

The Threshold Effect in Expected Volatility: A Model Based on Asymmetric Information

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This article develops theoretical insight into the threshold effect in expected volatility, which means that large shocks are less persistent in volatility than small shocks. The model uses the Kyle–Admati–Pfleiderer setup with liquidity traders, informed traders, and a market maker. Information is modeled as a GARCH process. It is shown that the GARCH process for information is transformed into a TARCH process (for “threshold GARCH”) for the market price changes. Working with information flows allows one to derive implications for trading volume and market liquidity which provide the basis for a more complete test of the model.

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This article provides theoretical insight into the threshold effect observed in the expected variance of stock returns. The threshold effect means that large shocks are less persistent in volatility than small shocks. In other words, after a big surprise market volatility increases, but not by much, and comes back quickly to its long-term level. Using a multiperiod model similar to Admati and Pfleiderer (1988) and inspired by Kyle (1985, section 2), it is shown that asymmetry of information among market participants is a source of the nonlinearity in the expected variance of risky asset price changes.

This article focuses mainly on the behavior of the variance of market price changes. More precisely, one is interested in the expected variance conditional on market participants' information. Engle (1982) introduced the statistical ARCH (for autoregressive conditional heteroscedastic) process to model the time-varying property of the second moment of a time series; the expected variance of future returns depends on the squared value of past innovations. Bollerslev (1986) generalized ARCH into GARCH, which also includes past variance for predicting future variance. Several explanations for these ARCH and GARCH effects can be put forth: the quality and quantity of information, the influence of economic variables, and the trading process itself. This class of statistical model was successfully applied to financial assets such as stocks, interest rates, and currency exchange rates.¹ ARCH and GARCH processes model the positive correlation observed in the size of returns and the clustering of large movements in prices. However, the failure of these models to take into account the influence of the sign or the size of past returns led to the development of asymmetric and threshold GARCH models. Volatility feedback modeled by Campbell and Hentschel (1992) and an argument based on the leverage effect discussed in Black (1976) can explain the sign effect (which means that bad news has a greater impact on volatility than good news). Until now, however, no theoretical work has been done for the threshold effect in the expected variance.

Classic and threshold GARCH models and characterizations of the threshold effect are reviewed in Section 1. Section 2 presents the theoretical model. The market environment is described as follows: trading for a single risky asset takes place in a centralized market at discrete intervals; investors are differentiated according to their motive for trading: liquidity or information. Liquidity-motivated investors trade for reasons exogenous to information about the firm; liquidity traders have no choice with regard to the timing of their trades, and their asset demand is completely exogenous. Information-motivated

¹ A review of the theory and empirical evidence of GARCH effects can be found in Bollerslev, Chou, and Kroner (1992).

investors trade to benefit from private information about the firm's value; informed traders maximize their expected trading profit. The information process is assumed to be a GARCH. This reflects the fact that information tends to arrive in waves. The specification of the information process is different from that in the articles based on the seminal model proposed by Kyle (1985), which assume a constant variance for information as in Back (1993) and Foster and Viswanathan (1993b). A competitive market maker absorbs the net demand for the risky asset at an efficient price. It is shown that there is a unique linear Nash equilibrium where informed traders take into account the strategy of the market maker, and vice versa. In this model, both classes of market participants explicitly take into account the time-varying behavior of the variance of information by computing their expectations and forming their strategies.

In Section 3, it is shown that the variance of equilibrium market price changes also exhibits a time-varying behavior. However, the process for price changes and the process for information are different. The difference is due to the presence of informed traders. When the number of informed traders is exogenous and constant over time, a GARCH process for market price changes is still obtained, but the degree of persistence of the latest informational shock in market variance depends on the number of informed traders. For a given number of informed traders, however, large shocks are as persistent as small shocks.

In Section 4, the case of endogenous acquisition of information is considered. When information is acquired at a cost, and the number of informed traders is endogenously determined, the degree of persistence depends on the size of the shocks: large shocks are less persistent than small shocks. The GARCH process for information is transformed into a TARCH process (for threshold GARCH) for market price changes.

Working with assumptions on information flows, the model has a much wider array of implications than was first derived from statistical models. In Section 5, it is shown that trading volume and market liquidity also present a time-varying behavior with thresholds. Cross-restrictions with market variance are also offered for testing the model.

1. The Threshold Effect in Expected Variance

1.1 GARCH and threshold GARCH models

The time-varying behavior of the expected variance denoted by h_t can be written as

$$h_t \equiv \text{Var}_{t-1}(r_t) = \alpha_0 + F(e_{t-1}) + \beta_1 h_{t-1}, \quad (1)$$

where the variable r_t represents the asset return, the variable e_t is the error term defined by $r_t - E_{t-1}(r_t)$, and the function F is the relation between the latest innovation and the expected variance of future returns. Parameters α_0 and β_1 , and function F have to be empirically estimated. The innovation e_t is assumed to be drawn from a conditional normal distribution with zero mean and variance h_t .

The GARCH(1,1) model, which is most used in empirical studies, corresponds to the case $F(e_{t-1}) = \alpha_1 e_{t-1}^2$. Coefficient α_1 represents the degree of persistence of the latest innovation in the expected variance of future returns. In the GARCH model, the degree of persistence is constant and independent of the size of the past shock e_{t-1} . Small and large shocks are equally persistent in expected variance.

Threshold models have been developed to take into account the effect of the size on expected volatility. Basically, a more sophisticated function F is used in Equation (1). Different models have been proposed in the literature; they are listed by category in Table 1. All these models work with the price process but differ in the way the nonlinearity is captured by incorporating the past price innovations in volatility. Although market prices are likely to be informative, the role of information itself and market participants' trading decisions as a primary source of market movements should be emphasized and modeled.

A useful concept for a better understanding of these models is the News Impact Curve (NIC) proposed by Friedman and Laibson (1989) and Engle and Ng (1993). The NIC graphically represents the function F and shows how new information e_{t-1} affects the next period variance h_t , holding constant the information dated $t-2$ and earlier. The slope of the NIC is equal to the degree of persistence α , empirically estimated. The steeper the NIC, the higher the persistence of the past shock in the expected variance. For the GARCH model, the NIC is a straight line and the slope of the curve is equal to the estimate of the coefficient α_1 . A threshold effect corresponds to a nonconstant degree of persistence and thus to a nonlinear NIC.

1.2 Characterizations of the threshold effect in expected variance

The influence of extreme shocks and the dynamics of future expected variance characterize the threshold effect. The influence of the largest shocks is reflected by a negative relation between the extreme movements and the expected variance given by a GARCH model. In Appendix A, the estimation of a GARCH(1,1) model is reported for high-frequency data (half-hour price changes in the Standard & Poor's 500 index futures traded on the Chicago Mercantile Exchange over the period 1986 to 1990). The three coefficients of the variance equation

Table 1
GARCH and TARCH models

Reference	Model	Function F used in the specification of volatility: $h_t = \alpha_0 + F(e_{t-1}) + \beta_1 h_{t-1}$
Engle (1982)	ARCH and GARCH (variance)	$F(e_{t-1}) = \alpha_1 e_{t-1}^2$ No threshold.
Bollerslev (1986)	TARCH (standard deviation, sign effect)	$F(e_{t-1}) = \alpha_1^+ D_{t-1}^+ e_{t-1} - \alpha_1^- D_{t-1}^- e_{t-1}$ D_{t-1}^+ and D_{t-1}^- are dummy variables loaded on $] -\infty, 0]$ and $]0, +\infty[$, respectively. Threshold: 0.
Rabemananjara and Zakoian (1991) Zakoian (1990)	Qualitative TARCH (standard deviation, sign and size effects)	$F(e_{t-1}) = \alpha_1 D_{1,t-1} + \alpha_2 D_{2,t-1} + \alpha_3 D_{3,t-1} + \dots$ $D_{1,t-1}, D_{2,t-1}, D_{3,t-1}, \dots$ are dummy variables loaded on the intervals $] -\infty, a_1[$, $]a_1, a_2[$, $]a_2, a_3[$, $\dots$, respectively. Thresholds: $a_1, a_2, a_3, \dots$
Glosten, Jagannathan, and Runkle (1993)	TARCH (variance, sign effect)	$F(e_{t-1}) = \alpha_1 e_{t-1}^2 + \alpha_1^- D_{t-1}^- e_{t-1}^2$ D_{t-1}^- is a dummy variable loaded if the past shock e_{t-1} is negative. Threshold: 0.
Engle and Ng (1993)	TARCH (variance, sign and size effects)	$F(e_{t-1}) = \alpha_2^- N_{1,t-1}(e_{t-1} + \sigma) + \alpha_1^- N_{1,t-1} e_{t-1} + \alpha_2^+ P_{1,t-1} e_{t-1} + \alpha_2^+ P_{2,t-1}(e_{t-1} - \sigma)$ $N_{1,t-1}, N_{1,t-1}, P_{1,t-1}$ and $P_{2,t-1}$ are dummy variables loaded on the intervals $] -\infty, -\sigma[$, $] -\infty, 0]$, $]0, +\infty[$ and $] +\sigma, +\infty[$, respectively. Thresholds: $-\sigma, 0, \sigma$ (more thresholds can be used).
Longin and Solnik (1995)	TARCH (variance, sign and size effects)	$F(e_{t-1}) = \alpha_2^- N_{2,t-1}(e_{t-1}^2 - \sigma^2) + \alpha_1^- N_{1,t-1} e_{t-1}^2 + \alpha_1 + P_{1,t-1} e_{t-1}^2 + \alpha_2^+ P_{2,t-1}(e_{t-1}^2 - \sigma^2)$ $N_{2,t-1}, N_{1,t-1}, P_{1,t-1}$ and $P_{2,t-1}$ are dummy variables loaded on the intervals $] -\infty, -\sigma^2[$, $] -\infty, 0]$, $]0, +\infty[$ and $] \sigma^2, +\infty[$, respectively. Thresholds: $-\sigma^2, 0$ and σ^2 .
Friedman and Laibson (1989)	MARCH (variance, size effect)	$F(e_{t-1}) = \alpha_1 (D_{1,t-1} \sin(a e_{t-1}^2) + D_{2,t-1})$. The parameter a is empirically estimated.
Model used in the empirical study	TARCH (variance, size effect)	$F(e_{t-1}) = \alpha_1 e_{t-1}^2 + \gamma_1 D_{1,t-1}(e_{t-1}^2 - \sigma^2)$ $D_{1,t-1}$ is a dummy variable loaded for large squared shocks on $[\sigma^2, +\infty[$. Threshold: σ^2 .

