

Transaction Costs and Asset Prices: A Dynamic Equilibrium Model

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In this article we study the effects of transaction costs on asset prices. We assume an overlapping generations economy with a riskless, liquid bond, and many risky stocks carrying proportional transaction costs. We obtain stock prices and turnover in closed form. Surprisingly, a stock's price may increase in transaction costs, and a more frequently traded stock may be less adversely affected by an increase in transaction costs. Calculations based on the "marginal" investor overestimate the effects of transaction costs. For realistic parameter values, transaction costs have very small effects on stock prices but large effects on turnover.

Transaction costs such as bid-ask spreads, brokerage commissions, market impact costs, and transaction taxes, are important in many financial markets.¹ Moreover, considerable attention has focused on their effects on asset prices. A recent example is the debate on how prices of NYSE stocks would be affected if a

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¹ For a complete description of transaction costs see Amihud and Mendelson (1991). For evidence on the magnitude of transaction costs see Aiyagari and Gertler (1991).

transaction tax is imposed.² Transaction costs have also been proposed as an explanation for various asset pricing puzzles such as the equity premium puzzle [Mehra and Prescott (1985)]³ and the small stock puzzle [Banz (1981) and Reinganum (1981)].⁴

Although transaction costs are mentioned in many asset pricing debates, they are generally absent from asset pricing models. Some articles introduce transaction costs, but assume that assets are identical in all other dimensions [see Amihud and Mendelson (1986), Aiyagari and Gertler (1991), Huang (1997), and Vayanos and Vila (1997)].⁵ This assumption is restrictive. In the equity premium puzzle for instance, stocks are riskier than bonds, while in the small stock puzzle, small stocks are not perfectly correlated with large stocks. Some other articles relax the assumption that assets differ only in transaction costs, assuming instead a riskless, perfectly liquid bond, and a risky stock that carries transaction costs [see Constantinides (1986), Duffie and Sun (1990), Davis and Norman (1990), Grossman and Laroque (1990), Dumas and Luciano (1991), Fleming et al. (1992), Shreve and Soner (1992), and Schroder (1997)]. These articles treat asset prices as exogenous and determine the optimal investment policy. They also compare the rate of return on the stock to the rate of return that a perfectly liquid stock should have, so that the investor is indifferent between the two stocks. However, the difference between the two rates can be interpreted as the effect of transaction costs, only when the investor is constrained to choose one of the two stocks and cannot diversify. In addition, the analysis is not done in a general equilibrium setup where both stocks are present and prices are endogenous. Heaton and Lucas (1996) endogenize asset prices in a one bond/one stock model, but have to resort to numerical methods.

In this article we develop a general equilibrium model with transaction costs. We assume a riskless, perfectly liquid bond with a constant rate of return, and *many* risky stocks that carry proportional transaction costs. Trade occurs because there are overlapping generations of

² Amihud and Mendelson (1986) regress cross-sectional asset returns on bid-ask spreads and betas. Using these results, Amihud and Mendelson (1990) argue that a .5% tax would decrease prices by 13.8%. Barclay, Kandel, and Marx (1997) study changes in bid-ask spreads associated with the use of odd-eighths quotes in the NASDAQ, and the migration of stocks from NASDAQ to NYSE or AMEX. Using their results, they argue that a tax would have much smaller effects. Umlauf (1993), Campbell and Froot (1994), and Stulz (1994) present and discuss empirical evidence on the effects of transaction taxes in Sweden and the U.K.

³ Aiyagari and Gertler (1991) and Heaton and Lucas (1996) examine whether transaction costs explain part of the difference in rates of return between bonds and less liquid stocks.

⁴ Schultz (1983), Stoll and Whaley (1983), Amihud and Mendelson (1986), and Constantinides (1986) examine whether transaction costs explain the difference in rates of return between large stocks and less liquid small stocks.

⁵ In Amihud and Mendelson assets are identical because agents are risk neutral, while in Aiyagari and Gertler, Huang, and Vayanos and Vila assets are identical because they are all riskless.

investors who buy the assets when born and slowly sell them until they die. The model is very tractable and stock prices are obtained in closed form.

Our results are surprising and contrary to “conventional wisdom” about the effects of transaction costs on asset prices. First, the price of a stock may increase in its transaction costs. An increase in transaction costs has two opposing effects on the stock’s demand. On the one hand, investors buy fewer shares, but on the other, they hold them for longer periods. We show that either effect can dominate.

According to conventional wisdom, the effect of transaction costs on the stock price depends on the stock’s minimum holding period, that is, the holding period of the marginal investor. For the marginal investor to be induced to buy the stock, the price has to fall by the present value of transaction costs that he, and all future marginal investors, incur (the “PV term”).⁶ Our second result is, however, that the PV term overstates the effect of transaction costs on the stock price. The reason is that, with transaction costs, investors hold the stock for longer periods. Therefore the marginal investor holds fewer shares and requires a smaller *risk* premium. The difference between the effect of transaction costs and the PV term is important and should not be neglected in practical applications. For small transaction costs ϵ , the effect of transaction costs is of order ϵ , while the PV term is of order $\sqrt{\epsilon}$.

Our third result is about how the effect of transaction costs on the stock price depends on the stock’s characteristics. According to conventional wisdom, a more liquid stock or a riskier stock is more adversely affected by an increase in its transaction costs. Indeed, since such a stock is traded more frequently, the holding period of the marginal investor is shorter and the PV term larger. We show, however, that a more frequently traded stock may be less adversely affected by an increase in its transaction costs. The reason is that the marginal investor greatly reduces his stock holdings and requires a smaller risk premium. An empirical implication of our results is that a cross-sectional regression of asset returns on transaction costs should include the following two terms with opposite signs: (i) a nonlinear term in transaction costs⁷ and (ii) an interaction term between transaction costs and asset risk.

Our fourth result concerns the effect of a change in one stock’s transaction costs on the price of another stock. According to conventional wisdom, if transaction costs of a stock decrease, the price of a

⁶ Amihud and Mendelson (1986) and Vayanos and Vila (1997) show that when assets are identical except for transaction costs, the price indeed falls by the PV term.

⁷ Amihud and Mendelson (1986) allow for the effect of transaction costs to be nonlinear.

less liquid but correlated stock should increase, since agents do not need to trade it as much. We show, however, that the price may decrease. This result suggests that the introduction of a low transaction cost derivative (such as an option or a futures contract) may decrease the price of the underlying asset.⁸

In addition to stock prices, we study stock turnover and obtain it in closed form. A stock's turnover decreases in the stock's transaction costs and increases in the transaction costs of other stocks. One would expect turnover to increase in the stock's risk, since investors would face a higher cost of deviating from their optimal, no transaction cost portfolio. We show, however, that turnover may decrease.

Finally, we calibrate the model. For realistic parameter values, transaction costs have very small effects on stock prices but large effects on investors' trading strategies and turnover. In addition, turnover is small, because it is generated only by life-cycle effects. Our results are very similar to Constantinides (1986) and Barclay, Kandel, and Marx (1997).

The rest of the article is structured as follows: in Section 1 we present the model. In Section 2 we study agents' optimization problem and market clearing, and determine a simple set of conditions that are sufficient for equilibrium. In Section 3 we construct the equilibrium in the benchmark case where there are no transaction costs. In Section 4 we construct the equilibrium in the case where there are transaction costs. We then study the effects of transaction costs on stock prices and turnover. In Section 5 we extend the analysis to the case where the short-sale constraint for some stocks is binding. In Section 6 we calibrate the model. Section 7 concludes, and most proofs are in the Appendix.

1. The Model

We consider a continuous-time overlapping generations economy. Time, t , goes from $-\infty$ to ∞ . There is a continuum of agents. Each agent lives for an interval of length T . Between times t and $t + dt$, dt/T agents are born and dt/T die. The total population is thus 1.

1.1 Financial assets

Agents can invest in $N + 1$ financial assets. The $N + 1$ st asset is a riskless and perfectly liquid bond. We assume that its rate of return is constant and equal to r . We also assume that the bond can be sold short. The first N assets are risky stocks. Stock i pays dividends at rate

⁸ For a survey of the literature on the effects of derivatives, see Chapter 2 of Allen and Gale (1994).

$D_{i,t}$, which follows an Ornstein–Uhlenbeck process

$$dD_{i,t} = -\kappa_i(D_{i,t} - \bar{D}_i)dt + \sigma_i db_{i,t}. \quad (1)$$

The constants $\bar{D}_i$, κ_i , and σ_i are the long-run mean, reversion rate, and instantaneous standard deviation, respectively. The process $b_t = (b_{1,t}, \dots, b_{N,t})$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The standard filtration of b_t is $F = \{\mathcal{F}_t : t \in (-\infty, \infty)\}$. The instantaneous covariance between $b_{i,t}$ and $b_{j,t}$ is $\rho_{i,j}$, and $\rho_{i,i}$ is normalized to 1. The total number of shares (i.e., the supply) of stock i is $s_i > 0$, and the price at time t is $P_{i,t}$. Stock i is not perfectly liquid, but carries transaction costs that are proportional to the number of shares traded. The costs of buying or selling x_i shares are $\epsilon_i x_i$, with $\epsilon_i \geq 0$. We assume that stocks cannot be sold short. Our results hold, however, even when the short-sale constraint is not binding. The shares of stock i held by an agent of age t are $x_{i,t}$.⁹ We assume that the process $x_{i,t}$ is adapted to F , absolutely continuous, and that its derivative¹⁰ is square-integrable, that is,

$$E \left(\int_0^T \left(\frac{dx_{i,t}}{dt} \right)^2 dt \right) < \infty.$$

The agent's wealth, W_t , is defined as the sum of the values of the stock and bond portfolio, that is,

$$W_t \equiv \sum_{i=1}^N P_{i,t} x_{i,t} + M_t. \quad (2)$$

1.2 Endowments and preferences

Agents are born with an endowment y of a consumption good, and do not receive any other endowment over their lifetime. They derive utility from lifetime consumption. Consumption takes the form of a flow c_t for $t \in [0, T]$ and a “gulp” C_T for $t = T$. We assume that the process c_t is adapted to F , continuous, and square-integrable, that is,

$$E \left(\int_0^T c_t^2 dt \right) < \infty.$$

⁹ We consider an agent born at time 0 and use t for both time and the agent's age.

¹⁰ Since $x_{i,t}$ is absolutely continuous, it has a derivative almost everywhere and is the integral of its derivative.

Utility over consumption is exponential, that is,

$$u(c_t, C_T) = - \int_0^T e^{-ac_t - \beta t} dt - e^{-AC_T - \beta T}. \quad (3)$$

We assume a “smooth” utility $e^{-AC_T - \beta T}$, rather than the constraint $C_T \geq 0$ that is more standard and corresponds to $A = \infty$, for technical reasons. We assume, however, that A is large in order to obtain qualitatively similar results to the case $A = \infty$.

The assumptions that are essential to the tractability of the model are the following. First, the riskless rate is constant. Second, dividends follow Ornstein–Uhlenbeck processes and are thus normal. Third, utility over consumption is exponential. Fourth, transaction costs are proportional to the number of shares, rather than the dollar value. These assumptions turn out to imply two key properties of the equilibrium, namely that stock prices are linear in dividends and that agents’ stock holdings are deterministic.

The assumptions of constant riskless rate, normal dividends, and exponential utility are admittedly special. However, they are almost standard in market microstructure theory.¹¹ They are also used in dynamic asset pricing theory¹² since they are one of the very few sets of assumptions under which the CAPM holds. This article shows that, under such standard assumptions, the effects of transaction costs can be contrary to conventional wisdom.

2. Optimization and Market Clearing

In this section we study agents’ optimization problem and market clearing, and determine a simple set of conditions that are sufficient for equilibrium.

2.1 Optimization

We first state the optimization problem. We then determine a set of conditions that are sufficient for optimality.

2.1.1 The optimization problem. To simplify the optimization problem, we immediately assume the equilibrium dynamics of the

¹¹ For a survey of market microstructure theory see O’Hara (1995).

¹² Stapleton and Subrahmanyam (1978) assume a constant riskless rate, normal dividends, and exponential utility and, like us, obtain linear prices and deterministic stock holdings. However, they assume no transaction costs and a finite, discrete-time economy where all agents are born at time 0.

stock prices, $P_{i,t}$. In equilibrium $P_{i,t}$ is given by

$$P_{i,t} = \bar{P}_i + \frac{D_{i,t} - \bar{D}_i}{r + \kappa_i}. \quad (4)$$

The stock price, $P_{i,t}$, is a long-run mean, $\bar{P}_i$, plus a linear function of the deviation of the dividend, $D_{i,t}$, from its own long-run mean, $\bar{D}_i$. The sensitivity of $P_{i,t}$ with respect to $D_{i,t}$ is $1/(r + \kappa_i)$, and is decreasing in the riskless rate, r , and the reversion rate, κ_i . This is intuitive: if r and κ_i are high, an increase in the current dividend has a small effect on the present value of future dividends. The instantaneous standard deviation of $P_{i,t}$ is $\sigma_i/(r + \kappa_i)$. By appropriately redefining a share of each stock we can change σ_i , and adopt the normalization

$$\frac{\sigma_i}{r + \kappa_i} = 1. \quad (5)$$

An agent chooses stock holdings, $x_{i,t}$, and consumption, c_t , to maximize expected utility. The dynamics of his wealth, W_t , are given by

$$dW_t = \sum_{i=1}^N (D_{i,t} dt + dP_{i,t}) x_{i,t} + rM_t dt - c_t dt - \sum_{i=1}^N \epsilon_i \left| \frac{dx_{i,t}}{dt} \right| dt. \quad (6)$$

Using Equations (1), (2), (4), and (5), we can simplify the wealth dynamics to

$$\begin{aligned} dW_t = & \sum_{i=1}^N (\bar{D}_i - r\bar{P}_i) x_{i,t} dt + \sum_{i=1}^N x_{i,t} db_{i,t} + rW_t dt - c_t dt \\ & - \sum_{i=1}^N \epsilon_i \left| \frac{dx_{i,t}}{dt} \right| dt. \end{aligned} \quad (7)$$

The first term on the right-hand side corresponds to the excess return on the stock portfolio relative to the bond. The second term corresponds to the portfolio's risk. The third term corresponds to the return if all wealth were invested in the bond. The fourth and fifth terms correspond to the consumption and transaction costs, respectively. Notice that, due to the dividend and price dynamics [Equations (1) and (4), respectively], the dividends, $D_{i,t}$, do not enter in the wealth dynamics [Equation (7)]. Therefore stock holdings, $x_{i,t}$, will be independent of $D_{i,t}$. Since utility is exponential, $x_{i,t}$ will also be independent of wealth, W_t . The $x_{i,t}$'s will only be functions of the agent's age, and thus deterministic. This is a key simplifying property of the equilibrium.

At time 0 the agent buys $x_{i,0}$ shares of stock i . His bond holdings thus are

$$M_0 = y - \sum_{i=1}^N (P_{i,0} + \epsilon_i) x_{i,0}. \quad (8)$$

Equations (2) and (8) imply that the agent's initial wealth is

$$W_0 = y - \sum_{i=1}^N \epsilon_i x_{i,0}, \quad (9)$$

that is, initial wealth, W_0 , is equal to the endowment, y , minus the transaction costs. At time T the agent sells $x_{i,T}$ shares of stock i . Equation (2) implies that the consumption gulp is

$$C_T = M_T + \sum_{i=1}^N (P_{i,T} - \epsilon_i) x_{i,T} = W_T - \sum_{i=1}^N \epsilon_i x_{i,T}. \quad (10)$$

Therefore the agent's optimization problem, $(\mathcal{P})$, is

$$\sup_{(x_{i,t}, c_t)} -E \left(\int_0^T e^{-ac_t - \beta t} dt + e^{-AC_T - \beta T} \right)$$

subject to

$$\begin{aligned} dW_t = & \sum_{i=1}^N ((\bar{D}_i - r\bar{P}_i) x_{i,t} dt + x_{i,t} db_{i,t}) + rW_t dt - c_t dt \\ & - \sum_{i=1}^N \epsilon_i \left| \frac{dx_{i,t}}{dt} \right| dt, \end{aligned}$$

$$W_0 = y - \sum_{i=1}^N \epsilon_i x_{i,0},$$

$$C_T = W_T - \sum_{i=1}^N \epsilon_i x_{i,T},$$

and the short-sale constraint $x_{i,t} \geq 0$.

2.1.2 The optimality conditions. We now study the optimization problem $(\mathcal{P})$. The main tool for studying consumption/investment problems with transaction costs is dynamic programming. Dynamic programming is used, for instance, in all the articles that assume one stock [see Constantinides (1986), Duffie and Sun (1990), David and Norman (1990), Grossman and Laroque (1990), Dumas and Luciano

(1991), Fleming et al. (1992), Shreve and Soner (1992), and Schroder (1997)]. With many stocks, however, dynamic programming becomes very complex. The state space becomes large, since one state variable must be introduced for each stock. We will study $(\mathcal{P})$ using the calculus of variations instead of dynamic programming. More precisely, we will determine a set of sufficient conditions for a control $(x_{i,t}, c_t)$ to be locally optimal. Since $(\mathcal{P})$ is concave, the control $(x_{i,t}, c_t)$ will be globally optimal. We determine the sufficient conditions in Proposition 1, proven in Appendix A. Before stating the proposition we define $\gamma = 1 - ra/A$, and A_t by

$$A_t = \frac{ra}{1 - \gamma e^{-r(T-t)}}. \quad (11)$$

A_t corresponds to the coefficient of absolute risk aversion of an agent of age t . Since A is large, $\gamma \in (0, 1)$ and A_t increases with age t . Agents are more risk averse when old because with a shorter horizon their consumption reacts more to an adverse price shock.

