when the spot price takes the highest value $\overline{m}d$ and the futures price takes the lowest value d, the constraints for investing $\frac{f_t}{\overline{m}-1}$ of the total wealth by the short seller ensures no defaults on any contract.[12] Therefore, for $z_t = -1$, speculator t's demand for futures contracts in time period s, denoted as $q_s^t(p_s^f)$, is[13]

$$q_s^t(p_s^f) = \begin{cases} 0 & \text{if } p_s^f < b_s^t, \\ \frac{1}{\overline{m}-1}\left[-\frac{f_t V_{s-1}^t}{p_s^f}, 0\right] & \text{if } p_s^f = b_s^t, \\ \frac{1}{\overline{m}-1}\left[-\frac{f_t V_{s-1}^t}{p_s^f}, -\frac{f_t V_{s-1}^t}{p_s^f + d}\right) & \text{if } p_s^f = b_s^t + rd, \\ & \quad \text{where } r = 1, 2, \ldots. \end{cases}$$

A negative number of futures contracts indicates the quantity of short sales. See Figure 2 for a plot of such a demand curve. The above demand functions are similar to those used by Feiger (1978).

Obviously, if a speculator's wealth is zero then this speculator will not be able to continue to participate in the markets in any of the future time periods, since this speculator is not able to buy or sell any contracts in any of the future time periods. In other words, if a speculator's wealth is zero in one time period, then this speculator is considered to exit the economy at the end of that time period.

Therefore, at the end of time period s, where $s \geq t$, the wealth of

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[11] Ensuring no defaults can also be accomplished if the spot price is constrained such that it cannot exceed twice that of the futures price. This latter assumption is similar to that used by Feiger (1978). Either assumption leads to the same conclusions.

[12] More precisely, consider

$$q_s^t = -k\frac{f_t V_{s-1}^t}{p_s^f}, \quad \text{for } k > 0; \tag{1$'$}$$

and

$$p_s \leq \overline{m}d \tag{2$'$}$$

and

$$p_s^f \geq d. \tag{3$'$}$$

To ensure that short sellers honor their transactions it is assumed that the speculative wealth $f_t V_s^t = (p_s - p_s^f)q_s^t + f_t V_{s-1}^t \geq 0$ (4′). Therefore, with Equations (1′), (2′), and (3′), Equation (4′) is ensured when $k \leq \frac{1}{\overline{m}-1}$. Suppose this constraint is not imposed for short sellers and suppose that k is set equal to one. Then in the extreme case of $p_s^f = d$ and $p_s = \overline{m}d$ it would follow that $f_t V_s^t = (p_s - p_s^f)q_s^t + f_t V_{s-1}^t < 0$ for $\overline{m} > 2$. This implies that the short seller is not able to honor his or her contracts.

[13] For similar reasons described in note 10, a steplike demand curve is used for any seller.

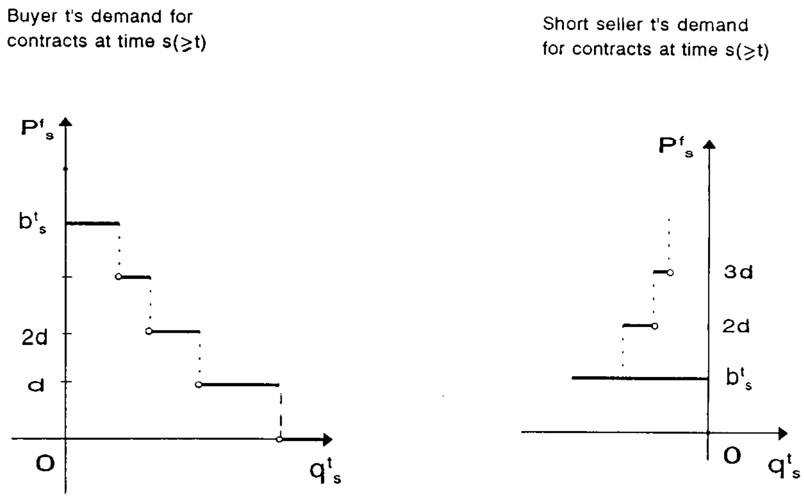

Figure 2
Speculators' demands for contracts

speculator t is

$$V_s^t = (p_s - p_s^f)q_s^t + V_{s-1}^t.$$

1.3 The futures market equilibrium

The futures market structure is the standard Walrasian mechanism. Denote the net aggregate demand at time s as $Q_s^f(p_s^f)$. Hence

$$Q_s^f\left(p_s^f\right) = \sum_{t=1}^{s-1} q_s^t(p_s^f) + q_s^s(p_s^f).$$

Since the spot price is at least d, to be consistent, the futures price is assumed to be greater than or equal to d. Furthermore, this regulation prevents short selling at zero price which could cause unlimited defaults. Under the Walrasian market structure, the futures market clearing price p_s^f is set to fulfill the condition that the net aggregate demand is greater than or equal to the aggregate number of contracts offered for sale by producers. That is,

$$p_s^f = \max\left[\max\left\{p_s^f : Q_s^f\left(p_s^f\right) \geq S_s^f\right\}, d\right].$$

There may be multiple solutions to $\{p_s^f : Q_s^f(p_s^f) \geq S_s^f\}$. Choosing the equilibrium solution with the highest futures price is arbitrary. Nevertheless, whatever equilibrium is chosen, the results of the article are unchanged. Notice that in the early time periods of the market process, if a buyer has not yet entered, then no futures market transaction takes place. If a buyer

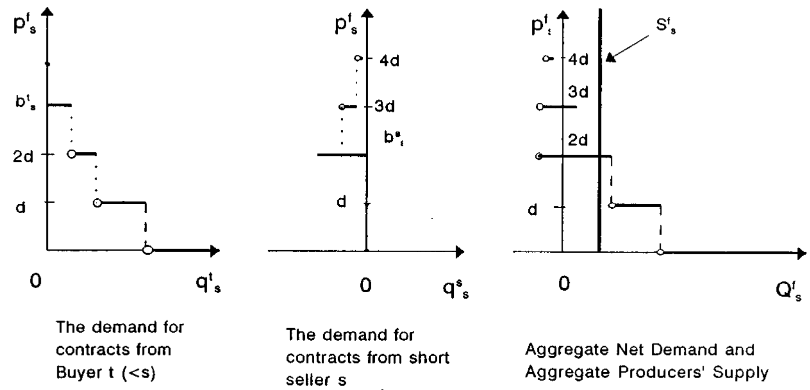

Figure 3
An example of futures market equilibrium price—two speculators in the futures market

has entered and if the producers' supply in the futures market exceeds the speculators' net demand at all positive prices (usually occurring in the earlier time periods), then the futures market price is regulated to be d, and the futures market demand is delivered to the buyer at price d and the excess of the producers' supply is returned to the producers (to be sold at the end of the time period at the spot price).