Lists the GARCH and TARCH models with some of their characteristics: the variable modeled (variance or standard deviation of returns), the effect(s) captured (sign and/or size), the function F , which links the expected market variance h_t to the latest innovation e_{t-1} , and the thresholds.

are all significant. For example, the degree of persistence of the past shock α_1 is equal to 0.089 with a standard error of 0.002. With the time series of the expected variances given by this model, a series of standardized innovations v_t equal to $e_t/\sqrt{b_t}$ is constructed. If the GARCH model is well specified, then the variable v_t should not depend on variables known at time $t-1$, in particular it should not depend on the size of the past innovation. To test this proposition, Engle and Ng (1993) suggested the following regression:

$$v_t^2 = a + b^{10}D_{t-1}^{10} + b^{90}D_{t-1}^{90} + u_t, \quad (2)$$

where D^{10} and D^{90} are dummy variables that are loaded if the past innovation e_{t-1} is in the first and last decile, respectively, of the unconditional distribution. No threshold effect corresponds to the null hypothesis: $b^{10} = 0.00$ and $b^{90} = 0.00$. Empirically one finds: $b^{10} = -0.585$ ($t = -9.70$) and $b^{90} = -0.269$ ($t = -6.61$). After a large shock, volatility is significantly lower than that predicted by the GARCH model.

This suggests the estimation of a TARCH model to capture the nonlinearity in the variance. A model developed by Longin and Solnik (1995) is used here. The new equation for the conditional variance is given as follows:

$$b_t \equiv \text{Var}_{t-1}(r_t) = \alpha_0 + \alpha_1 e_{t-1}^2 + \gamma_1 D_{t-1}(e_{t-1}^2 - \sigma^2) + \beta_1 b_{t-1}, \quad (3)$$

where the variable D is a dummy, which takes into account the size of the past innovation; D_{t-1} is equal to 1 if the latest squared shock e_{t-1}^2 is greater than the unconditional variance of returns σ^2 taken as a threshold, and 0 otherwise.² The GARCH model correspond to the constrained case: $\gamma_1 = 0$. A threshold effect corresponds to a negative value for the parameter γ_1 which measures the difference in persistence between small and large shocks. The estimation of the TARCH model reported in Appendix A shows that there is a nonlinearity in the expected variance and that this nonlinearity is well captured by introducing a threshold in the variance equation (although the true process may not be exactly a threshold process).³ The degree of persistence for small shocks measured by α_1 is equal to 0.128, compared with 0.089 for the GARCH model. The difference in persistence be-

² σ^2 corresponds to the unconditional mean of the random squared return e_{t-1}^2 . A shock e_{t-1} is classified as a large (small) shock if e_{t-1}^2 stands above (below) its mean σ^2 . If σ^2 is exactly known, it is an exogenous threshold and maximum likelihood estimators of model parameters are efficient. The choice of σ^2 as a threshold provides a reliable statistic for γ_1^T since this coefficient is estimated with more than 33% of observations (assuming the conditional normality for the distribution of returns). If σ^2 is not known exactly, but has to be estimated from data, model parameters may not be estimated efficiently.

³ A better estimate of the actual variance (and a better fit of the actual NIC) could be obtained by estimating a TARCH model with several thresholds denoted by $\Delta_1, \Delta_2, \dots, \Delta_{J-1}, \Delta_J$. The variance

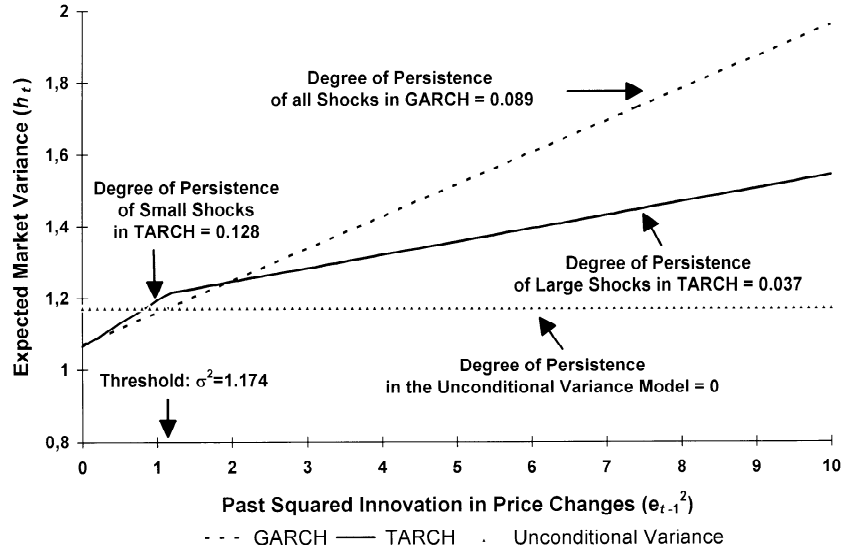


Figure 1
News impact curves (Standard & Poor's 500 index futures, 1986–1990)

This figure represents the NIC linking the past squared innovation in the process of price changes to the expected variance obtained from a GARCH model (dashed line) and a TARCH model (solid line). The dotted line corresponds to the unconditional variance. The slope of the NIC is equal to the degree of persistence of the past innovation in the variance model. Data used are intraday price changes in the S&P 500 futures index for the period 1986 to 1990.

tween small and large shocks measured by γ_1 is significantly negative: -0.091 with a t -ratio equal to -10.57 . These estimates allow one to compare the different degrees of persistence: 0.128 for small shocks and 0.037 for large shocks. A likelihood ratio test confirms the importance of the threshold effect: with a statistic equal to 169.12 with a p -value less than 0.001 , the GARCH model is strongly rejected in favor of the TARCH model [a result similar to those of Friedman and Laibson (1989) and Friedman (1992)]. This is illustrated in Figure 1.

Another characterization of the threshold effect can be seen in terms of the dynamics of future expected variance. As the variance process is stationary, shocks to volatility (although persistent) are not permanent.

equation would be

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \sum_{i=1}^I \gamma_i D_{i,t-1} (e_{t-1}^2 - \Delta_i) + \beta_1 h_{t-1}, \quad (3)$$

where D_i are dummy variables; $D_{i,t-1}$ is equal to 1 if the past squared shock e_{t-1}^2 is greater than the threshold Δ_i . The parameters γ_i measure the difference in persistence between shocks of different sizes. Equation (3) is a special case of Equation (3') corresponding to $I = 1$ and $\Delta_1 = \sigma^2$.

After a shock, volatility comes back to its long-term level, that is, there is mean reversion in volatility. For a GARCH model, all shocks to volatility decay geometrically at the rate $(\alpha_1 + \beta_1)$. For a TARCH model, between time t and time $t + 1$, small and large shocks decay at different rates: $(\alpha_1 + \beta_1)$ and $(\alpha_1 + \gamma_1 + \beta_1)$. Empirically, the decay rate for small shocks is equal to 0.994 and near unity; this suggests that small price movements are associated with the permanent part of the variance process. As the decay rate for large shocks is equal to 0.903, large price movements could be associated with the transitory part of the process. When a threshold effect follows a large shock, one expects volatility of future returns to come back quickly to its long-term level [a result also observed in the implied volatility in option prices by Schwert (1990) and Engle and Mustapha (1992)].