Proposition 1. *Consider N continuous and piecewise C^1 functions $x_{i,t}$ in $[0, T]$. Suppose that for each $x_{i,t}$ there exist $0 \leq t_i < \hat{t}_i \leq T$ such that*

- (i) $x_{i,t} = x_{i,0} > 0 \ \forall t \in [0, t_i]$, $dx_{i,t}/dt < 0 \ \forall t \in [t_i, \hat{t}_i]$, and $x_{i,t} = 0 \ \forall t \in (\hat{t}_i, T]$,
- (ii)

$$\int_0^{t_i} \left(\bar{D}_i - r\bar{P}_i + r\epsilon_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) e^{-rt} dt - 2\epsilon_i = 0, \quad (12)$$

(iii)

$$\bar{D}_i - r\bar{P}_i + r\epsilon_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} = 0 \quad \forall t \in [t_i, \hat{t}_i], \quad (13)$$

- (iv) $A_t \sum_{j=1}^N \rho_{i,j} x_{j,t}$ increases $\forall t \in [0, t_i]$,
- and

(v)

$$\bar{D}_i - r\bar{P}_i + r\epsilon_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \leq 0 \quad \forall t \in (\hat{t}_i, T]. \quad (14)$$

Then there exists c_t , such that $(x_{i,t}, c_t)$ solves $(\mathcal{P})$.

Proposition 1 provides a set of sufficient conditions for stock holdings, $x_{i,t}$, to be part of an optimal control. Condition (i) states that holdings of stock i are constant in an interval $[0, t_i]$, strictly decreasing

in $[t_i, \hat{t}_i]$, and 0 in $(\hat{t}_i, T]$. The agent thus buys stock i when born and sells it slowly between t_i and $\hat{t}_i$. He sells the stock as he gets older because he becomes more risk averse. He starts selling at t_i rather than 0, that is, not immediately after he buys, because of transaction costs. If $\hat{t}_i < T$, the agent does not hold stock i in an interval $(\hat{t}_i, T]$ of positive length. In fact, the short-sale constraint for stock i is binding in this interval. In the next sections we show that the short-sale constraint is binding if stock i is highly correlated with other stocks that have larger transaction costs.

Conditions (ii)–(v) are first-order conditions. Condition (ii) is a first-order condition for the shares of stock i , $x_{i,0}$, that the agent buys when born. This Condition states that the agent's payoff does not change in the first order if he buys some additional shares at time 0 and sells them at time t_i . Indeed, we can write condition (ii) as

$$\int_0^{t_i} \left(\bar{D}_i - r\bar{P}_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) e^{-rt} dt - \epsilon_i (1 + e^{-rt_i}) = 0. \quad (15)$$

The first term inside the integral, $\bar{D}_i - r\bar{P}_i$, corresponds to the increase in the excess return on the stock portfolio from buying the additional shares. The second term corresponds to the increase in risk. It is the product of stock i 's contribution to portfolio risk, $\sum_{j=1}^N \rho_{i,j} x_{j,t}$, times the coefficient of absolute risk aversion, A_t . The last term corresponds to the transaction costs incurred at times 0 and t_i . All terms are discounted at interest rate r .

Condition (iii) is a first-order condition for the shares of stock i , $x_{i,t}$, that the agent holds at time $t \in [t_i, \hat{t}_i]$, that is, when he is selling the stock. This condition states that the agent's payoff does not change in the first order if he sells some shares at time $t + dt$ rather than at time t . Indeed, multiplying by dt , we can write Condition (iii) as

$$(\bar{D}_i - r\bar{P}_i)dt - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} dt + r\epsilon_i dt = 0.$$

The first term corresponds to the increase in excess return from holding the shares until time $t + dt$. The second term corresponds to the increase in risk. The last term corresponds to the savings from incurring the transaction cost at time $t + dt$ rather than at time t . Condition (iv) ensures that the agent's payoff decreases if he buys some additional shares at time 0 and sells them at time $t < t_i$, or if he sells some shares at time $t < t_i$ rather than at time t_i . Finally, Condition (v) ensures that the agent's payoff decreases if he sells some shares during the "short-sale interval" $(\hat{t}_i, T]$ rather than at time $\hat{t}_i$.

2.2 Market clearing

We first study market clearing in a “stock” sense and then in a “flow” sense. The market for stock i clears in a “stock” sense if total holdings at a given point in time are equal to the stock’s supply. To compute total holdings, we note that there are dt/T agents with age between t and $t + dt$, and each holds $x_{i,t}$ shares. Therefore the market clears in a “stock” sense if

$$\int_0^T x_{i,t} \frac{dt}{T} = s_i. \quad (16)$$

The market for stock i clears in a “flow” sense if the number of shares bought in a given time interval is equal to the number of shares sold. If Equation (16) is satisfied, the market clears in a “stock” sense at each point in time, and thus automatically clears in a “flow” sense. A more direct and perhaps more intuitive way to show that the market clears in a “flow” sense is to compute the number of shares of stock i bought and sold between times t and $t + dt$. For the number of shares bought, we note that the buyers are the agents born between t and $t + dt$. Since there are dt/T such agents and each buys $x_{i,0}$ shares, the number of shares bought is $x_{i,0}dt/T$. For the number of shares sold, we note that the sellers are the agents who at time t have age greater than t_i . Since at time $t + dt$ these agents have age greater than $t_i + dt$, they collectively own $x_{i,t_i}dt/T$ fewer shares. Therefore the number of shares sold is $x_{i,t_i}dt/T$ and is equal to the number of shares bought. The turnover of stock i , V_i , is defined as the stock’s trading volume between times t and $t + dt$, divided by dt and by the stock’s supply. It is equal to

$$V_i = \frac{x_{i,0}}{Ts_i}. \quad (17)$$

To construct the equilibrium in the next sections, we determine numbers $\bar{P}_i$ and functions $x_{i,t}$ that satisfy the following simple set of sufficient conditions. First, Conditions (i)–(v) of Proposition 1, and second, the market-clearing conditions [Equation (16)].

3. Equilibrium Without Transaction Costs

In this section we study the benchmark case where the ϵ_i ’s are zero. We construct the equilibrium in Proposition 2. Before stating the proposition, we define the function $f(t)$ by

$$f(t) = \frac{ra}{T} \left(\frac{t}{A_t} + \int_t^T \frac{ds}{A_s} \right). \quad (18)$$

The function $f(t)$ corresponds to total stock holdings when agents start selling at time t .

Proposition 2. Define $\bar{P}_i$ by

$$\bar{P}_i = \frac{\bar{D}_i}{r} - \frac{a \sum_{j=1}^N \rho_{i,j} S_j}{f(0)} \quad (19)$$

and $x_{i,t}$ by

$$x_{i,t} = s_i \frac{ra}{f(0)A_t}. \quad (20)$$

Then the conditions of Proposition 1 and the market-clearing conditions hold.

Proof. Equation (11) implies that the $x_{i,t}$'s are C^1 in $[0, T]$. Condition (i) of Proposition 1 holds since the $x_{i,t}$'s are strictly decreasing in $[0, T]$ and thus $t_i = 0$ and $\hat{t}_i = T$. The only other condition of Proposition 1 left to check is Condition (iii). Equations (19) and (20) imply that it holds. Finally, Equations (18) and (20) imply that the market-clearing conditions [Equation (16)] hold. ■

The equilibrium has a very simple form. Consider first Equation (20) which gives agents' stock holdings. To obtain stock holdings at age s , we simply need to multiply stock holdings at age t by A_t/A_s . Therefore all agents hold the same stock portfolio, which is the market portfolio. Agents buy the market portfolio when born and sell it slowly until they die. Turnover is the same for all stocks.¹³ Indeed, the definition of turnover, that is, Equation (17), and Equation (20) imply that the turnover of stock i is

$$V_i = \frac{x_{i,0}}{Ts_i} = \frac{ra}{Tf(0)A_0}.$$

Equation (19) gives stock prices and has the CAPM flavor. Stock i 's price, $\bar{P}_i$, is the present value of the dividend, $\bar{D}_i$, minus a risk premium. The risk premium depends on the coefficient of absolute risk aversion, a , and the stock's systematic risk, that is, its covariance with the market portfolio, $\sum_{j=1}^N \rho_{i,j} S_j$.

¹³ This result is consistent with Lo and Wang (1997), who show that all assets have the same turnover under two-fund separation.

4. Equilibrium With Transaction Costs

In this section we study the case where the ϵ_i 's are not zero and the short-sale constraint is not binding. In the next section we extend the analysis to the case where the short-sale constraint is binding.

To construct the equilibrium, we make three assumptions. The first two assumptions ensure that the short-sale constraint is not binding. The constraint will be binding if stocks are highly correlated but differ substantially in transaction costs. Indeed, because of the high correlation, the gains to diversification are small. The optimal investment is determined by transaction cost considerations, with low transaction cost stocks being sold first and the other stocks afterward. Assumption 1 ensures that there are gains to diversification, that is, stocks are not perfectly correlated.

Assumption 1. *The correlation matrix*

$$\begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix} \quad (21)$$

is positive definite.

Assumption 2 ensures that the gains to diversification are larger than the differences in transaction costs. For notational simplicity, we state the assumption in terms of the stock prices, $\bar{P}_i$, that are endogenous and defined in Proposition 3.

Assumption 2. *For $\bar{P}_i$'s defined by Equation (28), the column vector*

$$\begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{D}_1 - r\bar{P}_1 + r\epsilon_1 \\ \bar{D}_N - r\bar{P}_N + r\epsilon_N \end{pmatrix} \quad (22)$$

is strictly positive.

To motivate the vector (22), assume that stocks are highly positively correlated but differ substantially in transaction costs. Assume, for instance, that stock 1 has the lowest transaction costs. Since stocks are highly positively correlated, the nondiagonal terms of the correlation matrix are negative and large. Since, in addition, stock 1 has lower transaction costs than the other stocks, the first term of the vector (22) is negative and Assumption 2 is violated. Assumption 2 is satisfied in

the no transaction costs case. Indeed, Equation (19) implies that

$$\frac{f(0)}{ra} \begin{pmatrix} \bar{D}_1 - r\bar{P}_1 \\ \vdots \\ \bar{D}_N - r\bar{P}_N \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix}.$$

Therefore the vector (22) is simply $(ra/f(0))(s_1, \dots, s_N) > 0$. By continuity, Assumption 2 is always satisfied for small transaction costs.

Our third assumption is

Assumption 3. For any $\{b_1, \dots, b_i\} \subseteq \{1, \dots, N\}$, the nondiagonal terms of the matrix

$$\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \quad (23)$$

are negative.

Assumption 3 implies that the stocks are substitutes in the following strong sense. Suppose that there are no transaction costs and that agents can only invest in stocks $b_1, \dots, b_i$ and in the bond. The demand for a stock has then to increase in the price of another stock. Indeed, Equation (13) implies that the demand of an agent of age t is given by

$$\frac{1}{A_t} \begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{D}_{b_1} - r\bar{P}_{b_1} \\ \vdots \\ \bar{D}_{b_i} - r\bar{P}_{b_i} \end{pmatrix}.$$

An implication of Assumption 3, obtained by setting $i = 2$, is that $\rho_{j,k} \geq 0$, that is, stocks are positively correlated. Assumption 3 is satisfied, for instance, when $\rho_{j,k} = \rho \geq 0, \forall j, k, j \neq k$. Besides being quite realistic, Assumption 3 ensures that stock holdings are (weakly) decreasing with age, a key simplifying property of the equilibrium.¹⁴ In Section 4.1 we construct the equilibrium, and in Sections 4.2 and 4.3 we study the effects of transaction costs on stock prices and turnover.

4.1 Construction of the equilibrium

We first give some useful definitions. We define the function $g(t)$ by

$$g(t) = \int_0^t \left(1 - \frac{A_s}{A_t}\right) e^{-rs} ds. \quad (24)$$

¹⁴ Increasing risk aversion ensures decreasing stock holdings only when there is one stock. Indeed, suppose that there are two stocks and that Assumption 3 is violated, that is, the stocks are negatively correlated. Agents' holdings of the less liquid stock remain unchanged for a while because of transaction costs. Since agents become more risk averse with age, they hedge their holdings of this stock by increasing their holdings of the more liquid stock.

The function $g(t)$ corresponds to the benefit of buying an additional share at time 0 and selling it at t , gross of transaction costs. We also define the function $I(\zeta, b, \{b_1, \dots, b_i\})$ for $\zeta = (\zeta_1, \dots, \zeta_N)$, $\{b_1, \dots, b_i\} \subseteq \{1, \dots, N\}$, and $b \in \{1, \dots, N\}$, by

$$I(\zeta, b, \{b_1, \dots, b_i\}) = \zeta_b - (\rho_{b,b_1}, \dots, \rho_{b,b_i}) \times \begin{pmatrix} 1 & \cdot & \rho_{b_1,b_i} \\ \cdot & & \cdot \\ \rho_{b_1,b_i} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_{b_1} \\ \cdot \\ \zeta_{b_i} \end{pmatrix}. \quad (25)$$

The vector ζ in this definition will be the transaction costs vector $\epsilon = (\epsilon_1, \dots, \epsilon_N)$, or a linear combination of the correlation vectors $\rho_{\cdot,j} = (\rho_{1,j}, \dots, \rho_{N,j})$.

We now construct the t_i 's, that is, the times at which the agents start selling the stocks or, equivalently, the stocks' minimum holding periods. Without loss of generality, we assume that stock 1 minimizes

$$\frac{\epsilon_k}{\sum_{j=1}^N \rho_{k,j} s_j}$$

over all stocks k . Therefore it has small transaction costs, ϵ_1 , and large systematic risk, $\sum_{j=1}^N \rho_{1,j} s_j$. Stock 1 is the first stock that agents start selling, and t_1 is defined by

$$\frac{g(t_1)}{f(t_1)} = \frac{2}{ra} \frac{\epsilon_1}{\sum_{j=1}^N \rho_{1,j} s_j}.$$

Agents start selling stock 1 first because this is the cheapest way to reduce portfolio risk. Similarly, we assume that stock 2 minimizes

$$\frac{\epsilon_k - \rho_{k,1} \epsilon_1}{\sum_{j=1}^N \rho_{k,j} s_j - \rho_{k,1} \sum_{j=1}^N \rho_{1,j} s_j}$$

over all stocks $k \geq 2$. Therefore, in addition to having small transaction costs and large systematic risk, it has a small correlation with stock 1. Stock 2 is the second stock that agents start selling, and t_2 is defined by

$$\frac{g(t_2)}{f(t_2)} = \frac{2}{ra} \frac{\epsilon_2 - \rho_{2,1} \epsilon_1}{\sum_{j=1}^N \rho_{2,j} s_j - \rho_{2,1} \sum_{j=1}^N \rho_{1,j} s_j}.$$

To generalize this construction, we set $S_i = \{1, \dots, i\}$ and $S_0 = \emptyset$, and assume that stock i minimizes

$$\frac{I(\epsilon, k, S_{i-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i-1})} \quad (26)$$

over all stocks $k \geq i$. Stock i is the i 'th stock that agents start selling, and t_i is defined by

$$\frac{g(t_i)}{f(t_i)} = \frac{2}{ra} \frac{I(\epsilon, i, S_{i-1})}{I(\sum_{j=1}^N \rho_{.,j} S_j, i, S_{i-1})}. \quad (27)$$

In Proposition 3 we construct the stock prices, $\bar{P}_i$, and stock holdings, $x_{i,t}$. Proposition 3 is proven in Appendix B.

Proposition 3. *If A is large, then $0 \leq t_1 \leq \dots \leq t_N < T$. Define $\bar{P}_i$ by*

$$\bar{P}_i = \frac{\bar{D}_i}{r} + \epsilon_i - a \sum_{k=1}^i \frac{I(\sum_{j=1}^N \rho_{.,j} S_j, k, S_{k-1})}{f(t_k)} \frac{I(\rho_{.,i}, k, S_{k-1})}{I(\rho_{.,k}, k, S_{k-1})} \quad (28)$$

and $x_{i,t}$ recursively by

$$\begin{pmatrix} x_{1,t} \\ \vdots \\ x_{N,t} \end{pmatrix} = \frac{1}{A_t} \begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{D}_1 - r\bar{P}_1 + r\epsilon_1 \\ \vdots \\ \bar{D}_N - r\bar{P}_N + r\epsilon_N \end{pmatrix} \quad (29)$$

for $t \in [t_N, T]$ and

$$x_{k,t} = x_{k,t_k} \quad k \geq i,$$

$$\begin{aligned} \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{i-1,t} \end{pmatrix} &= \frac{1}{A_t} \begin{pmatrix} 1 & \cdot & \rho_{1,i-1} \\ \cdot & \cdot & \cdot \\ \rho_{i-1,1} & \cdot & 1 \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} \bar{D}_1 - r\bar{P}_1 + r\epsilon_1 \\ \vdots \\ \bar{D}_{i-1} - r\bar{P}_{i-1} + r\epsilon_{i-1} \end{pmatrix} \\ &\quad - \begin{pmatrix} 1 & \cdot & \rho_{1,i-1} \\ \cdot & \cdot & \cdot \\ \rho_{i-1,1} & \cdot & 1 \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} \rho_{1,i} & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{i-1,i} & \cdot & \rho_{i-1,N} \end{pmatrix} \begin{pmatrix} x_{i,t} \\ \vdots \\ x_{N,t} \end{pmatrix} \end{aligned} \quad (30)$$

for $t \in [t_{i-1}, t_i]$ ($t_0 = 0$). Then all conditions of Proposition 1 and the market-clearing conditions hold.