The demand and supply orders made by speculators are executed as follows. At time period s, at the futures market clearing price p_s^f, $Q_s^f(p_s^f) \geq S_s^f$ or $p_s^f = d$. For the short sellers with reservation bids below or equal to the market clearing price p_s^f, their supply orders at the market clearing price p_s^f are executed at the market clearing price p_s^f and their wealth is exhausted; for the buyers with reservation bids above the market clearing price, their demand orders at a price of d above the market clearing price p_s^f are filled at p_s^f; and the remaining supply in the market is allocated to buyers with reservation bids greater than the market clearing price p_s^f and to all speculators with reservation bids equal to the market clearing price p_s^f.[14] One example of a futures market equilibrium is shown in Figure 3. As can be seen, the futures price p_s^f will be a function of the realizations of $z_1, z_2, \ldots, z_s$; and $f_1, f_2, \ldots, f_s$; and v_k^t for all $t \leq s$ and all $k \in [t, s]$; and $p_1, p_2, \ldots, p_s$; and producers' supplies $S_1^f, S_2^f, \ldots, S_s^f$.

The purpose of this article is to show that with probability 1, the proportion of time that the futures price equals the fundamental value (the spot price) converges to one as time goes to infinity. In the limit, the futures price reflects only accurate information about the spot price and ignores

[14] The precise way that the remaining supply is allocated has no effect on the article's results.

noise. Consequently, no individual speculator can cause the futures price to systematically deviate away from the spot price in the limit.

2. Convergence of the Futures Market to Efficiency

This section shows that with the above market selection process, with probability 1, the proportion of time that the futures price equals the fundamental value (the spot price) converges to one as time goes to infinity. This is established in Theorem 3. To show Theorem 3, first of all, the article shows that with probability 1, the proportion of time that the futures price is below the spot price converges to zero as time goes to infinity. This is proven in Theorem 1. Second, Theorem 2 shows that with probability 1, the proportion of time that the futures price is above the spot price converges to zero as time goes to infinity. Finally, Theorem 3 is established by using Theorems 1 and 2.[15]

The article begins by proving Theorem 1, which shows that with probability 1, the proportion of time that the futures price is below the spot price converges to zero as time goes to infinity. The intuition goes as follows. Suppose Theorem 1 is not true, then with a strictly positive probability, there must exist a subsequence such that up to any point of time on this subsequence, the proportion of time that the futures price is below the spot price is bounded away from zero. Given the continuous spectrum of speculators' all possible characteristics, with probability 1 there is one buyer, say buyer I, who enters the market with a fraction of wealth allocated for speculation of at least $\frac{\overline{m}}{\overline{m}+1}$ and who predicts exactly right the spot price with a probability (θ_2^I) sufficiently larger than the probability (θ_1^I) of predicting above the spot price every time period. That is, over an infinite period of time, eventually there will be entry into the market by some buyer I who has a rather large probability of predicting the spot price accurately and a rather low probability of overpredicting it. Since short sellers always honor their contracts and since the spot price never goes below d, buyer I's wealth is always positive. This prevents buyer I's wealth from being stuck at zero. Since buyer I makes gains if he or she predicts exactly right the spot price at a time period when the futures price is below the spot price and potentially makes a loss if he or she overpredicts the spot price at a time period when the futures price is above the spot price, the relative magnitudes of θ_2^I and θ_1^I, together with the existence of the aforementioned subsequence, ensure that buyer I's wealth grows at least exponentially. (This growth of

[15] Although the model of this article is formulated in a particular manner, the conclusions do not rely on the range of trading types. The proof can be modified to provide the same conclusions if all speculators are buying contracts in some time periods and in other time periods selling contracts [see Luo (1995b)]. Furthermore, the results of the model would also hold with continuous rather than discrete futures and spot prices.

wealth comes from a constant redistribution of wealth away from producers and relatively noisy traders. In other words, there is a natural selection of information leading to this wealth redistribution, which in turn causes exponential growth of buyer I's wealth.) However, the total wealth injected into the market and shared by all speculators in the market can only be from producers' losses or new entrants' injections of new wealth. Producers' losses in each time period can be no more than $(\overline{m}-1)d\overline{S}$. The new entrant's injection of wealth is V_0 in each time period. Therefore the total wealth of all speculators in the market grows at most arithmetically. This contradicts the fact that buyer I's wealth grows at least exponentially with a strictly positive probability.

Before proving Theorem 1, an indicator variable for describing whether at time t the futures price is below the spot price is defined.

Definition 1. *Define random variables l_t, where $t = 1, 2, \ldots,$ as*

$$l_t = \begin{cases} 1 & \text{if } p_t^f < p_t \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *With probability 1, the proportion of time that the futures price is below the spot price converges to zero as time goes to infinity. That is,*

$$\Pr\left\{\lim_{T\to\infty}\left(\frac{\#\left\{t \le T : p_t^f < p_t\right\}}{T}\right) = 0\right\} = 1.$$

Proof. Using the l_t notation, Theorem 1 can be restated as

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\sum_{t=1}^{T} l_t}{T}\right) = 0\right) = 1. \tag{2}$$

This is shown by way of contradiction. Suppose Equation (2) is not true, then with a strictly positive probability, there exists an $\epsilon_0 \in (0, \frac{1}{2}]$ and there exists a subsequence $T_1, T_2, \ldots, T_s, \ldots$ such that for any s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \ge 2\epsilon_0$. This together with Proposition 1 in Appendix A implies that

$$\lim_{s\to\infty}\Pr\left(\exists \text{ a buyer } I < s \ s.t. \left\{z_I = 1, (\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0),\ f_I \ge \underline{f}\right\} |G\right) > 0, \tag{3}$$

where $\frac{\overline{m}}{\overline{m}+1} < \underline{f} < 1$, $\Theta(\epsilon_0) = \{(\theta_1, \theta_2) : \theta_2 \ge 1-\epsilon_0, \theta_1 < \frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})}(\theta_2 - (1-\epsilon_0))\}$ and G represents the event that there exists an $\epsilon_0 \in (0, \frac{1}{2}]$ and there exists a subsequence $T_1, T_2, \ldots, T_s, \ldots$ such that for any s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$.

