Stock Price Volatility in a Multiple Security Overlapping Generations Model

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A number of empirical studies have reached the conclusion that stock price volatility cannot be fully explained within the standard dividend discount model. This article proposes a resolution based upon a model that contains both a random supply of risky assets and finitely lived agents who trade in a multiple security environment. As the analysis shows there exist 2^K equilibria when K securities trade. The low volatility equilibria have properties analogous to those found in the infinitely lived agent models of Campbell and Kyle (1991) and Wang (1993, 1994). In contrast, the high-volatility equilibria have very different characteristics. Within the high-volatility equilibria very large price variances can be generated with very small supply shocks. Adding securities to the economy further reduces the required supply shocks. Using previously established empirical results the model can reconcile the data with supply shocks that are less than 10% as large as observed return shocks. These results are shown to hold even when the dividend process is mean reverting.

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Articles by Shiller (1981a), West (1988), and Mankiw, Romer, and Shapiro (1985, 1991) have shown that the standard dividend discount model cannot explain the observed price to dividend volatility ratio in the aggregate stock market index. These findings are apparently robust to both the empirical technique used and the assumed underlying dividend process. Campbell and Kyle (CK) (1993) suggest that noise traders may provide the answer. Noise trader demand adds volatility to the price beyond that induced by dividends, which helps the model explain the empirical findings. Nevertheless, they find that rationalizing the data is not easy. Estimates of their most general model indicate that, relative to changes in fundamentals, stock prices are either 5.77 or 18.51 times too volatile (depending on whether one fixes or estimates the interest rate). Clearly if the noise traders have sufficiently volatile demands then their presence can explain these results. However, this leaves open the question as to whether other models will also fit the data, and if these models can do so with smaller noise shocks. As Wang (1993, 1994) shows, introducing asymmetric information helps in this regard. Still the volatility of the market remains proportional to the amount of noise trading. More noise trading implies more volatility which means that potentially large amounts of system noise may be needed to explain the data.

The model presented in this article seeks to explain the observed dividend and stock price volatilities in a model with (i) homogeneous information, (ii) rational agents, and (iii) as little unobservable noise as possible. At the very least a model with these three elements seems like an appropriate place to start when searching for an explanation of the price-dividend volatility paradox. The first two elements constitute the traditional basis for most financial models, and are the assumptions that lead to the conclusion that stock prices are too volatile relative to dividends in the first place.[1] The third element allows some deviation from the traditional paradigm, but in a way that seems empirically relevant. Most people are probably willing to accept the idea that security supplies vary unpredictably over time. On the other hand, it seems unlikely that they vary by a great deal from period to period. Thus if a model wishes to use random supplies as an explanation for some phenomenon, it seems incumbent upon it to do so with supply shocks that have a relatively small variance. This model also allows one to see how much of the stock price volatility can be explained within a fairly traditional framework that includes small supply shocks.

[1] One might also wish to start with a model that contains homogeneously informed agents since, as Subrahmanyam (1991) indicates, one expects asymmetric information to play a diminished role when pricing aggregate indices. To the degree that asymmetric information arrives in the form of idiosyncratic shocks they will tend to cancel out when pricing the aggregate index.

With this base, one can then see how much of the remaining volatility should be attributed to asymmetric information, irrational agents, or other market imperfections.

The model yields three primary findings. First, their exist two equilibria per security each with very different volatility characteristics. The low-volatility equilibria have traits similar to those found in models with infinitely lived agents such as CK and Wang (1993, 1994). However, the high-volatility equilibria generally have the opposite comparative statics. As an example, increasing the variance of the market's supply noise increases the price variance in the low-volatility equilibrium but decreases it in the high-volatility equilibrium. Thus the model offers the hope that one can explain the apparently large discrepancies between dividend and price variances without resorting to large fluctuations in unobservable quantities. Second, increasing the number of securities in the economy will generally reduce the aggregate supply volatility needed to explain any particular level of price volatility. This reduction can be rather substantial. Roughly, if there are K independent risky securities in the economy then the standard deviation of the aggregate supply noise needs to be only $1/K$ as large as in a single security model.

The third insight derived from the article is that different securities can trade in environments with different volatility characteristics. Roughly the public has beliefs over a set of mutual funds. These funds are formed via a singular decomposition of a matrix derived from a combination of the variance-covariance matrices governing individual stock payoffs. Depending upon these beliefs different stocks will show varying levels of volatility.

The fact that overlapping generations models can produce multiple equilibria, with varying levels of volatility, goes back to Azariadis (1981). He examines a production economy where the young use labor to produce a consumption good desired only by the old. The analysis shows that the introduction of a random coordinating variable can result in price uncertainty for the consumption good. However, the results derive from the inability of investors to store the consumption good over time. An interest-bearing storage technology (a riskless bond in most financial settings) eliminates the extraneous market volatility, since the young will invest, rather than sell, their output if prices are below average. In contrast the model presented in this article produces "excess" stock market volatility within a more traditional financial framework. Production occurs via the corporate sector, and investors can store the consumption good via a riskless bond if they choose to do so.

In order to find overlapping generations models that include an interest-bearing storage technology and produce stock prices that will violate Shiller's inequality bounds, one must turn to the more recent models of price bubbles. Tirole (1985) and Jackson (1994) show that a price bubble can exist if traders believe that it will grow at the rate of interest. (Otherwise traders will refuse to hold the bubble.) Thus if bubbles fully explain stock price volatility, then over time the dividend-price ratio should go toward zero. However, it is not apparent that this trend exists within the long-run data. So while there may exist periods of time during which speculative bubbles arise, other explanations may still prove useful. This article helps to fill this gap by introducing volatility shocks that arise from the reluctance of risk-averse traders to absorb supply shocks. As a result returns do not need to include a speculative return component, and thus the model does not necessarily predict any long-term trends will exist in the dividend-price ratio data.[2]

Among recent overlapping generations models those by De Long et al. (1990) and Bhushan, Brown, and Mello (1997) are the closest in spirit to the current work. All three models use an overlapping generations framework, in which agents are born, invest, realize their returns, consume, and then die. They also share the standard market microstructure assumptions that all agents have exponential utilities and risky assets have normally distributed payoffs. However, the articles differ in terms of the number of securities individuals may trade and the modeling of each agent's behavior. In both De Long et al. (1990) and Bhushan, Brown, and Mello (1997) the economy contains one risky and one risk-free security. Traders are either rational or "irrational" in that they misinterpret a signal that they receive about the stock's future value. De Long et al. (1990) show that these demand functions can induce prices to exhibit more volatility than a rational expectations model might otherwise predict. However, Bhushan, Brown, and Mello (1997) show that the existence and properties of the volatile equilibrium in De Long et al. (1990) depend rather critically on the assumptions regarding how the irrational investors behave. In contrast, this article considers an economy with a number of risky securities, and assumes that every agent maximizes a well-defined utility function and correctly carries out any calculation.

In addition to the previously cited infinite horizon models there exists another literature that examines the issue of stock market volatility in a finite horizon setting. Romer (1993) shows that if some traders

[2] Another difference between the current model and the price bubble literature can be found in the assumptions needed to produce their respective results. For a bubble to grow at the rate of interest the overall economy must grow fast enough to support this as a rational belief. In contrast, the volatility in the current model can occur within an economy that does not grow, or even shrinks. Again these differences arise from the source of the results as explained above.

are unsure about the quality of information held by others small fundamental shocks can lead to large price changes. Eden and Jovanovic (1994) use asymmetric information about future production to generate their results. A recent article by Kraus and Smith (1994) shows that disparate beliefs are sufficient to induce stock price volatility without fundamental value volatility. Allen and Gale (1994) show that restrictions on borrowing and costly market participation can leave the market so illiquid that small demand shocks will generate large price swings. Allen and Gorton (1993) show that agency issues between investors and mutual fund managers can also lead to potentially high volatility levels within a finite horizon model. While these articles highlight factors that may influence stock price movements, testing them will require the development of new datasets that include the requisite information. Furthermore, while these models clearly produce short-run volatility it is not clear what long-run volatility patterns they will produce. For example, if they apply to events over 1 week then yearly data will reflect the sum of these events. Since agents begin and end each week with homogeneous information, annual observations can only contain excess volatility to the extent that the final week contains excess volatility. The other 51 events must enter the annual price series as standard white noise news induced price shocks since they become common knowledge between observations. By contrast, the current approach is designed to provide an explanation for long-run data that can be calibrated against previous empirical results and tested against readily available datasets.[3]

This article is organized as follows. Section 1 presents the formal model. Section 2 provides empirical implications. Section 3 generalizes the dividend and supply processes to allow for mean reversion and discusses the impact on the article's primary results. This section of the article also relates the model to more recent empirical findings. Section 4 concludes.

