Debt in Industry Equilibrium

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This article shows (1) how entry and exit of firms in a competitive industry affect the valuation of securities and optimal capital structure, and (2) how, given a trade-off between tax advantages and agency costs, a firm will optimally adjust its leverage level after it is set up. We derive simple pricing expressions for corporate debt in which the price elasticity of demand for industry output plays a crucial role. When a firm optimally adjusts its leverage over time, we show that total firm value comprises the value of discounted cash flows assuming fixed capital structure, plus a continuum of options for marginal increases in debt.

Valuation theory in financial economics is generally concerned with obtaining restrictions on the joint distribution of asset returns from assumptions on payoffs or preferences. Few attempts have been made to link the pricing of securities to characteristics of the factor and product markets in which firms operate.

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Making such a link is difficult because it requires explicit modeling of a full industry equilibrium.

The impact of debt in industry equilibrium with imperfectly competitive firms has been investigated in a series of articles by Brander and Lewis (1986, 1988) and Maksimovic (1988, 1990). These articles focus primarily on the way in which debt financing may enable firms to commit to particular ex post behavior in imperfectly competitive markets. Maksimovic and Zechner (1991) also study debt and industry equilibrium but are mainly concerned with agency costs. With the exception of Maksimovic (1988), all these models are static, and it is not obvious how they could be employed for pricing assets.

In this article, we study the interaction between bond-pricing, industry equilibrium, and optimal capital structure. In particular, we derive explicit solutions for the value of corporate bonds that depend in a simple, transparent manner on the elasticity of demand for total industry output. We also show how firms will optimally adjust their leverage levels, both when they are set up and subsequently. In the process, we obtain a dynamic theory of optimal leverage.

Valuing debt without allowing for dynamic adjustments in leverage appears quite restrictive. In the absence of leverage adjustment costs and restrictive convenants on existing debt, firms may wish to adjust their debt levels on a continuous basis. Brennan and Schwartz (1984) model such behavior. Fischer, Heinkel, and Zechner (1989a, 1989b) analyze optimal periodic adjustments of leverage when recapitalization costs are proportional to bonds issued. As the work of Kane, Marcus, and MacDonald (1984, 1985) shows, the residual value of the firm in the event of bankruptcy or when debt falls due depends on the ability of the firm's owners to adjust the level of leverage at that point.

Our treatment of releverage behavior resembles that of Fischer, Heinkel, and Zechner in that we examine optimal changes in leverage in the presence of adjustment costs. However, the cost structure we assume is quite different from theirs, since we suppose that hold-out problems or possible legal restrictions make it prohibitively expensive to reduce leverage, but that increases in debt are costless. Roe (1987) documents how hard it is to renegotiate public bond issues in the United States outside formal bankruptcy proceedings.[1]

The approach we take in this article builds on recent work by Leahy (1993). Leahy shows how Dixit's (1989) continuous-time real-investment model may be extended to study a competitive industry

[1] Also see Gertner and Scharfstein (1991) for an extremely interesting analysis of exit consent renegotiations of public bond issues.

equilibrium with entry and exit by firms triggered by output price movements. Leahy's model, like that of Dixit, assumes pure equity financing of the firm. For industries in which expenditure on fixed capital is important, this assumption seems unattractive. Differences between corporate and personal tax rates, together with bankruptcy costs, will in principle make financial structure matter for firm valuation.

Section 1 of the article sets out a simple model of a single firm that finances an irreversible investment with a mixture of debt and equity. We demonstrate in this context the solution methods that will be employed throughout the article and derive some preliminary results on optimal capital structure, agency costs, and incentives to adjust leverage levels.[2]

Section 2 of the article embeds our model of a single firm in an industry equilibrium. We show that free entry and exit impose reflecting barriers on the price of the industry's output, affecting the valuation of corporate debt and equity and the optimal capital structure decision.

Our pricing expressions for corporate debt have the interesting feature that, when equity-holders are cash-constrained, the bonds become riskless as the elasticity of output demand approaches zero. The reason is that, for low price elasticities, very few firms have to leave the industry in order to prevent the output price dropping below the exit trigger level p_b. Hence, for a given firm, bankruptcy is a low-probability event, and the bonds resemble the safe asset.