Equations (29) and (30) give agents' stock holdings for $t \in [t_N, T]$ and $t \in [0, t_N]$, respectively. For $t \in [0, t_1]$ holdings of all stocks are constant. For $t \in [t_1, t_2]$ holdings of stocks 2 to N are constant, while holdings of stock 1 decrease. Agents use only stock 1 to reduce port-

folio risk. More generally, for $t \in [t_{i-1}, t_i]$ holdings of stocks i to N are constant, while holdings of stocks 1 to $i - 1$ decrease. Finally, for $t \in [t_N, T]$ holdings of all stocks decrease. With transaction costs, the composition of the stock portfolio depends on age. Low transaction cost stocks, held for short periods, are mainly in the portfolios of young agents, while high transaction cost stocks, held for long periods, are in the portfolios of old agents. Equation (28) gives stock prices in closed form. In Sections 4.2 and 4.3 we use Equations (28), (29), and (30) to study the effects of transaction costs on stock prices and turnover.

4.2 Stock prices

In this section we study how a stock's price depends on its and other stocks' transaction costs. Propositions 4 and 5 examine how the stock's price depends on its transaction costs, and how the effect of transaction costs depends on the stock's characteristics. Both propositions are proven in Appendix C. Before stating the propositions we define T_1 by $(2 + \gamma) - 3\gamma e^{-rT_1} - rT_1 = 0$, and T_2 by $2\gamma e^{-rT_2} - 1 = 0$.¹⁵

Proposition 4. *If $T \leq T_1$, then $\bar{P}_i$ decreases in ϵ_i . If $T > T_1$, then $\bar{P}_i$ increases for small ϵ_i and decreases for large ϵ_i .*

The surprising result of Proposition 4 is that a stock's price does not always decrease in its transaction costs. If T is large and ϵ_i small, it increases. To provide an intuition for Proposition 4 we assume that there is only one stock and that transaction costs increase from ϵ_1 to $\epsilon'_1 > \epsilon_1$. In Figure 1 we plot agents' stock holdings as a function of age for ϵ_1 and the corresponding equilibrium price $\bar{P}_1$ (solid line) and for ϵ'_1 and $\bar{P}_1$ (dashed line). When transaction costs increase, agents buy fewer shares, that is, $x'_{1,0} < x_{1,0}$. However, they hold these shares for longer periods, that is, they hold x shares for $t'_{1,x} > t_{1,x}$. Total stock holdings, the area under each curve, may thus increase or decrease. Stock holdings increase when T is large because agents are selling during a very long period. Therefore the effect that they sell more slowly dominates the effect that they buy fewer shares.

Proposition 5. *If $T \leq T_2$, then $\partial \bar{P}_i / \partial \epsilon_i$ increases in t_i . If $T > T_2$, then $\partial \bar{P}_i / \partial \epsilon_i$ decreases for small ϵ_i and may increase for large ϵ_i .*

Proposition 5 shows that a stock with a shorter minimum holding period, t_i , is not always more adversely affected by an increase in its transaction costs. If T is large and ϵ_i small, it is less adversely affected.

¹⁵ It is easy to check that both equations have a unique positive solution for any value of $\gamma = 1 - ra/A \in (0, 1)$.

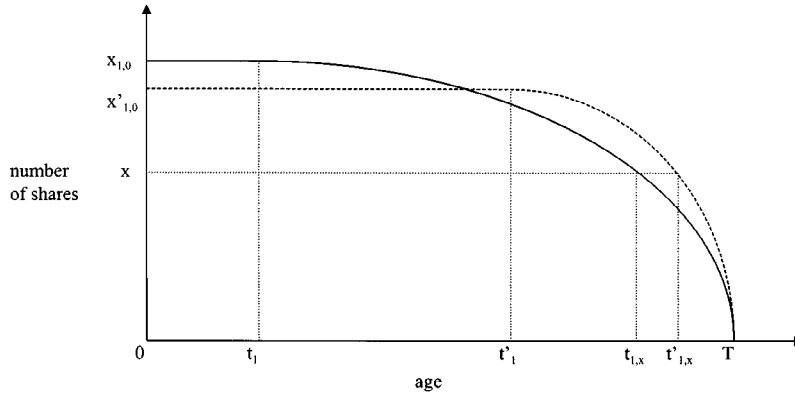


Figure 1
Transaction costs and stock holdings

Stock holdings as a function of age. The solid line corresponds to low transactions costs and the dashed line to high transactions costs. When transaction costs increase, agents buy fewer shares, that is, $x'_{1,0} < x_{1,0}$. However, they hold these shares for longer periods, that is, they hold x shares for $t'_{i,x} > t_{i,x}$.

In Corollary B.2, proven in Appendix B, we show that a stock's t_i increases in its transaction costs, ϵ_i , and decreases in its supply, s_i , which is a measure of the stock's risk. Proposition 5 thus implies that more liquid stocks or riskier stocks may be less adversely affected by increases in their transaction costs. This result is surprising since these stocks are more frequently traded.

To provide an intuition for Proposition 5, we examine the first-order condition for the shares of stock i , $x_{i,0}$, that an agent buys when born [i.e., Equation (15)]. We can write this equation as

$$\bar{P}_i = \frac{\bar{D}_i}{r} - \epsilon_i \frac{1 + e^{-rt_i}}{1 - e^{-rt_i}} - \frac{\int_0^{t_i} A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} e^{-rt} dt}{1 - e^{-rt_i}}, \quad (31)$$

and interpret it as a *pricing* equation. Stock i 's price, $\bar{P}_i$, has to be such that the agent is induced to buy the marginal $x_{i,0}$ th share and hold it for $t \leq t_i$. Equation (31) is a valuable tool for providing intuition for our results.

Equation (31) shows that the price, $\bar{P}_i$, is the present value of the dividend, $\bar{D}_i$, minus two terms. The first term compensates the agent for transaction costs. We call this term the "PV term" and write it as

$$\epsilon_i(1 + 2e^{-rt_i} + 2e^{-2rt_i} + \dots).$$

This is equal to the present value of the following transaction costs. First, the costs that the agent incurs buying and selling the $x_{i,0}$ th share,

at times 0 and t_i . Second, the costs that the next agent who buys the share incurs, at times t_i and $2t_i$, and so on. According to conventional wisdom, the effect of transaction costs on the stock price is equal to the PV term.

The second term compensates the agent for buying a risky stock, and is thus a risk premium. The risk premium increases in the stock's contribution to portfolio risk, $\sum_{j=1}^N \rho_{i,j} x_{j,t}$, for $t \leq t_i$, and in the coefficient of absolute risk aversion, A_t . Very importantly, the risk premium changes with transaction costs. Indeed, the lifetime pattern of stock holdings changes. As Figure 1 shows, agents buy fewer shares and hold them for longer periods. Therefore they hold fewer shares for $t \leq t_i$, and the risk premium decreases. The PV term thus overstates the effect of transaction costs on the stock price. The difference between the effect of transaction costs and the PV term is important and should not be neglected in practical applications. Equation (28) shows that the effect of transaction costs is of order ϵ , while the PV term is of order $\sqrt{\epsilon}$.¹⁶ The PV term measures correctly the effect of transaction costs in two special cases. First, when agents are risk neutral [Amihud and Mendelson (1986)] and second, when stocks are riskless [Vayanos and Vila (1997)]. In both cases the risk premium is zero.

Equation (31) shows why a stock with a shorter minimum holding period, t_i , may be less adversely affected by an increase in its transaction costs. On the one hand, the PV term increases more in response to an increase in transaction costs. On the other hand, agents greatly reduce their stock holdings during the shorter minimum holding period. Therefore the risk premium decreases more, and the overall effect is ambiguous.

Corollary 1 and Proposition 6 examine how a stock's price, $\bar{P}_i$, depends on the transaction costs of the other stocks, ϵ_k . Corollary 1 examines the case $i < k$ and is proven in Appendix B, while Proposition 6 examines the case $i > k$ and is proven in Appendix C.

Corollary 1. $\bar{P}_i$ is independent of ϵ_k for $i < k$.

Corollary 1 shows that a stock's price is independent of the transaction costs of less liquid stocks. If ϵ_k increases, agents buy fewer shares of stock k and hold them for longer periods. To adjust for that change, they buy more shares of stock i and hold them for shorter periods. Since stock i is more liquid, agents can exactly “undo” the change in their holdings of stock k and keep stock i 's contribution to portfolio risk, $\sum_{j=1}^N \rho_{i,j} x_{j,t}$, unchanged for each time t . [This is easiest

¹⁶ Equation (24) implies that $g(t)$ is of order t^2 for $t \sim 0$. Equation (27) implies then that t_i is of order $\sqrt{\epsilon}$. Therefore the PV term is of order $\sqrt{\epsilon}$.

to see in the extreme case where $\epsilon_i = 0$. In that case $t_i = 0$, and Equation (13) implies that $\sum_{j=1}^N \rho_{i,j} x_{j,t} = (\bar{D}_i - r\bar{P}_i)/A_t, \forall t$, independently of ϵ_k .] Therefore t_i , the PV term, the risk premium, and $\bar{P}_i$ remain unchanged.

Proposition 6. *If $T \leq T_2$, then $\bar{P}_i$ decreases in ϵ_k , for $i > k$. If $T > T_2$, then $\bar{P}_i$ increases for small ϵ_i, ϵ_k , and may decrease for large ϵ_i, ϵ_k .*

Proposition 6 shows that a stock's price does not always decrease in the transaction costs of more liquid stocks. If T is large and ϵ_i, ϵ_k small, it increases. To provide a simple intuition for this result, we note that if ϵ_k increases, stock i will be traded more frequently. Therefore, by Proposition 5, it may be more or less adversely affected by its own transaction costs, that is, its price may decrease or increase. We can also provide a more direct intuition using Equation (31). If ϵ_k increases, agents buy fewer shares of stock k and hold them for longer periods. Since stock i is less liquid, agents cannot "undo" the change in their holdings of stock k and start selling stock i earlier. Therefore t_i decreases (as we show in Corollary B.2) and the PV term increases. At the same time, since agents buy fewer shares of stock k and stock i is positively correlated with stock k , the risk premium decreases, and the overall effect is ambiguous.

A natural conjecture is that changes in transaction costs of the most liquid stocks have the strongest effect on the prices of the other stocks, since they affect most agents' portfolios. Corollary 2 shows that this conjecture is true in the special case $\rho_{i,j} = \rho \geq 0, \forall i, j, i \neq j$. The corollary follows from Proposition 6 and is proven in Appendix C.

Corollary 2. *Suppose that $\rho_{i,j} = \rho \geq 0, \forall i, j, i \neq j$. If $T \leq T_2$, then $\partial \bar{P}_i / \partial \epsilon_k$ increases in k for $k < i$. If $T > T_2$, then $\partial \bar{P}_i / \partial \epsilon_k$ decreases for small ϵ_i, ϵ_k , but may increase for large ϵ_i, ϵ_k .*

Corollary 2 shows that if the correlation between all stocks is the same, the effect on $\bar{P}_i$ of an increase in ϵ_k is stronger (either more negative or more positive) the more liquid k is. The corollary cannot be extended to the case where the correlations between stocks differ. We can choose, for instance, the most liquid stock to be independent of the other stocks, so that an increase in its transaction costs does not affect their price.

4.3 Stock turnover

In this section we study stock turnover, defined by Equation (17). We compute turnover in Proposition 7, proven in Appendix D.

Proposition 7. *The turnover of stock i is*

$$V_i = \frac{x_{i,0}}{Ts_i} = \frac{ra}{Ts_i} \left(\frac{1}{f(t_i)A_{t_i}} \frac{I(\sum_{j=1}^N \rho_{\cdot,j} S_j, i, S_{i-1})}{I(\rho_{\cdot,i}, i, S_{i-1})} - \sum_{j=i+1}^N \frac{1}{f(t_j)A_{t_j}} \frac{I(\sum_{j=1}^N \rho_{\cdot,j} S_j, j, S_{j-1})}{I(\rho_{\cdot,j}, j, S_{j-1})} \frac{I(\rho_{\cdot,j}, i, S_{j-1} \setminus \{i\})}{I(\rho_{\cdot,i}, i, S_{j-1} \setminus \{i\})} \right). \quad (32)$$

Proposition 8 examines how a stock's turnover depends on its and other stocks' transaction costs. The proposition is proven in Appendix D.

Proposition 8. *V_i decreases in ϵ_i and increases in ϵ_k for $k \neq i$*

The intuition for Proposition 8 is simple. If ϵ_i increases, V_i decreases since agents buy fewer shares of stock i and hold them for longer periods. If ϵ_k increases, agents buy fewer shares of stock k and hold them for longer periods. To adjust for that change, they buy more shares of stock i and sell them faster, that is, V_i increases. The result that V_i increases in ϵ_k depends crucially on Assumption 3, that is, that stocks are substitutes.

We also examine how V_i depends on stock i 's risk, as measured, for instance, by the stock's supply, s_i . If s_i increases, t_i decreases, that is, agents start selling the stock earlier. One would thus expect V_i to increase. However, V_i may decrease, if stock i is correlated with other stocks that have higher transaction costs. To economize on trading these stocks, agents hold a large fraction of stock i initially and a small fraction later. If s_i increases, gains to diversification become more important than economizing on transaction costs, and agents have to hold stock i for longer periods. Therefore V_i may decrease. Using Equation (32), one can check that V_i increases with s_i when, for instance, stock i is independent of the other stocks, while V_i decreases when there are two stocks, $i = 1$, $\rho_{1,2} > 0$, and $\epsilon_1 = 0$.

5. Binding Short-Sale Constraint

In this section we extend the analysis to the case where the short-sale constraint is binding. For simplicity we assume that there are only two positively correlated stocks, and set $\rho_{1,2} = \rho \in [0, 1]$. Assumption 1' ensures that the short-sale constraint is binding.

Assumption 1'. Suppose that $\rho < 1$ and

$$\frac{\epsilon_i}{s_i + \rho s_j} \leq \frac{\epsilon_j}{s_j + \rho s_i}$$

for $i \neq j \in \{1, 2\}$. Then, for $\bar{P}_i$ and $\bar{P}_j$ defined by Equation (28),

$$\bar{D}_i - r\bar{P}_i + r\epsilon_i - \rho(\bar{D}_j - r\bar{P}_j + r\epsilon_j) \leq 0. \quad (33)$$

Assumption 1' is the "opposite" of Assumptions 1 and 2 of Section 4, that is, the parameters s_1 , s_2 , ϵ_1 , ϵ_2 , and $\rho \in [0, 1]$ satisfy Assumption 1' if and only if they do not satisfy Assumptions 1 or 2.¹⁷ In section 5.1 we construct the equilibrium, and in section 5.2 we study the effects of transaction costs on stock prices.

5.1 Construction of the equilibrium

Without loss of generality, we assume that

$$\frac{\epsilon_1}{s_1 + \rho s_2} \leq \frac{\epsilon_2}{s_2 + \rho s_1}, \quad (34)$$

that is, stock 1 has a smaller ratio of transaction costs to systematic risk than stock 2. Stock 1 is the first stock that the agents start selling. Equation (34) is the same as in Section 4. However, now it implies that $\epsilon_1 \leq \epsilon_2$, that is, stock 1 has smaller transaction costs than stock 2.¹⁸ In Proposition 9 we construct the stock prices and stock holdings. Proposition 9 is proven in Appendix E.

Proposition 9. If A is large, the equations

$$g(t_1)(\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) - 2\epsilon_1 = 0, \quad (35)$$

$$g(t_2)(\bar{D}_2 - r\bar{P}_2 + r\epsilon_2) - 2\epsilon_2 - \rho(g(\hat{t}_1)(\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) - 2\epsilon_1) = 0, \quad (36)$$

¹⁷ The only nonobvious step in the proof is that if s_1 , s_2 , ϵ_1 , ϵ_2 , and $\rho \in [0, 1]$ do not satisfy Assumption 2, they satisfy Assumption 1'. Equation (28) implies that

$$\bar{D}_j - r\bar{P}_j + r\epsilon_j - \rho(\bar{D}_i - r\bar{P}_i + r\epsilon_i) = ar \frac{s_j(1 - \rho^2)}{f(t_j)} > 0.$$

Since Assumption 2 is not satisfied, Equation (33) must be satisfied.

¹⁸ When $\rho = 1$, this is obvious. When $\rho < 1$, it is less obvious and follows from Assumption 1'. The proof is available upon request.

$$\frac{\bar{D}_1 - r\bar{P}_1 + r\epsilon_1}{A_{\hat{t}_1}} - \rho \frac{\bar{D}_2 - r\bar{P}_2 + r\epsilon_2}{A_{t_2}} = 0, \quad (37)$$

$$\frac{f(t_1) - f(\hat{t}_1)}{ra} (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) = s_1, \quad (38)$$

and

$$\frac{f(t_2)}{ra} (\bar{D}_2 - r\bar{P}_2 + r\epsilon_2) = s_2, \quad (39)$$

have a solution, $(t_1, \hat{t}_1, t_2, \bar{P}_1, \bar{P}_2)$, such that $0 \leq t_1 < \hat{t}_1 \leq t_2 < T$. Define $x_{2,t}$ by

$$x_{2,t} = \frac{\bar{D}_2 - r\bar{P}_2 + r\epsilon_2}{A_t} \quad \forall t \in [t_2, T] \quad \text{and} \quad x_{2,t} = x_{2,t_2} \quad \forall t \in [0, t_2). \quad (40)$$

Define also $x_{1,t}$ by

$$\begin{aligned} x_{1,t} &= 0 \quad \forall t \in [\hat{t}_1, T], \\ x_{1,t} &= \frac{\bar{D}_1 - r\bar{P}_1 + r\epsilon_1}{A_t} - \rho x_{2,t} \quad \forall t \in [t_1, \hat{t}_1), \quad \text{and} \\ x_{1,t} &= x_{1,t_1} \quad \forall t \in [0, t_1). \end{aligned} \quad (41)$$

Then all conditions of Proposition 1 and the market-clearing conditions hold.

Equations (40) and (41) give agents' holdings of stocks 2 and 1, respectively. Agents start selling stock 1 first at time t_1 . By time $\hat{t}_1$ they own no shares of stock 1 and the short-sale constraint becomes binding. Agents start selling stock 2 at time t_2 . The stock prices, $\bar{P}_1$ and $\bar{P}_2$, are jointly determined with the times t_1 , $\hat{t}_1$, and t_2 , by the nonlinear system of Equations (35)–(39).