Most of the models described in this section apply a statistical model directly to stock returns. Although some statistical assumptions are always necessary, a more satisfactory approach would require a statistical model for other variables (information, dividend, interest rates, etc.), with the behavior of stock prices emerging from the solution of an economic model. This article takes a step in that direction by specifying first the process of information, and then by deriving the process of market prices. The next section presents a theoretical model for which the effects discussed above can be included. Moving the assumption of the GARCH process back from prices to information also allows one to address a much wider array of implications (such as the relation between market volatility, trading volume, and market liquidity) than those first explored in the statistical literature.

2. The Basic Model

The trading environment, the information process, and the market participants are described first. In Proposition 1, it is shown that there is a unique linear Nash equilibrium.

2.1 Trading environment and information

The market environment is modeled as follows: trading for a single risky asset takes place in a centralized market at discrete intervals. There are T trading dates. At time T , the shareholders receive the liquidation value of the asset. The fundamental value of the asset V is exogenously determined and modeled as follows:

$$V_T = V_0 + \sum_{t=1}^T \delta_t, \quad (4)$$

where $(\delta_t)_{t=1\dots T}$ are random variables that represent the changes in the equilibrium value V over time. Each variable δ is conditionally normally distributed with zero mean; δ_t may be considered as a piece of information about the value of the asset. The variance of information, which is allowed to vary over time, is explicitly modeled. The expected variance of δ_t , denoted by σ_t^2 , depends on the value of the most recent information δ_{t-1} , and on past variance σ_{t-1}^2 . A parsimonious representation is the GARCH(1,1) model:⁴

$$\sigma_t^2 \equiv \text{Var}_{t-1}(\delta_t) = \alpha_0 + \alpha_1 \delta_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \quad (5)$$

Information about the asset's payoff enters the model from a public source and from a private signal observed by some participants. Information is gradually revealed to all market participants. The piece δ_t is revealed to the public between time $t-1$ and time t . However, some traders—called hereafter informed traders—know information δ_t just before time $t-1$ and can use this information to their advantage at the trading session at time $t-1$. Private information is short lived: informed traders can use it at one trading session only. At time $t-1$, informed traders have a perfect and correct knowledge of information δ_t . Their information contains no noise.⁵ All market participants believe the information process to be as described in Equation (5), and have homogeneous assessments of the parameters.

This type of process is relevant to this study. First, it includes previous models: assumptions made by Admati and Pfleiderer (1988) about the information process, and by Kyle (1985) about the final asset payoff, corresponding to the case $\alpha_0 = 1$, $\alpha_1 = 0$, and $\beta_1 = 0$. In these models, the variance of information is constant. Second, a GARCH process reflects quite well the fact that information tends to come in waves. This assumption finds some empirical support: Engle et al. (1990) emphasized the influence of the dynamics of the news process on the volatility of exchange rates, and Lamoureux and Lastrapes (1990) observed a daily time dependence in the rate of information arrival to the market for individual stocks of the New York Stock Exchange. The assumption of a time-varying process for information is also related to the justification of a subordinated process for speculative prices, introduced by Clark (1973, p. 137): "The different evolution of price series on different days is due to the fact that information is available to traders at a varying rate. On days when no new information is available, trading is slow, and the price process evolves slowly.

⁴ Foster and Viswanathan (1993a) made a similar assumption for the information process by taking a process related to the exponential GARCH developed by Nelson (1990).

⁵ This assumption has also been made by Kyle (1985) and Foster and Viswanathan (1993b).

On days when new information violates old expectations, trading is brisk, and the price process evolves much faster.”

The varying rate of information arrival implies that the economic time reflecting the real activity of agents differs from the calendar time. As more information arrives, the economic time accelerates, the market price moves, and trading increases. Clark used this positive relation between trading volume and market volatility to take trading volume as a proxy for the directing process of market prices. Although there is no apparent causality between market volatility and trading volume, Clark found empirically a positive correlation between the two variables.

In this article the assumption of a time-varying economic time is moved to the deep parameters of the model. This leads to a much wider array of implications than were first explored in the literature: not only the time-varying behavior of market volatility and the presence of thresholds, but also a volatility-liquidity relation and a volatility-volume relation will be derived from this economic model and thus will be given an economic significance.

2.2 Market participants

There are three types of market participants: liquidity traders, informed traders, and a market maker. Liquidity traders have a demand for the risky asset that is determined by exogenous reasons. For example, they sell assets for their current consumption or buy assets to invest their revenues and transfer wealth over time. The demand from liquidity traders is represented by a normal random variable y_t , which has a mean of zero and a variance ϕ^2 assumed to be constant over time.⁶ All liquidity traders are nondiscretionary: they trade immediately and cannot act strategically.

Informed investors trade to benefit from their informational advantage. As in Kyle (1985), insiders send market orders; they cannot make their quantity conditional on price (limit orders are not allowed). Each informed trader assumed to be risk-neutral submits an order x_t^i , which depends on his private information. His trading strategy is an optimal response to the other market participants’ (other informed traders and the market maker) strategies. An informed trader chooses the quantity to trade given the market-maker pricing rule P_t and the expected demand from other traders, such that he maximizes the expected value of his trading profit. A trader informed just before time t trades at time t and holds his position until the final time T . At time T , there are no trades and investors receive dividends equal to V_T paid by the

⁶ See Back (1993) for a model with a time-varying variance of liquidity trading.

firm. The expected gain of an informed trader is only related to the information that he received. The whole demand from n_t informed traders at time t is noted as x_t .

The total demand w_t is thus $x_t + y_t$. All orders to trade are brought to a market maker who acts as an investor of last resort. She tries to match one order with another and absorbs (buys or sells) the demand w_t . The market maker is assumed to be risk neutral and to make zero expected profit under the pressure of competition. As in Kyle (1985), all trading is done at an efficient price P_t set by the market maker. To set the price P_t , the market maker uses all her information: the past values of information and the past and current values of the order flow. Prices are not posted, but rather are determined after informed traders and liquidity traders have submitted their orders. The market maker observes the imbalance w_t but ignores the nature of the orders (liquidity-driven or information-driven).

2.3 Equilibrium

Proposition 1. *Assuming normality for the demand from liquidity traders and for information,⁷ there is a unique linear Nash equilibrium. The price schedule and the quantity traded by each informed trader are given by*

$$P_t = V_t + \lambda_t w_t \text{ and } x_t^i = \tau_t^i \delta_{t+1}. \quad (6)$$

The parameters λ and τ , defining the pricing rule of the market maker and the strategies of the informed traders, are

$$\lambda_t = \frac{1}{n_t + 1} \sqrt{\frac{n_t \sigma_{t+1}^2}{\phi^2}} \quad \text{and} \quad \tau_t^i = \tau_t = \sqrt{\frac{\phi^2}{n_t \sigma_{t+1}^2}}, \quad (7)$$

where λ and τ depend on the expected variance of information.

Proof in Appendix B.

This result is quite similar to that of Admati and Pfleiderer (1988) and Foster and Viswanathan (1993b). Differences arise from the time-varying behavior of the variance of information σ_t^2 only. Both informed traders and the market maker take this into account by computing their anticipations and forming their strategies. The parameter λ_t , used by the market maker to set the clearing price, is the usual inverse measure of the market depth or liquidity. It increases with the ratio of expected variance of information to variance of liquidity trad-

⁷ Foster and Viswanathan (1993b) generalized this result by showing that the linear form of the decision rules is maintained if the distribution belongs to the elliptically contoured class.

ing σ_t^2/ϕ^2 . Assuming a constant variance of liquidity trading, a high ratio σ_t^2/ϕ^2 means a high variance of information, which means a high probability for a big news item to appear in the future, and then a high expected trading profit for the informed traders, and finally, a high loss for the market maker due to informed trading (adverse selection problem faced by the market maker). The market maker is thus more sensitive to the order flow. She increases λ_t to balance expected losses from the trades with informed traders and expected gains from trades with liquidity traders.⁸ An increase in λ_t means a less liquid market at time t . The parameter τ_t , defining informed traders' strategy, decreases with the expected variance of information σ_t^2 . Informed traders, whose demand is directly related to their private information, tend to limit their impact on the price as the market is expected to be less liquid.

3. Exogenous Acquisition of Information

In this section, informed traders are exogenously informed. Proposition 2 derives the equation for the expected variance of market price changes. Propositions 2.1 and 2.2 deal with special cases.

The variable b_t denotes the expected variance of the price change $(P_t - P_{t-1})$ computed just before time $t - 1$. To compute the expected variance b_t , the most recent public information known to all market participants is used: δ_{t-1} . It is not possible to make the variance conditional on the price change $(P_{t-1} - P_{t-2})$, as is done in empirical studies using statistical GARCH models, because the variable P_{t-1} is not known to investors before time $t - 1$. Also, it is the variance of price changes that is studied here, although empirical studies use logarithmic price changes or percentage returns, which are closely related but not identical to price changes.