When equity-holders are not cash-constrained and debt is risky, individual firms entering the industry have an incentive to "undercut" other firms by adopting marginally lower leverage. In equilibrium, this incentive forces down the level of borrowing until all debt is fully collateralized, that is, at closure equal in value to the residual value of firms' assets.

Section 3 relaxes the assumption that the leverage level is fixed after the firm is set up. We characterize a firm's optimal dynamic releverage policy, assuming that increasing leverage is costless but that unlevering can only be accomplished through liquidation. We justify the latter assumption by supposing that free-rider behavior by

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[2] The framework of Section 1 resembles models employed in a number of recent articles on corporate bond pricing, namely Leland (1994), Mella-Barral and Perraudin (1993), and Fries, Mella-Barral, and Perraudin (1997). An important feature of these models is that unlike earlier work by, for example, Merton (1974), they suppose that bankruptcy is triggered by equity-holders' choice rather than net-worth constraints. Leland (1994a) argues persuasively that bond covenants concerning net worth are hard to enforce, and that it is more reasonable to think of equity-holders as effectively choosing the bankruptcy point through their decision when to cease meeting coupon payments.

individual bondholders prevents any capital restructuring agreement that involves reductions in leverage.

We show that the value of the firm in these circumstances equals that of its discounted cash flows assuming a fixed capital structure, *plus* the value of a continuum of options to increase leverage. These options are exercised successively as the output price hits new peaks.

1. A Simple Benchmark Model

1.1 Basic assumptions

We begin by describing a simple model that demonstrates our solution techniques and establishes a benchmark. This model will abstract from industry equilibrium effects in that firm bankruptcies will not affect the basic state variable, the price of the firm's output.

Suppose that a firm may issue debt or equity to finance the purchase of capital costing K. We shall initially assume that no adjustment in the level of leverage is possible after the firm is set up. At the end of this section and then at more length in later sections, we shall return to this point.

The firm's technology is such that output per period equals unity and the firm faces a flow of costs w (assumed to be constant through time) as long as production continues. If p_t is the price of the firm's output, net earnings equal

$$(1-\tau)(p_t - w), \tag{1}$$

where τ is the corporate income tax rate. Implicitly, we assume here that the firm's dividend policy is fixed. One should regard our analysis below of optimal investment and bankruptcy decisions as conditional on this. Also, suppose that p_t is a geometric Brownian motion:

$$dp_t = p_t \mu \, dt + p_t \sigma \, dB_t, \tag{2}$$

for constants μ and σ and a standard Brownian motion B_t.

If the output price falls far enough, earnings will eventually turn negative, obliging equity-holders to cover operating losses through capital injections.[3] Bankruptcy will be triggered by the equity-holders' decision to cease injecting funds.[4]

A crucial question for valuing debt is as follows: *what is the residual value of the firm after bankruptcy*? To emphasize the wide applicability

[3] We abstract here from the fact that the marginal tax rate for negative earnings may be zero. In practice, there are many ways in which companies can use tax losses to offset other income, including the sale of subsidiaries embodying tax losses and leasing transactions. It is therefore probably more realistic to assume that τ remains unchanged when earnings turn negative than to assume that it drops to zero.

[4] Black and Cox (1976) and Leland (1994) model bankruptcy in a similar way.

of our results, we develop our models with a general residual value function $X(p)$. The only restriction we place on the function $X(p)$ is that it be nondecreasing in its argument.

Assumption 1. *The residual value of the firm after bankruptcy is a function $X(p_t)$ of the output price, where $X'(p) \geq 0$ for all $p \geq 0$.*

Our approach is sufficiently general to encompass various assumptions made in past studies. For example, classic studies of debt valuation, such as Black and Cox (1976) and Merton (1974), have supposed that after bankruptcy, bondholders receive the value of the firm's assets that are assumed to follow a simple geometric Brownian motion. On the other hand, Mello and Parsons (1992) assume that when bankruptcy occurs, bondholders receive the pure equity value of the firm, that is, a basic asset value following a geometric Brownian motion, as in Merton (1974), *plus* the option value associated with liquidation possibilities.[5]

In fact, neither the Black-Cox-Merton nor the Mello-Parsons approach is ideal. Bondholders taking control after bankruptcy may relever a firm in a variety of ways. The options associated with releverage possibilities are likely to form an important part of the firm's value and hence affect the premiums on corporate debt well before bankruptcy occurs. Our approach is sufficiently general to allow for such options.