1. Mathematical Model with Random Walk Processes

1.1 Agents and assets

The economy contains $K+1$ assets. Asset $K+1$ represents a riskless bond that pays r units per period. The price of the bond constitutes the numeraire for the economy and thus always sells for a price of 1. The remaining K assets represent stocks. In period t the stocks pay a

[3] In some studies observations are set 10 years apart. For example, see Shiller (1981b).

dividend $D_t = \{D_{1t}, \ldots, D_{Kt}\}'$ of

$$D_t = D_{t-1} + \delta_t \tag{1}$$

where $\delta_t = \{\delta_{1t}, \ldots, \delta_{Kt}\}'$ equals a period t shock. For simplicity, this formulation assumes that dividends follow a random walk. Section 3 will explore what happens under more general conditions. The δ_t shocks are normally distributed with means of zero, and a variance-covariance matrix of Σ_δ.

As in most market microstructure models, the supply of stock $N_t = \{N_{1t}, \ldots, N_{Kt}\}$ varies over time according the following equation,[4]

$$N_t = N_{t-1} + \eta_t. \tag{2}$$

The $\eta_t = \{\eta_{1t}, \ldots, \eta_{Kt}\}'$ represent normally distributed shocks with means of zero and variance-covariance matrix Σ_η.[5] There are several ways one can interpret Equation (2). For example, the capital base of the economy may change unexpectedly as the population creates and destroys assets. Whatever the cause, the resulting analysis will remain basically unchanged, so long as market clearing requires demand to equal the supply given by Equation (2).

While stocks in the economy live on forever, the individual investors do not. An investor born in period t comes endowed with shares of the bond and a personal allocation of the period t supply shock η_t. Since the initial endowments do not impact the equilibrium, assume for simplicity that it is spread equally among the newly born population.

Agents live for two periods.[6] An agent born in period t can buy and sell securities in period t; there are no short-sale constraints so negative positions are possible. The model assumes that there exists an atomless continuum of investors with unit mass. This assumption implies that each trader acts as a price taker, and in fact no individual can move prices. After trade ends, the economy moves into period $t+1$. At this point in time agents receive their period $t+1$ payoff from the securities they own. Next the agents sell all of their securities and consume the single consumption good. Utility derives from an

[4] Among the many other articles that analyze economies with a random stock supply see Bray (1981), Diamond and Verrecchia (1981), Glosten (1989), Bhattacharya and Spiegel (1991), Paul (1995), and Wang (1994).

[5] As with the dividend process, Section 3 considers what happens under more general processes for the supply shocks.

[6] None of the model's qualitative results depend upon the assumption that traders live for only two periods. High- and low-volatility equilibria exist even when individuals live for one unit of time and can trade continuously. The only essential element when passing from discrete to continuous time is that traders face a constant amount of volatility over their lives. See De Long et al. (1990) for a more detailed discussion of this issue.

exponential utility function over final period consumption, with risk aversion parameter θ.

1.2 Trading and timing of the dividends

At the start of period t all shares pay D_t and bonds r to their owners. After these payouts, trading takes place. Designate $X_t(i) = \{X_{1t}(i), \ldots, X_{Kt}(i)\}'$ as the period t demands of individual i, and $P_t = \{P_{1t}, \ldots, P_{Kt}\}'$ as the market clearing prices. At the beginning of period $t+1$ the trader receives dividends from his stocks and interest from his bonds, he then sells off his stock portfolio to fund consumption. Thus i's final wealth equals

$$W_{t+1}(i) = X_t(i)'(P_{t+1} + D_{t+1}) + (1+r)b_t(i) \tag{3}$$

where $b_t(i)$ represents his position in the bond. To conserve on notation, the article suppresses the variable i from here on in.

Equation (3) presents the terminal (period $t+1$) wealth of an investor born in period t. This investor's primary trading decision takes place in period t, when he faces the budget constraint

$$X_t'P_t + b_t = W_t, \tag{4}$$

where W_t equals the value of the trader's initial endowment.

Plugging the budget constraint of Equation (4) into Equation (3) allows one to rewrite W_{t+1} as

$$W_{t+1} = X_t'(P_{t+1} - P_t + D_{t+1} - rP_t) + (1+r)W_t. \tag{5}$$

Thus each trader born in period t seeks to maximize his expected utility over X_t given the terminal wealth relationship in the above equation.

1.3 Equilibrium

Following standard solution techniques, one now conjectures a form for the price function and then verifies that if everybody believes the postulated form, it will in fact hold. In this case there are two state vectors N and D, and thus one expects prices to have the linear form

$$P_t = AN_t + BD_t. \tag{6}$$

The $K \times K$ matrices A and B map the state of nature into prices. Since this article only examines the set of stationary equilibria the conjecture contained in Equation (6) is that these matrices do not vary over time. One can now employ Equation (6) in Equation (5) to produce an equation for each trader's terminal wealth in terms of the matrices

A and B

$$W_{t+1} = X_t'\{A\eta_{t+1} + B\delta_{t+1} + D_t + \delta_{t+1} - rP_t\} + (1+r)W_t. \quad (7)$$

Standard arguments found elsewhere in the exponential-normal literature imply that given Equation (7) each trader will seek to

$$\max_{X_t} X_t'[D_t - rP_t] + (1+r)W_t - .5\theta X_t'\{A'\Sigma_\eta A(I+B)'\Sigma_\delta(I+B)\}X_t. \quad (8)$$

This leads to the first-order conditions,

$$X_t = \theta^{-1}[A\Sigma_\eta A' + (I+B)\Sigma_\delta(I+B)']^{-1}[D_t - rAN_t - rBD_t] \quad (9)$$

where Equation (6) has been used to eliminate P_t from Equation (9). Since the traders have unit mass the left-hand side of Equation (9) equals the aggregate level of demand. In equilibrium, prices must set supply (N_t) equal to demand (X_t). Imposing this requirement and then matching terms leaves one with the equilibrium equations

$$D_t - rBD_t = 0 \quad (10)$$

and

$$N_t = -\frac{r}{\theta}[A\Sigma_\eta A' + (I+B)\Sigma_\delta(I+B)']^{-1}AN_t. \quad (11)$$

For Equation (10) to hold for all D, it must follow that $B = \frac{1}{r}I$. Using this result to eliminate B from Equation (11) produces a quadratic matrix equation describing A. Some manipulation of Equation (11) yields the following lemma.

Lemma 1. *The matrix A mapping the supply of shares to prices is symmetric and has the following solution*

$$A = -\frac{1}{2}\frac{r}{\theta}\Sigma_\eta^{-1} + \left[\frac{1}{4}\left(\frac{r}{\theta}\right)^2\Sigma_\eta^{-2} - \left(\frac{1+r}{r}\right)^2\Sigma_\eta^{-1/2}\Sigma_\delta\Sigma_\eta^{-1/2}\right]^{1/2}. \quad (12)$$

Proof. See the Appendix.