5.2 Stock prices

To study the effects of transaction costs on stock prices, we have to solve the nonlinear system of Equations (35)–(39). We can obtain closed-form solutions only for small transaction costs. In addition we have to assume $\rho = 1$, since if $\rho < 1$, the short-sale constraint is not binding for small transaction costs. For large transaction costs or $\rho < 1$, we solve the system numerically and obtain similar results. In Proposition 10, proven in Appendix F, we determine $\bar{P}_1$ and $\bar{P}_2$ for small transaction costs, $\rho = 1$, and, for simplicity, $A = \infty$, that is, $\gamma = 1$.

Proposition 10. Suppose that ϵ is small, $\rho = 1$, and $A = \infty$. The stock prices, $\bar{P}_1$ and $\bar{P}_2$, are given by

$$\begin{aligned} \bar{P}_1 = & \frac{\bar{D}_1}{r} - \frac{a(s_1 + s_2)}{f(0)} + \epsilon_1 \left(1 - \frac{2(1 - e^{-rT})}{rTf(0)} - \frac{2(\frac{T-t^*}{T} - \frac{1-e^{-r(T-t^*)}}{rT})}{(1 - e^{-rt^*})f(0)} \right) \\ & + \epsilon_2 \frac{2(\frac{T-t^*}{T} - \frac{1-e^{-r(T-t^*)}}{rT})}{(1 - e^{-rt^*})f(0)} + o(\epsilon), \end{aligned} \quad (42)$$

and

$$\begin{aligned} \bar{P}_2 = & \frac{\bar{D}_2}{r} - \frac{a(s_1 + s_2)}{f(0)} + \epsilon_1 \left(\frac{2(\frac{t^*}{T} - \frac{e^{-r(T-t^*)} - e^{-rT}}{rT})}{(1 - e^{-rt^*})f(0)} - \frac{2(1 - e^{-rT})}{rTf(0)} \right) \\ & + \epsilon_2 \left(1 - \frac{2(\frac{t^*}{T} - \frac{e^{-r(T-t^*)} - e^{-rT}}{rT})}{(1 - e^{-rt^*})f(0)} \right) + o(\epsilon), \end{aligned} \quad (43)$$

where t^* is defined by

$$\frac{f(t^*)}{f(0)} = \frac{s_2}{s_1 + s_2}. \quad (44)$$

In Corollary 3, proven in Appendix F, we examine whether the price of a stock decreases in its transaction costs.

Corollary 3. If $T \leq T_1$, then $\bar{P}_1$ and $\bar{P}_2$ decrease in ϵ_1 and ϵ_2 , respectively. If $T > T_1$, then $\bar{P}_1$ increases in ϵ_1 only if t^* is sufficiently large, and $\bar{P}_2$ increases in ϵ_2 only if t^* is sufficiently small.

Comparing Corollary 3 with Proposition 4, we conclude that the conditions for a stock's price to decrease in its transaction costs are weaker. The stock price can decrease even when $T > T_1$ and ϵ is small, provided that the stock is in small supply. Indeed, Equation (44) implies that t^* is large if s_1 is sufficiently larger than s_2 . To provide an intuition for why the stock price decreases when the stock is in small supply, assume that the stocks are identical, so that $\bar{P}_1 = \bar{P}_2$. Assume also that s_2 is small and that $\bar{P}_2$ increases in response to an increase in ϵ_2 . Since s_2 is small, the increase in ϵ_2 does not affect $\bar{P}_1$. Since, in addition, the stocks are perfectly correlated, investing in stock 1 dominates investing in stock 2. Therefore agents do not invest in stock 2, which contradicts equilibrium.

In Corollary 4, proven in Appendix F, we examine whether more liquid stocks are more adversely affected by increases in their transaction costs.

Table 1
Transaction costs and the stock's rate of return

$\epsilon_1/\bar{D}_1$	0	0.05	0.1	0.15	0.2	0.25	0.3
r_1	10%	10.034%	10.073%	10.12%	10.16%	10.21%	10.26%

Corollary 4. *If $T \leq 1.08/r$, then $\partial \bar{P}_1/\partial \epsilon_1 \leq \partial \bar{P}_2/\partial \epsilon_2$. If $T > 1.08/r$, then there exist some t^* such that the opposite holds.*

Corollary 4 shows that the more liquid stock 1 is always more adversely affected by an increase in its transaction costs, if $T \leq 1.08/r$. This condition is weaker than the one in Proposition 5, which was $T \leq \log(2)/r = 0.69/r$. An intuition for why the condition is weaker is the following. When the short-sale constraint is binding, the stocks are sold in sequence: the more liquid stock 1 is sold first, and the less liquid stock 2 is sold second. In response to a small change in transaction costs, prices adjust so that stocks are still sold in the same sequence. Therefore the lifetime pattern of stock holdings does not change much (relative to when the short-sale constraint is not binding). Stock holdings during a stock's minimum holding period, as well as the risk premium, do not change much either, and the change in the PV term dominates.

6. Calibration

We now study the effects of transaction costs for realistic parameter values. We assume that there is only one stock. We set agents' "economic" lifetimes, T , to 40 years, and the riskless rate, r , to 5%. We set A to ∞ , and choose the coefficient of absolute risk aversion, a , using the "average" rate of return on the stock, $r_1 = \bar{D}_1/\bar{P}_1$. We assume that a is such that, in the no transaction costs case, $r_1 = 10\%$.

In Table 1 we study the effects of transaction costs on the stock's rate of return. In the first row are the transaction costs, expressed as a fraction of $\bar{D}_1$, and in the second row is the stock's rate of return. To express transaction costs as a function of $\bar{P}_1$, we need to multiply $\epsilon_1/\bar{D}_1$ by r_1 , which is close to 10%. If, for instance, $\epsilon_1/\bar{D}_1 = 0.2$, then $\epsilon_1/\bar{P}_1$ is close to 0.02. Table 1 shows that transaction costs have a very small effect on the stock's rate of return. If, for instance, transaction costs increase from 0 to 2% of the stock price, the rate of return increases by 16 basis points.

In Table 2 we study the effects of transaction costs on the stock's minimum holding period and turnover. In the first row are the transaction costs, expressed as a fraction of $\bar{D}_1$. In the second row is the stock's minimum holding period. In the third row is the stock's turnover, and in the fourth row is the percentage reduction in turnover,

Table 2
Transaction costs and the stock's minimum holding period and turnover

$\epsilon_1/\bar{D}_1$	0	0.05	0.1	0.15	0.2	0.25	0.3
t_1	0	6.6	9.1	10.9	12.8	13.6	14.6
V_1	0.0381	0.0360	0.0352	0.0346	0.0342	0.0338	0.0334
Reduction	0	5.40%	7.50%	9.03%	10.28%	11.34%	12.26%

relative to the no transaction costs case. Table 2 shows that transaction costs have a large effect on the stock's minimum holding period and turnover. If, for instance, transaction costs increase from 0 to 2% of the stock price, agents wait for 12.8 years before starting to sell, and turnover decreases by 10.28%. However, turnover is small: even without transaction costs it is 3.81%. Turnover is small because it is generated only by life-cycle effects. Since turnover is small, it is perhaps not surprising that transaction costs have very small effects on stock returns.

Constantinides (1986) assumes that trading is generated by portfolio rebalancing. He finds very small effects of transaction costs on asset returns but large effects on agents' trading strategies. Our results are thus very similar to his. Aiyagari and Gertler (1991) and Heaton and Lucas (1996) assume that trading is generated by labor income shocks and consumption smoothing. They find larger effects of transaction costs on asset returns. Huang (1997) assumes that trading is generated by random liquidity shocks. He shows that transaction costs have larger effects when liquidity shocks are random than when they are predictable.

In our example, transaction costs increase the rate of return, that is, decrease the stock price. The value of T above which transaction costs increase the stock price is $T_1 = 56.4$. The value of T above which the surprising results of Propositions 5 and 6 hold is smaller however: $T_2 = 13.9$.

7. Concluding Remarks

In this article we develop a general equilibrium model with transaction costs. We assume a riskless, perfectly liquid bond with a constant rate of return and many risky stocks that carry proportional transaction costs. Trade occurs because there are overlapping generations of investors who buy the assets when born and slowly sell them until they die. The model is very tractable and stock prices are obtained in closed form. Our results are surprising and contrary to "conventional wisdom" about the effect of transaction costs on asset prices. First, the price of a stock may increase in its transaction costs. Second, the effect of transaction costs is smaller than the present value of transac-

tion costs incurred by a sequence of marginal investors. Third, a more frequently traded stock may be less adversely affected by an increase in its transaction costs. Fourth, a stock's price may decrease when the transaction costs of a more liquid and correlated stock decrease. For realistic parameter values, transaction costs have very small effects on asset returns but large effects on investors' trading strategies and turnover.

The analysis can be extended in a number of realistic directions. First, transaction costs can be assumed fixed, rather than proportional to the number of shares. We conjecture that this is feasible because stock prices will still be linear in dividends and agents' stock holdings deterministic. Second, transaction costs can be endogenized as a function of turnover. Since turnover and transaction costs will be mutually reinforcing, multiple equilibria may exist, with interesting welfare implications. The analysis may be relevant for the government bond market, where very similar bonds have very different transaction costs and turnover.

Appendix A: Proof of Proposition 1

We first prove two useful facts in Lemma A.1.

Lemma A.1. *Consider N continuous functions $x_{i,t}$, and two adapted, square-integrable processes $b_{1,t}$, $b_{2,t}$, in $[0, T]$. Then*

$$\begin{aligned} & E \left(e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^T b_{1,t} dt + \int_0^T b_{2,t} db_{i,t} \right) \right) \\ &= E \left(\int_0^T e^{-\int_0^t \sum_{j=1}^N x_{j,s} db_{j,s} + \frac{1}{2} \int_0^t \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds} \right. \\ &\quad \times \left. \left(b_{1,t} - b_{2,t} \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) dt \right). \end{aligned} \quad (\text{A.1})$$

If $b_{1,t}$, $b_{2,t}$ are (deterministic) functions in $[0, T]$, then

$$\begin{aligned} & E \left(e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^T b_{1,t} dt + \int_0^T b_{2,t} db_{i,t} \right) \right) \\ &= e^{\frac{1}{2} \int_0^T \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds} \int_0^T \left(b_{1,t} - b_{2,t} \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) dt. \end{aligned} \quad (\text{A.2})$$

Proof. We first prove Equation (A.1). The process

$$T \rightarrow e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^T b_{1,t} dt + \int_0^T b_{2,t} db_{i,t} \right)$$

is an Itô process. Itô's lemma implies that the drift term is

$$\begin{aligned} & e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(b_{1,T} - b_{2,T} \sum_{j=1}^N \rho_{i,j} x_{j,T} \right) \\ & + \frac{1}{2} e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^T b_{1,t} dt + \int_0^T b_{2,t} db_{i,t} \right) \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,T} x_{k,T} \end{aligned}$$

Denoting the left-hand side of Equation (A.1) by f_T , we thus have

$$\begin{aligned} \frac{df_T}{dT} &= E \left(e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(b_{1,T} - b_{2,T} \sum_{j=1}^N \rho_{i,j} x_{j,T} \right) \right) \\ &+ \frac{1}{2} f_T \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,T} x_{k,T}. \end{aligned} \quad (\text{A.3})$$

Denoting the right-hand side of Equation (A.1) by g_T , and inverting the expectation and the integral, we get

$$\begin{aligned} g_T &= \int_0^T e^{\frac{1}{2} \int_t^T \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds} \\ &\times E \left(e^{-\int_0^t \sum_{j=1}^N x_{j,s} db_{j,s}} \left(b_{1,t} - b_{2,t} \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) \right) dt. \end{aligned}$$

Differentiating with respect to T , we get

$$\begin{aligned} \frac{dg_T}{dT} &= E \left(e^{-\int_0^T \sum_{j=1}^N x_{j,t} db_{j,t}} \left(b_{1,T} - b_{2,T} \sum_{j=1}^N \rho_{i,j} x_{j,T} \right) \right) \\ &+ \frac{1}{2} g_T \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,T} x_{k,T}. \end{aligned} \quad (\text{A.4})$$

Equations (A.3) and (A.4), together with the fact that $f_0 = g_0 = 0$, imply that $f_T = g_T$. For Equation (A.2), we use Equation (A.1), invert the expectation and the integral, and use

$$E \left(e^{-\int_0^t \sum_{j=1}^N x_{j,s} db_{j,s}} \right) = e^{\frac{1}{2} \int_0^t \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds}. \quad \blacksquare$$

We now prove the proposition.

Proof. Consider a control $(x_{i,t}, c_t)$, an alternative control $(x_{i,t} + b_{i,t}, c_t + b_{c,t})$, and their convex combination $(x_{i,t} + \alpha b_{i,t}, c_t + \alpha b_{c,t})$. Denote by $U(\alpha)$ the utility associated to the convex combination and to a particular sample path. If $U(\alpha)$ is concave in α and has a right derivative $U'(0)$ for $\alpha = 0$ (as it will) then

$$U'(0) \geq U(1) - U(0)$$

and

$$EU'(0) \geq EU(1) - EU(0). \quad (\text{A.5})$$

Therefore a control $(x_{i,t}, c_t)$ such that $EU'(0)$ is negative for all $(b_{i,t}, b_{c,t})$ is optimal. To prove the proposition, we use this result and proceed in three steps. First, we compute $U'(0)$ assuming that the $x_{i,t}$'s are deterministic and satisfy condition (i) of the proposition. Second, we define the consumption c_t . Third, we use conditions (ii)–(v) and show that $EU'(0)$ is negative.

Determination of $U'(0)$

We first compute $U(\alpha)$. Equations (7), (9), and (10) imply that the consumption gulp at time T for the control $(x_{i,t} + \alpha b_{i,t}, c_t + \alpha b_{c,t})$ is $C_T(x_{i,t} + \alpha b_{i,t}, c_t + \alpha b_{c,t})$, where the function $C_T(x_{i,t}, c_t)$ is defined by

$$\begin{aligned} C_T(x_{i,t}, c_t) &= ye^{rT} + \int_0^T \sum_{i=1}^N (\bar{D}_i - r\bar{P}_i) x_{i,t} e^{r(T-t)} dt \\ &\quad + \int_0^T \sum_{i=1}^N x_{i,t} e^{r(T-t)} db_{i,t} \\ &\quad - \int_0^T c_t e^{r(T-t)} dt - \int_0^T \sum_{i=1}^N \epsilon_i \left| \frac{dx_{i,t}}{dt} \right| e^{r(T-t)} dt \\ &\quad - \sum_{i=1}^N \epsilon_i x_{i,0} e^{rT} - \sum_{i=1}^N \epsilon_i x_{i,T}. \end{aligned} \quad (\text{A.6})$$

Therefore

$$U(\alpha) = - \int_0^T e^{-a(c_t + \alpha b_{c,t}) - \beta t} dt - e^{-AC_T(x_{i,t} + \alpha b_{i,t}, c_t + \alpha b_{c,t}) - \beta T}. \quad (\text{A.7})$$

Equation (A.6) implies that $U(\alpha)$ is concave in α and has a right derivative for $\alpha = 0$. Equations (A.6), (A.7), and condition (i) of the

proposition, imply that

$$\begin{aligned}
 U'(0) = & \int_0^T a e^{-a c_t - \beta t} b_{c,t} dt + A e^{-A C_T - \beta T} \\
 & \times \left(\int_0^T \sum_{i=1}^N (\bar{D}_i - r \bar{P}_i) b_{i,t} e^{r(T-t)} dt \right. \\
 & + \int_0^T \sum_{i=1}^N b_{i,t} e^{r(T-t)} db_{i,t} - \int_0^T b_{c,t} e^{r(T-t)} dt \\
 & - \sum_{i=1}^N \epsilon_i \left(\int_0^{t_i} \left| \frac{db_{i,t}}{dt} \right| e^{r(T-t)} dt - \int_{t_i}^{\hat{t}_i} \frac{db_{i,t}}{dt} e^{r(T-t)} dt \right. \\
 & \left. \left. + \int_{\hat{t}_i}^T \left| \frac{db_{i,t}}{dt} \right| e^{r(T-t)} dt + b_{i,0} e^{rT} + b_{i,T} \right) \right) \quad (A.8)
 \end{aligned}$$

Consumption

We define the consumption flow by

$$\begin{aligned}
 c_t = & c_0 + \frac{(r - \beta)t}{a} + \int_0^t \frac{A_s}{a} \sum_{i=1}^N x_{i,s} db_{i,s} \\
 & + \frac{1}{2} \int_0^t \frac{A_s^2}{a} \sum_{i=1}^N \sum_{j=1}^N \rho_{i,j} x_{i,s} x_{j,s} ds, \quad (A.9)
 \end{aligned}$$

and the consumption gulp by $A e^{-A C_T} = a e^{-a c_T}$, that is,

$$C_T = \frac{a}{A} c_T + \frac{1}{A} \log \frac{A}{a}. \quad (A.10)$$

The first term in Equation (A.9) is the consumption at time 0, which we determine later. Consumption at time t is different from consumption at time 0 because of the second, third, and fourth terms. The second term corresponds to intertemporal substitution, and its sign depends on the comparison between the interest rate, r , and the discount rate, β . The third term corresponds to unexpected wealth changes between times 0 and t . Equation (7) implies that the unexpected wealth change at time $0 \leq s \leq t$ is $\sum_{i=1}^N x_{i,s} db_{i,s}$. The change in consumption flow is a fraction A_s/a of the wealth change. To motivate the term A_s/a , we note that the change in the consumption gulp is a fraction a/A of the consumption flow change, by Equation (A.10). The definition of A_s , that is, Equation (11), ensures that the present value of the consumption changes is equal to the wealth change. Finally, the fourth term corresponds to precautionary savings. Since consumption at

time t is uncertain as of time 0, the agent prefers it to be greater in expectation than consumption at time 0. The fourth term is simply equal to the variance of the third term, times $a/2$.