As well as specifying the residual value function $X(p_t)$ to price debt in industry equilibrium, one needs to make assumptions about what bankruptcy implies for the output level of the firm. Specifically, if a firm ceases to produce when it goes bankrupt, this may affect industry output and feed back into the profitability of other corporations.

The models we develop below will vary according to whether industry output falls when firms go bankrupt. In the benchmark model of this section, industry output is assumed not to fall, and hence the price process p_t can be treated as fully exogenous.

Finally, note that we do not allow the equity-holders in our model to suspend production temporarily. Such "moth-balling" of productive activities and the real option values that it can induce have been studied by Brennan and Schwartz (1984) and Mello and Parsons (1992).

[5] To anticipate our results somewhat, note that in our framework, Merton's approach would correspond to assuming that upon bankruptcy, bondholders receive the unlimited liability value of the firm's income stream, that is, $X(p_t) = (1 - \tau)[p_t/(r - \mu) - (w/r)]$. On the other hand, the equivalent of Mello and Parsons' approach in our framework would be $X(p_t) = (1-\tau)[p_t/(r-\mu)-(w/r)]-(1-\tau)(p_t/p_l)^{\lambda_1}[p_l/(r-\mu)-(w/r)]$, where $p_l = -\lambda_1/(1-\lambda_1)w(r-\mu)/r$ and λ is as defined in Proposition 1.

1.2 Debt and equity valuation

Let us suppose that the firm is financed with equity and infinite maturity debt, paying a fixed coupon b. If agents are risk-neutral,[6] and r is the constant risk-free interest rate,[7] the firm's total equity and debt values V_t and L_t must satisfy

$$rV_t = (1-\tau)(p_t - w - b) + \frac{d}{d\Delta} E_t V_{t+\Delta}\bigg|_{\Delta=0}, \tag{3}$$

and

$$rL_t = b + \frac{d}{d\Delta} E_t L_{t+\Delta}\bigg|_{\Delta=0}, \tag{4}$$

where b is the flow of coupon payments on the firm's debt and E_t is the conditional expectations operator. In words, the expected return from investing amounts V_t or L_t in the safe asset must equal the expected return from investing in the firm's equity or debt. For simplicity, we assume that, at a personal level, all income is taxed at the same rate, and hence personal tax rates drop out of the above equilibrium conditions. Note that we also assume in the above equations that interest payments are tax-deductible. This assumption introduces a tax advantage to leverage.

If V_t and L_t are twice-continuously differentiable functions of the state variable p_t, we may apply Ito's lemma to obtain

$$rV(p_t) = (1-\tau)(p_t - w - b) + V'(p_t)p_t\mu + V''(p_t)p_t^2\frac{\sigma^2}{2}, \tag{5}$$

and

$$rL(p_t) = b + L'(p_t)p_t\mu + L''(p_t)p_t^2\frac{\sigma^2}{2}. \tag{6}$$

The appropriate boundary conditions are as follows. If the firm is closed at some price p_b, then, in the absence of arbitrage, $V(p_b) = 0$ and $L(p_b) = X(p_b)$. As $p_t \to \infty$, the possibility of closure plays a smaller and smaller role in valuation, so V and L must approach the

[6] This does not represent much of a loss of generality since an economy with risk-averse agents may be transformed into one with risk neutrality by the change of measure arguments suggested by Harrison and Kreps (1979). The significant restriction one has to make is to suppose that the financial structure of the firm does not affect the Radon Nikodym derivative for this change of measure; that is, the state prices are invariant to the firm's financing decisions. Adopting this assumption, one should interpret the term $p_t\mu$ in Equation (2) as the risk-adjusted drift and not the actual instantaneous mean of the process.