Equation (12) provides the heart of the model. Notice that Σ_δ does not always multiply Σ_η. This implies that stock price volatility can exist even in the absence of dividend variability. Using Equation (12) and the solution for B one finds that the variance-covariance price matrix equals

$$\begin{aligned}\Sigma_p &= \frac{1}{2}\left(\frac{r}{\theta}\right)^2 \Sigma_\eta^{-1} \\ &\quad + \frac{r}{\theta}\Sigma_\eta^{-1/2}\left(\frac{1}{4}\left(\frac{r}{\theta}\right)^2 I - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{1/2}\Sigma_\delta\Sigma_\eta^{1/2}\right)^{1/2} \Sigma_\eta^{-1/2} \\ &\quad - (2+r)\Sigma_\delta, \qquad (13)\end{aligned}$$

after some algebra. In their study of a single-security model with noise traders De Long et al. (1990) also find that prices can vary even when dividends do not. What makes the result presented here unique is that it occurs within an environment in which all agents maximize well-defined utility functions and trade in a variety of securities.

1.4 Single-stock economy

Since single-security economies have received so much attention in the literature, it is useful to analyze this case to provide a touchstone with the rest of the literature. In this case A becomes a scalar and takes on the somewhat simpler form

$$A = -\frac{1}{2}\frac{r}{\theta}\sigma_\eta^{-2} \pm \sigma_\eta^{-2}\sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\eta^2\sigma_\delta^2}, \qquad (14)$$

where, following notational standards, σ_η^2 and σ_δ^2 have replaced Σ_η and Σ_δ to signify that they are now scalar variables.

Notice that A has two solutions corresponding to two possible equilibrium beliefs agents may hold. To see the correspondence consider the case where σ_δ^2 equals zero, so that the stock pays a constant dividend every period. In this case the *positive* root of the equation sets A equal to zero. Since A and B both equal zero, stock prices do not move at all. Of course this is what occurs in most models. If dividends are constant forever, and this is common knowledge, then in general one expects the price of the stock to remain constant. However, the *negative* root of A corresponds to an economy where prices change over time even though dividends do not, a divergence from what one normally expects to see in a stock market model with fully rational agents.

The two equilibria in the model correspond to two possible self-fulfilling beliefs agents may have about stock prices. Roughly, stock prices will be "excessively" volatile if people believe they will be. To see why beliefs are so important consider an agent's decision problem when $\sigma_\delta^2 = 0$. If this agent lives in the positive root economy then he believes that he knows with certainty next period's stock price.

Since the investor also knows the stock's dividend payout, he views the stock as a risk-free asset. Thus people in the economy willingly provide a perfectly elastic demand schedule for the stock at a price equal to D/r, just as in standard introductory finance texts. However, if agents think stock prices will be volatile then a different picture emerges. Because agents do not believe that they can perfectly forecast prices, they no longer voluntarily provide infinite levels of liquidity to the market. Consider what happens in the economy under the negative root for A when a large supply shock hits the market. If an agent agrees to purchase the stock he now takes on risk. The larger the supply shock investors must absorb, the larger the compensation level they will demand. In the stock market increased compensation levels derive from lower stock prices, and so large supply shocks must be associated with low stock prices. For similar reasons small supply shocks must lead to high prices. In the end, prices become random and negatively correlated with the supply of equity in the economy. Thus if agents think that prices are volatile, then their actions will make these beliefs self-fulfilling.

Within the empirical excess volatility literature the basic issue revolves around the fact that stock price volatility exceeds dividend volatility by several orders of magnitude. The next theorem shows that this model can harmonize these two contradictory facts.

Theorem 1. *As the variance of the supply noise goes to zero* ($\sigma_\eta^2 \to 0$)*, the variance of the per period price change* ($\Delta P = P_t - P_{t-1}$) *goes to infinity under the negative root equilibrium.*

Proof. The variance of the per period price change equals $A^2\sigma_\eta^2 + B^2\sigma_\delta^2$. Plugging in the solutions from Equations (14) and (10) produces

$$\sigma_p^2 = \left(-\frac{1}{2}\frac{r}{\theta} - \sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\eta^2\sigma_\delta^2}\right)^2 \sigma_\eta^{-2} + \frac{1}{r^2}\sigma_\delta^2 \tag{15}$$

after some minor algebra. To prove the theorem, let σ_η^2 go to zero. Now expand the term in brackets and pull out a σ_η^{-2} to get

$$\sigma_p^2 = \left(\frac{1}{2}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2\sigma_\eta^2\sigma_\delta^2 + \frac{r}{\theta}\sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2\sigma_\eta^2\sigma_\delta^2}\right)\sigma_\eta^{-2} + \frac{1}{r^2}\sigma_\delta^2. \tag{16}$$

As σ_η^2 goes to zero the first term on the right-hand side goes to $\frac{(r/\theta)^2}{\sigma_\eta^2}$

which goes to infinity. Since the second term on the right-hand side does not depend upon σ_η^2, the price variance must go to infinity. ∎

Theorem 1 shows that a very volatile price series can be compatible with an economy consisting of fully rational maximizing agents. In fact, the more volatile the price series, the lower the required volatility on the supply series in the negative root equilibrium.

Theorem 1 offers the possibility that small supply shocks can induce large price changes. However, the question then arises as to whether or not the beliefs needed to support the high-volatility equilibrium are "reasonable?" The answer is yes, in that for the model's equilibrium to hold investors only need to form beliefs about the variance of future stock prices. In Equation (8) (the investor's maximization problem) the term in braces equals the variance-covariance matrix of the innovations in the stock price vector. In equilibrium the realized price series must produce data that matches these beliefs, and these beliefs only concern the variance-covariance matrix of the prices and *not* the cause of the volatility. Thus investors do not need to know the model that produces the prices that they are observing. They only need to believe that the price volatility matrix equals the summed terms in Equation (8)'s curly braces. This opens up the possibility that investors within the economy may be puzzled by the volatility of prices relative to dividends. (Just as econometricians have been.) Imagine a world where people do not know about the model and have not thought about the impact that random supplies may have on prices. Such people will believe that in a world of rational agents the variance of prices divided by the dividend variance should equal $1/r^2$. Yet when they look at the realized prices they observe that the actual ratio in the data is quite a bit larger. While they may find this puzzling they will also find that the phenomena does not disappear (in other words it appears to derive from equilibrium forces). Nevertheless, they can still select their optimal portfolios and these portfolios will solve the problem in Equation (8) since their decision only depends upon their beliefs about the price variance. Thus the equilibrium can hold even though no agent in the economy knows about the model driving the observed price series.

While Theorem 1 provides a mapping from supply variance into price variance, econometric data does not contain the supply variance or the selected root from Equation (14) as observable quantities. Instead researchers have information about prices, dividends, and individual levels of risk aversion. What the next proposition shows is that the observable data can pin down the unobservable information.