Before showing that $EU'(0)$ is negative, we need to check that the control $(c_{i,t}, x_{i,t})$ satisfies Equations (7), (9), and (10). This is equivalent to checking that

$$C_T = C_T(x_{i,t}, c_t), \quad (\text{A.11})$$

where C_T and $C_T(x_{i,t}, c_t)$ are defined by Equations (A.10) and (A.6), respectively. The definitions of $C_T(x_{i,t}, c_t)$, c_t , and C_T imply that Equation (A.11) holds if c_0 is appropriately chosen and if

$$\begin{aligned} \frac{a}{A} \int_0^T \frac{A_t}{a} \sum_{i=1}^N x_{i,t} db_{i,t} &= \int_0^T \sum_{i=1}^N x_{i,t} e^{r(T-t)} db_{i,t} \\ &\quad - \int_0^T \left(\int_0^t \frac{A_s}{a} \sum_{i=1}^N x_{i,s} db_{i,s} \right) e^{r(T-t)} dt. \end{aligned} \quad (\text{A.12})$$

Integrating by parts the last term and using the definition of A_t , that is, Equation (11), it is easy to check that Equation (A.12) holds.

$EU'(0)$ is negative

We now use Equation (A.8) and conditions (ii)–(v) of the proposition to show that $EU'(0) \leq 0$. We first consider the terms in $b_{c,t}$. Using the definition of C_T , we can write these terms as

$$E \left(\int_0^T a e^{-ac_t - \beta t} b_{c,t} dt \right) - E \left(a e^{-ac_T - \beta T} \int_0^T b_{c,t} e^{r(T-t)} dt \right). \quad (\text{A.13})$$

The definition of c_t implies that Equation (A.13) is 0 if

$$\begin{aligned} E \left(\int_0^T e^{-\int_0^t A_s \sum_{i=1}^N x_{i,s} db_{i,s} + \frac{1}{2} \int_0^t A_s^2 \sum_{i=1}^N \sum_{j=1}^N \rho_{i,j} x_{i,s} x_{j,s} ds} b_{c,t} e^{-rt} dt \right) \\ = E \left(\int_0^T e^{-\int_0^t A_t \sum_{i=1}^N x_{i,t} db_{i,t}} \int_0^T b_{c,t} e^{-rt} dt \right). \end{aligned} \quad (\text{A.14})$$

Equation (A.14) follows from Equation (A.1).

We now consider the terms in $b_{i,t}$. The definitions of c_t and C_T imply that these terms have the same sign as

$$\begin{aligned} E \left\{ e^{-\int_0^T A_t \sum_{i=1}^N x_{i,t} db_{i,t}} \left(\int_0^T (\bar{D}_i - r \bar{P}_i) b_{i,t} e^{r(T-t)} dt \right. \right. \\ \left. \left. + \int_0^T b_{i,t} e^{r(T-t)} db_{i,t} - \epsilon_i \left(\int_0^{t_i} \left| \frac{db_{i,t}}{dt} \right| e^{r(T-t)} dt \right) \right) \right\} \end{aligned}$$

$$- \int_{t_i}^{\hat{t}_i} \frac{db_{i,t}}{dt} e^{r(T-t)} dt + \int_{\hat{t}_i}^T \left| \frac{db_{i,t}}{dt} \right| e^{r(T-t)} dt \\ + b_{i,0} e^{rT} + b_{i,T} \Big) \Bigg) \Bigg\}. \quad (\text{A.15})$$

Using integration by parts

$$\begin{aligned} & - \int_{t_i}^{\hat{t}_i} \frac{dh_{i,t}}{dt} e^{r(T-t)} dt + \int_{\hat{t}_i}^T \left| \frac{dh_{i,t}}{dt} \right| e^{r(T-t)} dt \\ & \geq - \int_{t_i}^T \frac{dh_{i,t}}{dt} e^{r(T-t)} dt \\ & = -(b_{i,T} - b_{i,t_i} e^{r(T-t_i)}) - \int_{t_i}^T r b_{i,t} e^{r(T-t)} dt. \end{aligned}$$

This fact implies that Equation (A.15) is negative if

$$E \left\{ e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^T (\bar{D}_i - r\bar{P}_i) b_{i,t} e^{-rt} dt + \int_0^T b_{i,t} e^{-rt} db_{i,t} - \epsilon_i \left(\int_0^{t_i} \left| \frac{db_{i,t}}{dt} \right| e^{-rt} dt + b_{i,0} + b_{i,t_i} e^{-rt_i} - \int_{t_i}^T r b_{i,t} e^{-rt} dt \right) \right) \right\} \quad (\text{A.16})$$

is negative. We decompose this into

$$E\left(e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_{t_i}^T (\bar{D}_i - r\bar{P}_i + r\epsilon_i) b_{i,t} e^{-rt} dt + \int_{t_i}^T b_{i,t} e^{-rt} db_{i,t} \right) \right) \quad (\text{A.17})$$

and

$$E \left\{ e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^{t_i} (\bar{D}_i - r\bar{P}_i) b_{i,t} e^{-rt} dt + \int_0^{t_i} b_{i,t} e^{-rt} db_{i,t} - \epsilon_i \left(\int_0^{t_i} \left| \frac{db_{i,t}}{dt} \right| e^{-rt} dt + b_{i,0} + b_{i,t_i} e^{-rt_i} \right) \right) \right\}. \quad (\text{A.18})$$

Equation (A.17) corresponds to the interval $[t_i, T]$. We will show that it is negative using conditions (iii) and (v). Equation (A.18) corresponds to the interval $[0, t_i]$. We will show that it is negative using conditions (ii), (iii), and (iv).

Interval $[t_i, T]$

Using Equation (A.1), we can write Equation (A.17) as

$$E \left(\int_{t_i}^T e^{-\int_0^t A_s \sum_{j=1}^N x_{j,s} db_{j,s} + \frac{1}{2} \int_t^T A_s \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds} \right. \\ \left. \times \left(\bar{D}_i - r\bar{P}_i + r\epsilon_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) b_{i,t} dt \right). \quad (\text{A.19})$$

Since $x_{i,t} + b_{i,t} \geq 0$ and $x_{i,t} = 0$ for $t \in (\hat{t}_i, T]$, $b_{i,t} \geq 0$ for $t \in (\hat{t}_i, T]$. This fact and conditions (iii) and (v) imply that Equation (A.19) is negative.

Interval $[0, t_i]$

Using integration by parts

$$\int_0^{t_i} (\bar{D}_i - r\bar{P}_i) b_{i,t} e^{-rt} dt + \int_0^{t_i} b_{i,t} e^{-rt} db_{i,t} \\ = \left(\int_0^{t_i} (\bar{D}_i - r\bar{P}_i) e^{-rt} dt + \int_0^{t_i} e^{-rt} db_{i,t} \right) b_{i,0} \\ + \int_0^{t_i} \left(\int_t^{t_i} (\bar{D}_i - r\bar{P}_i) e^{-rs} ds + \int_t^{t_i} e^{-rs} db_{i,s} \right) \frac{db_{i,t}}{dt} dt.$$

This fact, together with

$$b_{i,t_i} = b_{i,0} + \int_0^{t_i} \frac{db_{i,t}}{dt} dt$$

imply that Equation (A.18) can be written as

$$E \left\{ e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^{t_i} (\bar{D}_i - r\bar{P}_i) e^{-rt} dt \right. \right. \\ \left. \left. + \int_0^{t_i} e^{-rt} db_{i,t} - \epsilon_i (1 + e^{-rt_i}) \right) b_{i,0} \right\} \\ + E \left\{ e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^{t_i} \left(\int_t^{t_i} (\bar{D}_i - r\bar{P}_i) e^{-rs} ds \right. \right. \right. \\ \left. \left. + \int_t^{t_i} e^{-rs} db_{i,s} - \epsilon_i e^{-rt_i} \right) \frac{db_{i,t}}{dt} dt \right. \\ \left. \left. - \int_0^{t_i} \epsilon_i \left| \frac{db_{i,t}}{dt} \right| e^{-rt} dt \right) \right\}. \quad (\text{A.20})$$

Using Equation (A.2), we can write the first term of Equation (A.20) as

$$e^{-\frac{1}{2} \int_0^T A_t \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,t} x_{k,t} dt} \times \left(\int_0^{t_i} \left(\bar{D}_i - r\bar{P}_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) e^{-rt} dt - \epsilon_i (1 + e^{-rt_i}) \right) b_{i,0}.$$

This is 0 because of condition (ii). We will show that the second term of Equation (A.20) is

$$E \left\{ \int_0^{t_i} e^{-\int_0^t A_s \sum_{j=1}^N x_{j,s} db_{j,s} + \frac{1}{2} \int_0^t A_s \sum_{j=1}^N \sum_{k=1}^N \rho_{j,k} x_{j,s} x_{k,s} ds} \left[\frac{db_{i,t}}{dt} \left(\int_t^{t_i} \left(\bar{D}_i - r\bar{P}_i - A_s \sum_{j=1}^N \rho_{i,j} x_{j,s} \right) e^{-rs} ds - \epsilon_i e^{-rt_i} \right) - \epsilon_i \left| \frac{db_{i,t}}{dt} \right| e^{-rt} \right] dt \right\}. \quad (\text{A.21})$$

To obtain the terms in $\bar{D}_i - r\bar{P}_i$ and ϵ_i , we use Equation (A.1). To obtain the term in $A_s \sum_{j=1}^N \rho_{i,j} x_{j,s}$, we need to compute

$$E \left\{ e^{-\int_0^T A_t \sum_{j=1}^N x_{j,t} db_{j,t}} \left(\int_0^{t_i} \left(\int_t^{t_i} e^{-rs} db_{i,s} \right) \frac{db_{i,t}}{dt} dt \right) \right\}. \quad (\text{A.22})$$

Inverting expectation and integral and rearranging, we get

$$\int_0^{t_i} \left(E \left(e^{-\int_0^t A_s \sum_{j=1}^N x_{j,s} db_{j,s}} \frac{db_{i,t}}{dt} e^{-\int_t^T A_s \sum_{j=1}^N x_{j,s} db_{j,s}} \int_t^{t_i} e^{-rs} db_{i,s} \right) \right) dt.$$

Since the process $db_{i,t}/dt$ is adapted, we get

$$\int_0^{t_i} \left(E \left(e^{-\int_0^t A_s \sum_{j=1}^N x_{j,s} db_{j,s}} \frac{db_{i,t}}{dt} \right) E \left(e^{-\int_t^T A_s \sum_{j=1}^N x_{j,s} db_{j,s}} \int_t^{t_i} e^{-rs} db_{i,s} \right) \right) dt.$$

Using Equation (A.2) to compute the second expectation, and inverting back expectation and integral, we obtain the term $A_s \sum_{j=1}^N \rho_{i,j} x_{j,s}$ in Equation (A.21). To show that Equation (A.21) is negative, we distinguish two cases.

Case 1. $dh_{i,t}/dt \geq 0$

The term in square brackets has the same sign as

$$\int_t^{t_i} \left(\bar{D}_i - r\bar{P}_i - A_s \sum_{j=1}^N \rho_{i,j} x_{j,s} \right) e^{-rs} ds - \epsilon_i e^{-rt_i} - \epsilon_i e^{-rt}. \quad (\text{A.23})$$

The derivative of Equation (A.23) with respect to t has the same sign as

$$- \left(\bar{D}_i - r\bar{P}_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) + r\epsilon_i.$$

Condition (iv) implies that this expression increases in t , and thus Equation (A.23) is convex in t . Since Equation (A.23) is 0 for $t = 0$ [because of condition (ii)] and is $-2\epsilon_i e^{-rt_i} \leq 0$ for $t = t_i$, it is negative.

Case 2. $dh_{i,t}/dt < 0$

The term in square brackets has the same sign as

$$- \int_t^{t_i} \left(\bar{D}_i - r\bar{P}_i - A_s \sum_{j=1}^N \rho_{i,j} x_{j,s} \right) e^{-rs} ds + \epsilon_i e^{-rt_i} - \epsilon_i e^{-rt}. \quad (\text{A.24})$$

The derivative of Equation (A.24) with respect to t has the same sign as

$$\left(\bar{D}_i - r\bar{P}_i - A_t \sum_{j=1}^N \rho_{i,j} x_{j,t} \right) + r\epsilon_i.$$

Condition (iv) implies that this expression decreases in t . Since it is 0 at $t = t_i$, [because of condition (iii)] it is positive in $[0, t_i]$. Therefore Equation (A.24) increases and, since it is 0 for $t = t_i$, it is negative. ■

Appendix B: Proof of Proposition 3

We first prove Lemmas B.1–B.5 and Corollary B.1. The lemmas and the corollary establish some matrix algebra results. These results are needed when there are two or more stocks, and are used in Appendixes B, C, and D.

Lemma B.1. *Suppose that $\{b_1, \dots, b_i\} \subseteq S_N$ and $j \leq i$. Then*

$$\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_{b_1} \\ \cdot \\ \zeta_{b_i} \end{pmatrix} \right)_j = \frac{I(\zeta, b_j, \{b_1, \dots, b_i\} \setminus \{b_j\})}{I(\rho, b_j, b_j, \{b_1, \dots, b_i\} \setminus \{b_j\})}. \quad (\text{B.1})$$

Proof. By rearranging rows and columns, we can assume that $j = i$. Using the “block inverse” formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \quad (\text{B.2})$$

for A , the submatrix formed by rows and columns 1 to $i - 1$, we find that

$$\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_{b_1} \\ \cdot \\ \zeta_{b_i} \end{pmatrix} \right)_i = \frac{I(\zeta, b_i, \{b_1, \dots, b_{i-1}\})}{I(\rho, b_i, b_i, \{b_1, \dots, b_{i-1}\})}$$

that is, Equation (B.1) for $j = i$. ■

Lemma B.2. Suppose that $\{b_1, \dots, b_i\} \subseteq S_N$, $j \neq k$ and $j, k \leq i$. Then

$$\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \right)_{k, k} = \frac{1}{I(\rho, b_k, b_k, \{b_1, \dots, b_i\} \setminus \{b_k\})} \quad (\text{B.3})$$

and

$$\begin{aligned} & \left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \right)_{j, k} \\ &= - \frac{I(\rho, b_k, b_j, \{b_1, \dots, b_i\} \setminus \{b_j, b_k\})}{I(\rho, b_k, b_k, \{b_1, \dots, b_i\} \setminus \{b_k\}) I(\rho, b_j, b_j, \{b_1, \dots, b_i\} \setminus \{b_j, b_k\})} \quad (\text{B.4}) \end{aligned}$$

Proof. By rearranging rows and columns, we can assume that $k = i$. We use Equation (B.2) for A , the submatrix formed by rows and columns 1 to $i - 1$, and get

$$\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \right)_{i, i} = \frac{1}{I(\rho, b_i, b_i, \{b_1, \dots, b_{i-1}\})}$$

that is, Equation (B.3) for $k = i$, and

$$\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_i} \\ \cdot & \cdot & \cdot \\ \rho_{b_i, b_1} & \cdot & 1 \end{pmatrix}^{-1} \right)_{j, i}$$

$$\begin{aligned}
&= -\frac{1}{I(\rho_{\cdot, b_i}, b_i, \{b_1, \dots, b_{i-1}\})} \\
&\quad \times \left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_{i-1}} \\ \cdot & \cdot & \cdot \\ \rho_{b_{i-1}, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{b_1, b_i} \\ \cdot \\ \rho_{b_{i-1}, b_i} \end{pmatrix} \right)_j \quad (\text{B.5})
\end{aligned}$$

Lemma B.1 implies that

$$\begin{aligned}
&\left(\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_{i-1}} \\ \cdot & \cdot & \cdot \\ \rho_{b_{i-1}, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{b_1, b_i} \\ \cdot \\ \rho_{b_{i-1}, b_i} \end{pmatrix} \right)_j \\
&= \frac{I(\rho_{\cdot, b_i}, b_j, \{b_1, \dots, b_{i-1}\} \setminus \{b_j\})}{I(\rho_{\cdot, b_j}, b_j, \{b_1, \dots, b_{i-1}\} \setminus \{b_j\})}. \quad (\text{B.6})
\end{aligned}$$

Equations (B.5) and (B.6) imply Equation (B.4) for $k = i$. $\blacksquare$

Corollary B.1 contains some immediate and useful implications of Lemma B.2.

Corollary B.1. *Suppose that $\{b_1, \dots, b_i\} \subseteq S_N$, $j \neq k$ and $j, k \leq i$. Then*

$$I(\rho_{\cdot, b_k}, b_k, \{b_1, \dots, b_i\} \setminus \{b_k\}) > 0 \quad (\text{B.7})$$

$$\begin{pmatrix} 1 & \cdot & \rho_{b_1, b_{i-1}} \\ \cdot & \cdot & \cdot \\ \rho_{b_{i-1}, b_1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{b_1, b_i} \\ \cdot \\ \rho_{b_{i-1}, b_i} \end{pmatrix} \geq 0 \quad (\text{B.8})$$

$$I(\rho_{\cdot, b_k}, b_j, \{b_1, \dots, b_i\} \setminus \{b_j, b_k\}) \geq 0 \quad (\text{B.9})$$

Proof. Equation (B.7) follows from Equation (B.3) of Lemma B.2 and the fact that the matrix in the statement of the lemma is positive definite. Equation (B.8) follows from Assumption 3, Equation (B.5), and Equation (B.7). Equation (B.9) follows from Assumption 3, Equation (B.4) of Lemma B.2, and Equation (B.7). $\blacksquare$

Lemma B.3. *Suppose that for the vector $\zeta = (\zeta_1, \dots, \zeta_N)$ we have*

$$\begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & \cdot & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_1 \\ \cdot \\ \zeta_N \end{pmatrix} > 0. \quad (\text{B.10})$$

Then

$$\begin{pmatrix} 1 & \cdot & \rho_{1,i} \\ \cdot & & \cdot \\ \rho_{i,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_1 \\ \cdot \\ \zeta_i \end{pmatrix} > 0. \quad (\text{B.11})$$

for all $i \leq N$.