[7] We assume that $r > \mu$. One may show that, otherwise, claims to the firm's earning stream would be of infinite value, which is obviously inconsistent with equilibrium.

unlimited liability values of the income streams:

$$\lim_{p\to\infty} V(p) = E_t\left\{\int_t^\infty \exp[-r(s-t)](1-\tau)(p_s - w - b)\,ds\right\}, \quad (7)$$

$$\lim_{p\to\infty} L(p) = E_t\left\{\int_t^\infty \exp[-r(s-t)]b\,ds\right\}. \quad (8)$$

The right-hand-side terms may readily be calculated as $\lim_{p\to\infty} V(p) = (1-\tau)[p_t/(r-\mu) - (w+b)/r]$ and $\lim_{p\to\infty} L(p) = b/r$.

Finally, since equity-holders may always cover operating losses by injecting capital,[8] the closure price p_b is effectively their choice. Hence, p_b will be determined so as to maximize the value of equity-holders' claims, $V(p_t)$. As Dixit (1992) and Dumas (1992) show, the relevant condition determining p_b is a smooth-pasting equation for total equity at the trigger price, that is, $V'(p_b) = 0$.[9] Applying the above boundary conditions and supposing that $V(p_t)$ and $L(p_t)$ satisfy Equations (5) and (6), one may obtain explicit solutions for equity and bond values.

Proposition 1. *Let the firm's output price be exogenously given and equal to the process given in Equation (2). If the flow of coupon payments is fixed at b, and debt is risky in that the residual firm value at closure is less than the face value of the debt, the values of the firm's equity and bonds, L_t and V_t, satisfy*

$$\begin{aligned} V(p_t) &= (1-\tau)\left(\frac{p_t}{r-\mu} - \frac{w+b}{r}\right) \\ &\quad - (1-\tau)\left(\frac{p_t}{p_b}\right)^{\lambda_1}\left(\frac{p_b}{r-\mu} - \frac{w+b}{r}\right), \end{aligned} \quad (9)$$

and

$$L(p_t) = \frac{b}{r} + \left(\frac{p_t}{p_b}\right)^{\lambda_1}\left[X(p_t) - \frac{b}{r}\right], \quad (10)$$

where λ_1 is the negative root of the quadratic equation $\lambda(\lambda-1)\sigma^2/2+$

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[8] Share offerings are equivalent, in this context, to capital injections since existing equity-holders could always buy the new issue.

[9] In optimal stopping problems of this sort, a necessary condition for optimality is that the derivatives with respect to the state variable of the optimizing agent's value function and of the payoff received after stopping are equal at the trigger level.

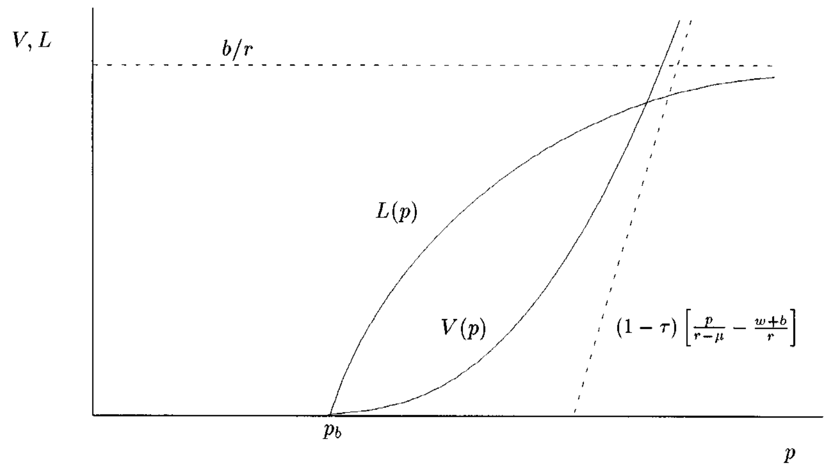

Figure 1
A simple benchmark model
Debt and equity values $L(p)$ and $V(p)$ appear as functions of the firm's output price. The postbankruptcy residual value $X(p)$ is assumed to be zero, so V and L equal zero at the bankruptcy trigger price p_b. Optimal choice of p_b by equity-holders implies that $V'(p_b) = 0$.