Proposition 1. *Assume that an econometrician knows the risk aversion of the population, the variance of the dividend series, and the*

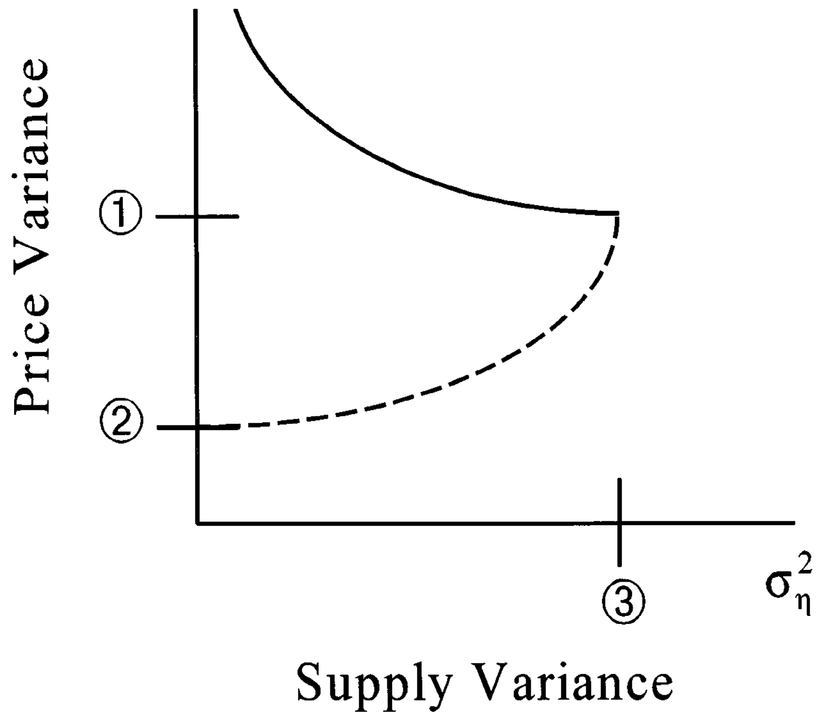

Figure 1
$① = \left[\left(\frac{1+r}{r}\right)^2 + \frac{1}{r^2}\right]\sigma_\delta^2$, $② = \sigma_\delta^2/r^2$, and $③ = .25\left(\frac{r}{\theta}\right)\left(\frac{1+r}{r}\right)^{-2}\sigma_\delta^{-2}$.

variance of the stock's price. Then there exists only one value for the supply noise and one equilibrium consistent with the available data if $\sigma_p^2 \geq \sigma_\delta^2/r^2$. *If* $\sigma_p^2 < \sigma_\delta^2/r^2$ *then the model cannot explain the data.*

Proof. See the appendix.

The proof of Proposition 1 shows that within the positive root equilibrium the price variance increases in the supply variance. In contrast, for the negative root equilibrium exactly the opposite relationship holds. Furthermore, the largest possible price variance in the positive root equilibrium exactly equals the smallest possible price variance in the negative root equilibrium. Figure 1 shows both the position and general shape of the curves describing the two equilibria.

Proposition 1 also shows that the model can be tested empirically. While the model can, in principle, rationalize any variance for the price series that exceeds the variance of the dividend series divided by r^2, it can only do so via a particular supply noise. Thus one can think of this model as testable in the same sense one tests models like the consumption CAPM. If estimates of the model's unobservable parameters seem unreasonable, then people conclude that the model does not generate the observed data. Section 2 provides some insight into this issue by showing that the model can produce reasonable estimates for the supply variance based upon the parameters produced in past empirical studies.

1.5 Variance in a multiple security economy

While single-security models provide numerous insights, there exist many issues that can only be addressed in a multiple security environment. This section examines the way stocks interact within the set of potential equilibria.

While a single-security economy contains two potential equilibria, the multiple security economy contains 2^K potential equilibria. Naturally they derive from the same source: the square root term in the solution to A. If a linear equilibrium exists the term under the square root in Equation (12) is a symmetric positive definite matrix. Write

$$\Gamma' \Lambda \Gamma = \frac{1}{4}\left(\frac{r}{\theta}\right)^2 \Sigma_\eta^{-2} - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{-1/2} \Sigma_\delta \Sigma_\eta^{-1/2}, \tag{17}$$

where Γ represents an orthonormal matrix of eigenvectors and Λ a diagonal matrix of eigenvalues. Because Equation (17) appears within a square root in the solution for A, one can restate the far right term in A [from Equation (12)] as $\Gamma' \Lambda^{1/2} \Gamma$. This implies that their exist two potential values for each diagonal element of $\Lambda^{1/2}$ that are compatible with the equilibrium requirements. With K securities there exist K diagonal elements, and so there must be 2^K possible solutions, each corresponding to a unique equilibrium. One thus has the following proposition.

Proposition 2. *With K securities in random supply there exist 2^K equilibria. These equilibria correspond to the set of feasible eigenvalues for the matrix in Equation (17).*

In the single-security case the equilibrium in the economy depends upon the selected root for Equation (14). When the market contains numerous securities the square root of the eigenvalues within Λ play an analogous role. As one might suspect, the volatility conclusions from the single-security case also generalize.

Proposition 3. *Consider an equilibrium, and the variance of any portfolio under that equilibrium. Then switching any eigenvalue in the matrix* $\Lambda^{1/2}$ *from negative to positive reduces the portfolio's variance.*

Proof. See the appendix.

Clearly society would like to operate in an economy where all the selected roots from $\Lambda^{1/2}$ are positive. If some of the roots are negative, then their exists an incentive for some organization to try and change the equilibrium. However, this may be rather difficult since the equilibria do not, in general, associate each eigenvalue with a single security. Rather an eigenvalue determines the volatility of a mutual fund with portfolio weights determined by the associated eigenvector. To change the equilibrium an organization must somehow alter the behavior of an entire mutual fund and not just a single security. Unless one can credibly accomplish this task, investors will not believe that they are in a new equilibrium and so the old equilibrium will prevail. It seems that once a high-volatility equilibrium gets started it may be very difficult to alter it.

While stock price volatility depends upon the selected equilibrium, production by the corporate sector does not. In fact, the model's assumptions force all of the price volatility to derive from beliefs and not an interaction of beliefs and production. However, while corporate production does not depend upon the selected equilibrium, the economy-wide store of the risk-free asset does. In period t the total value of the available risky asset equals $P_t N_t$, an amount that the young generation must turn over to the old generation via a transfer of the risk-free asset. If P_t differs across equilibria then so will the amount of the risk-free asset given to the older generation to consume. As a result, while stock price volatility does not manifest itself in corporate production, it still influences aggregate consumption via fluctuations in the population's holdings of the risk-free security. This imposes another potential impediment to switching among equilibria. While ex ante everybody may prefer the low-volatility equilibrium, ex post different generations will have opposing interests depending upon how switching the equilibrium will influence the set of current prices.

Returning again to Equation (17), it also shows how two securities with identical dividend streams can trade at different prices and exhibit different price volatilities. A simple examination makes it clear that Σ_δ, the dividend variance-covariance matrix, can be singular. Only the supply shock matrix Σ_η needs to be of full rank. Thus, depending upon the supply shocks, one can create an equilibrium in which two stocks with perfectly correlated dividend streams have different moments associated with their prices.[7] Initially, it might seem that if the two stocks have different supply levels then in some sense they are not identical. However, this is only true to a limited degree. In principle, since both stocks have identical dividend streams, one can divide the supply shocks up in any manner one wishes, without changing the fundamentals of the economy. Thus in this case the correlation between the supply shocks is in some sense arbitrary. Nevertheless, if the dividend streams are identical and the prices are not, does this open up an arbitrage opportunity? No, providing investors have finite lives. An infinitely lived individual can buy the cheap stock, sell the expensive stock and make a risk-free profit. However, in any finite length of time an investor that tries this risks having the prices move further in the wrong direction. Thus stocks that have identical dividend streams but different pricing patterns provide people with a profit opportunity, but one that contains risk. The equilibrium equations tell us the degree to which prices can diverge and still produce demands that balance out the supplies.

There already exist several models that can potentially reconcile the empirical volatility results through the introduction of "noise" from various sources. One contribution of this article is to show, in Theorem 1, that a model with fully rational agents can produce the requisite variances with small supply noise levels. That theorem used Equation (15) to derive its results and was based on the assumption that the market contains only one risky security. As the next set of results show, in a multiple security environment, Equation (15) can vastly overstate the required aggregate supply noise. Propositions 4 and 5 show that under fairly mild assumptions, as the number of securities grows to infinity the supply variance of the market portfolio needed to reconcile any particular aggregate price and dividend variance goes to zero. Lemma 2 then shows, under the assumption that all securities have independent identically distributed payoffs and supplies, that if there exist K securities then the value of σ_η^2 derived via Equation (15) will overstate the required aggregate supply variance by a factor of K^2.