Notice that since short sellers always honor their contracts and since the spot price never goes below d, the wealth of buyer I is always positive. (The importance of buyer I's wealth remaining positive in any time period is that if buyer I's wealth is zero in one time period, then buyer I will not be able to buy futures contracts in any subsequent time period and his or her wealth will be zero in all subsequent time periods.)

Proposition 2 in Appendix A uses the characteristics of buyer I, $(\theta_1^I, \theta_2^I, \theta_3^I, z_I, f_I)$, and uses Equation (3) to show that the probability that buyer I's average wealth over time on a subsequence $T_1, T_2, \ldots, T_s, \ldots$ is unbounded, is bounded away from zero. That is, for any $F > 0$,

$$\lim_{s \to \infty} \Pr\left(\frac{V_{T_s}^I}{T_s} \geq F\right) > 0.$$

However, this contradicts the fact that the average wealth of any individual speculator over time, including buyer I, must be strictly bounded from above by a constant $(\overline{m} - 1)d\overline{S} + V_0$. The reason is as follows. The new wealth at time t, which could be injected into the futures market and which is potentially shared by all speculators including buyer I, must be transferred from the producer or brought in by the new entrant at time t. The wealth which could come from the producers at time t is $(p_t - p_t^f)S_t^f$, which is bounded from above by $(\overline{m} - 1)d\overline{S}$. The wealth that a new entrant brings in is V_0. Therefore at time t the total wealth that could be injected into the futures market and is shared by existing speculators in the market must be bounded from above by $(\overline{m} - 1)d\overline{S} + V_0$. It follows that the aggregate wealth of all speculators in the market at the end of time period T_s, where $s = 1, 2, \ldots$, must be bounded from above by $T_s\left((\overline{m} - 1)d\overline{S} + V_0\right)$. This further implies that the average total wealth of all speculators over time at the end of time period T_s, where $s = 1, 2, \ldots$, must be bounded from above by $(\overline{m} - 1)d\overline{S} + V_0$. Hence, the average wealth of any individual speculator over time, including buyer I, at the end of time period T_s, where $s = 1, 2, \ldots$, must be bounded from above by $(\overline{m} - 1)d\overline{S} + V_0$. ■

The following will show that with probability 1, the proportion of time that the futures price equals the spot price converges to one as time goes to infinity. To show that, given the result in Theorem 1, it needs to be shown that with probability 1, the proportion of time that the futures price is above the spot price converges to zero as time goes to infinity. This is proven in Theorem 2. The proof of Theorem 2 follows exactly the same logic as the proof of Theorem 1.

Theorem 2, like Theorem 1, is shown by way of contradiction. Assuming that the result of Theorem 2 is not true, one can show that with a strictly positive probability, there exists an entering seller with certain characteristics. This seller never exits the economy. Following the same logic, one can show that this seller's average wealth over time is unbounded with a positive probability, which produces a contradiction.

Theorem 2. *With probability 1, the proportion of time that the futures price is above the spot price converges to zero as time goes to infinity. That is,*

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\#\left\{t \le T : p_t^f > p_t\right\}}{T}\right) = 0\right) = 1.$$

Proof. See Appendix C for proof. ■

Finally, using Theorem 1 and Theorem 2, Theorem 3 can be established as follows.

Theorem 3. *With probability 1, the proportion of time that the futures price equals the spot price converges to one as time goes to infinity. That is,*

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\#\left\{t \le T : p_t^f = p_t\right\}}{T}\right) = 1\right) = 1.$$

Proof. Noticing that for any $T \ge 1$,

$$\frac{\#\left\{t \le T : p_t^f < p_t\right\}}{T} + \frac{\#\left\{t \le T : p_t^f = p_t\right\}}{T} + \frac{\#\left\{t \le T : p_t^f > p_t\right\}}{T} = 1,$$

and using Theorem 1 and Theorem 2, Theorem 3 follows. ■

The result of Theorem 3 suggests that in the limit, the futures price fully reflects accurate information in the futures market. In the short run, the futures price reflects all the beliefs of speculators, but in the long run, the futures price reflects only accurate information and eliminates noise.

Furthermore, one can also show that each individual speculator's logarithm of wealth is infinitesimally small relative to the market aggregate of logarithms of wealth. This is formally stated in Corollary 1.

Corollary 1. *With probability 1, the ratio between the logarithm of wealth of speculator t, $t \ge 1$, at the end of time period T ($\ln V_T^t$) and the aggregate of logarithms of wealth of all speculators at the end of time period T ($\sum_{k=1}^{T} \ln V_T^k$) converges to zero as $T \to \infty$. That is,*

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\ln V_T^t}{\sum_{k=1}^{T} \ln V_T^k}\right) = 0\right) = 1.$$

Proof. See Appendix D for the proof. ■

3. Conclusions

This article shows that, due to natural selection, information efficiency can be achieved even if the agents are not assumed to be rational. In the long run, the impact of irrational traders on market prices becomes negligible. This is consistent with the conjecture made by Friedman (1953) and in contrast with the assumptions posited by Fischer and Verrechia (1997) or Kyle and Wang (1997) [who appeal to the psychology literature, e.g., Kahneman, Slovic, and Tversky (1982)]. In this article each speculator is assumed to be randomly endowed with its own trading type, its own fraction of wealth allocated for speculation, and its own inherent distribution of prediction error with respect to the spot price. Those who act upon better predictions accumulate wealth at the expense of trading counterparts acting on less reliable predictions. With ongoing entry of such speculators into the economy over time, the market selection process redistributes wealth constantly among the speculators. Consequently, this constant redistribution process causes the futures price to reflect more accurate predictions with higher and higher weights as time goes by. In the short run, the futures price merely aggregates all noise speculators' beliefs or predictions. In the long run, with probability 1, the proportion of time that the futures price equals the spot price converges to one as time goes to infinity. In the limit, the futures price reflects only accurate information and ignores noise. Furthermore, the logarithm of each individual speculator's wealth is infinitesimally small relative to the market aggregate of logarithms of all speculators' wealth.