Proof. Equation (B.10) is equivalent to

$$\begin{pmatrix} \zeta_1 \\ \cdot \\ \zeta_N \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \rho_{1,N} \\ \cdot & & \cdot \\ \rho_{N,1} & \cdot & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \cdot \\ \eta_N \end{pmatrix} \quad (\text{B.12})$$

where $\eta_i > 0$. Multiplying the vector of the first i components of Equation (B.12) by the inverse matrix of

$$\begin{pmatrix} 1 & \cdot & \rho_{1,i} \\ \cdot & & \cdot \\ \rho_{i,1} & \cdot & 1 \end{pmatrix} \quad (\text{B.13})$$

we get

$$\begin{aligned} \begin{pmatrix} 1 & \cdot & \rho_{1,i} \\ \cdot & & \cdot \\ \rho_{i,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_1 \\ \cdot \\ \zeta_i \end{pmatrix} &= \begin{pmatrix} \eta_1 \\ \cdot \\ \eta_i \end{pmatrix} + \begin{pmatrix} 1 & \cdot & \rho_{1,i} \\ \cdot & & \cdot \\ \rho_{i,1} & \cdot & 1 \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} \rho_{1,i+1} & \cdot & \rho_{1,N} \\ \cdot & & \cdot \\ \rho_{i,i+1} & \cdot & \rho_{i,N} \end{pmatrix} \begin{pmatrix} \eta_{i+1} \\ \cdot \\ \eta_N \end{pmatrix} \end{aligned} \quad (\text{B.14})$$

Equation (B.8) of Corollary B.1 implies that all elements of the matrix multiplying $(\eta_{i+1}, \dots, \eta_N)$ are positive. Since the η_i 's are strictly positive, the right-hand side of the above equation is strictly positive. ■

Lemma B.4. Consider two vectors $\zeta = (\zeta_1, \dots, \zeta_N)$ and $\theta = (\theta_1, \dots, \theta_N)$ and suppose that $I(\zeta, k, S_{i-1}) > 0$ for all i and $k \geq i$. Then

$$\frac{I(\theta, i, S_{i-1})}{I(\zeta, i, S_{i-1})} \leq \frac{I(\theta, k, S_{i-1})}{I(\zeta, k, S_{i-1})} \Rightarrow \frac{I(\theta, k, S_{i-1})}{I(\zeta, k, S_{i-1})} \leq \frac{I(\theta, k, S_i)}{I(\zeta, k, S_i)} \quad (\text{B.15})$$

for $k > i$.

Proof. Using Equation (B.2) for Equation (B.13) and for A , the submatrix formed by rows 1 to $i-1$ and columns 1 to $i-1$, it is a matter

of algebra to check that

$$I(\zeta, k, S_i) = I(\zeta, k, S_{i-1}) - I(\zeta, i, S_{i-1}) \frac{I(\rho_{\cdot, k}, i, S_{i-1})}{I(\rho_{\cdot, i}, i, S_{i-1})} \quad (\text{B.16})$$

and similarly for θ . Lemma B.4 follows then from Equations (B.7) and (B.9) of Corollary B.1. $\blacksquare$

Lemma B.5. *Suppose that for the vectors $\zeta = (\zeta_1, \dots, \zeta_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ we have $I(\zeta, i, S_{i-1}) = \eta_i \forall i$. Then*

$$\zeta_i = \sum_{k=1}^i \eta_k \frac{I(\rho_{\cdot, i}, k, S_{k-1})}{I(\rho_{\cdot, k}, k, S_{k-1})} \quad \forall i. \quad (\text{B.17})$$

The converse also holds.

Proof. Equation (B.16) implies that

$$\begin{aligned} I(\zeta, i, S_{i-1}) &= I(\zeta, i, S_{i-2}) - I(\zeta, i-1, S_{i-2}) \frac{I(\rho_{\cdot, i}, i-1, S_{i-2})}{I(\rho_{\cdot, i-1}, i-1, S_{i-2})} \\ &= I(\zeta, i, S_{i-3}) - I(\zeta, i-2, S_{i-3}) \frac{I(\rho_{\cdot, i}, i-2, S_{i-3})}{I(\rho_{\cdot, i-2}, i-2, S_{i-3})} \\ &\quad - I(\zeta, i-1, S_{i-2}) \frac{I(\rho_{\cdot, i}, i-1, S_{i-2})}{I(\rho_{\cdot, i-1}, i-1, S_{i-2})} \\ &= \dots = I(\zeta, i, S_0) - \sum_{k=1}^{i-1} I(\zeta, k, S_{k-1}) \frac{I(\rho_{\cdot, i}, k, S_{k-1})}{I(\rho_{\cdot, k}, k, S_{k-1})}. \end{aligned}$$

Therefore

$$I(\zeta, i, S_{i-1}) = \zeta_i - \sum_{k=1}^{i-1} I(\zeta, k, S_{k-1}) \frac{I(\rho_{\cdot, i}, k, S_{k-1})}{I(\rho_{\cdot, k}, k, S_{k-1})}. \quad (\text{B.18})$$

Suppose that $I(\zeta, i, S_{i-1}) = \eta_i \forall i$. Equation (B.18) implies Equation (B.17). Conversely suppose that Equation (B.17) holds $\forall i$. Using Equation (B.18) and proceeding recursively, we find that $I(\zeta, i, S_{i-1}) = \eta_i \forall i$. $\blacksquare$

We now prove Proposition 1 and Corollaries B.2 and 1.

Proof. We first show that the t_i 's are well defined by Equation (27) and that $0 \leq t_1 \leq \dots \leq t_N < T$. We then show that the conditions of Proposition 1 and the market-clearing conditions hold. At the end of the proof, we motivate our construction of the equilibrium.

The t_i 's

We first show that for $k \geq i$, $I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i-1}) > 0$. Noting that $I(\rho_{\cdot,j}, k, S_{i-1}) = 0$ for $j \in S_{i-1}$, we get

$$I\left(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i-1}\right) = I(\rho_{\cdot,k}, k, S_{i-1}) s_k + \sum_{j \in \{i, \dots, N\} \setminus \{k\}} I(\rho_{\cdot,j}, k, S_{i-1}) s_j. \quad (\text{B.19})$$

Equations (B.7) and (B.9) of Corollary B.1 and the fact that $s_i > 0$ imply that the right-hand side of Equation (B.19) is strictly positive.

We now show that

$$0 \leq \frac{I(\epsilon, 1, S_0)}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, i, S_0)} \leq \dots \leq \frac{I(\epsilon, N, S_{N-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, N, S_{N-1})}. \quad (\text{B.20})$$

Consider $i \leq i' < k$. Using Equation (26) for stocks i' and k , we get

$$\frac{I(\epsilon, i', S_{i'-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, i', S_{i'-1})} \leq \frac{I(\epsilon, k, S_{i'-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i'-1})}.$$

Lemma B.4 implies then that

$$\frac{I(\epsilon, k, S_{i'-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i'-1})} \leq \frac{I(\epsilon, k, S_{i'})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i'})}. \quad (\text{B.21})$$

Using Equation (26) for stocks i and k , and Equation (B.21) for $i \leq i' < k$, we get

$$\begin{aligned} \frac{I(\epsilon, i, S_{i-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, i, S_{i-1})} &\leq \frac{I(\epsilon, k, S_{i-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{i-1})} \\ &\leq \frac{I(\epsilon, k, S_{k-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, k, S_{k-1})}. \end{aligned}$$

Finally, it is immediate to check that $g(t)$ is strictly increasing and is 0 at $t = 0$, and that $f(t)$ is strictly positive and decreasing. As a result, $g(t)/f(t)$ is strictly increasing and is 0 at $t = 0$. Since $g(T) > 0$ and $f(T) = 0$ for $A = \infty$, then

$$\frac{g(T)}{f(T)} > \frac{2}{ra} \frac{I(\epsilon, N, S_{N-1})}{I(\sum_{j=1}^N \rho_{\cdot,j} s_j, N, S_{N-1})} \quad (\text{B.22})$$

is satisfied for large A . Equations (27), (B.20), and (B.22) imply then that the t_i 's are well defined and that $0 \leq t_1 \leq \dots \leq t_N < T$.

Conditions hold

It is immediate to check that the $x_{i,t}$'s are continuous and piecewise C^1 at $[0, T]$.

Condition (i)

Assumption 2 implies that Equation (B.10) holds for $\zeta = \bar{D} - r\bar{P} + r\epsilon$. Lemma B.3 and Equations (29) and (30) imply then that $dx_{i,t}/dt < 0$ in $[t_i, T]$. Since $x_{i,t} = x_{i,t_i}$ in $[0, t_i]$, condition (i) holds.

Condition (iii)

The definition of the $x_{i,t}$'s makes it clear that condition (iii) holds.

Condition (ii)

We first write Equation (28) as

$$\bar{D}_i - r\bar{P}_i + r\epsilon_i = ra \sum_{k=1}^i \frac{I(\sum_{j=1}^N \rho_{\cdot,j} S_j, k, S_{k-1})}{f(t_k)} \frac{I(\rho_{\cdot,i}, k, S_{k-1})}{I(\rho_{\cdot,k}, k, S_{k-1})}. \quad (\text{B.23})$$

Equation (B.23) and Lemma B.5 imply that

$$\frac{f(t_i)}{ra} I(\bar{D} - r\bar{P} + r\epsilon, i, S_{i-1}) = I\left(\sum_{j=1}^N \rho_{\cdot,j} S_j, i, S_{i-1}\right). \quad (\text{B.24})$$

Equations (27) and (B.24) imply

$$g(t_i)I(\bar{D} - r\bar{P} + r\epsilon, i, S_{i-1}) - 2I(\epsilon, i, S_{i-1}) = 0. \quad (\text{B.25})$$

To show that condition (ii) holds, we proceed recursively. We assume that it holds for stocks 1 to $i-1$ and show that it holds for stock i (i can be 1). Combining condition (ii) for stock $k < i$ and condition (iii) for stock k and $t \in [t_k, t_i]$, we find

$$\int_0^{t_i} \left(\bar{D}_k - r\bar{P}_k + r\epsilon_k - A_t \sum_{j=1}^N \rho_{k,j} x_{j,t} \right) e^{-rt} dt - 2\epsilon_k = 0. \quad (\text{B.26})$$

Using condition (iii) for stock k and $t = t_i$, we can write Equation (B.26) as

$$\int_0^{t_i} \left((\bar{D}_k - r\bar{P}_k + r\epsilon_k) \left(1 - \frac{A_t}{A_{t_i}} \right) - A_t \sum_{j=1}^N \rho_{k,j} (x_{j,t} - x_{j,t_i}) \right) e^{-rt} dt - 2\epsilon_k = 0.$$

Noting that $x_{j,t} = x_{j,t_i}$ for $j \geq i$ and $t \in [0, t_i]$, and using the definition

of $g(t)$, that is, Equation (24), we can write the above equation as

$$g(t_i)(\bar{D}_k - r\bar{P}_k + r\epsilon_k) - \int_0^{t_i} A_t \sum_{j=1}^{i-1} \rho_{k,j}(x_{j,t} - x_{j,t_i}) e^{-rt} dt - 2\epsilon_k = 0. \quad (\text{B.27})$$

Proceeding for stock i as we did for stock k , we conclude that proving condition (ii) for stock i is equivalent to proving

$$g(t_i)(\bar{D}_i - r\bar{P}_i + r\epsilon_i) - \int_0^{t_i} A_t \sum_{j=1}^{i-1} \rho_{i,j}(x_{j,t} - x_{j,t_i}) e^{-rt} dt - 2\epsilon_i = 0. \quad (\text{B.28})$$

Using Equation (B.27) for $k < i$ ($i - 1$ equations) to eliminate

$$\int_0^{t_i} A_t (x_{k,t} - x_{k,t_i}) e^{-rt} dt$$

for $k < i$ ($i - 1$ unknowns) in Equation (B.28), we conclude that Equation (B.28) is equivalent to Equation (B.25).

Condition (iv)

To show that $A_t \sum_{j=1}^N \rho_{i,j} x_{j,t}$ increases in $[0, t_i]$, we will show that $A_t \sum_{j=1}^N \rho_{i,j} x_{j,t}$ increases in $[t_{k-1}, t_k]$ for $k \leq i$. Equation (30) implies that it is equal to a constant term plus

$$A_t \left\{ -(\rho_{i,1}, \dots, \rho_{i,k-1}) \begin{pmatrix} 1 & \cdot & \rho_{1,k-1} \\ \cdot & & \cdot \\ \rho_{k-1,1} & \cdot & 1 \end{pmatrix}^{-1} \right. \\ \left. \times \begin{pmatrix} \rho_{1,k} & \cdot & \rho_{1,N} \\ \cdot & & \cdot \\ \rho_{k-1,k} & \cdot & \rho_{k-1,N} \end{pmatrix} \begin{pmatrix} x_k \\ \cdot \\ x_N \end{pmatrix} + \sum_{j=k}^N \rho_{i,j} x_j \right\}.$$

The coefficient of $x_j > 0$ for $j \geq k$ in the above expression is

$$A_t \left\{ \rho_{i,j} - (\rho_{i,1}, \dots, \rho_{i,k-1}) \begin{pmatrix} 1 & \cdot & \rho_{1,k-1} \\ \cdot & & \cdot \\ \rho_{k-1,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{1,j} \\ \cdot \\ \rho_{k-1,j} \end{pmatrix} \right\} \\ = A_t I(\rho_{\cdot,j}, i, S_{k-1}).$$

Equation (B.9) of Corollary B.1 implies that $I(\rho_{\cdot,j}, i, S_{k-1}) \geq 0$. Since A_t is increasing, the above expression is increasing.

Market-Clearing Conditions

Finally, we show the market-clearing conditions. We will show that

$$\sum_{j=1}^N \rho_{i,j} \int_0^T x_{j,t} \frac{dt}{T} = \sum_{j=1}^N \rho_{i,j} s_j \quad \forall i \quad (\text{B.29})$$

and conclude, from Assumption 1, that $\int_0^T x_{i,t} dt / T = s_i, \forall i$. We also proceed recursively, assume that Equation (B.29) holds for stocks 1 to $i-1$ and show that it holds for stock i .

We write condition (iii) for stock $k < i$ as

$$\frac{\bar{D}_k - r\bar{P}_k + r\epsilon_k}{A_t} - \sum_{j=1}^N \rho_{k,j} x_{j,t} = 0.$$

We integrate this equation from $t_i \geq t_k$ to T and add to it the same equation for $t = t_i$ and multiplied by t_i . Using also the definition of $f(t)$, that is, Equation (18), we get

$$\begin{aligned} \frac{f(t_i)}{ra} (\bar{D}_k - r\bar{P}_k + r\epsilon_k) &= \sum_{j=1}^N \rho_{k,j} \left(\frac{t_i}{T} x_{j,t_i} + \int_{t_i}^T x_{j,t} \frac{dt}{T} \right) \\ &= \sum_{j=1}^N \rho_{k,j} \left(\int_0^T x_{j,t} \frac{dt}{T} - \int_0^{t_i} (x_{j,t} - x_{j,t_i}) \frac{dt}{T} \right). \end{aligned}$$

Since $x_{j,t} = x_{j,t_i}$ for $j \geq i$ and $t \in [0, t_i]$, we can write the above equation as

$$\begin{aligned} \frac{f(t_i)}{ra} (\bar{D}_k - r\bar{P}_k + r\epsilon_k) &= \sum_{j=1}^N \rho_{k,j} \int_0^T x_{j,t} \frac{dt}{T} \\ &\quad - \sum_{j=1}^{i-1} \int_0^{t_i} \rho_{k,j} (x_{j,t} - x_{j,t_i}) \frac{dt}{T}. \quad (\text{B.30}) \end{aligned}$$

Since Equation (B.29) holds for stocks 1 to $i-1$, we can write Equation (B.30) as

$$\frac{f(t_i)}{ra} (\bar{D}_k - r\bar{P}_k + r\epsilon_k) = \sum_{j=1}^N \rho_{k,j} s_j - \sum_{j=1}^{i-1} \int_0^{t_i} \rho_{k,j} (x_{j,t} - x_{j,t_i}) \frac{dt}{T}. \quad (\text{B.31})$$

Proceeding for stock i as we did for stock k , we conclude that proving

Equation (B.29) for stock i is equivalent to proving

$$\frac{f(t_i)}{ra}(\bar{D}_i - r\bar{P}_i + r\epsilon_i) = \sum_{j=1}^N \rho_{i,j} S_j - \sum_{j=1}^{i-1} \int_0^{t_i} \rho_{i,j} (x_{j,t} - x_{j,t_i}) \frac{dt}{T}. \quad (\text{B.32})$$

Using Equation (B.31) for $k < i$ ($i - 1$ equations) to eliminate

$$\int_0^{t_i} (x_{k,t} - x_{k,t_i}) \frac{dt}{T}$$

for $k < i$ ($i - 1$ unknowns) in Equation (B.32), we conclude that Equation (B.32) is equivalent to Equation (B.24). ■

We can now work backwards and motivate our construction of the equilibrium. First, we fix the t_i 's and the $\bar{P}_i$'s, and define the $x_{i,t}$'s from condition (iii). Second, we derive Equations (B.25) and (B.24) using condition (ii) and market clearing, respectively, and proceeding as in the proof. Third, we determine the t_i 's and the $\bar{P}_i$'s. To determine the t_i 's we simply divide Equation (B.25) by Equation (B.24). To determine the $\bar{P}_i$'s we solve Equation (B.24) (N equations) using Lemma B.5.