Proposition 2. *Assuming a GARCH(1,1) process for information given by $\sigma_t^2 = \alpha_0 + \alpha_1\delta_{t-1}^2 + \beta_1\sigma_{t-1}^2$, the process for the expected variance of market price changes is given by*

$$b_t \equiv \text{Var}_{t-1}(P_t - P_{t-1}) = \alpha_{0,t}^* + \alpha_{1,t}^*\delta_{t-1}^2 + \beta_{1,t}^*b_{t-1}, \quad (8)$$

where the coefficients $\alpha_{0,t}^*$, $\alpha_{1,t}^*$, and $\beta_{1,t}^*$ given in Appendix B vary over time and depend on the number of informed traders at time $t -$

⁸ Unlike Kyle (1985), the market liquidity (inversely related to λ_t) changes over time. Even if liquidity is expected to be higher in the future (i.e., $E_{t-1}(\lambda_{t+1}) < \lambda_t$), informed traders cannot spread their orders over several trading sessions to benefit from a better liquidity. As information δ_{t+1} is revealed to all market participants between time t and time $t+1$, it is profitable for informed traders to trade on their private information at time t only. As information δ_{t+1} is short lived, trades after time t are done on public information and lead to zero expected profit.

1 and time t , n_{t-1} , and n_t . The process of market price changes is thus a GARCH process with time-varying parameters. The degree of persistence of information in market variance $\alpha_{1,t}^*$ is equal to

$$\alpha_{1,t}^* = \alpha_1 \left[\frac{1}{n_{t-1} + 1} + \frac{n_t}{n_t + 1} (\alpha_1 + \beta_1) \right]. \quad (9)$$

Proof in Appendix B.

The process of the variance is not completely identified until the process followed by the number of informed traders is specified. Two special cases, for which the conditional variance has a simple form, are discussed below: the case of no informed traders and the case of a constant, positive number of informed traders.

Proposition 3. *No informed traders: $n_t = 0$. The process of market price changes is a GARCH process identical to the information process:*

$$h_t = \alpha_0 + \alpha_1 \delta_{t-1}^2 + \beta_1 h_{t-1} = \alpha_0 + \alpha_1 (P_{t-1} - P_{t-2})^2 + \beta_1 h_{t-1}. \quad (10)$$

If there are no informed traders, the competitive risk-neutral market maker sets the price equal to the expected value of the asset. The market maker knows the true value of the asset with a lag of one period. The market maker does not lose any more money because of the presence of informed traders. The price change is equal to the new piece of information, and thus the process of the conditional variance of the price change is exactly equal to the process of the conditional variance of the information. The degree of persistence of the latest informational shock δ_{t-1} is the same in variance of information σ_t^2 and in market variance h_t ; it is equal to α_1 .

Proposition 4. *A constant number of informed traders: $n_t = n$. The variance of market price changes follows a GARCH process, whose degree of persistence depends on the number of informed traders:*

$$h_t = \alpha_0 \frac{1 + n(1 + \alpha_1)}{1 + n} + \alpha_1 \frac{1 + n(\alpha_1 + \beta_1)}{1 + n} \delta_{t-1}^2 + \beta_1 h_{t-1}. \quad (11)$$

With exogenous acquisition of information and a constant, positive, number of informed traders, a GARCH process similar to that for information is still obtained. However, the two processes do not have the same coefficients; in particular, the degree of persistence of the latest information into volatility is different. This difference arises from the presence of informed traders. As shown by Equation (11), the persistence decreases with the number of informed traders from α_1 to $\alpha_1(\alpha_1 + \beta_1)$, values obtained for two extreme cases: no informed traders and a number of informed traders growing toward infinity. The more informed traders are, the less persistent the latest informa-

tion into market variance. Thus, after a large informational shock, the variance of market price changes is lower than the variance of information and, conversely, after a small informational shock, the variance of market price changes is higher than the variance of information.

Two points must be considered to explain the decreasing relation between the degree of persistence and the number of informed traders: the quantity of private information incorporated in market prices and the mean reversion in the conditional variance of information. In this model of trading, a part of private information is incorporated in the market price before it becomes public. The more informed traders are, the more information is transmitted through the trading process and the more informative the prices.

The level of market volatility relative to the level of information volatility is determined by the relative size of the variance of the two pieces of information and by the importance of the two pieces in the market price change. To quantify this statement let us consider the following equation:

$$\text{Var}_{t-1}(P_t - P_{t-1}) = \frac{1}{n+1} \text{Var}_{t-1}(\delta_t^2) + \frac{n}{n+1} \text{Var}_{t-1}(\delta_{t+1}^2). \quad (12)$$

Equation (12) relates the variance of market price changes to the variance of the different pieces of information incorporated in the prices at time $t-1$ and time t . The price change $(P_t - P_{t-1})$ is partly due to private information δ_t incorporated at time $t-1$ and partly to private information δ_{t+1} incorporated at time t . The relative importance of the two pieces of information δ_t and δ_{t+1} is measured by the two weights $1/(n+1)$ and $n/(n+1)$, respectively. With no informed traders ($n = 0$), the price change $(P_t - P_{t-1})$ is equal to δ_t , and the weights are equal to 1 and 0. With a constant number of informed traders equal to n , both informational shocks influence the change in price: $n/(n+1)\%$ of δ_t is incorporated at time $t-1$ and $1/(n+1)\%$ of δ_t is incorporated at time t , and similarly, $n/(n+1)\%$ of δ_{t+1} is incorporated at time t and $1/(n+1)\%$ of δ_{t+1} is incorporated later at time $t+1$.

The second point relevant for the explanation is the mean reversion in the conditional variance of information. Shocks to volatility decay at the rate $(\alpha_1 + \beta_1)$ toward the unconditional variance level σ^2 . In a GARCH model, after a large shock (defined by the inequality: $\delta_{t-1}^2 \geq \sigma^2$), the variance of information is expected to be high next period [$\text{Var}_{t-1}(\delta_t) \geq \sigma^2$], to remain high later on [$\text{Var}_{t-1}(\delta_{t+1}) \geq \sigma^2$], but then to come back (i.e., to decrease) to its long-term level σ^2 [$\text{Var}_{t-1}(\delta_t) \geq \text{Var}_{t-1}(\delta_{t+1})$]. Similarly, after a small shock (defined by the inequality: $\delta_{t-1}^2 < \sigma^2$), the variance of information is expected to be low next period [$\text{Var}_{t-1}(\delta_t) < \sigma^2$], to remain low later on

$[\text{Var}_{t-1}(\delta_{t+1}) < \sigma^2]$, but then to come back (i.e., to increase) to its long-term level $\sigma^2[\text{Var}_{t-1}(\delta_t) < \text{Var}_{t-1}(\delta_{t+1})]$.

The two points detailed above explain the decreasing relation between the degree of persistence and the number of informed traders. According to Equation (12), after a large informational shock, the variance of market price changes b_t is lower than the variance of information σ_t^2 since the latter tends to decrease; and after a small informational shock, the variance of market price changes b_t is higher than the variance of information σ_t^2 since the latter tends to increase.

In the special cases studied in this section, GARCH processes are still obtained. For a given number of informed traders, the degree of persistence is the same for all shocks: large shocks are as persistent as small shocks. In Figure 2, the NICs for the case of a constant number of informed traders corresponding to $n = 0, 1, 3$, and 20 are drawn. An ARCH(1) process is used and defined as follows: $\sigma_t^2 = 0.5 + 0.5\delta_{t-1}^2$, which corresponds to $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta_1 = 0$, and implies an unconditional variance $\sigma^2 = 1$. The NICs associated with these processes have different slopes but are still straight lines. A change in the number of informed traders n leads to a rotation of the NIC around the fixed point (σ^2, σ^2) . In Figure 3, the decreasing relation between the degree of persistence and the number of informed traders is represented. Although there is no threshold effect in a model with exogenous acquisition of information, we can guess from Figure 3 how such an effect can appear in a more sophisticated framework with an endogenous, time-varying number of informed traders: if there were few informed traders after a small informational shock, then the degree of persistence would be high (upper part of the graph), and if there were many informed traders after a large informational shock, then the degree of persistence would be low (lower part of the graph). In the next section, it will be shown that there is indeed an increasing relation between the number of informed traders and the size of the past informational shock when the acquisition of information is endogenous. In that case, a threshold effect appears—the degree of persistence depends on the size of the shock: large shocks are less persistent than small shocks.