$\lambda\mu = r$, and where the close-down price p_b is defined as

$$p_b = -\frac{\lambda_1}{1-\lambda_1}\frac{r-\mu}{r}(w+b). \tag{11}$$

The proofs of this and subsequent propositions are sketched in the Appendix.[10]

Figure 1 depicts the solution given in Proposition 1 for the simple case in which $X(p) = 0$ for all p. Note that the equity and bond values approach the unlimited liability values as $p \to \infty$, that $V(p)$ smooth-pastes at p_b, and that $L(p)$ hits $X(p)$ (in this case, the horizontal axis) obliquely.

1.3 Optimal leverage

How then is the level of leverage determined? First, suppose that equity-holders have access to unlimited external resources, for example, because outside equity is available in any quantity. In this case, leverage will be set to maximize the unconstrained ex ante value of the firm. Let $W(p_t, b)$ denote the total value of the firm for some

[10] If the assumption that $X(p) \geq 0$ for all $p \geq 0$ is relaxed, then debt and equity values equal the solutions in Proposition 1, except that the closure price is the maximum of p_b (as defined in the proposition) and of $\sup(p\colon X(p) < 0)$. Subsequent propositions may be similarly generalized.

leverage level b. Then

$$W(p_t, b) = \left[(1-\tau)\left(\frac{p_t}{r-\mu} - \frac{w}{r}\right) + \frac{b\tau}{r}\right] + \left(\frac{p_t}{p_b}\right)^{\lambda_1} \times \left\{X(p_b) - \left[(1-\tau)\left(\frac{p_t}{r-\mu} - \frac{w}{r}\right) + \frac{b\tau}{r}\right]\right\}. \quad (12)$$

Suppose that the firm is set up when the output price is at some initial level p_i. The capital structure will then be chosen so as to maximize $W(p_i, b)$. In particular, the optimal b will be the root of the first order condition $\partial W(p_i, b^*)/\partial b = 0$.

Proposition 2. *Suppose that equity-holders have access to unlimited external resources. Let the firm's output price be exogenously given and equal to the process given in Equation (2). If the flow of coupon payments is fixed at b, and debt is risky in that the face value of debt exceeds the residual assets available at closure, the optimal capital structure conditional on some initial price p_i is the root of the equation*

$$\frac{\tau}{r}\left[1 - \left(\frac{p_i}{p_b}\right)^{\lambda_1}\right] = -\frac{\lambda_1}{w+b}\frac{b}{r}\left(\frac{p_i}{p_b}\right)^{\lambda_1} + \frac{1}{w+b}\left[X'(p_b)p_b - \lambda_1 X(p_b)\right], \quad (13)$$

where p_b is defined as in Proposition 1. When $X(p) = 0$ for all p, one may show that Equation (13) has a unique solution and that this is an optimum.

The optimal degree of leverage indicated by Equation (13) sets the probability-weighted marginal tax benefit $\tau/r[1 - (p_t/p_b)^{\lambda_1}]$ equal to the net discounted marginal bankruptcy cost stemming from an inefficient choice of closure point by equity-holders who select p_b to maximize $V(p_t)$ rather than $W(p_t)$. Of the two terms on the right-hand side of Equation (13), the first measures the effect of additional leverage in accelerating bankruptcy and hence the loss of the tax shield. The second reflects the fact that as debt increases, equity-holders may precipitate bankruptcy and hence switch to the residual value function $X(p_t)$ at an inefficient time.

Figure 2 illustrates the initial leverage choice decision. In the top panel, $W_b \equiv \partial W(p_i, b)/\partial b$ is drawn as an increasing function of p_i for two different coupon flows, b^L and b^H. The points at which the schedules cross the horizontal give the prices p_i^L and p_i^H, for which such leverage would be optimal, while their intersections with the BB schedule indicate the prices at which these levels of debt would