There are two routes that lead to the conclusion that as the number of securities goes to infinity the supply noise needed to produce any particular variance in the price innovations goes to zero. One can either place some restrictions on the form of the variance-covariance matrix describing the supply noise or the dividend shocks. Proposition 4 takes the former tack and Proposition 5 the latter. For the

[7] At least two other articles in this genre have produced the same result. In both DeLong et al. (1990) and Bhushan, Brown, and Mello (1997) there are two risk-free assets with identical payoffs. Within both models one risk-free asset has a constant price and the other a volatile price.

limiting results that follow let K both index a particular economy and represent the number of risky securities. Further, let n_j represent the jth element of N at time t and assume that at a particular time t

Assumption 1. $\sum_{j=1}^{K} n_j = 1$ *for all* K, *and* $n_j \geq 0$ *for all* j.

Assumption 2. *That the*

$$\lim_{K \to \infty} \max\{n_1, \ldots, n_K\} \to 0. \tag{18}$$

Assumption 3. $N' \Sigma_\delta N = k_\delta$, *where* k_δ *is a constant independent of* K.

Assumption 4. $N' \Sigma_p N = k_p$, *where* k_p *is a constant independent of* K.

Assumption 5. $k_p \geq \frac{1}{r^2} k_\delta$.[8]

Assumption 1 acts to normalize the number of shares and their sign.[9] Assumption 2 states that as the number of securities goes to infinity any one firm becomes only a small part of the economy. Assumptions 3 and 4 hold the aggregate dividend and price variance constant. Finally, Assumption 5 states that stock prices exhibit "excess" volatility.

An econometrician cannot observe the supply noise directly. Thus a reasonable starting point for empirical research may be to assume that the variance-covariance matrix of the security supply is a diagonal matrix proportional to I. If this assumption describes the economy then the next proposition shows that under Assumptions 1 through 5 the aggregate supply noise goes to zero as K goes to infinity.

Proposition 4. *Assume that one can write* Σ_η *as* $\sigma_\eta^2 I$, *where* σ_η^2 *is a scalar. Then under assumptions 1 through 5* $N' \Sigma_\eta N = \sigma_\eta^2$ *goes to zero as K goes to infinity.*

Proof. From Equation (12) an equilibrium exists if and only if

$$\frac{1}{4}\left(\frac{r}{\theta}\right)^2 I - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{1/2} \Sigma_\delta \Sigma_\eta^{1/2} \tag{19}$$

is a positive semidefinite matrix. Under the assumption that $\Sigma_\eta = \sigma_\eta^2 I$,

[8] In Assumptions 1 through 5 the date index t has been suppressed for notational simplicity.

[9] If the supply of a particular stock is currently negative, one can simply change the sign on the dividend and the stock simultaneously, and thereby leave the economy unchanged.

Equation (19) simplifies to

$$\frac{1}{4}\left(\frac{r}{\theta}\right)^2 I - \left(\frac{1+r}{r}\right)^2 \sigma_\eta^2 \Sigma_\delta. \tag{20}$$

If Equation (20) is a positive semidefinite matrix then pre- and post-multiplying by N must produce a nonnegative number. Using Assumptions 1 and 2 one can prove that $N'N$ goes to zero as K goes to infinity. Since Σ_δ is a positive definite matrix, and by Assumption 3 $_{N}{}'\Sigma_\delta N$ does not depend on K, one must have that σ_η^2 goes to zero. Since $_{N}{}'\Sigma_\eta N = \sigma_\eta^2$, this proves that the aggregate supply variance goes to zero if an equilibrium exists. Finally, given the restrictions imposed by Assumption 5, one can always select a value for σ_η^2 that satisfies Assumption 4 and causes an equilibrium to exist by solving Equation (13) and picking positive or negative eigenvalues as needed. Since the proof follows along the same lines as the proof for Proposition 1 it is not repeated here. ∎

While Proposition 4 allows for a general dividend covariance matrix it imposes assumptions on the supply's covariance structure. Since the supply cannot be observed directly it may be possible to construct a more robust statistical test by restricting Σ_δ instead as in the next proposition.

Proposition 5. *Assume that as K goes to infinity the smallest eigenvalue of the matrix Σ_δ approaches a number strictly greater than zero. Then under Assumptions 1 through 5 the supply variance of the market portfolio $N'\Sigma_\eta N$ goes to zero.*

Proof. An equilibrium exists if and only if the matrix of Equation (19) is positive semidefinite, and thus

$$\lim_{K\to\infty} N'\Sigma_\eta^{1/2}\Sigma_\delta\Sigma_\eta^{1/2}N \to 0 \tag{21}$$

since Assumptions 1 and 2 imply $N'N$ goes to zero as K goes to infinity. Let $Y = \Sigma_\eta^{1/2}N$. Then using Equation (21) one has

$$N'\Sigma_\eta^{1/2}\Sigma_\delta\Sigma_\eta^{1/2}N = Y'\Sigma_\delta Y \geq \nu_{\min} Y'Y \tag{22}$$

where $\nu_{\min}$ represents the smallest eigenvalue of Σ_δ. By assumption $\nu_{\min}$ approaches a strictly positive number as K goes to infinity. Thus Equation (21) can only hold if $Y'Y$ goes to zero, which proves the proposition since $Y'Y = N'\Sigma_\eta N$. ∎

Propositions 4 and 5 show that within a multiple security environment very small shocks can produce large price changes. To obtain some intuition as to just how quickly the supply shock declines, consider what happens as one divides the economy's risky capital asset into K identically sized independent pieces. As the following lemma shows this divides the required supply variance by K^2.

Lemma 2. *Assume $\Sigma_\eta = \sigma_\eta^2 I$, $\Sigma_\delta = \sigma_\delta^2 I$, and $n_i = 1/K$ in addition to Assumptions 1 through 5. As K increases assume that the equilibrium only uses either the positive or negative roots from the matrix $\Lambda^{1/2}$. Then $\sigma_\delta^2 = Kk_\delta$ and if $\sigma_\eta^2 = \hat{\sigma}_\eta^2$ equals the solution in the 1 security case then $\sigma_\eta^2 = \hat{\sigma}_\eta^2/K$ in the K security case. The latter result implies that the supply variance of the market portfolio equals $\hat{\sigma}_\eta^2/K^2$.*

Proof. Since $\Sigma_\delta = \sigma_\delta^2 I$ and $n_i = 1/K$ the dividend variance will equal k_δ if $\sigma_\delta^2 = Kk_\delta$. Use this to substitute out σ_δ^2 in Equation (13). Next replace σ_η^2 with $\hat{\sigma}_\eta^2/K$ to verify that this keeps $N'\Sigma_P N$ constant. Finally, with $n_i = 1/K$, and $\sigma_\eta^2 = \hat{\sigma}_\eta^2/K$ it follows that the supply variance of the market portfolio equals $\hat{\sigma}_\eta^2/K^2$. ∎

Lemma 2 shows that in practice a very small aggregate supply variance can easily produce a very large price variance. Empirically the lemma indicates that estimates of the model based only on the aggregate market index will probably overestimate the supply variance by a factor of perhaps several hundred.