4. Endogenous Acquisition of Information

In this section, the number of informed traders is endogenous. At time $t - 1$ and time t , the acquisition of pieces of information δ_t and δ_{t+1} is costly. The cost is assumed to be constant and is denoted by c . As in Admati and Pfleiderer (1988), a trader decides to acquire private information if the expected trading profit exceeds the cost of acquisition. At equilibrium, under the pressure of competition, an informed trader

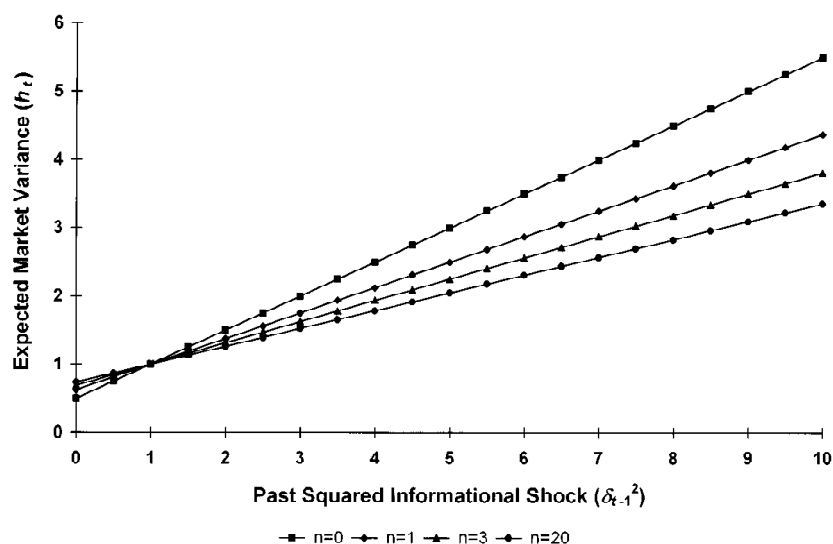


Figure 2
News impact curves (exogenous acquisition of information)

This figure represents the NIC linking the past squared informational shock to the expected variance of market price changes, obtained from a theoretical model of asymmetric information with a constant number of traders exogenously informed. Various numbers of informed traders are considered. The NICs are straight lines with a slope (equal to the degree of persistence) that is constant for a given number of informed traders in the market. The choice of the deep parameters of the model $(\alpha_0, \alpha_1, \beta_1, \phi)$ is $(0.5, 0.5, 0, 5)$.

expects to make zero net profit: the gains from his trading activity are balanced by the cost of information. The equilibrium number of informed traders at time $t - 1$ is given by

$$n_{t-1} = \operatorname{argmax} \left\{ n; \frac{\pi_{t-1}}{n} > c \right\}, \quad (13)$$

where π_{t-1} is the expected trading profit made by informed traders as a group. At time $t - 1$, this number is known and deterministic. A similar equation gives the number of informed traders at time t , n_t . From the time $t - 1$, this number is not known but the market participants can compute its expected value.

As the market maker is making zero profit on average, the gains of informed traders are equal to the losses suffered by liquidity traders equal to $\lambda_{t-1}\phi^2$. Using Equation (7), the trading profit of the group of informed traders is equal to $\phi\sigma_t\sqrt{n_{t-1}}/(n_{t-1} + 1)$, product of the liquidity-trading standard deviation ϕ multiplied by the expected standard deviation of information σ_t and by a function of the number of informed traders in the market. The expected trading profit of informed traders is positively linked to liquidity-trading volatility (the

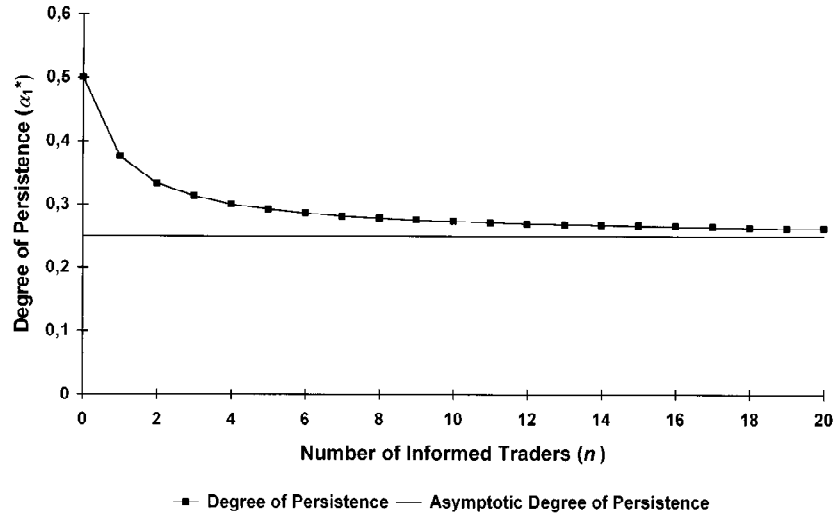


Figure 3

Degree of persistence (exogenous acquisition of information)

This figure represents the decreasing relation between the degree of persistence of the past squared informational shock in expected market variance and the number of informed traders. Each point corresponds to a model with a constant number of informed traders. The choice of the deep parameters of the model $(\alpha_0, \alpha_1, \beta_1, \phi)$ is $(0.5, 0.5, 0, 5)$.

higher the activity of liquidity traders, the easier for informed traders to hide their activity from the market maker), positively related to the expected volatility of information (the higher the volatility of information, the bigger the probability of getting a large piece of information), and negatively linked to the number of informed traders (competition among informed traders). After rearrangement, n_{t-1} is characterized as the greatest integer that satisfies the following inequality:

$$n_{t-1}(n_{t-1} + 1)^2 < \frac{\phi^2 \sigma_t^2}{c^2}. \quad (14)$$

A similar inequality characterizes the random number of informed traders at time t , n_t .

The values of the number of informed traders cannot be easily computed from this inequality. Moreover, the computation of the variance of market price changes is complicated by the conditionality of these numbers. These problems cannot be solved analytically. However, qualitative results, which explain the threshold effect in market volatility, can be derived. Numerical results complete the qualitative propositions.

1. The greater the latest shock of information, the greater the ex-

pected variance of information. This follows directly from the assumption about information volatility, which is modeled as a GARCH process.

2. The greater the expected variance of information, the greater the number of informed traders. The intuition behind this result is that the probability of obtaining important information in the future is high if the expected variance of information is high. If a big news item is expected, the trading profit of the group of informed traders is expected to be large, and many traders will be willing to acquire this information. At equilibrium, the number of informed traders increases so that the trading profit per informed trader is equal to the cost of acquiring private information.

3. The greater the number of informed traders, the lower the persistence of the latest informational shock in market volatility. In Section 3, it was shown that the degree of persistence depends on the number of informed traders. When the number of informed traders is low, the degree of persistence is high; and conversely, when the number of informed traders is high, the degree of persistence is low. However, the equations leading to these results were derived with exogenous acquisition of information, and are now invalid since the number of informed traders is endogenously determined and depends especially on past information via its expected variance. Even when the number of informed traders at time $t - 1$ is known, the number of informed traders at time t is not known because it is a random variable whose realization depends on the value of δ_{t+1} . Further results are obtained by numerical computations (see Appendix B for details). For particular values of information δ_{t-1} , the equilibrium value of the number of informed traders at time $t - 1$ and the expected number of informed traders at time t are computed numerically. These two functions are represented in Figure 4. As previously noted, both functions are increasing with the size of information δ_{t-1} . After a small informational shock, there are few informed traders at time $t - 1$ and few traders are expected to be informed later on. However, more informed traders are expected at time t than at time $t - 1$ because the variance of information is expected to go back (i.e., to increase) to its long-term level. After a large informational shock, there are many informed traders at time $t - 1$ and many traders are expected to be informed later on. However, fewer informed traders are expected at time t than at time $t - 1$ because the variance of information is expected to go back (i.e., to decrease) to its long-term level. In other words, the number of informed traders is persistent over time and mean reverts toward its long-term level. This is because the number of informed traders at time $t - 1$ is a function of the size of the past informational shock, which shows both persistence and mean reversion. The discontinuity

Threshold Effect in Expected Volatility

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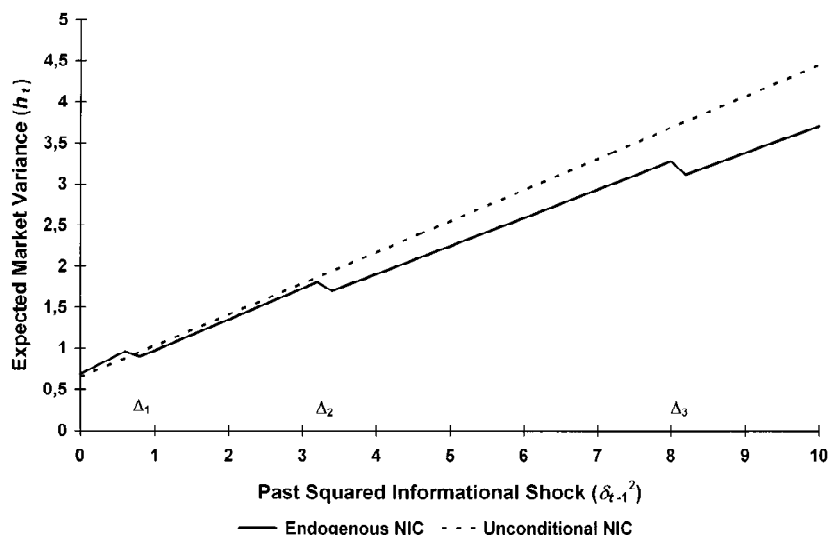