Appendix A

Proposition 1. *If, with a strictly positive probability, there exists an* $\epsilon_0 \in (0, \frac{1}{2}]$ *and a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for all* s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$, *then*

$$\lim_{s\to\infty} \Pr\left(\exists \text{ a buyer } I < s \quad s.t. \ \left\{z_t = 1, (\theta_1^t, \theta_2^t) \in \Theta(\epsilon_0),\ f_t \geq \underline{f}\right\} |G \right) > 0,$$

where $\frac{\overline{m}}{\overline{m}+1} < \underline{f} < 1$, $\Theta(\epsilon_0) = \{(\theta_1, \theta_2) : \theta_2 \geq 1 - \epsilon_0, \theta_1 < \frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})} (\theta_2 - (1 - \epsilon_0))\}$, *and* G *represents the event, that there exists an* $\epsilon_0 \in (0, \frac{1}{2}]$ *and a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for all* s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$.

Proof. If, with a strictly positive probability, there exists an $\epsilon_0 \in (0, \frac{1}{2}]$ and a subsequence $T_1, T_2, \ldots, T_s, \ldots$ such that for all s,

$$\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0, \tag{4}$$

then, define a random variable R_t for $t = 1, 2, \ldots,$ as

$$R_t = \begin{cases} 1 & \text{if } z_t = 1, (\theta_1^t, \theta_2^t) \in \Theta(\epsilon_0),\ f_t \geq \underline{f} \\ 0 & \text{otherwise,} \end{cases}$$

where $\frac{\overline{m}}{\overline{m}+1} < \underline{f} < 1$ and $\Theta(\epsilon_0) = \{(\theta_1, \theta_2) : \theta_2 \geq 1 - \epsilon_0, \theta_1 < \frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})}(\theta_2 - (1 - \epsilon_0))\}$. Since the random vectors $(\theta_1^t, \theta_2^t, \theta_3^t)$, (z_t), and (f_t) are mutually independent for a fixed t and the random vector $(\theta_1^t, \theta_2^t, \theta_3^t, z_t, f_t)$ is independently and identically distributed across speculators $t = 1, 2, \ldots,$ it follows that $\{R_t\}_{t \geq 1}$ is independently and identically distributed. Therefore $\Pr(R_t = 1) = \Pr(z_t = 1, (\theta_1^t, \theta_2^t) \in \Theta(\epsilon_0),\ f_t \geq \underline{f}) = \Pr(z_t = 1) \Pr((\theta_1^t, \theta_2^t) \in \Theta(\epsilon_0)) \Pr(f_t \geq \underline{f})$. Notice that since z_t has support at 1, (θ_1^t, θ_2^t) has support over $\Theta(\epsilon_0)$ and f_t has support over the interval $(0, 1]$, it follows that $\Pr(R_t = 1) > 0$, for all t. This further implies that

$$\sum_{t=1}^{\infty} \Pr(R_t = 1) = \infty,$$

and using the Second Borel Cantelli Lemma [see Billingsley (1986, p. 83)],

$$\Pr\left(\sum_{t=1}^{\infty} R_t = \infty\right) = 1.$$

This further implies that

$$\lim_{s \to \infty} \Pr(\text{there exists a buyer, say buyer } I,\ I < s,\ s.t.\ R_I = 1) = 1. \tag{5}$$

Equations (4) and (5) imply that

$$\lim_{s \to \infty} \Pr(\exists \text{ buyer } I, I < s, s.t.\ R_I = 1 \,|\, G) > 0,$$

where G represents the event that there exists an $\epsilon_0 \in (0, \frac{1}{2}]$ and a subsequence $T_1, T_2, \ldots, T_s, \ldots$ such that for all s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$.

Proposition 1 follows. ∎

The following definitions are used in all the following propositions.

Define m_k and a_k, for $k = 1, 2, \ldots,$ as

$$m_k = \begin{cases} 1 \text{ if } p_k^f = p_k \\ 0 \text{ otherwise} \end{cases} ; \; a_k = \begin{cases} 1 \text{ if } p_k^f > p_k \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 2. *If*

$$\lim_{s\to\infty} \Pr\Big(\exists \text{ a buyer } I < s \text{ s.t. } \Big\{z_I = 1, (\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0), \; f_I \geq \underline{f}\Big\} \,|G\Big) > 0, \tag{6}$$

where $\frac{\overline{m}}{\overline{m}+1} < \underline{f} < 1$, $\Theta(\epsilon_0) = \{(\theta_1, \theta_2) : \theta_2 \geq 1 - \epsilon_0, \theta_1 < \frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})}(\theta_2 - (1-\epsilon_0))\}$ *and G represents the event that there exists an* $\epsilon_0 \in (0, \frac{1}{2}]$ *and there exists a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for any* s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$, *then for any* $F > 0$,

$$\lim_{s\to\infty} \Pr\left(\frac{V_{T_s}^I}{T_s} \geq F\right) > 0.$$

Proof. Before the proof begins, some notation is needed. Define another two indicator variables x_k^I and g_k^I, where $k \geq I$, for buyer I as

$$x_k^I = \begin{cases} 1 \text{ if } v_k^I > 0 \\ 0 \text{ otherwise;} \end{cases} \quad g_k^I = \begin{cases} 1 \text{ if } v_k^I = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Consider buyer I. Buyer I's wealth at the end of time period T_s is $V_{T_s}^I$ and

$$\theta_2^I \geq 1 - \epsilon_0 \text{ and } \theta_1^I < \frac{\ln\left(\frac{\underline{f}(\overline{m}+1)}{\overline{m}}\right)}{\ln\left(\frac{\overline{N}+1}{\underline{f}}\right)}\left(\theta_2^I - (1-\epsilon_0)\right). \tag{7}$$

Since the prediction error v_k^I are independently and identically distributed across time (k), x_k^I are independently and identically distributed across time periods (k) and g_k^I are independently and identically distributed across time periods (k), using the strong law of large numbers, it follows that

$$\frac{\sum_{k=I}^{T_s} x_k^I}{T_s - I + 1} \to \theta_1^I \; a.s. \text{ and } \frac{\sum_{k=I}^{T_s} g_k^I}{T_s - I + 1} \to \theta_2^I \; a.s. \text{ and } \frac{\ln V_0}{T_s - I + 1} \to 0.$$