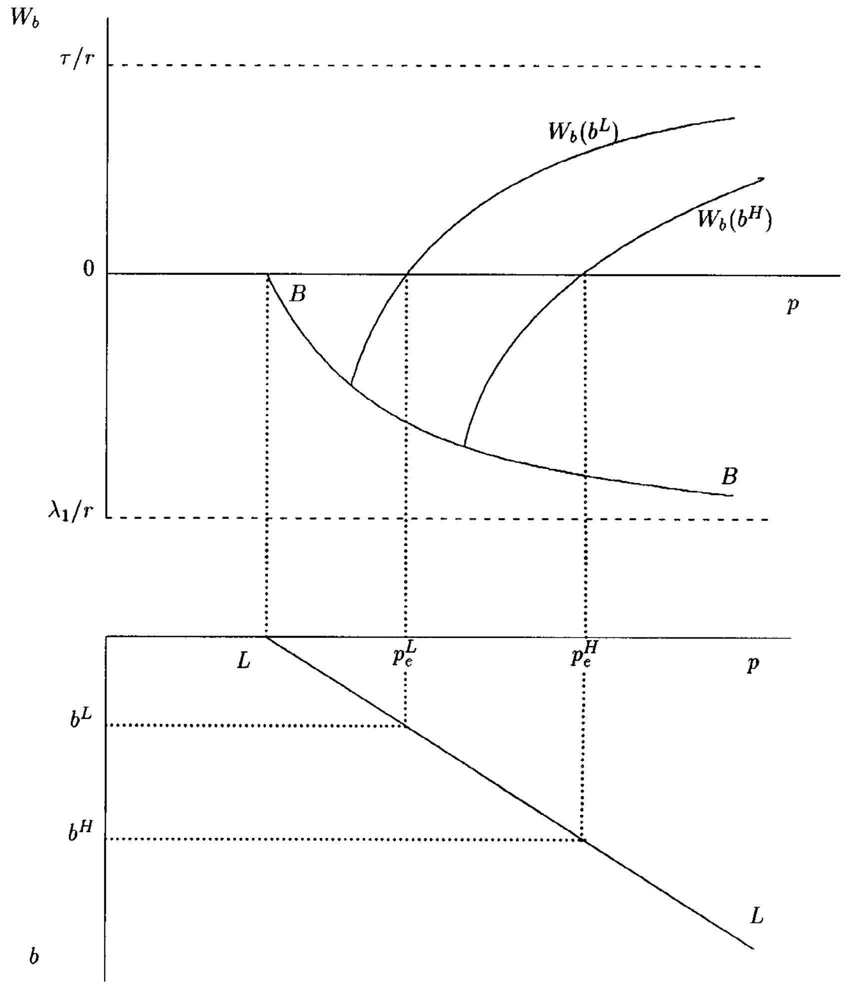

Figure 2
Optimal initial leverage choice
The optimal coupon b^* is shown for different entry prices p_e as the zeros of the derivative of total firm value with respect to the coupon W_b. Entry prices and optimal coupon levels are positively correlated.

trigger bankruptcy .[11] The lower panel of Figure 2 shows how optimal leverage increases with the entry price. Later in the article where we allow for increases in leverage, we find that the option of waiting shifts the LL schedule to the right.

[11] BB itself crosses the horizontal axis at $-\lambda_1/(1-\lambda_1)(r-\mu)/rw$, which is the closure price for the pure equity firm when $X(p) = 0$ for all p.

How is the leverage decision affected if the total amount of equity capital is constrained? Several recent articles have studied capital structure under this assumption [see, for example, Aghion and Bolton (1992), Anderson and Sundaresan (1996), and Hart and Moore (1994)]. Such situations can arise if firm profits are unverifiable by holders of external equity. Equity claims then can only be held by firm insiders who combine ownership with control of the firm's operations. Within our framework, one may suppose that equity-holders have a maximum sum J to invest in the firm and that $L(p_i, b^*) + J < K$, where K is the initial capital cost of establishing the firm, and b^* is the level of leverage that maximizes ex ante firm value. If there exists a $\bar{b}$ such that $L(p_i, \tilde{b}) + J \geq K$, then the firm will be set up but with a suboptimally high level of debt. In particular, b will be the root of $L(p_i, \tilde{b}) + J = K$.

1.4 Agency costs

Note that the closure price $\bar{p}_b$ that maximizes the ex ante value of the firm is not equal to the liquidation price p_b chosen by equity-holders. This leads to an agency cost. If equity-holders could commit before the firm is set up to close the firm at $\bar{p}_b$, the ex ante value of the firm would be higher. The debt remains fairly priced since bondholders correctly anticipate that equity-holders will close the firm at p_b, but this outcome is still inefficient. The agency cost, which resembles the agency costs investigated by, for example, Myers (1977), may be calculated as follows.