In Corollary B.2 we study how the minimum holding period of a stock depends on its and other stocks' transaction costs, and on its supply.

Corollary B.2. *The minimum holding period of stock i , t_i , increases in ϵ_i , is independent of ϵ_k for $i < k$, decreases in ϵ_k for $i > k$, and decreases in s_i .*

Proof. We use Equation (27) and the fact that $g(t)/f(t)$ is strictly increasing. To prove the comparative statics with respect to the transaction costs, we note that $I(\epsilon, i, S_{i-1})$ increases in ϵ_i , is independent of ϵ_k for $i < k$, and, from Equation (B.8) of Corollary B.1, decreases in ϵ_k for $i > k$. To prove that t_i decreases in s_i , we use Equation (B.19) and Equation (B.7) of Corollary B.1. ■

Finally, we prove Corollary 1.

Proof. We use Equation (28) and the result of Corollary B.2 that t_i is independent of ϵ_k for $i < k$. ■

Appendix C: Proofs of Propositions 4, 5, and 6

We first prove Lemma C.1.

Lemma C.1. *Consider the function $b(t)$ defined by*

$$b(t) = \frac{2f'(t)}{r(g'(t)f(t) - g(t)f'(t))} + 1. \quad (\text{C.1})$$

If $T \leq T_2$, $b(t)$ is increasing, while if $T > T_2$, it is decreasing for small t and may then become increasing. If $T \leq T_1$, $b(t)$ is negative, while if $T > T_1$, it is positive for small t and then negative.

Proof. Using the definitions of $f(t)$ and $g(t)$, that is, Equations (18) and (24), we get

$$f'(t) = \frac{d(1/A_t)}{dt} \frac{rat}{T} \quad (\text{C.2})$$

and

$$g'(t) = -\frac{d(1/A_t)}{dt} \int_0^t A_s e^{-rs} ds. \quad (\text{C.3})$$

Plugging back into Equation (C.1), and simplifying by ra/T and $d(1/A_t)/dt$, we get

$$b(t) = -\frac{2}{r \left(\int_0^t e^{-rs} ds + \frac{\int_0^t A_s e^{-rs} ds}{t} \int_t^T \frac{ds}{A_s} \right)} + 1. \quad (\text{C.4})$$

Using Equation (C.4) and the definition of A_t , that is, Equation (11), we get

$$b(0) = \frac{rT - (2 + \gamma) + 3\gamma e^{-rT}}{rT - \gamma(1 - e^{-rT})}$$

and

$$b(T) = -\frac{1 + e^{-rT}}{1 - e^{-rT}}.$$

$b'(t)$ has the same sign as the derivative of

$$\int_0^t e^{-rs} ds + \frac{\int_0^t A_s e^{-rs} ds}{t} \int_t^T \frac{ds}{A_s}.$$

It is easy to check that this derivative has the same sign as

$$A_t t e^{-rt} - \int_0^t A_s e^{-rs} ds = \int_0^t s \frac{d(A_s e^{-rs})}{ds} ds.$$

If $T \leq T_2$ then $2\gamma e^{-rT} - 1 \geq 0$. In this case $A_t e^{-rt}$ is increasing and so is $b(t)$. If $T > T_2$, then $2\gamma e^{-rT} - 1 < 0$. In this case $A_t e^{-rt}$ is decreasing for small t and may then become increasing. The same is true for $b(t)$. If $T \leq T_1$, then $b(0) \leq 0$. $b(t)$ is either increasing, or decreasing and then possibly increasing. Since $b(T) < 0$, $b(t)$ is negative. If $T > T_1$, then $b(0) > 0$. Since $b(T) < 0$, $b(t)$ is decreasing and then possibly increasing. Therefore it is positive for small t and then negative. ■

We now prove Proposition 4.

Proof. Equation (27) implies that

$$\frac{\partial(1/f(t_i))}{\partial I(\epsilon, i, S_{i-1})} = -\frac{f'(t_i)}{f(t_i)^2} \frac{1}{(g/f)'(t_i)} \frac{2}{r \alpha I(\sum_{j=1}^N \rho_{.,j} S_j, i, S_{i-1})}. \quad (\text{C.5})$$

Differentiating Equation (28) with respect to ϵ_i and using Equation (C.5) we find

$$\frac{\partial \bar{P}_i}{\partial \epsilon_i} = \sum_{j=1}^i \frac{2f'(t_j)}{rf(t_j)^2} \frac{1}{(g/f)'(t_j)} \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_i} \frac{I(\rho_{.,i}, j, S_{j-1})}{I(\rho_{.,j}, j, S_{j-1})} + 1.$$

Noting that $I(\epsilon, j, S_{j-1})$ is independent of ϵ_i for $j < i$, and using the definition of $b(t)$ we can write the above equation as

$$\frac{\partial \bar{P}_i}{\partial \epsilon_i} = b(t_i). \quad (\text{C.6})$$

Proposition 4 follows then from Lemma C.1, and the fact that t_i is small for ϵ_i small and vice versa. ■

We now prove Proposition 5.

Proof. The proposition follows from Equation (C.6) and Lemma C.1. ■

We now prove Proposition 6.

Proof. Differentiating Equation (28) with respect to ϵ_k , $k < i$ and using Equation (C.5) we find

$$\frac{\partial \bar{P}_i}{\partial \epsilon_k} = \sum_{j=1}^i \frac{2f'(t_j)}{rf(t_j)^2} \frac{1}{(g/f)'(t_j)} \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \frac{I(\rho_{.,i}, j, S_{j-1})}{I(\rho_{.,j}, j, S_{j-1})}$$

that we can write, using the definition of $b(t)$, as

$$\begin{aligned} \frac{\partial \bar{P}_i}{\partial \epsilon_k} &= \frac{2f'(t_k)}{rf(t_k)^2} \frac{1}{(g/f)'(t_k)} \frac{\partial}{\partial \epsilon_k} \sum_{j=1}^i I(\epsilon, j, S_{j-1}) \frac{I(\rho_{\cdot, i}, j, S_{j-1})}{I(\rho_{\cdot, j}, j, S_{j-1})} \\ &\quad + \sum_{j=1}^i (b(t_j) - b(t_k)) \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \frac{I(\rho_{\cdot, i}, j, S_{j-1})}{I(\rho_{\cdot, j}, j, S_{j-1})}. \end{aligned} \quad (\text{C.7})$$

Using Lemma B.5 for $\zeta = \epsilon$ and η defined by $\eta_i = I(\epsilon, i, S_{i-1})$, we conclude that

$$\frac{\partial}{\partial \epsilon_k} \sum_{j=1}^i I(\epsilon, j, S_{j-1}) \frac{I(\rho_{\cdot, i}, j, S_{j-1})}{I(\rho_{\cdot, j}, j, S_{j-1})} = \frac{\partial \epsilon_i}{\partial \epsilon_k} = 0.$$

Noting also that $I(\epsilon, j, S_{j-1})$ is independent of ϵ_k for $j < k$ we can write Equation (C.7) as

$$\frac{\partial \bar{P}_i}{\partial \epsilon_k} = \sum_{j=k+1}^i (b(t_j) - b(t_k)) \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \frac{I(\rho_{\cdot, i}, j, S_{j-1})}{I(\rho_{\cdot, j}, j, S_{j-1})}. \quad (\text{C.8})$$

Equation (B.8) of Corollary B.1 implies that $\partial I(\epsilon, j, S_{j-1})/\partial \epsilon_k \leq 0$ for $j > k$. Equations (B.7) and (B.9) of Corollary B.1 imply that

$$I(\rho_{\cdot, i}, j, S_{j-1})/I(\rho_{\cdot, j}, j, S_{j-1}) \geq 0.$$

Therefore Proposition 6 follows from Lemma C.1. $\blacksquare$

Finally, we prove Corollary 2.

Proof. We first note that $\rho_{i,j} = \rho \geq 0 \ \forall i, j, i \neq j$, implies that $\partial I(\epsilon, j, S_{j-1})/\partial \epsilon_k$ is independent of k for $k < j$. Using Equation (C.8) for k and $k+1 < i$ and the fact above, we get

$$\begin{aligned} \frac{\partial \bar{P}_i}{\partial \epsilon_k} - \frac{\partial \bar{P}_i}{\partial \epsilon_{k+1}} &= (b(t_{k+1}) - b(t_k)) \\ &\quad \times \sum_{j=k+1}^i \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \frac{I(\rho_{\cdot, i}, j, S_{j-1})}{I(\rho_{\cdot, j}, j, S_{j-1})}. \end{aligned} \quad (\text{C.9})$$

Corollary 2 follows then from Equation (C.9) and Lemma C.1. $\blacksquare$

Appendix D: Proofs of Propositions 7 and 8

We start with Proposition 7.

Proof. We first derive a linear system of N equations for the N unknowns $x_{i,0}$. We then show that its solution are the $x_{i,0}$'s of the proposition.

The linear system

To derive the i th equation of the system, we use condition (iii) of Proposition 1. Noting that $x_{j,t_i} = x_{j,0}$ for $j \geq i$, we can write the condition for stock $k < i$ and $t = t_i$ as

$$\sum_{j=1}^{i-1} \rho_{k,j} x_{j,t_i} + \sum_{j=i}^N \rho_{k,j} x_{j,0} = \frac{\bar{D}_k - r\bar{P}_k + r\epsilon_k}{A_{t_i}}. \quad (\text{D.1})$$

We can also write condition (iii) for stock i and $t = t_i$ as

$$\sum_{j=1}^{i-1} \rho_{i,j} x_{j,t_i} + \sum_{j=i}^N \rho_{i,j} x_{j,0} = \frac{\bar{D}_i - r\bar{P}_i + r\epsilon_i}{A_{t_i}}. \quad (\text{D.2})$$

Using Equation (D.1) for $k < i$ ($i - 1$ equations) to eliminate x_{k,t_i} for $k < i$ ($i - 1$ unknowns) in Equation (D.2), we find

$$\sum_{j=i}^N I(\rho_{\cdot,j}, i, S_{i-1}) x_{j,0} = \frac{I(\bar{D} - r\bar{P} + \epsilon, i, S_{i-1})}{A_{t_i}}.$$

Using Equation (B.24), we can write the above equation as

$$\sum_{j=i}^N I(\rho_{\cdot,j}, i, S_{i-1}) x_{j,0} = \frac{r a I(\sum_{j=1}^N \rho_{\cdot,j} S_j, i, S_{i-1})}{f(t_i) A_{t_i}}. \quad (\text{D.3})$$

Equation (D.3) is the i th equation of our linear system.

The solution

We now show that the $x_{i,0}$'s defined by Equation (32) are the solution to the linear system. We plug the $x_{i,0}$'s in Equation (D.3), and show that the coefficient of $1/f(t_j)A_{t_j}$ in the resulting equation is 0 for $j \geq i$. This is obvious for $j = i$. For $j > i$ the coefficient is

$$I(\rho_{\cdot,j}, i, S_{i-1}) = \sum_{j'=i}^{j-1} I(\rho_{\cdot,j'}, i, S_{i-1}) \frac{I(\rho_{\cdot,j}, j', S_{j-1} \setminus \{j'\})}{I(\rho_{\cdot,j'}, j', S_{j-1} \setminus \{j'\})}. \quad (\text{D.4})$$

Using Lemma B.1, we can write Equation (D.4) as

$$\begin{aligned} I(\rho_{\cdot,j}, i, S_{i-1}) &= (0, \dots, 0, I(\rho_{\cdot,i}, i, S_{i-1}), \dots, I(\rho_{\cdot,j-1}, i, S_{i-1})) \\ &\quad \times \begin{pmatrix} 1 & \cdot & \rho_{1,j-1} \\ \cdot & \cdot & \cdot \\ \rho_{j-1,1} & \cdot & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{1,j} \\ \cdot \\ \rho_{j-1,j} \end{pmatrix}. \end{aligned} \quad (\text{D.5})$$

Using Equation (B.2) for A , the submatrix formed by rows and columns 1 to $i - 1$, we can write Equation (D.5) as

$$I(\rho_{\cdot,j}, i, S_{i-1}) = (I(\rho_{\cdot,i}, i, S_{i-1}), \dots, I(\rho_{\cdot,j-1}, i, S_{i-1}))$$

$$\begin{pmatrix} I(\rho_{\cdot,i}, i, S_{i-1}) & \dots & I(\rho_{\cdot,j-1}, i, S_{i-1}) \\ I(\rho_{\cdot,i}, j-1, S_{i-1}) & \dots & I(\rho_{\cdot,j-1}, j-1, S_{i-1}) \end{pmatrix}^{-1} \begin{pmatrix} I(\rho_{\cdot,j}, i, S_{i-1}) \\ I(\rho_{\cdot,j}, j-1, S_{i-1}) \end{pmatrix}. \quad (\text{D.6})$$

The definition of $I(\zeta, b, \{b_1, \dots, b_i\})$, that is, Equation (25), implies that the matrix in Equation (D.6) is symmetric. Therefore the equation obviously holds. ■

We now prove Proposition 8.

Proof. We first define the function $l(t)$ by

$$l(t) = \frac{f'(t)A_t + f(t)\frac{dA_t}{dt}}{A_t^2(g'(t)f(t) - g(t)f'(t))}. \quad (\text{D.7})$$

Proceeding as in the proof of Lemma C.1, we get

$$l(t) = \frac{1}{\frac{t \int_0^t e^{-rs} ds}{\int_t^T \frac{ds}{A_s}} + \int_0^t A_s e^{-rs} ds}.$$

Therefore $l(t)$ is positive and decreasing.

Equation (27) implies that

$$\frac{\partial(1/f(t_i)A_{t_i})}{\partial I(\epsilon, i, S_{i-1})} = -l(t_i) \frac{2}{raI(\sum_{j=1}^N \rho_{\cdot,j} s_j, i, S_{i-1})}. \quad (\text{D.8})$$

We use Equation (D.8) to calculate $\partial V_i / \partial \epsilon_k$. Equations (32) and (D.8) imply that

$$\begin{aligned} \frac{\partial V_i}{\partial \epsilon_k} = & -\frac{2}{Ts_i} \left(\frac{l(t_i)}{I(\rho_{\cdot,i}, i, S_{i-1})} \frac{\partial I(\epsilon, i, S_{i-1})}{\partial \epsilon_k} \right. \\ & \left. - \sum_{j=i+1}^N \frac{l(t_j)}{I(\rho_{\cdot,j}, j, S_{j-1})} \frac{I(\rho_{\cdot,j}, i, S_{j-1} \setminus \{i\})}{I(\rho_{\cdot,i}, i, S_{j-1} \setminus \{i\})} \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \right). \quad (\text{D.9}) \end{aligned}$$

Equation (B.8) of Corollary B.1 implies that $\partial I(\epsilon, j, S_{j-1}) / \partial \epsilon_i \leq 0$ for $j > i$. Since $\partial I(\epsilon, i, S_{i-1}) / \partial \epsilon_i > 0$ and $l(t) > 0$, $\partial V_i / \partial \epsilon_i < 0$.

We now show that $\partial V_i / \partial \epsilon_k \geq 0$ for $k < i$. The proof for $k > i$ is similar. Dividing by $l(t_i)$, and using the facts that $l(t)$ is positive and

decreasing, and that $\partial I(\epsilon, j, S_{j-1})/\partial \epsilon_k \leq 0$ for $j \geq i > k$, it suffices to show that

$$\frac{1}{I(\rho_{\cdot, i}, i, S_{i-1})} \frac{\partial I(\epsilon, i, S_{i-1})}{\partial \epsilon_k} - \sum_{j=i+1}^N \frac{1}{I(\rho_{\cdot, j}, j, S_{j-1})} \frac{I(\rho_{\cdot, j}, i, S_{j-1} \setminus \{i\})}{I(\rho_{\cdot, i}, i, S_{j-1} \setminus \{i\})} \frac{\partial I(\epsilon, j, S_{j-1})}{\partial \epsilon_k} \leq 0. \quad (\text{D.10})$$

The definition of $I(\zeta, b, \{b_1, \dots, b_i\})$ and Lemma B.1 imply that Equation (D.10) is equivalent to

$$\frac{1}{I(\rho_{\cdot, i}, i, S_{i-1})} \frac{I(\rho_{\cdot, i}, k, S_{i-1} \setminus \{k\})}{I(\rho_{\cdot, k}, k, S_{i-1} \setminus \{k\})} - \sum_{j=i+1}^N \frac{1}{I(\rho_{\cdot, j}, j, S_{j-1})} \frac{I(\rho_{\cdot, j}, i, S_{j-1} \setminus \{i\})}{I(\rho_{\cdot, i}, i, S_{j-1} \setminus \{i\})} \frac{I(\rho_{\cdot, j}, k, S_{j-1} \setminus \{k\})}{I(\rho_{\cdot, k}, k, S_{j-1} \setminus \{k\})} \geq 0. \quad (\text{D.11})$$

To show that Equation (D.11) holds, we first establish a useful identity. We compute the term in the k th row and i th column of the inverse matrix of

$$\begin{pmatrix} 1 & \cdot & \rho_{1,j} \\ \cdot & & \cdot \\ \rho_{j,1} & \cdot & 1 \end{pmatrix}$$

for $j > i$, in two ways. First, we use Lemma B.2. Second, we use Equation B.2 for A , the submatrix formed by rows and columns 1 to $j-1$, Lemma B.1 and Lemma B.2. Since we should get the same result, the following identity holds

$$\begin{aligned} & \frac{I(\rho_{\cdot, i}, k, S_j \setminus \{i, k\})}{I(\rho_{\cdot, i}, i, S_j \setminus \{i\}) I(\rho_{\cdot, k}, k, S_j \setminus \{i, k\})} \\ &= \frac{I(\rho_{\cdot, i}, k, S_{j-1} \setminus \{i, k\})}{I(\rho_{\cdot, i}, i, S_{j-1} \setminus \{i\}) I(\rho_{\cdot, k}, k, S_{j-1} \setminus \{i, k\})} \\ & - \frac{1}{I(\rho_{\cdot, j}, j, S_{j-1})} \frac{I(\rho_{\cdot, j}, i, S_{j-1} \setminus \{i\})}{I(\rho_{\cdot, i}, i, S_{j-1} \setminus \{i\})} \frac{I(\rho_{\cdot, j}, k, S_{j-1} \setminus \{k\})}{I(\rho_{\cdot, k}, k, S_{j-1} \setminus \{k\})}. \quad (\text{D.12}) \end{aligned}$$