Figure 5

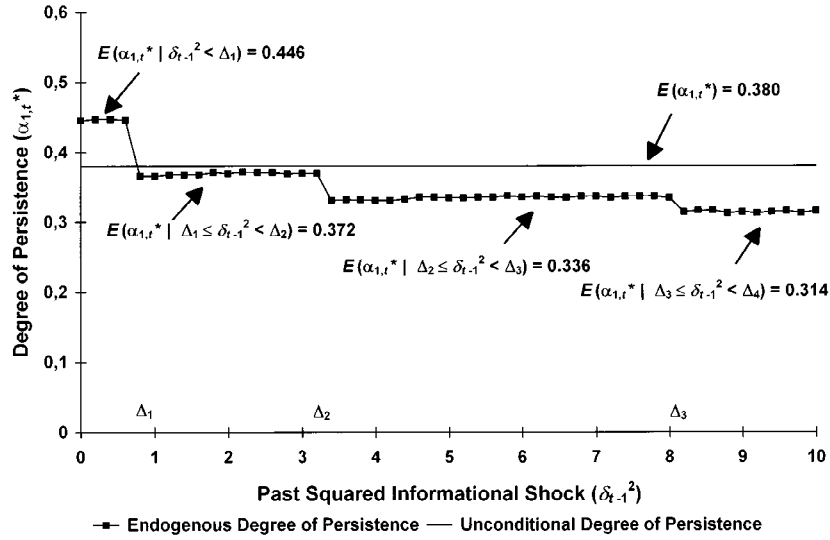
News impact curves (endogenous acquisition of information)

This figure represents the NIC (solid line) linking the past squared informational shock to expected volatility of market price changes obtained from a model of asymmetric information whose traders' decision to become informed is endogenous in the model. The degree of persistence (graphically represented by the slope of the NIC) is a function of the past squared informational shock: large shocks are less persistent than small shocks. The dashed line corresponds to the NIC with an unconditional degree of persistence.

Proposition 5. Suppose a GARCH(1,1) process for information given by $\sigma_t^2 = \alpha_0 + \alpha_1 \delta_{t-1}^2 + \beta_1 \sigma_{t-1}^2$. With endogenous acquisition of information, holding information prior to $t - 2$ constant at its unconditional level, there exists a sequence of thresholds $\Delta_1, \Delta_2, \dots$, such that

if $0 \leq |\delta_{t-1}| < \Delta_1$, then $n_{t-1} = N$, $\alpha_{1,t}^* \approx \alpha_1^*(N)$ and $b_t \approx \alpha_0^*(N) + \alpha_1^*(N)\delta_{t-1}^2$,
 if $\Delta_1 \leq |\delta_{t-1}| < \Delta_2$, then $n_{t-1} = N + 1$, $\alpha_{1,t}^* \approx \alpha_1^*(N + 1)$ and $b_t \approx \alpha_0^*(N + 1) + \alpha_1^*(N + 1)\delta_{t-1}^2$,
 if $\vdots \leq \vdots < \vdots$, then $\vdots$, $\vdots$ and $\vdots$;

hence, the number of informed traders at time $t - 1$ is an increasing step function of information; the degree of persistence is a decreasing, discontinuous function of information approximately equal to a constant [denoted $\alpha_1^*(N + i)$] on the interval $[\Delta_i, \Delta_{i+1}[$, and the expected variance of the future market price change, given past information, is a discontinuous function of information, approximately linear on the interval $[\Delta_i, \Delta_{i+1}[$, where the discontinuities occur at the thresholds Δ_i , $i = 1, 2, \dots$.

**Figure 6****Degree of persistence (endogenous acquisition of information)**

This figure shows that the degree of persistence in volatility of market price changes is a decreasing function of the past squared informational shock. The degree of persistence α_1 of a GARCH model described by Equation (1) would capture the unconditional level of persistence over time $E(\alpha_{1,t}^*)$. A TARCH model, as described in Footnote 3, with several thresholds equal to those given by the theoretical model $\Delta_1, \Delta_2, \dots$ would capture the entire dynamics of persistence.

In Figure 6, the choice of the parameters $(\alpha_0, \alpha_1, \beta_1, \phi, c)$ implies that there is always at least one informed trader (since $\sqrt{\phi^2(\alpha_0 + \beta_1\sigma^2)} > c$). The first threshold Δ_1 is equal to $\sqrt{(4c^2/\phi^2 - \alpha_0 - \beta_1\sigma^2)/\alpha_1}$, the second threshold Δ_2 is $\sqrt{(18c^2/\phi^2 - \alpha_0 - \beta_1\sigma^2)/\alpha_1}$, and the k th threshold Δ_k is $\sqrt{(k(k+1)^2c^2/\phi^2 - \alpha_0 - \beta_1\sigma^2)/\alpha_1}$. If the variance of liquidity trading was higher, the cost of acquiring information lower, or the variance of information higher, then the minimum number of informed traders in the market at any time could be greater than one.

Figure 6 gives an economic insight into the empirical results obtained with statistical GARCH and TARCH models. In a GARCH model, the persistence in expected market variance is modeled by one parameter only: α_1 . It corresponds to the unconditional value of the random, time-varying persistence $\alpha_{1,t}^*$ given by the theoretical model. In the TARCH model proposed in Section 1, the persistence is captured by two parameters: α_1 for small shocks and the sum $(\alpha_1 + \gamma_1)$ for large shocks in Equation (3). Small and large shocks are separated by a threshold set by the econometrician at σ^2 . Using the theoretical model, the coefficient α_1 corresponds to the value of the degree of persistence conditional on a small shock $E(\alpha_{1,t}^* | \delta_{t-1}^2 < \sigma^2)$, and the

sum $(\alpha_1 + \gamma_1)$ corresponds to the value of the degree of persistence conditional on a large shock $E(\alpha_{1,t}^* | \delta_{t-1}^2 \geq \sigma^2)$. According to the theoretical model, the sum $(\alpha_1 + \gamma_1)$ is less than α_1 and thus γ_1 is negative, as found empirically. Note that a TARCH model, as described in Footnote 3, with several thresholds equal to those given by the theoretical model $\Delta_1, \Delta_2, \dots$ would capture at best the dynamics of persistence. This work has some theoretical and empirical implications for further research in the modeling of market volatility: although this gives some economic content to the models with several thresholds (by relating the value of these thresholds to the deep parameters of the model: variance of information, variance of liquidity trading, and cost of acquiring information), it shows that the persistence may be modeled in a simple way, in terms of the number of parameters to estimate, by using a parametric, decreasing function for the function F in Equation (1).

5. Further Testable Implications of the Model

In this section, further implications of the model are derived: the magnitude of the persistence in expected market variance is linked to the firm size; cross-restrictions between trading volume, market liquidity, and market volatility are also offered to test the model further. Finally, the research design for a more complete test, using the method of simulated moments, is outlined.

5.1 Persistence and firm size

In the financial industry, firms are followed by investment analysts whose number varies from firm to firm. As argued in Brennan, Jegadeesh, and Swaminathan (1993), if the number of analysts can be regarded as a proxy for the number of informed traders, then the speed of adjustment of the firm's stock price to new information is related to the firm size. As is now shown, the degree of persistence in expected market variance is indeed inversely related to the speed at which new information is incorporated in the market price: the lower the degree of persistence, the higher the speed of adjustment. Let us consider first the case of exogenous acquisition of information. With no informed traders, the piece of information δ_{t-1} is incorporated in the market price at time t and its persistence in expected market variance is equal to α_1 . If there are informed traders, a part of δ_{t-1} is incorporated in the market price earlier at time $t - 1$, and, from Equation (11), its persistence in expected market variance is lower than α_1 . With endogenous acquisition of information, the number of informed traders varies over time as information is available at a varying rate. As in the exogenous case, when the unconditional number

of informed traders is high, the unconditional degree of persistence in expected market variance is low. Such a result motivates a cross-sectional study that involves estimating a GARCH model for stocks issued by firms of different sizes: assuming the same characteristics for the information process for small and large firms, the unconditional degree of persistence α_1 , estimated by a GARCH model, should be higher for small firms than for large firms. A study by Conrad, Gultekin, and Kaul (1991) confirms this implication of the model: using firms traded on the American and New York Stock Exchanges for the period 1962 to 1988, they estimated univariate GARCH processes for portfolios grouping firms of different sizes; they found a degree of persistence equal to 0.190 for the 100 smallest firms, 0.159 for the 100 intermediate firms, and 0.114 for the 100 largest firms.

5.2 The volatility-volume relation

Trading volume is defined in the same way as by Admati and Pfleiderer (1988). Specifically, expected trading volume in each period is computed as half of the orders from the informed traders, plus half of the orders from liquidity traders, plus half of the orders traded with the market maker (noncrossed orders): $\frac{1}{2}\phi(1 + \sqrt{n_{t-1}} + \sqrt{1 + n_{t-1}})$.

As for the process of the variance, the process of trading volume is not completely identified until the process followed by the number of informed traders is specified. When there are no informed traders, the market maker absorbs the demand from liquidity traders only, and the volume is then equal to ϕ . When the number of informed traders is constant over time, expected trading volume is also constant, although the variance of information changes. This is due to the fact that informed traders modify the intensity with which they trade on the basis of their private information to limit their impact on market liquidity (an informed trader's order is proportional to δ_t/σ_t). There is a positive relation between the number of informed traders and expected trading volume: the higher the number of informed traders, the more aggressive informed traders trade as a group (competition among themselves), and the higher the expected trading volume. In a model with endogenous acquisition of information, expected trading volume is positively linked to the value of the past squared informational shock via the number of informed traders: the larger the past informational shock, the higher the expected variance of information, the higher the number of informed traders at equilibrium (see Section 4), and the higher the expected trading volume.