This together with Equation (7) implies that if

$$\lim_{s\to\infty} \Pr\Big(\exists \text{ a buyer } I < s \text{ s.t. } \Big\{z_I = 1, (\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0), \; f_I \geq \underline{f}\Big\} \,|G\Big) > 0,$$

where $\frac{\overline{m}}{\overline{m}+1} < \underline{f} < 1$, $\Theta(\epsilon_0) = \{(\theta_1, \theta_2) : \theta_2 \geq 1 - \epsilon_0, \theta_1 < \frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})}(\theta_2 - (1-\epsilon_0))\}$ and G represents the event that there exists an $\epsilon_0 \in (0, \frac{1}{2}]$ and there exists a subsequence $T_1, T_2, \ldots, T_s, \ldots$ such that for any s, $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$, then for some positive $\lambda > 0$, for $E = (\exists \text{ a buyer } I < s \ s.t.\ \{z_I = 1, (\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0), \quad f_I \geq \underline{f}\} \cap G)$,

$$\lim_{s\to\infty} \Pr\left(\left(\frac{\sum_{k=I}^{T_s} g_k^I}{T_s - I + 1} - (1-\epsilon_0)\right)\ln\left(\frac{\underline{f}(\overline{m}+1)}{\overline{m}}\right) + \frac{\ln V_0}{T_s - I + 1} + \frac{\sum_{k=I}^{T_s} x_k^I}{T_s - I + 1}\ln\left(\frac{\underline{f}}{\overline{N}+1}\right) \geq \lambda \,\middle|\, E\right) = 1. \tag{8}$$

Now, notice that

$$\sum_{k=I}^{T_s} x_k^I \geq \sum_{k=I}^{T_s}\left(a_k x_k^I\right). \tag{9}$$

In addition, since

$$\begin{aligned}
\sum_{k=I}^{T_s}\left(l_k g_k^I\right) &= \sum_{k=I}^{T_s} g_k^I - \sum_{k=I}^{T_s}\left(m_k g_k^I\right) - \sum_{k=I}^{T_s}\left(a_k g_k^I\right) \\
&\geq \sum_{k=I}^{T_s} g_k^I - \sum_{k=I}^{T_s} m_k - \sum_{k=I}^{T_s} a_k \\
&= \sum_{k=I}^{T_s} g_k^I - (T_s - I + 1) + \sum_{k=I}^{T_s} l_k \\
&= \left(\frac{\sum_{k=I}^{T_s} g_k^I}{T_s - I + 1} + \frac{\sum_{k=I}^{T_s} l_k}{T_s - I + 1} - 1\right)(T_s - I + 1),
\end{aligned}$$

if $\frac{\sum_{k=1}^{T_s} l_k}{T_s} \geq 2\epsilon_0$ then it follows that $\frac{\sum_{k=I}^{T_s} l_k}{T_s - I + 1} = (\frac{\sum_{k=1}^{T_s} l_k}{T_s} - \frac{\sum_{k=1}^{I-1} l_k}{T_s})(\frac{T_s}{T_s - I + 1}) > (2\epsilon_0 - \frac{\sum_{k=1}^{I-1} l_k}{T_s})(\frac{T_s}{T_s - I + 1}) > \epsilon_0$ (for a sufficiently large s). Therefore for a sufficiently large s,

$$\sum_{k=I}^{T_s} \left(l_k g_k^I\right) \geq \left(\frac{\sum_{k=I}^{T_s} g_k^I}{T_s - I + 1} + \epsilon_0 - 1 \right) (T_s - I + 1). \tag{10}$$

Notice that $\ln(\frac{f}{\overline{N}+1}) < 0$ and $\ln(\frac{f(\overline{m}+1)}{\overline{m}}) > 0$. Hence, Equations (8), (9), and (10) imply that

$$\lim_{s\to\infty} \Pr \left(\frac{\sum_{k=I}^{T_s} (a_k x_k^I)}{T_s - I + 1} \ln\left(\frac{f}{\overline{N}+1}\right) + \frac{\sum_{k=I}^{T_s} (l_k g_k^I)}{T_s - I + 1} \ln\left(\frac{f(\overline{m}+1)}{\overline{m}}\right) \right.$$

$$\left. + \frac{\ln V_0}{T_s - I + 1} \geq \lambda \,\middle|\, E \right) = 1.$$

This together with Equation (6) imply that

$$\lim_{s\to\infty} \Pr \left(\frac{\sum_{k=I}^{T_s} (a_k x_k^I)}{T_s - I + 1} \ln\left(\frac{f}{\overline{N}+1}\right) + \frac{\sum_{k=I}^{T_s} (l_k g_k^I)}{T_s - I + 1} \ln\left(\frac{f(\overline{m}+1)}{\overline{m}}\right) \right.$$

$$\left. + \frac{\ln V_0}{T_s - I + 1} \geq \lambda \right) > 0. \tag{11}$$

Now, the following uses Equation (11) to show that for any $F > 0$,

$$\lim_{s\to\infty} \Pr \left(\frac{V_{T_s}^I}{T_s} > F \right) > 0.$$

Consider buyer I's wealth at time period k. If $a_k = 1$ and $x_k^I = 1$ then $p_k^f > p_k$ and $b_k^I > p_k$, which further implies that $q_k^I \leq \frac{f_I V_{k-1}^I}{p_k^f}$. Hence buyer I potentially makes a loss (a negative profit) at time period k. At the end of time period k buyer I's wealth is

$$\begin{aligned} V_k^I &= V_{k-1}^I + q_k^I \left(p_k - p_k^f\right) \geq V_{k-1}^I \left(1 + \left(p_k - p_k^f\right) \frac{f_I}{p_k^f}\right) \\ &\geq \left(\frac{p_k}{p_k^f}\right) f_I V_{k-1}^I \geq \left(\frac{\underline{f}}{\overline{N}+1}\right) V_{k-1}^I. \end{aligned} \tag{12}$$

If $l_k = 1$ and $g_k^I = 1$ then $p_k^f < p_k$ and $b_k^I = p_k$, which further implies that $q_k^I \in (\frac{f_I V_{k-1}^I}{p_k^f + d}, \frac{f_I V_{k-1}^I}{p_k^f}]$. Hence buyer I makes a gain (a positive profit) at time period k. At the end of time period k buyer I's wealth is

$$\begin{aligned} V_k^I &= V_{k-1}^I + \left(p_k - p_k^f\right) q_k^I > V_{k-1}^I + \left(p_k - p_k^f\right) \frac{f_I V_{k-1}^I}{p_k^f + d} \\ &\geq V_{k-1}^I + d \left(\frac{f_I V_{k-1}^I}{\overline{m} d}\right) > \underline{f} V_{k-1}^I + d \left(\frac{\underline{f} V_{k-1}^I}{\overline{m} d}\right) \\ &= \left(\frac{\underline{f}\left(\overline{m}+1\right)}{\overline{m}}\right) V_{k-1}^I. \end{aligned} \tag{13}$$