Proposition 3. *Let the firm's output price be exogenously given and equal to the process given in Equation (2). If the flow of coupon payments is fixed at b, and debt is risky in that the face value exceeds the residual assets available at closure, the agency cost equals*

$$\bar{W}(p_i, b) - W(p_i, b), \tag{14}$$

where $\bar{W}$ is the same as in Equation (12), except that p_b is replaced by $\bar{p}_b$, where

$$\bar{p}_b = \frac{-\lambda_1}{1-\lambda_1}\frac{r-\mu}{r} \times \left\{ w - \frac{\tau b}{1-\tau} - \frac{r}{1-\tau}\frac{1}{\lambda_1}\left[\lambda_1 X(\bar{p}_b) - \bar{p}_b X'(\bar{p}_b)\right]\right\}. \tag{15}$$

Since $X'(p)$ is non-negative, by assumption, for all $p > 0$, it follows that $\bar{p}_b < p_b$.

The last sentence of the Proposition highlights the fact that the closure price that maximizes firm value, $\bar{p}_b$, is lower than that which would be chosen by equity-holders maximizing their own equity value, p_b. The reason is that equity-holders ignore (1) the tax shield that may be lost when the firm closes, and (2) do not internalize the residual value of the firm in their choice of closure time.

1.5 Incentives to adjust leverage

The last point we wish to make in this preliminary section relates to incentives for adjustments in capital structure after the firm is set up. Below we shall develop a dynamic theory of capital structure, but for the moment, we assume that the leverage level is fixed by the initial agreement between equity-holders and bondholders. The following proposition shows the degree to which this assumption is restrictive.

Proposition 4. *In the model of this section, excluding side payments, equity-holders will never wish to buy back marginal amounts of the firm's debt. Equally, in the absence of side payments, if $X(p_b)$ and $X'(p_b)$ are small, bondholders will never prefer that the firm issue more debt of the same seniority.*[12]

Equity-holders' reluctance to unlever at the margin reflects the fact that reducing leverage cuts the option value associated with limited liability implicit in the value of equity. If equity-holders could agree on side payments from the bondholders who remain after the debt buyback (the latter benefit because the firm moves further from the liquidation point and so might be willing to pay), they might wish to purchase marginal amounts of debt. Otherwise, only discrete reductions in leverage could be in the interests of equity-holders.

However, if some debt-holders try to free-ride, demanding special treatment in the buyback, the effect may be to make even discrete reductions in leverage too expensive as far as equity-holders are concerned. The standard solution to this problem is a "take it or leave it" offer conditional on everyone accepting. Such a strategy is not really implementable in practice since debt is often dispersed among many creditors and "take it or leave it" offers are not credible. This provides a justification for the assumption made here that firms cannot unlever.

The second part of Proposition 4 states that, as long as the residual postbankruptcy value of the firm is small and insensitive to the output price p_t-bondholders will tend to resist *increases* in leverage. Although bond convenants do commonly limit firms' ability to increase borrowing, we regard the assumption made in this section and the next that firms cannot raise their leverage as simply a benchmark case. The assumption is open to criticism, as precluding leverage

[12] Leland (1994) found a similar result in a related model.

adjustments may seriously impair the ex ante value of the firm by impeding optimal exploitation of tax advantages after the firm is set up. In Section 3, we show how this assumption may be relaxed by developing a theory of optimal dynamic releverage.

The above model has provided a benchmark case. We now wish to show how one can (1) endogenize the output price by embedding the above model in an industry equilibrium, and (2) incorporate optimal dynamic adjustments in leverage.

2. Industry Equilibrium

2.1 The output price in industry equilibrium

Consider an industry with a large number of identical firms all possessing the technology and cost structure described in the last section. Again, we limit attention to the case in which, once the firm is set up, changes in the level of leverage are impossible. First, let us examine the equilibrium that arises if equity-holders are cash-constrained as described in Section 1.3. In Section 2.4 below, we discuss how the industry equilibrium changes when this assumption is altered to permit equity-holders an unconstrained choice of capital structure.