A positive relation is also obtained between expected trading volume and expected market variance: both quantities are positively re-

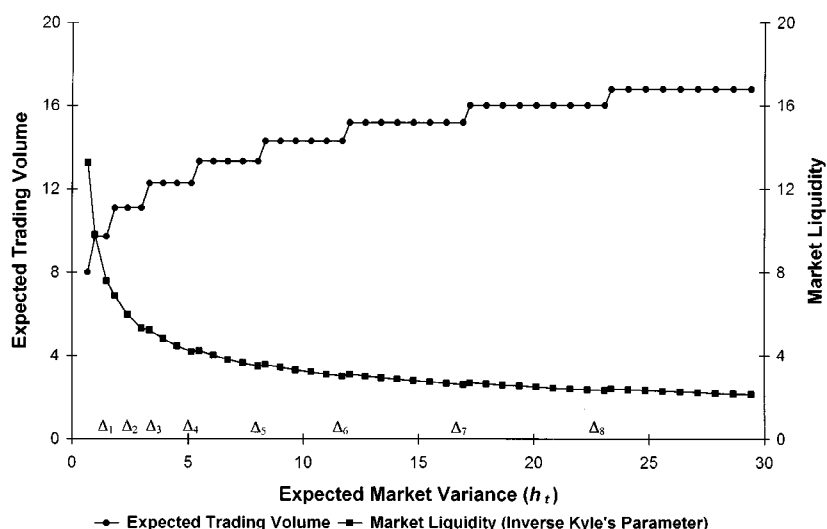
lated to the information flow rate.⁹ This relation is represented in Figure 7 and constitutes a testable implication to confront the model further with real data. It seems to be corroborated by the analysis of the volatility-volume relation by Gallant, Rossi, and Tauchen (1992). Using the S&P composite price index from 1928 to 1987 they found a positive relation between trading volume and market variance. Figure 7 also shows that the volatility-volume relation is concave. For very high levels of market volatility (during periods of market booms and crashes associated with big changes in the information environment), the relation is almost flat as the marginal impact of an increase in the number of informed traders on their total demand becomes smaller. This additional fact seems consistent with the study by Balduzzi, Kallal, and Longin (1996) on the breakdown of the price-volume relation. Using data for an NYSE stocks index for the period 1885 to 1990, they found a strong positive correlation (0.52) between middle-size returns and trading volume, and no significant correlation (0.06) between crashes (defined as negative returns under a threshold of four standard deviations) and trading volume.

In this model, expected market variance is positively related to expected trading volume. This correlation is not causal since the behavior of both variables is explained (caused) by another variable: the information flow rate. The model, however, justifies Clark's idea of taking the trading volume as a measure of the economic time.

5.3 The volatility-liquidity relation

In this model the concept of market liquidity is associated with market depth, measured by the inverse of Kyle's (1985) parameter λ given in Equation (7). In the model this parameter depends on the number of informed traders and on the ratio of expected variance of information to expected variance of liquidity trading. With exogenous acquisition of information, assuming the variance of liquidity trading is constant, after a large informational shock, information volatility increases (ARCH effect), the ratio of the amount of private information to the amount of liquidity trading increases, the market maker expects the trading profit made by the group of informed traders to increase, and then decreases the liquidity she offers by increasing the parameter λ . With endogenous acquisition of information, two effects work in opposite directions for the relation between market liquidity

⁹ This characterization of the volatility-volume relation adds to Foster and Viswanathan's (1993a) work. With a similar framework they found that innovations in trading volume and market variance are positively correlated. Here, a positive relation is found between the expectation of the two variables (conditional trading volume and conditional market variance).

**Figure 7****Trading volume, liquidity, and market variance (endogenous acquisition of information)**

This figure relates the expected market variance to two market microstructure variables: trading volume and market liquidity. The three variables are linked through the number of informed traders at equilibrium.

and the size of information: first, as with exogenous acquisition of information, market liquidity decreases with the size of information via the expected volatility of information, and second, market liquidity increases with the size of information via the number of informed traders which increases due to competition. Numerical results show that the first effect prevails: market liquidity decreases with the size of the past informational shock.

A positive relation is also obtained between expected market liquidity and expected market variance: both quantities are positively related to the information flow rate. In Figure 7 the relation between expected market variance and market liquidity is also represented. Such a relation constitutes another testable implication. Empirical results by Handa (1991) tend to support this proposition. Handa (1991) built a practical, observable measure of market liquidity by computing the market depth from the market-maker's quotes. Using transaction data for NYSE and AMEX stocks over the period 1988 to 1989, he found that market liquidity (measured by the market depth) is inversely related to the absolute size of the latest change in the bid-ask midpoint (taken as the fundamental asset value), and therefore to

market variance (as the bid-ask midpoint presents ARCH-like characteristics).

5.4 A more complete test of the model

In this model (as in models using Kyle's paradigm) the information process and the price process are distinguished. The information process is first specified, and the price process is then derived as a solution of an economic model. It makes sense to work with information flows since information constitutes the real news of financial markets.

Following the works by Foster and Viswanathan (1993a) and Bernhardt and Hughson (1995), one can work directly with information flows and to estimate the deep parameters of the model. The deep parameters of the model (coefficients of the variance of information α_0 , α_1 , and β_1 , liquidity trading variance ϕ^2 , and cost of acquiring private information c) could be directly inferred from observed data using the procedure of the method of simulated moments.¹⁰

The results obtained here suggest that market prices, trading volume, and market liquidity should be used for the observable variables because they are influenced by information flows. Market prices allow one mainly to test the hypothesis about the information variance process; trading volume tests the hypothesis about informed traders' demand; and market liquidity tests the hypothesis about the market-maker's behavior. As the model is a transactional level, high-frequency data should be used for the test. The model also suggests the use of serial and cross moments. The choice of serial moments is motivated by the persistence in the variables. Although information is short lived, the variance of information persists over time, and this induces a persistence in all derived variables: number of informed traders, expected market variance, expected trading volume, and market liquidity. Such a serial correlation may be intensified if information is long lived (i.e., if private information is not revealed to the public in the next period and can be used for several further trading sessions by the informed traders). Trading on long-lived information, considered in Kyle (1985) and Cho (1995), may add another source of serial correlation in price volatility.

The choice of cross-moments between market variance, trading volume, and market liquidity is motivated by the correlation found between these three variables in Sections 5.2 and 5.3.

¹⁰ Two other approaches have been proposed in the literature to test market microstructure models: Cho (1995) applies Kalman filtering to identify the deep parameters of a model with long-lived information but only one informed trader; and Easley, Kiefer, and O'Hara (1993) estimate the structural parameters defined as probabilities (the probability of an information event, of good news, and of the presence of an informed trader) by maximizing the likelihood of the model.

6. Conclusion

In this research a GARCH effect is included in the information process and, in equilibrium, manifests itself in the price process, but with different characteristics. For example, it is shown that with asymmetric information, the GARCH process for information is transformed into a TARCH process for market price changes. Although market variance is often used as a measure of information flows [as argued by Ross (1989) for a world without arbitrage opportunities], this article shows that market participants' trading activity may add characteristics in the observed price process that are not contained in the information process. Using a model of information also allows one to investigate other properties of financial markets, such as trading volume and market liquidity, that are related to information flows. Such an approach provides a wide array of testable propositions that could enhance our understanding of how financial markets work.

Appendix A: Estimation of GARCH and TARCH Models

To illustrate the threshold effect in expected market variance, GARCH and TARCH models for changes in the Standard & Poor's 500 index futures prices. The period of estimation is 1986 to 1990 and contains 17,680 observations. A detailed description of the dataset can be found in Becker, Finnerty, and Friedman (1993). Following Foster and Viswanathan (1993a), the observed price change is first whitened for different effects: nontrading periods such as overnight, weekends, and holidays, opening and closing time periods, and seasonal patterns both in the mean and variance of the price change. Standardized price changes are then used to estimate GARCH and TARCH equations.

A.1 Equation for the expected variance (GARCH):

$$h_t = 4.07 \cdot 10^{-2} + 0.089e_{t-1}^2 + 0.878h_{t-1},$$

(27.44) (56.51) (671.62)

$$\log \text{likelihood} = -20,964.75.$$

A.2 Equation for the expected variance (TARCH):

$$h_t = 4.91 \cdot 10^{-2} + 0.128e_{t-1}^2 - 0.091D_{1,t-1}(e_{t-1}^2 - \sigma^2) + 0.866h_{t-1},$$

(19.68) (33.61) (-10.57) (591.77)

$$\log \text{likelihood} = -20,880.19.$$

The threshold is defined as the unconditional variance denoted by σ^2 and is equal to 1.174.