2. Empirical Implications

2.1 Calibrating the model

Intuitively one would like the model to explain the data with "relatively small" supply shocks. However, providing a useful definition for "relatively small" turns out to be fairly complex. Since the present analysis seeks to explain market volatility it seems intuitive to compare the supply volatility to the wealth volatility induced from one share of stock. For notational consistency define π_t as the wealth from holding one share of stock this period (i.e., $\pi_t = P_t$) and π_{t+1} as the wealth from selling the share next period and cashing in its dividend (i.e., $\pi_{t+1} = P_{t+1} + D_{t+1}$). Further define σ_π^2 as the per period variance in π, so that $\sigma_\pi^2 = E(\pi_{t+1} - \pi_t)^2$ given π_t. At this point one may be tempted to compare σ_η directly with σ_π. Unfortunately a direct comparison is not productive since σ_η and σ_π are in inverse units to each other. To see this consider a unit of account such that there exists only one share of stock and fix both σ_η and σ_π. Now consider the same economy but split the stock into two shares. This doubles the number of shares which doubles σ_η and simultaneously cuts the dividend per share in half, thereby dividing σ_π in half. Thus depending on the normalization selected one can set σ_η/σ_π to any value. Fortunately one can easily circumvent this problem by normalizing with the appropriate units, thereby producing a scale-free test statistic

$$S = \frac{\sigma_\eta}{N}\frac{\pi}{\sigma_\pi} = \frac{\sigma_\eta}{N}\frac{P}{\sigma_\pi}. \tag{23}$$

If S equals one the supply shocks and wealth shocks have similar magnitudes. Larger values indicate that σ_η predominates, and smaller values that σ_π predominates. Since it seems unlikely that the aggregate supply shocks exceed the stock price shocks, a "good" model should produce fairly small values for S.

While Equation (23) takes care of the scaling problem it requires one to know both N and P, which for fairly obvious reasons do not appear among the statistics generally reported. However, this problem can be circumvented by requiring N and P to take on values that will induce the model to produce statistics that correspond with well-accepted properties associated with market returns. Under a standard CAPM-like model, with dividends that follow a random walk, $P_t = D_t/r^*$ where $r^* > r$ represents the dividend discount rate. One can then write the return variance σ_r^2 as

$$\sigma_r^2 = \frac{(1+r^*)^2\sigma_\delta^2}{D_t^2}. \tag{24}$$

Well-known estimates from Ibbotson Associates (1989) place σ_r^2 at .209 for annual data.[10] Thus filling in Equation (24) with σ_r^2 at .209 and a particular study's reported value for σ_δ^2 and r^* one obtains the implicit price scale that will induce the model to produce a sensible return series. The value of N in Equation (23) is somewhat simpler to find. Let W represent the dollar value of the wealth held by each trader in the stock market. Then by definition $W_t = P_t N$. Substituting this into Equation (23) eliminates N from the equation.

To obtain S in terms of observable variables one now only needs an estimate of σ_π. Since $\pi_{t+1} = P_{t+1} + D_{t+1}$ and $\pi_t = P_t$ it follows that

$$\sigma_\pi = \left(\frac{1+r^*}{r^*}\right)\sigma_\delta \tag{25}$$

since $P_t = D_t/r^*$.

[10] However, as the text will shortly explain, the S value does not depend upon σ_r^2.

Using Equations (24) and (25) along with the substitutions $N = W_t/P_t$ and $P_t = D_t/r^*$ the statistic S reduces to

$$S = \frac{\sigma_\eta \sigma_\delta (1 + r^*)}{W_t r^* \sigma_r^2}. \tag{26}$$

Even with the values of σ_δ and σ_P reported by a study the model cannot pin down σ_η without knowledge of θ. Under a mean-variance analysis a standard CAPM-type model will produce the first-order condition

$$r^* - r - \theta \sigma_\delta^2 (1 + r^*)^2 \frac{X_t}{r_s D_t} = 0. \tag{27}$$

Solving this for θ one obtains

$$\theta = \frac{r^*(r^* - r) D_t}{\sigma_\delta^2 (1 + r^*)^2 X_t}. \tag{28}$$

Now use Equation (24) to eliminate σ_δ from the above equation, along with $P_t = D_t/r^*$ and $W_t = P_t X_t$, to produce

$$\theta = \frac{r^* - r}{\sigma_r^2 W_t}. \tag{29}$$

Using the above value for θ, along with values from a study for σ_δ and σ_P, Equation (15) will yield the value of σ_η needed to reconcile the model with the reported empirical results. Notice that the implied risk aversion parameter is *not* scale-free. Moving from dollars to cents will multiply W_t by 100 while leaving all the other variables on the right-hand side of Equation (29) unchanged. Thus the value of θ depends inversely on the units of account selected. Normally people think of preference parameters as exogenous to the model and independent of the currency unit. But the parameter θ represents risk aversion per unit of consumption as measured in the unit of account. Changing the unit of account changes the risk aversion through the change in translation from currency to consumption units.

It may appear that the estimate of S will depend heavily upon the values used for σ_r^2 and W_t. In fact, S is invariant to changes in either of these parameters. To see why, notice that θ depends inversely upon both σ_r^2 and W_t and so does S. At the same time one can see from Equation (15) that σ_η depends inversely upon θ. Thus the changes induced to the model through both σ_r^2 and W_t leave S unchanged. Intuitively the reason this occurs is that S has been made into a scale-free statistic, and the values of σ_r^2 and W_t depend arbitrarily upon the selected scale.

Table 1
Model statistics implied by Shiller's (1981b) equation (I-3) empirical estimates

r	.04	.05	.06	.07
$1 - \sigma_\delta / r\sigma_p$	0.505	0.622	0.699	0.754
		S		
$K = 1$	0.552	0.502	0.458	0.422
$K = 30$	0.018	0.017	0.015	0.026

In all cases the estimates imply that the negative root equilibrium holds.
Row labeled r: interest rate used in each column. Row labeled $1 - \sigma_\delta / r\sigma_p$: fraction of the price variance that cannot be accounted for in the standard fixed supply dividend discount model. West [1988, Equation (17)] uses an identical measure multiplied by 100. Rows labeled S: numerical value of S under the given interest rate, the return standard deviation, and the number of traded securities under the assumptions of Lemma 2.

Among the many variance inequality tests of the efficient markets hypothesis that have been explored one of the earliest is

$$\sigma(\Delta P) \leq \sigma(\Delta D) / \sqrt{2r^3/(1+2r)} \tag{30}$$

which can be found in Shiller (1981b) with ΔD defined as $D_t - D_{t-1}$. Using 10-year intervals Shiller (1981b) finds that $\sigma(\Delta P) = 69.4$, and $\sigma(\Delta D) = 16.5$. Thus, in his test, markets can only be efficient if the annual interest rate remains below 3.2%. Since actual discount rates appear to lie above 4%, the model's predictions do not conform with the data.

Since the model presented in Section 1 predicts that stock price volatility has both a dividend and supply component, it can potentially reconcile Shiller's (1981b) estimates with an environment populated by rational traders. By adjusting σ_η Equation (15) can produce an expected price volatility equal to that reported in past empirical work given any previously estimated dividend variance. However, in order to calculate σ_η from Equation (15) and S from Equation (26) estimates of the dividend discount factor and the return variance are needed. Based upon the values used in West (1989), Mankiw, Romer, and Shapiro (1991), and Campbell and Kyle (1993) an appropriate discount factor seems to lie between 4% and 7%.

Table 1 provides values of the model's parameters that will produce data consistent with Shiller's (1981b) estimates in a one-security environment. However, as Lemma 2 shows, the resulting estimates of the aggregate supply standard deviation will exceed the supply standard deviation in a multiple security economy by a factor of K. Thus the single risky security aggregate supply standard deviation figures calculated in the table should be viewed as extremely conservative upper bounds. To provide a more realistic scenario figures are also provided under a 30-security economy based on the assumptions leading to Lemma 2. Employing the empirical estimates within the model requires that the economy operate under the negative root equilibrium. Thus it appears that the additional equilibrium found in this model provides a useful degree of freedom with which to explain the data.

Notice that the column labeled $1 - \sigma_\delta / r\sigma_p$ shows that dividend shocks explain less than half of the price volatility. Nevertheless, the calculated value of *S* lies well below 1. The model, therefore, explains over half the price volatility with relatively small supply shocks. Recall that supply shocks are generally taken to represent changes in human capital and other illiquid assets. As such it seems likely that the variance of these supply shocks should be small relative to the stock return shocks. Table 1 indicates that the model can produce estimates which conform to this intuition.