Equation (D.12) used for $j = i+1$ to N implies that Equation (D.11) is equivalent to

$$\frac{I(\rho_{\cdot, i}, k, S_N \setminus \{i, k\})}{I(\rho_{\cdot, i}, i, S_N \setminus \{i\}) I(\rho_{\cdot, k}, k, S_N \setminus \{i, k\})} \geq 0$$

which obviously holds. ■

Appendix E: Proof of Proposition 9

Existence of a solution

We first show that Equations (35)–(39) have a solution. For simplicity we assume that $A = \infty$. The proof for A large uses the fact that the functions A_t , $f(t)$, and $g(t)$ are close to their counterparts for $A = \infty$, and is available upon request. Eliminating $\bar{P}_1$ and $\bar{P}_2$, we get the following equations in t_1 , $\hat{t}_1$, and t_2

$$\frac{g(t_1)}{f(t_1) - f(\hat{t}_1)} = \frac{2\epsilon_1}{ras_1}, \quad (\text{E.1})$$

$$\frac{g(t_2)}{f(t_2)}s_2 - \rho \frac{g(\hat{t}_1)}{f(t_1) - f(\hat{t}_1)}s_1 = \frac{2(\epsilon_2 - \rho\epsilon_1)}{ra}, \quad (\text{E.2})$$

and

$$\frac{A_{t_2}f(t_2)}{s_2} - \rho \frac{A_{\hat{t}_1}(f(t_1) - f(\hat{t}_1))}{s_1} = 0. \quad (\text{E.3})$$

We treat $\hat{t}_1 \in (0, T)$ as a parameter, and define $t_1(\hat{t}_1)$ and $t_2(\hat{t}_1)$ by Equations (E.1) and (E.2), respectively. We assume first $\epsilon_1 > 0$. Since $g(0) = 0$, $g(t)$ is strictly increasing, $f(t)$ is strictly decreasing, and $\epsilon_1 > 0$, Equation (E.1) has a unique solution $t_1(\hat{t}_1) \in (0, \hat{t}_1)$, that is continuous in $\hat{t}_1$. Using Equation (E.1), we can write Equation (E.2) as

$$\frac{g(t_2)}{f(t_2)}s_2 = \rho \left(\frac{g(\hat{t}_1)}{g(t_1)} - 1 \right) \frac{2\epsilon_1}{ra} + \frac{2\epsilon_2}{ra} \geq 0. \quad (\text{E.4})$$

Since $g(T) > 0$ and $f(T) = 0$, Equation (E.4) has a solution $t_2(\hat{t}_1) \in [0, T)$ that is continuous in $\hat{t}_1$. If $\epsilon_1 = 0$, $t_1(\hat{t}_1) = 0$, and $t_2(\hat{t}_1)$ is defined instead from Equation (E.2).

We now plug $t_1(\hat{t}_1)$ and $t_2(\hat{t}_1)$ into Equation (E.3) and show that it has a solution, $\hat{t}_1$. For $\hat{t}_1$ close to 0, the left-hand side of Equation (E.3) is strictly positive. When $\hat{t}_1$ goes to T , $t_1(\hat{t}_1)$ and $t_2(\hat{t}_1)$ go to limits in $(0, T)$. Since A_t goes to ∞ the left-hand side of Equation (E.3) goes to $-\infty$. Therefore a solution to Equations (E.1)–(E.3) exists, as well as a solution to the initial equations.

Proof that $0 \leq t_1 < \hat{t}_1 \leq t_2 < T$

We only have to show that $\hat{t}_1 \leq t_2$. We assume first that $\rho < 1$. Defining t_1^* and t_2^* by Equation (27), and using Equation (28), it is easy to check that Assumption 1' implies that

$$f(t_1^*)\rho s_2 \geq f(t_2^*)(s_1 + \rho s_2). \quad (\text{E.5})$$

To prove that $\hat{t}_1 \leq t_2$, we will assume the opposite and prove that (i) $f(t_1)\rho s_2 < f(t_2)(s_1 + \rho s_2)$, (ii) $t_1 \leq t_1^*$, and (iii) $t_2 > t_2^*$. Since $f(t)$ is decreasing, these three facts will contradict Equation (E.5).

Proof that $f(t_1)\rho s_2 < f(t_2)(s_1 + \rho s_2)$

We write Equation (E.3) as

$$f(t_1) = f(\hat{t}_1) + \frac{A_{t_2}f(t_2)}{A_{\hat{t}_1}} \frac{s_1}{\rho s_2}. \quad (\text{E.6})$$

The result follows from our assumption that $\hat{t}_1 > t_2$ and the facts that $f(t)$ is strictly decreasing and A_t strictly increasing.

Proof that $t_1 \leq t_1^*$

Using Equation (E.6), our assumption that $\hat{t}_1 > t_2$, and the fact that $f(t)A_t$ is strictly increasing, we get

$$(f(t_1) - f(\hat{t}_1)) \frac{s_1 + \rho s_2}{s_1} < f(t_1).$$

The above inequality and Equation (E.1) imply that

$$\begin{aligned} \frac{g(t_1)}{f(t_1)} &\leq \frac{g(t_1)}{f(t_1) - f(\hat{t}_1)} \frac{s_1}{s_1 + \rho s_2} \\ &= \frac{2\epsilon}{2\epsilon} \end{aligned}$$

Conditions hold

We now show that the conditions of Proposition 1 and the market-clearing conditions hold. Equations (37), (40), and (41) imply that

$$x_{1,t} = (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) \left(\frac{1}{A_t} - \frac{1}{A_{\hat{t}_1}} \right), \quad \forall t \in [t_1, \hat{t}_1]. \quad (\text{E.8})$$

Therefore $x_{1,t}$ is continuous at $\hat{t}_1$, and $x_{1,t}$ and $x_{2,t}$ are piecewise C^1 . It is easy to check that conditions (i), (iii), (iv), and (v) of Proposition 1 hold.

Condition (ii)

We proceed as in the proof of Proposition 3. Proving condition (ii) for stock 1 is equivalent to proving

$$\int_0^{\hat{t}_1} (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) \left(1 - \frac{A_t}{A_{\hat{t}_1}} \right) e^{-rt} dt - 2\epsilon_1 = 0,$$

that is, Equation (35). Combining condition (ii) for stock 1, and condition (iii) for stock 1 and $t \in [t_1, \hat{t}_1]$, we get

$$\int_0^{\hat{t}_1} \left((\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) \left(1 - \frac{A_t}{A_{\hat{t}_1}} \right) - A_t(x_{1,t} - x_{1,\hat{t}_1}) \right) e^{-rt} dt - 2\epsilon_1 = 0. \quad (\text{E.9})$$

Proving condition (ii) for stock 2 is equivalent to proving

$$\int_0^{t_2} \left((\bar{D}_2 - r\bar{P}_2 + r\epsilon_2) \left(1 - \frac{A_t}{A_{t_2}} \right) - A_t \rho(x_{1,t} - x_{1,t_2}) \right) e^{-rt} dt - 2\epsilon_2 = 0. \quad (\text{E.10})$$

Using Equation (E.9) to eliminate

$$\int_0^{\hat{t}_1} A_t x_{1,t} e^{-rt} dt = \int_0^{\hat{t}_1} A_t (x_{1,t} - x_{1,\hat{t}_1}) e^{-rt} dt = \int_0^{t_2} A_t (x_{1,t} - x_{1,t_2}) e^{-rt} dt$$

in Equation (E.10), we conclude that Equation (E.10) is equivalent to Equation (36).

Market-clearing conditions

Finally, we show the market-clearing conditions. The definition of $x_{2,t}$ shows that the market-clearing condition of stock 2 is equivalent to Equation (39). Using Equation (E.8), it is easy to check that the market-clearing condition for stock 1 is equivalent to Equation (38).

Appendix F: Proof of Proposition 10

We first prove the proposition and then prove Corollaries 3 and 4.

Proof. Using the definition of t^* , that is, Equation (44), it is easy to check that Equations (35)–(39) (the equilibrium equations) are satisfied for $\epsilon_1 = \epsilon_2 = 0$, $t_1 = 0$, $\hat{t}_1 = t_2 = t^*$, and

$$\bar{D}_1 - r\bar{P}_1 = \bar{D}_1 - r\bar{P}_1 = \frac{ra(s_1 + s_2)}{f(0)}. \quad (\text{F.1})$$

We now study the equilibrium equations for ϵ small, assuming that t_1 , $\hat{t}_1$, t_2 , $\bar{P}_1$, and $\bar{P}_2$ are close to their $\epsilon = 0$ values. We first determine t_1 and $t_2 - \hat{t}_1$ from Equations (E.1) and (E.7), respectively. Since $g(0) = g'(0) = 0$, Equation (E.1) implies that

$$\frac{\frac{1}{2}g''(0)t_1^2}{f(0) - f(t^*)} = \frac{2\epsilon_1}{ras_1}. \quad (\text{F.2})$$

Equation (C.3), the definition of A_t , that is, Equation (11), and the fact that $\gamma = 1$ imply that

$$g''(0) = - \left. \frac{d(1/A_t)}{dt} \right|_{t=0} A_0 = \frac{re^{-rT}}{1 - e^{-rT}}. \quad (\text{F.3})$$

Using Equation (F.3) and the definition of t^* , we can write Equation (F.2) as

$$\frac{re^{-rT}t_1^2}{2(1 - e^{-rT})f(0)} = \frac{2\epsilon_1}{ra(s_1 + s_2)}. \quad (\text{F.4})$$

Equation (E.7) implies that

$$\frac{s_2}{f(t^*)A_{t^*}} \left. \frac{d(g(t)A_t)}{dt} \right|_{t=t^*} (t_2 - \hat{t}_1) = \frac{2(\epsilon_2 - \epsilon_1)}{ra}. \quad (\text{F.5})$$

Using the definition of $g(t)$, that is, Equation (24), we get

$$\frac{d(g(t)A_t)}{dt} = \frac{d}{dt} \int_0^t (A_t - A_s)e^{-rs} ds = \frac{dA_t}{dt} \frac{1 - e^{-rt}}{r}. \quad (\text{F.6})$$

The definition of t^* , and Equations (F.5) and (F.6), imply that

$$\frac{1}{f(0)A_{t^*}} \left. \frac{dA_t}{dt} \right|_{t=t^*} \frac{1 - e^{-rt^*}}{r} (t_2 - \hat{t}_1) = \frac{2(\epsilon_2 - \epsilon_1)}{ra(s_1 + s_2)}. \quad (\text{F.7})$$

We now come to $\bar{P}_1$ and $\bar{P}_2$, and first compute $\bar{P}_1 - \bar{P}_2$. We can write Equation (37) as

$$\bar{D}_2 - r\bar{P}_2 + r\epsilon_2 - (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) = (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) \left(\frac{A_{t_2}}{A_{\hat{t}_1}} - 1 \right). \quad (\text{F.8})$$

Using Equation (F.1) and the fact that $t_2 - \hat{t}_1$ is small, we can write Equation (F.8) as

$$\bar{D}_2 - r\bar{P}_2 + r\epsilon_2 - (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) = \frac{ra(s_1 + s_2)}{f(0)A_{t^*}} \frac{dA_t}{dt} \Big|_{t=t^*} (t_2 - \hat{t}_1). \quad (\text{F.9})$$

We now compute $\bar{P}_1$ and prove Equation (42). Adding Equations (38) and (39), we get

$$\begin{aligned} & (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1)(f(t_1) - f(\hat{t}_1) + f(t_2)) \\ & + (\bar{D}_2 - r\bar{P}_2 + r\epsilon_2 - (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1))f(t_2) = ra(s_1 + s_2). \end{aligned} \quad (\text{F.10})$$

Using Equation (F.9) and the fact that $f'(0) = 0$, we can write Equation (F.10) as

$$\begin{aligned} & (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) \left(f(0) + \frac{1}{2}f''(0)t_1^2 + f'(t^*)(t_2 - \hat{t}_1) \right) \\ & + \frac{ra(s_1 + s_2)}{f(0)A_{t^*}} \frac{dA_t}{dt} \Big|_{t=t^*} f(t^*)(t_2 - \hat{t}_1) = ra(s_1 + s_2). \end{aligned} \quad (\text{F.11})$$

Equation (C.2) and the definition of A_t imply that

$$f''(0) = \frac{d(1/A_t)}{dt} \Big|_{t=0} \frac{ra}{T} = -\frac{re^{-rT}}{T}. \quad (\text{F.12})$$

Noting that t_1 and $t_2 - \hat{t}_1$ are small, and using Equations (C.2), (F.1), and (F.12), we can write Equation (F.11) as

$$\begin{aligned} & (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1)f(0) \\ & + \frac{ra(s_1 + s_2)}{f(0)} \left(-\frac{re^{-rT}}{2T}t_1^2 + \frac{rat^*}{T} \frac{d(1/A_t)}{dt} \Big|_{t=t^*} (t_2 - \hat{t}_1) \right) \\ & + \frac{ra(s_1 + s_2)}{f(0)A_{t^*}} \frac{dA_t}{dt} \Big|_{t=t^*} f(t^*)(t_2 - \hat{t}_1) = ra(s_1 + s_2). \end{aligned} \quad (\text{F.13})$$

Using the fact that $d(1/A_t)/dt = -(dA_t/dt)/A_t^2$, and the definition of

$f(t)$, that is, Equation (18), we can write Equation (F.13) as

$$\begin{aligned} & (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1)f(0) \\ & + \frac{ra(s_1 + s_2)}{f(0)} \left(-\frac{re^{-rT}}{2T} t_1^2 + \frac{dA_t}{dt} \Big|_{t=t^*} \frac{1}{A_{t^*}} (t_2 - \hat{t}_1) \frac{ra}{T} \int_{t^*}^T \frac{ds}{A_s} \right) \\ & = ra(s_1 + s_2). \end{aligned} \quad (\text{F.14})$$

Equation (F.4) implies that

$$\frac{ra(s_1 + s_2)}{f(0)} \frac{re^{-rT}}{2T} t_1^2 = \frac{2\epsilon_1(1 - e^{-rT})}{T}. \quad (\text{F.15})$$

Equation (F.7) and the definition of A_t imply that

$$\begin{aligned} & \frac{ra(s_1 + s_2)}{f(0)} \frac{dA_t}{dt} \Big|_{t=t^*} \frac{1}{A_{t^*}} (t_2 - \hat{t}_1) \frac{ra}{T} \int_{t^*}^T \frac{ds}{A_s} \\ & = \frac{2r(\epsilon_2 - \epsilon_1)}{1 - e^{-rt^*}} \left(\frac{T - t^*}{T} + \frac{1 - e^{-r(T-t^*)}}{rT} \right). \end{aligned} \quad (\text{F.16})$$

Equation (42) follows from Equations (F.14), (F.15), and (F.16). To obtain Equation (43), we note that Equations (F.7) and (F.9) imply that

$$\bar{D}_2 - r\bar{P}_2 + r\epsilon_2 - (\bar{D}_1 - r\bar{P}_1 + r\epsilon_1) = \frac{2r(\epsilon_2 - \epsilon_1)}{1 - e^{-rt^*}}. \quad (\text{F.17})$$

Equation (43) follows from Equations (42) and (F.17). $\blacksquare$

We now prove Corollary 3.

Proof. We first study $\bar{P}_1$. The definitions of A_t and $f(t)$ imply that

$$f(0) = 1 - \frac{1 - e^{-rT}}{rT}. \quad (\text{F.18})$$

Therefore the coefficient of ϵ_1 in Equation (42) is

$$\frac{rT - 3(1 - e^{-rT})}{rTf(0)} - \frac{2\left(\frac{T-t^*}{T} - \frac{1 - e^{-r(T-t^*)}}{rT}\right)}{(1 - e^{-rt^*})f(0)}.$$

Since $x - (1 - e^{-x}) \geq 0$, the second term is positive. It is 0 for $t = T$, and goes to ∞ as t goes to 0. The result for $\bar{P}_1$ follows.

We now come to $\bar{P}_2$. Equation (F.18) implies that the coefficient of ϵ_2 in Equation (43) has the same sign as

$$(1 - e^{-rt^*})(rT - (1 - e^{-rT})) - 2(rt^* - (e^{-r(T-t^*)} - e^{-rT})). \quad (\text{F.19})$$

Equation (F.19) is 0 for $t = 0$, and is $-(1 + e^{-rT})(rT - (1 - e^{-rT})) \leq 0$ for $t = T$. Its derivative has the same sign as

$$rT - (1 - e^{-rT}) - 2(e^{rt} - e^{-r(T-2t)}). \quad (\text{F.20})$$

Equation (F.20) is $rT - 3(1 - e^{-rT})$ for $t = 0$, and is $rT - (1 - e^{-rT}) \geq 0$ for $t = T$. Its derivative has the same sign as $2e^{-r(T-t)} - 1$, and is either positive, or negative and then positive. Therefore if $rT - 3(1 - e^{-rT}) \leq 0$, Equation (F.20) is negative and then positive. Equation (F.19) is thus negative. If $rT - 3(1 - e^{-rT}) > 0$, Equation (F.20) is either positive, or positive then negative and then positive again. Therefore Equation (F.19) is positive and then negative. ■

The proof of Corollary 4 is very similar to the proof of the second part of Corollary 3. It is available upon request.

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