Appendix B: Proofs and Details of Numerical Computations

B.1 Proof of Proposition 1

The Nash equilibrium in the game between the market maker and informed traders is solved. Following Kyle (1985), normality for the demand from liquidity traders y_t and for information δ_{t+1} is assumed. Under this assumption, the price set by the market maker P_t is linear in the order flow observed at time t , w_t , and the quantity traded by each informed investor, x_t^i , is linear in their private information δ_{t+1} .

B.1.1 Demand of informed traders. The i th informed trader chooses x_t^i equal to $\tau_t^i \delta_{t+1}$. He chooses x_t^i , the amount to trade at time t , to maximize his expected trading profits. The informed trader uses all his information to compute the expectation. He maximizes as follows:

$$E(x_t^i(V_T - P_t) | w_1, w_2, \dots, w_{t-2}, w_{t-1}, \sigma_1, \delta_1, \delta_2, \dots, \delta_t, \delta_{t+1}). \quad (B1)$$

V_T is the random terminal liquidation value of the asset. P_t is the asset price set at time t by the market maker. P_t is a random variable for informed traders, because they do not know the demand from liquidity traders at time t , y_t , and thus neither do they know the total demand at time t , $w_t (= x_t + y_t)$. Given the linear pricing rule of the market maker and the conjecture made by the i th informed trader about the other $n_t - 1$ informed traders' market order, τ_t can be written

$$\tau_t = \frac{1}{\lambda_t(n_t + 1)}. \quad (B2)$$

B.1.2 Pricing rule of the market maker. Under the assumption of normality of the liquidity traders' demand and information, the market maker uses a linear pricing rule characterized by the parameter λ_t . The value of λ_t is determined for a given set of strategies by all informed traders. The condition of zero expected profit for the market maker is used.

$$E((V_T - P_t)w_t | w_1, w_2, \dots, w_{t-2}, w_{t-1}, \sigma_1, \delta_1, \delta_2, \dots, \delta_{t-1}, \delta_t) = 0. \quad (B3)$$

Equation (B3), δ_t contains some information about δ_{t+1} since the variance of the next piece of information depends on past information. By computing her expectations, the market maker takes into account this information. The following equation for λ_t is obtained

$$\lambda_t = \frac{\text{Cov}_t(\delta_{t+1}, w_t)}{\text{Var}_t(w_t)} = \frac{n_t \tau_t \text{Var}_t(\delta_{t+1})}{n_t^2 \tau_t^2 \text{Var}_t(\delta_{t+1}) + \phi^2}. \quad (B4)$$

The covariance and variance are conditional on the information of the market maker. The only relevant piece of information is the latest news δ_t , which is known to the public between time $t - 1$ and time t . The expected variance of future information δ_{t+1} is conditioned on the value of the latest piece of information δ_t . This expectation is denoted by σ_{t+1}^2 . Parameter λ_t can be written

$$\lambda_t = \frac{n_t \tau_t \sigma_{t+1}^2}{n_t^2 \tau_t^2 \sigma_{t+1}^2 + \phi^2}. \quad (\text{B5})$$

The system given by Equations (B2) and (B5) is now used to obtain the values of τ_t and λ_t , given in the text (Equation 7).

B.2 Proof of Proposition 2

The expected variance of the price change over the period $[t - 1, t]$, that is to say $(P_t - P_{t-1})$, is computed. The expectation of the variance is computed just before time $t - 1$. The expected variance of the price change $(P_t - P_{t-1})$ is called h_t and defined as

$$h_t \equiv \text{Var}((P_t - P_{t-1}) | \sigma_1, \delta_1, \delta_2, \dots, \delta_{t-2}, \delta_{t-1}). \quad (\text{B6})$$

The computation of h_t is detailed below. The price change is first decomposed:

$$P_t - P_{t-1} = \delta_t + \lambda_t w_t - \lambda_{t-1} w_{t-1}, \quad (\text{B7})$$

where the demand purchased by the market maker w_t is given by

$$\begin{aligned} \lambda_t w_t &= \lambda_t (y_t + n_t \tau_t \delta_{t+1}) \\ &= \frac{1}{n_t + 1} \sqrt{\frac{n_t \sigma_{t+1}^2}{\phi^2}} y_t + \frac{n_t}{n_t + 1} \delta_{t+1}. \end{aligned} \quad (\text{B8})$$

The price change is

$$\begin{aligned} P_t - P_{t-1} &= \frac{\delta_t}{n_{t-1} + 1} + \frac{n_t \delta_{t+1}}{n_t + 1} + \frac{1}{n_t + 1} \sqrt{\frac{n_t \sigma_{t+1}^2}{\phi^2}} y_t \\ &\quad - \frac{1}{n_{t-1} + 1} \sqrt{\frac{n_{t-1} \sigma_t^2}{\phi^2}} y_{t-1}. \end{aligned} \quad (\text{B9})$$

Now, the expectation of the square of the above equation, conditional on the knowledge of δ_{t-1} , is taken. The independence of the liquidity

trading y_t from the pieces of information is used to obtain

$$b_t = \frac{\sigma_t^2}{(n_{t-1} + 1)} + E \left[\left(\frac{n_t}{n_t + 1} \right)^2 \delta_{t+1}^2 | \delta_{t-1} \right] + E \left[\frac{n_t}{(n_t + 1)^2} \sigma_{t+1}^2 | \delta_{t-1} \right]. \quad (\text{B10})$$

As the number of informed traders is exogenously given, Equation (B10) can be written

$$b_t = \frac{s_t^2}{(n_{t-1} + 1)} + \left(\frac{n_t}{n_t + 1} \right)^2 E(\delta_{t+1}^2 | \delta_{t-1}) + \frac{n_t}{(n_t + 1)^2} E(\sigma_{t+1}^2 | \delta_{t-1}). \quad (\text{B11})$$

Two conditional expectations have to be computed; using the definition of the information process given by Equation (5), the two expectation terms are given by

$$E(\sigma_{t+1}^2 | \delta_{t-1}) = E(\alpha_0 + \alpha_1 \delta_t^2 + \beta_1 \sigma_t^2 | \delta_{t-1}) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2, \quad (\text{B12})$$

$$\begin{aligned} E(\delta_{t+1}^2 | \delta_{t-1}) &= E(E(\delta_{t+1}^2 | \delta_t) | \delta_{t-1}) \\ &= E(\sigma_{t+1}^2 | \delta_{t-1}) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2. \end{aligned} \quad (\text{B13})$$

By replacing these two expressions in the equation for b_t , one gets

$$b_t = \alpha_0 \frac{n_t}{n_t + 1} + \left(\frac{1}{n_{t-1} + 1} + \frac{n_t}{n_t + 1} (\alpha_1 + \beta_1) \right) \sigma_t^2. \quad (\text{B14})$$

The expression for σ_t^2 and the above relation at time $t - 1$ are now used to obtain the equation for the process of the conditional variance [Equation (8) in the text]:

$$h_t \equiv \text{Var}_{t-1}(P_t - P_{t-1}) = \alpha_{0,t}^* + \alpha_{1,t}^* \delta_{t-1}^2 + \beta_{1,t}^* h_{t-1}, \quad (\text{B15})$$

where

$$\begin{aligned} \alpha_{0,t}^* &= \alpha_0 \left(A_{0,t} + A_{1,t} - \beta_1 A_{0,t} \frac{A_{1,t}}{A_{1,t-1}} \right), \\ \alpha_{1,t}^* &= a_1 A_{1,t}, \text{ and } \beta_{1,t}^* = \beta_1 \frac{A_{1,t}}{A_{1,t-1}}. \end{aligned} \quad (\text{B16})$$

With

$$A_{0,t} = \frac{n_t}{n_t + 1} \text{ and } A_{1,t} = \frac{1}{n_{t-1} + 1} + \frac{n_t}{n_t + 1}(\alpha_1 + \beta_1). \quad (\text{B17})$$

B.3 Details of numerical computations

The details of the numerical computation of the expected variance of market price changes with endogenous acquisition of information are given below.

Equations (B1–B10) are still valid; but when the acquisition of information is endogenous, the number of informed traders is a random variable and Equations (B11–B17) are no longer valid. Equation (B10) is used to compute the expected volatility b_t . Expectations do not have simple analytic expressions because the variables n_{t-1} and n_t are derived from an inequality. The value of b_t is numerically computed by Monte Carlo simulations for a specific process of information and for specific parameters of the model (variance of liquidity traders' demand and cost of information) and for particular values of δ_{t-1} . For simulations and graphical representations, an ARCH(1) process is used and defined as follows: $\sigma_t^2 = 0.5 + 0.5\delta_{t-1}^2$, which corresponds to $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta_1 = 0$. Information δ_t is drawn from a normal distribution with mean zero and variance σ_t^2 . The simulations use 20,000 random draws. The unconditional variance is equal to one. To compute the number of informed traders at each date using Equation (13), the ratio of the cost of information to the variance of liquidity traders' demand (c^2/ϕ^2) is taken as equal to 1/25. It corresponds to an average number of informed traders equal to 1.50. Figures 4–7 reflect the choice of the parameters ($\alpha_0, \alpha_1, \beta_1, \phi, c$) equal to (0.5, 0.5, 0, 5, 1).

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