Therefore buyer I's wealth at the end of time period k is

$$V_k^I > \left(\frac{\underline{f}}{\overline{N}+1}\right)^{a_k x_k^I} \left(\frac{\underline{f}\left(\overline{m}+1\right)}{\overline{m}}\right)^{l_k g_k^I} V_{k-1}^I. \tag{14}$$

Using Equation (14), one can show inductively that

$$V_{T_s}^I > \left(\frac{\underline{f}}{\overline{N}+1}\right)^{\sum_{k=I}^{T_s}\left(a_k x_k^I\right)} \left(\frac{\underline{f}\left(\overline{m}+1\right)}{\overline{m}}\right)^{\sum_{k=I}^{T_s}\left(l_k g_k^I\right)} V_0. \tag{15}$$

Taking the logarithm of both sides of Equation (15), it follows that

$$\ln V_{T_s}^I > \left\{ \sum_{k=I}^{T_s} \left(a_k x_k^I\right) \ln\left(\frac{\underline{f}}{\overline{N}+1}\right) + \sum_{k=I}^{T_s} \left(l_k g_k^I\right) \ln\left(\frac{\underline{f}\left(\overline{m}+1\right)}{\overline{m}}\right) + \ln V_0 \right\},$$

which further implies that

$$\begin{aligned} \frac{\ln V_{T_s}^I}{T_s - I + 1} > \frac{1}{T_s - I + 1} &\left\{ \sum_{k=I}^{T_s} \left(a_k x_k^I\right) \ln\left(\frac{\underline{f}}{\overline{N}+1}\right) \right. \\ &\left. + \sum_{k=I}^{T_s} \left(l_k g_k^I\right) \ln\left(\frac{\underline{f}\left(\overline{m}+1\right)}{\overline{m}}\right) + \ln V_0 \right\}. \end{aligned}$$

Together with Equation (11), this implies that

$$\lim_{s\to\infty} \Pr\left(\frac{\ln V_{T_s}^I}{T_s - I + 1} > \lambda\right) > 0.$$

Therefore

$$\lim_{s\to\infty} \Pr\left(V_{T_s}^I > \exp\left((T_s - I + 1)\,\lambda\right)\right) > 0.$$

Since $\lim_{s\to\infty} \frac{\exp((T_s - I + 1)\lambda)}{T_s} = \lim_{s\to\infty} \{\lambda \exp\left((T_s - I + 1)\,\lambda\right)\} = \infty$, it follows that for any $F > 0$,

$$\lim_{s\to\infty} \Pr\left(\frac{V_{T_s}^I}{T_s} > F\right) > 0. \qquad \blacksquare$$

Appendix B

Proposition 3. *If with a strictly positive probability, there exists an* $\epsilon_0' \in (0, \frac{1}{2}]$ *and a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for all* s, $\frac{\sum_{k=1}^{T_s} a_k}{T_s} \geq 2\epsilon_0'$, *then*

$$\lim_{s\to\infty} \Pr\left(\exists \text{ a seller } I',\ I' < s,\ s.t.\{\, z_{I'} = -1,\ (\theta_2^{I'}, \theta_3^{I'}) \in \Theta'(\epsilon_0'),\ f_{I'} \leq \overline{f}\} \,\middle|\, G'\right) > 0,$$

where $0 < \overline{f} < 1$, $\Theta'(\epsilon_0') = \{(\theta_2, \theta_3) : \theta_2 \geq 1 - \epsilon_0', \theta_3 < \frac{\ln(\frac{f_{I'}}{\frac{2}{m} - 1} + 1)}{\ln(\frac{1}{1-\overline{f}})} (\theta_2 - (1 - \epsilon_0'))\}$ *and* G' *represents the event, that there exists an* $\epsilon_0' \in (0, \frac{1}{2}]$ *and a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for all* s, $\frac{\sum_{k=1}^{T_s} a_k}{T_s} \geq 2\epsilon_0'$.

Proof. This proposition uses exactly the same logic as Proposition 1. For the ease of notation and description of the proof, redefine R_t as follows. Define a random variable R_t', where $t = 1, 2, \ldots,$ as

$$R_t' = \begin{cases} 1 & \text{if } z_t = -1,\ (\theta_2^t, \theta_3^t) \in \Theta'(\epsilon_0'),\ f_t \leq \overline{f} \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \overline{f} < 1$ and $\Theta'(\epsilon_0') = \{(\theta_2, \theta_3) : \theta_2 \geq 1 - \epsilon_0', \theta_3 < \frac{\ln(\frac{f_{I'}}{\frac{2}{m} - 1} + 1)}{\ln(\frac{1}{1-\overline{f}})} (\theta_2 - (1 - \epsilon_0'))\}$. The proof can be done by following through the proof of Proposition 1 after changing some notation. In the proof of Proposition 1, replace l_k with a_k and replace R_t with R_t'. As a result, replace $z_t = 1$, $(\theta_1^t, \theta_2^t) \in \Theta(\epsilon_0)$, and $f_t \geq \underline{f}$ with $z_t = -1$, $(\theta_2^t, \theta_3^t) \in \Theta'(\epsilon_0')$, and $f_t \leq \overline{f}$, respectively. Replace buyer I with seller I. Replace I with I' and replace G with G'. Replace z_t has support at 1 with z_t has support at -1. ∎