Naturally, over time Shiller's (1981b) statistical model has been superceded by more powerful second-generation tests. These tests have confirmed Shiller's conclusions and generally produce even stronger rejections of the standard dividend discount model. Since the model can reconcile Shiller's results with values of *S* under .03 (with 30 securities) it produces even smaller *S* values when applied to the second-generation tests. Overall then, the model shows that a combination of relatively small random supply shocks and finitely lived agents can produce stock market data similar to what we observe.

3. A Trend Stationary Model

3.1 Modified model and its solution

The empirical literature has generally sought to reconcile the stock market dividend and price data by assuming that dividends follow a random walk. Thus the same assumptions were employed in Section 1. However, there are two reasons for considering a model with a mean reverting dividend process. First, a number of statistical studies, such as DeJong and Whiteman (1991), have come to the conclusion that dividends are mean reverting. Thus one would hope that adding this feature to a model will not invalidate its conclusions. Second, a mean reverting process makes it *more* difficult to reconcile the dividend-price data under the standard dividend discount model. This naturally raises the question as to how well the current model performs under a mean reverting dividend process. As the analysis in this section shows, under the volatile equilibrium, making either or both the dividends or the supply of the risky asset mean reverting increases the sensitivity of stock prices to supply shocks.

To relax the random walk assumptions used so far, assume that dividends are generated via the following formula

$$D_t = D_{t-1} - \beta(D_{t-1} - \bar{D}) + \delta_t, \tag{31}$$

and the stock supply by

$$N_t = N_{t-1} - \alpha(N_{t-1} - \bar{N}) + \eta_t. \tag{32}$$

Here $\bar{D}$ represents a vector of constants toward which dividends move over time, while $\bar{N}$ represents a similar vector for the stock supply. The terms β and α are scalar constants that determine the speed at which the processes revert toward their stationary values. Setting β or α to zero causes Equations (31) and (32) to revert back to their respective form in Section 1.

Since the analysis remains basically identical to that found in Section 1, only the results are presented here. The equation relating prices to the underlying state variables now includes a constant component $(K \times 1)$ vector C, and thus prices equal

$$P_t = AN_t + BD_t + C. \tag{33}$$

Repeating the steps found earlier one finds that A, B, and C satisfy the following equations:

$$\begin{aligned} A = & -\frac{1}{2}\left(\frac{\alpha + r}{\theta}\right)\Sigma_\eta^{-1} \\ & + \left(\frac{1}{4}\left(\frac{\alpha + r}{\theta}\right)^2 \Sigma_\eta^{-2} - \left(\frac{1 + r}{\beta + r}\right)^2 \Sigma_\eta^{-1/2}\Sigma_\delta\Sigma_\eta^{-1/2}\right)^{1/2}, \end{aligned} \tag{34}$$

$$B = \frac{1 - \beta}{\beta + r} I, \tag{35}$$

and

$$\begin{aligned} C = & \frac{(1 + r)\beta}{(\beta + r)r}\bar{D} \\ & + \frac{\alpha}{r}\left[-\frac{1}{2}\left(\frac{\alpha + r}{\theta}\right)\Sigma_\eta^{-1/2} \right. \\ & \left. + \left(\frac{1}{4}\left(\frac{\alpha + r}{\theta}\right)^2 \Sigma_\eta^{-2} - \left(\frac{1 + r}{\beta + r}\right)^2 \Sigma_\eta^{-1/2}\Sigma_\delta\Sigma_\eta^{-1/2}\right)^{1/2}\right]\bar{N}. \end{aligned} \tag{36}$$

While C seems to add a great deal of complexity to the solution, it drops out of the variance-covariance calculations. Thus, as the next subsection will show, a trend stationary dividend does not manifestly change the model's qualitative properties.

3.2 Implied price variance

The variance-covariance matrix of prices can be calculated as

$$E(P_{t+1}-P_t)(P_{t+1}-P_t)' = A\lfloor\Sigma_\eta+\alpha^2(N_t-\bar{N})(N_t-\bar{N})'\rfloor A' + B\lfloor\Sigma_\delta+\beta^2(D_t-\bar{D})(D_t-\bar{D})'\rfloor B'. \quad (37)$$

To obtain the formula only in terms of endogenous variables one can use Equations (34) and (35) to eliminate A and B from Equation (37). Notice that increasing α and β from 0 does not materially alter the equilibrium's properties. This becomes even more apparent in the scalar case. When there exists only one risky stock, A reduces to

$$A=-\frac{1}{2}\left(\frac{\alpha+r}{\theta}\right)\sigma_\eta^{-2}\pm\sigma_\eta^{-2}\sqrt{\frac{1}{4}\left(\frac{\alpha+r}{\theta}\right)^2-\left(\frac{1+r}{\beta+r}\right)^2\sigma_\delta^2\sigma_\eta^2}. \quad (38)$$

Because the basic structure of Equation (38) is identical to Equation (14) the conclusions reached in earlier sections remain unchanged. However, allowing for mean reversion allows one to prove the following additional results.

Proposition 6. *Within the volatile equilibrium, increasing the speed of mean reversion in either the dividend or supply process increases the sensitivity of the stock price to supply shocks. Formally,* $\partial A/\partial\alpha < 0$ *and* $\partial A/\partial\beta < 0$ *under the negative root equilibrium.*

4. Conclusion

The relative volatility of stock prices to the dividend process has produced a long string of papers. No doubt this will continue into the distant future. This article shows that one can produce large price-dividend standard deviation ratios in a model with fully rational agents. More importantly one can produce these statistics with a rather modest variance in the unobservable supply of stock. The model can fit reported relative price-dividend standard deviation ratios with supply shocks that are generally less than 10% as large as the price changes being explained. This seems like a useful trait for any model to possess if one suspects that the capital asset supply variance is much smaller than the aggregate stock price variance.

Another feature brought out by the analysis is the potential importance a multiple security environment can play in the market's aggregate volatility. Simply put, the greater the number of securities the smaller the required supply variance to generate any particular variance for the aggregate price index.

Appendix: Proofs for Claims in the Main Body of the Text

Lemma 1. *The matrix A mapping the supply of shares to prices is symmetric and has the following solution:*

$$A = -\frac{1}{2}\frac{r}{\theta}\Sigma_\eta^{-1} + \left[\frac{1}{4}\left(\frac{r}{\theta}\right)^2 \Sigma_\eta^{-2} - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{-1/2}\Sigma_\delta \Sigma_\eta^{-1/2}\right]^{1/2}. \quad \text{(A1)}$$

Proof. To prove A is symmetric, note that from Equation (11) A solves

$$A\Sigma_\eta A' + \frac{r}{\theta}A + \left(\frac{1+r}{r}\right)^2 \Sigma_\delta = 0. \quad \text{(A2)}$$

The first and third terms are symmetric since Σ_η and Σ_δ are symmetric variance-covariance matrices. Since r and θ are scalars, A must be symmetric. This allows one to replace the A' term with A in Equation (A2). Now pre- and postmultiply by $\Sigma_\eta^{1/2}$ and let $Y = \Sigma_\eta^{1/2} A \Sigma_\eta^{1/2}$, thereby producing

$$Y^2 + \frac{r}{\theta}Y + \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{1/2}\Sigma_\delta \Sigma_\eta^{1/2} = 0, \quad \text{(A3)}$$

which is a quadratic matrix equation. To solve for Y, complete the square by adding and subtracting $\frac{1}{4}(\frac{r}{\theta})^2 I$. After some minor algebra this generates

$$\left(Y + .5\frac{r}{\theta}I\right)^2 = .25\left(\frac{r}{\theta}\right)^2 I - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{1/2}\Sigma_\delta \Sigma_\eta^{1/2}, \quad \text{(A4)}$$

and allows one to conclude

$$Y = -\frac{1}{2}\frac{r}{\theta}I + \left(\frac{1}{4}\left(\frac{r}{\theta}\right)^2 I - \left(\frac{1+r}{r}\right)^2 \Sigma_\eta^{1/2}\Sigma_\delta \Sigma_\eta^{1/2}\right)^{1/2}. \quad \text{(A5)}$$