Proposition 4. *If*

$$\lim_{s\to\infty} \Pr\left(\exists \text{ a seller } I' < s \text{ s.t. } \left\{z_{I'} = -1, (\theta_2^{I'}, \theta_3^{I'}) \in \Theta'(\epsilon_0'),\ f_{I'} \geq \overline{f}\right\} \middle| G'\right) > 0, \tag{16}$$

where $0 < \overline{f} < 1$, $\Theta'(\epsilon_0') = \{(\theta_2, \theta_3) : \theta_2 \geq 1-\epsilon_0', \theta_3 < \frac{\ln(\frac{f_{I'}}{\overline{m}^2-1}+1)}{\ln(\frac{1}{1-\overline{f}})}(\theta_2-(1-\epsilon_0'))\}$ *and* G' *represents the event that there exists an* $\epsilon_0' \in (0, \frac{1}{2}]$ *and there exists a subsequence* $T_1, T_2, \ldots, T_s, \ldots$ *such that for any* s, $\frac{\sum_{k=1}^{T_s} a_k}{T_s} \geq 2\epsilon_0'$, *then for any* $F > 0$,

$$\lim_{s\to\infty} \Pr\left(\frac{V_{T_s}^{I'}}{T_s} \geq F\right) > 0.$$

Proof. The proof is identical to the proof of Proposition 2 of Appendix A with the following changes. Replace buyer I with seller I', replace I with I'. Inside the definition of x_k^I, replace $\nu_k^I > 0$ with $\nu_k^{I'} < 0$, and replace x_k^I with $x_k^{I'}$. Replace θ_2^I and θ_1^I with $\theta_2^{I'}$ and $\theta_3^{I'}$, respectively. Replace $\frac{\ln(\frac{\underline{f}(\overline{m}+1)}{\overline{m}})}{\ln(\frac{\overline{N}+1}{\underline{f}})}$ with $\frac{\ln(\frac{f_{I'}}{\overline{m}^2-1}+1)}{\ln(\frac{1}{1-\overline{f}})}$ and replace ϵ_0 with ϵ_0'. Replace $z_I = 1, (\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0)$, $f_I \geq \underline{f}$, and G with $z_{I'} = -1, (\theta_2^{I'}, \theta_3^{I'}) \in \Theta'(\epsilon_0')$, $f_{I'} \leq \overline{f}$, and G', respectively. Replace the definition of $\underline{f}$, $\Theta(\epsilon_0)$, and G with the definition of $\overline{f}$, $\Theta'(\epsilon_0')$, and G' which are defined in Proposition 3 of Appendix B. Furthermore, $\frac{\underline{f}}{\overline{N}+1}$ is replaced with $1 - \overline{f}$ and $\frac{\underline{f}(\overline{m}+1)}{\overline{m}}$ is replaced with $\frac{f_{I'}}{\overline{m}^2-1} + 1$. l_k are replaced with a_k and a_k is replaced with l_k.

The paragraph starts with the sentence "Consider buyer I''s wealth at time period k" and ending with Equation (13) is replaced with the following:

Now, if $l_k = 1$ and $x_k^{I'} = 1$ then $p_k^f < p_k$ and $b_k^{I'} < p_k$, which further implies that $q_k^{I'} \geq -\frac{1}{(\overline{m}-1)} \frac{f_{I'} V_{k-1}^{I'}}{p_k^f}$. Hence, short seller I' potentially makes a loss (a negative profit) at time period k. At the end of time period k the wealth of short-seller I' is

$$V_k^{I'} = V_{k-1}^{I'} + q_k^{I'}\left(p_k - p_k^f\right) \geq \left(1 + \left[\frac{f_{I'}}{\overline{m}-1} - \frac{p_k}{p_k^f}\frac{f_{I'}}{\overline{m}-1}\right]\right) V_{k-1}^{I'}$$

$$\geq \left(1+\left[\frac{f_{I'}}{\overline{m}-1}-\left(\frac{\overline{m}d}{d}\right)\frac{f_{I'}}{\overline{m}-1}\right]\right)V_{k-1}^{I'}$$
$$= (1-f_{I'})V_{k-1}^{I'} \geq (1-\overline{f})V_{k-1}^{I'}.$$

If $a_k = 1$ and $g_k^{I'} = 1$ then $p_k^f > p_k$ and $b_k^{I'} = p_k$, which further implies that $q_k^{I'} = -\frac{1}{(\overline{m}-1)}[\frac{f_{I'}V_{k-1}^{I'}}{p_k^f}, \frac{f_{I'}V_{k-1}^{I'}}{p_k^f+d})$. Hence, short-seller I' makes a gain (a positive profit) at time period k. At the end of time period k, the wealth of short-seller I' is

$$V_k^{I'} = V_{k-1}^{I'} + \left(p_k - p_k^f\right)q_k^{I'} \geq \left(1+\left[\frac{f_{I'}}{\overline{m}-1}-\frac{p_k}{p_k^f}\frac{f_{I'}}{\overline{m}-1}\right]\right)V_{k-1}^{I'}$$
$$\geq \left(1+\left[\frac{f_{I'}}{\overline{m}-1}-\left[\frac{\overline{m}}{\overline{m}+1}\right]\frac{f_{I'}}{\overline{m}-1}\right]\right)V_{k-1}^{I'}$$
$$= \left(\frac{f_{I'}}{\overline{m}^2-1}+1\right)V_{k-1}^{I'}. \qquad \blacksquare$$

Appendix C

The Proof of Theorem 2. The proof is identical to the proof of Theorem 1 with the following changes: l_t and l_k are replaced with a_t and a_k, respectively. Replace ϵ_0 with ϵ_0'. Replace "Proposition 1 in Appendix A" with "by Proposition 3 in Appendix B" and as a result, replace buyer I with seller I', replace I with I', and replace $z_I = 1$, $(\theta_1^I, \theta_2^I) \in \Theta(\epsilon_0)$, $f_I \geq \underline{f}$, and G with $z_{I'} = -1$, $(\theta_2^{I'}, \theta_3^{I'}) \in \Theta'(\epsilon_0')$, $f_{I'} \leq \overline{f}$, and G', respectively. Replace the definition of $\underline{f}$, $\Theta(\epsilon_0)$, and G with the definition of $\overline{f}$, $\Theta'(\epsilon_0')$, and G' which are defined in Proposition 3 of Appendix B. Replace $\left(\theta_1^I, \theta_2^I, \theta_3^I, z_I, f_I\right)$ with $\left(\theta_1^{I'}, \theta_2^{I'}, \theta_3^{I'}, z_{I'}, f_{I'}\right)$.