Finally, substitute out for Y by using the relationship $Y = \Sigma_\eta^{1/2} A \Sigma_\eta^{1/2}$, and then pre- and postmultiply by $\Sigma_\eta^{-1/2}$ to finish the proof. ■

Proposition 1. *Assume that an econometrician knows the risk aversion of the population, the variance of the dividend series, and the variance of the stock's price. Then there exists only one value for the* *supply noise and one equilibrium consistent with the available data if $\sigma_p^2 \geq \sigma_\delta^2/r^2$. If $\sigma_p^2 < \sigma_\delta^2/r^2$ then the model cannot explain the data.*

Proof. The proposition can be proved in three steps. The first step shows that in the positive root equilibrium σ_p^2 increases in σ_η^2. The second step shows that in the negative root equilibrium σ_p^2 decreases in σ_η^2. Finally, the third step shows that the largest value σ_p^2 takes on in the positive root equilibrium equals the smallest value it takes on in the negative root equilibrium.

Step 1, begin with the positive root equilibrium. In this case the price variance can be written as

$$\sigma_p^2 = \left(\frac{1}{2}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\eta^2 \sigma_\delta^2 - \frac{r}{\theta}\sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\eta^2 \sigma_\delta^2}\right)\sigma_\eta^{-2} + \frac{1}{r^2}\sigma_\delta^2, \tag{A6}$$

which is identical to the negative root of Equation (16) except for the sign in front of the square root symbol. One can simplify the algebra by using the precision of η rather than its variance. Thus let $\rho_\eta = \sigma_\eta^{-2}$ and substitute into Equation (A6), and then differentiate Equation (A6) with respect to ρ_η producing

$$\frac{\partial \sigma_p^2}{\partial \rho_\eta} = \frac{1}{2}\left(\frac{r}{\theta}\right)^2 - \frac{\frac{1}{2}\frac{r}{\theta}\left[\frac{1}{2}\left(\frac{r}{\theta}\right)^2 \rho_\eta - \left(\frac{1+r}{r}\right)^2 \sigma_\delta^2\right]}{\sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 \rho_\eta^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\delta^2 \rho_\eta}}. \tag{A7}$$

While Equation (A7) cannot immediately be signed for all ρ_η it can be signed at the lower bound of ρ_η. To generate a real solution to Equation (14) the term inside the radical sign must be positive. Therefore

$$\rho_\eta \in \left[4\left(\frac{r}{\theta}\right)^{-2}\left(\frac{1+r}{r}\right)^2 \sigma_\delta^2, \infty\right]. \tag{A8}$$

Now divide the top and bottom of the last term in Equation (A7) by ρ_η and then evaluate as ρ_η goes to infinity. After some algebra, this produces

$$\lim_{\rho_\eta \to \infty} \frac{\partial \sigma_p^2}{\partial \rho_\eta} \propto -\frac{r}{\theta} < 0. \tag{A9}$$

Thus at ρ_η's upper bound $\partial\sigma_p^2/\partial\rho_\eta$ is negative. To prove this holds for all ρ_η, consider the second derivative

$$\frac{\partial^2 \sigma_p^2}{\partial \rho_\eta^2} \propto \left(\frac{1+r}{r}\right)^4 \sigma_\delta^4 > 0, \tag{A10}$$

which arises after some extensive but straightforward algebra. Since $\partial\sigma_p^2/\partial\rho_\eta < 0$ at the upper bound for ρ_η and since $\partial^2\sigma_p^2/\partial\rho_n^2 > 0$ for all ρ_η, it follows that $\partial\sigma_p^2/\partial\rho_\eta < 0$ for all ρ_η. Translating back from ρ_η to σ_η^2, one has now shown that $\partial\sigma_p^2/\partial\sigma_\eta^2 > 0$ for all σ_η^2.

Step 2 is to consider the negative root equilibrium. Again use ρ_η and differentiate Equation (14), yielding

$$\frac{\partial \sigma_p^2}{\partial \rho_\eta} = \frac{1}{2}\left(\frac{r}{\theta}\right)^2 + \frac{\frac{r}{\theta}\left[\frac{1}{2}\left(\frac{r}{\theta}\right)^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\delta^2\right]}{\sqrt{\frac{1}{4}\left(\frac{r}{\theta}\right)^2 \rho_\eta^2 - \left(\frac{1+r}{r}\right)^2 \sigma_\delta^2 \rho_\eta}}. \tag{A11}$$

Since the term inside the square root must be positive, and the bracketed term in the numerator is bigger then the term inside the square root, one immediately has that $\partial\sigma_p^2/\partial\rho_\eta > 0$. This in turn implies that $\partial\sigma_p^2/\partial\sigma_\eta^2 < 0$ for all permissible σ_η^2.

Step 3 is to consider the value σ_p^2 in the positive root equilibrium. Since its value increases in σ_η^2 the maximal value of σ_p^2 must occur at the maximal value for σ_η^2. Plugging this in and using $\bar{\sigma}_\eta^2 = \frac{1}{4}(\frac{r}{\theta})^2(\frac{1+r}{r})^{-2}\sigma_\delta^{-2}$ produces

$$\sigma_p^2\Big|_{\sigma_\eta^2=\bar{\sigma}_\eta^2} = \left(\frac{1+r}{r}\right)^2 \sigma_\delta^2 + \frac{\sigma_\delta^2}{r^2}. \tag{A12}$$

Under the negative root equilibrium, the smallest value of σ_p^2 must occur at the maximal value σ_η^2 since in the negative root equilibrium σ_p^2 declines in σ_η^2. Plugging this into Equation (14) produces an equation identical to Equation (A12). Thus each equilibrium covers a disjoint set of values for σ_p^2. ■

Proposition 3. *Consider an equilibrium and the variance of any portfolio under that equilibrium. Then switching any eigenvalue in the matrix $\Lambda^{1/2}$ from negative to positive reduces the portfolio's variance.*

Proof. Let X represent an arbitrary portfolio. Then the variance of that portfolio can be written as

$$\sigma_X^2 = X'\left[A\Sigma_\eta A' + \frac{1}{r^2}\Sigma_\delta\right]X. \tag{A13}$$

Substituting in the solution for A from Equation (12) plus extensive manipulation produces

$$\sigma_X^2 = X' \left[\frac{1}{2} \left(\frac{r}{\theta} \right)^2 \Sigma_\eta^{-1} + \left[\frac{1}{r^2} - \left(\frac{1+r}{r} \right)^2 \right] \Sigma_\delta \right] X - \frac{r}{\theta} X' \Gamma' \Lambda \Gamma X, \tag{A14}$$

whose size needs to be determined. The first term on the right-hand side does not depend upon the selected equilibrium. Thus one only needs to consider the far right term. Let γ_i represent the ith eigenvector in the matrix Γ. Then one can write any X as

$$X = \sum_{i=1}^{K} w_i \gamma_i \tag{A15}$$

where the w_i represent weights. Substituting in for X in Equation (A14), the far right-hand term becomes

$$\frac{r}{\theta} X' \Gamma' \Lambda \Gamma X = \sum_{i=1}^{K} \lambda_i w_i^2 \tag{A16}$$

after making use of the fact that the columns of Γ are orthonormal vectors. Note that the w_i in Equation (A16) are squared, and Equation (A16) enters Equation (A14) with a negative sign. Thus switching any λ_i from *negative* to *positive* must reduce the variance of the portfolio X. ∎

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