Replace the paragraph beginning with "Notice that since . . . " with "Notice that since the fraction of wealth of short-seller I' spent on speculation is strictly less than 1, the wealth of short-seller I' is always positive. (The importance of short-seller I''s wealth remaining positive in any time period, is that if short-seller I''s wealth is zero in one time period, then short-seller I' will not be able to sell futures contracts in any subsequent time period and his or her wealth will be zero in all subsequent time periods.)" Replace the words "Proposition 2 in Appendix A" with "Proposition 4 in Appendix B". $\blacksquare$

Appendix D

The Proof of Corollary 1. If for $t \geq 1$,

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\ln V_T^t}{T}\right) = 0\right) = 1 \tag{17}$$

and if there exists a $u < +\infty$ such that

$$\Pr\left(\lim_{T\to\infty}\sup\left(\frac{\sum_{k=1}^{T}\ln V_T^k}{T}\right) < u\right) = 1, \tag{18}$$

then the result in Corollary 1 directly follows from the above equations. Now the following will show that Equations (17) and (18) are true.

The Proof of Equation (17). Consider speculator t's wealth at the end of time period k (i.e., V_k^t).

If $m_k = 1$, then $p_k^f = p_k$. Hence,

$$V_k^t = V_{k-1}^t. \tag{19}$$

If $a_k = 1$, then $p_k^f > p_k$. Since $V_k^t = V_{k-1}^t + q_k^t(p_k - p_k^f)$, using buyer t's or seller t's demand schedule, one can show that

$$\frac{p_k}{p_k^f}V_{k-1}^t < V_k^t < \left(1 + \frac{1}{\overline{m}-1}\, f_t\left(1 - \frac{p_k}{p_k^f}\right)\right)V_{k-1}^t.$$

This further implies that

$$\frac{1}{\overline{N}+1}V_{k-1}^t < V_k^t < \left(1 + \frac{1}{\overline{m}-1}\right)V_{k-1}^t. \tag{20}$$

If $l_k = 1$, then $p_k^f < p_k$. Since $V_k^t = V_{k-1}^t + q_k^t(p_k - p_k^f)$, using buyer t's or seller t's demand schedule, one can show that

$$(1 - f_t)\,V_{k-1}^t < V_k^t < \left(1 + \frac{p_k}{p_k^f}\right)V_{k-1}^t.$$

This further implies that

$$(1 - f_t)\,V_{k-1}^t < V_k^t < (1 + \overline{m})V_{k-1}^t. \tag{21}$$

Therefore Equations (19), (20), and (21) imply that at time k,

$$(1 - f_t)^{l_k}\left(\frac{1}{\overline{N}+1}\right)^{a_k}V_{k-1}^t < V_k^t < \left(1 + \frac{1}{\overline{m}-1}\right)^{a_k}(1+\overline{m})^{l_k}\,V_{k-1}^t. \tag{22}$$

Using Equation (22), one can show inductively that

$$(1-f_t)^{\sum_{k=t}^{T} l_k}\left(\frac{1}{\overline{\overline{N}}+1}\right)^{\sum_{k=t}^{T} a_k} V_0 < V_T^t < \left(1+\frac{1}{\overline{m}-1}\right)^{\sum_{k=t}^{T} a_k}(1+\overline{m})^{\sum_{k=t}^{T} l_k} V_0.$$

Since $f_t < 1$, $(1-f_t) > 0$, the above equation further implies that

$$\begin{aligned}&\frac{\sum_{k=t}^{T} l_k}{T}\ln(1-f_t) + \frac{\sum_{k=t}^{T} a_k}{T}\ln\left(\frac{1}{\overline{\overline{N}}+1}\right) + \frac{\ln V_0}{T}\\ &\quad < \frac{\ln V_T^t}{T} < \frac{\sum_{k=t}^{T} a_k}{T}\ln\left(1+\frac{1}{\overline{m}-1}\right) + \frac{\sum_{k=t}^{T} l_k}{T}\ln(1+\overline{m}) + \frac{\ln V_0}{T}.\end{aligned}$$

Since $\lim_{T\to\infty}(\frac{\ln V_0}{T}) = 0$, using Theorem 1 and Theorem 2, the above equation implies that

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\ln V_T^t}{T}\right) = 0\right) = 1.$$

Proof of Equation (18). Consider the aggregate wealth of all speculators at the end of time period T, ($i.e.$, $\sum_{k=1}^{T} V_T^k$). The aggregate wealth either comes from the initial wealth each speculator brings into the market or the gains or losses all speculators have made from the producers up to the end of time period T. That is,

$$\frac{\sum_{k=1}^{T} V_T^k}{T} = \frac{T V_0 + \sum_{k=1}^{T}\left(S_k^f(p_k - p_k^f)\right)}{T}.$$

Since $l_k + m_k + a_k = 1$, the above equation implies that

$$\begin{aligned}\frac{\sum_{k=1}^{T} V_T^k}{T} = V_0 &+ \frac{\sum_{k=1}^{T}\left(S_k^f(p_k-p_k^f)l_k\right)}{T} + \frac{\sum_{k=1}^{T}\left(S_k^f(p_k-p_k^f)m_k\right)}{T}\\ &+ \frac{\sum_{k=1}^{T}\left(S_k^f(p_k-p_k^f)a_k\right)}{T}.\end{aligned} \tag{23}$$

If $l_k = 1$, then $p_k^f < p_k$; which further implies that

$$0 \le S_k^f(p_k - p_k^f) \le \overline{S}(\overline{m}d - d). \tag{24}$$

If $m_k = 1$, then $p_k^f = p_k$; hence

$$S_k^f(p_k - p_k^f) = 0. \tag{25}$$

If $a_k = 1$, then $p_k^f > p_k$; hence

$$\overline{S}(d - \overline{m}d) \leq S_k^f(p_k - p_k^f) \leq 0. \tag{26}$$

Therefore, using Equations (24), (25), and (26), Equation (23) implies that

$$V_0 + \overline{S}(d - \overline{m}d)\frac{\sum_{k=1}^{T} a_k}{T} \leq \frac{\sum_{k=1}^{T} V_T^k}{T} \leq V_0 + \overline{S}(\overline{m}d - d)\frac{\sum_{k=1}^{T} l_k}{T}.$$

It follows that

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\sum_{k=1}^{T} V_T^k}{T}\right) = V_0\right) = 1. \tag{27}$$

Since one can always redenominate wealth in units such that $V_0 < 1$, without loss of generality, assume that $V_0 < 1$. Since

$$\frac{\sum_{k=1}^{T} \ln V_T^k}{T} \leq \ln\left(\frac{\sum_{k=1}^{T} V_T^k}{T}\right),$$

using Equation (27), Equation (18) follows, where $u = \ln V_0$. ■

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