Assume that the inverse demand function for industry output is

$$p_t = x_t D(q_t), \tag{16}$$

where q_t is the number of firms active at time t, which will turn out to be an endogenous variable in this model. (For simplicity, we take q_t to be a stochastic process with continuous rather than integer support.) Since each firm produces a unit of output, q_t is also the total output of the industry. Assume for simplicity that the good is not storable, so that industry output equals supply. Suppose that $D(.)$ is a continuously differentiable, monotonically declining function and x_t is an exogenous stochastic process reflecting taste shocks. Specifically, we assume that x_t is a geometric Brownian motion with constant parameters μ and σ. Application of Ito's lemma and the assumption (to be justified below) that q_t is a finite variation process yields

$$dp_t = p_t D'(q_t)/D(q_t)\,dq_t + \mu p_t\,dt + \sigma p_t\,dB_t, \tag{17}$$

where B_t is a standard Brownian motion. We suppose that there is an infinite pool of potential firms and that, apart from a lump sum cost of entry, they may enter the industry costlessly. Output increases if prices rise to such a point that it is profitable for some of these firms to enter. Since all firms are identical, and entry is free apart from fixed capital costs, the price process can never pass the level at which entry is profitable for an individual firm. Hence, the term $p_t D'(q_t)/D(q_t)$ acts like a "stochastic regulator," keeping p_t below the entry price p_e. A similar argument applies for the price p_b at which firms wish to leave the industry. Thus as in Leahy (1993), the price process may be thought of as a doubly reflected geometric Brownian motion with barriers p_b and p_e; that is,

$$dp_t = \mu p_t dt + \sigma p_t dB_t + dM_t^b + dM_t^e, \tag{18}$$

where M_t^b and M_t^e are monotonically increasing and decreasing stochastic regulator processes. Formally, M_t^b and M_t^e may be written as

$$M_t^e = \sup_{0 \le \tau \le t} [\min[p_e - (p_\tau - p_b) - M_t^b,\ 0]], \tag{19}$$

$$M_t^b = \sup_{0 \le \tau \le t} (\min(p_\tau - p_b - M_t^e,\ 0)), \tag{20}$$

[See Harrison (1985)] V_t and L_t continue to satisfy the differential Equations (5) and (6), but the relevant boundary conditions are now somewhat different.

2.2 Industry equilibrium boundary conditions

First, consider the value of equity. To rule out arbitrage, it must be the case, as before, that $V(p_b) = 0$. Instead of the asymptotic condition on V for high prices, however, the second boundary condition for V is now $V'(p_e) = 0$. To understand this condition, note that as p_t hits p_e, the behavior of the process changes, with upward movements in the infinite variation term $p_t \sigma\, dB_t$ being cancelled by offsetting movements in the stochastic regulator. The solution can satisfy the equilibrium condition given in Equation (3) both at p_e and at points close to p_e only if the infinite variation part of the equity value $V'(p_t)\sigma p_t dB_t$ vanishes, that is, if $V'(p_e) = 0$.

Turning now to the bond value L_t at p_e, we again have a reflecting barrier condition, in this case $L'(p_e) = 0$. At the lower price barrier p_b, however, the condition is much less standard. We shall suppose that when p_t hits p_b, each firm has an equal chance of leaving the industry. (Other assumptions are possible and we comment further below.) Now, for each firm, when p_t hits p_b, the probability of liquidation equals the proportion of firms that exit, dq_t/q_t. The expected capital loss to bondholders due to liquidation is therefore $L(p_b)dq_t/q_t$. Balancing this must be some probability of capital gains in bond values in the event that the output price rises.

To express this balancing of expected gains and losses more formally, recall that $p_t = x_t D(q_t)$ and that when regulation occurs, p_t is fixed. Hence, we can write output quantity changes in the neighborhood of p_b in terms of changes in the p_b stochastic regulator as follows:

$$dM_t^b = \frac{p_t D'(q_t)}{D(q_t)} dq_t. \tag{21}$$

The marginal cost of applying the p_b stochastic regulator (because of expected capital losses) may therefore be written as

$$-\frac{dq_t}{q_t}(L_t - X_t) = -\frac{D(q_t)}{D'(q_t)q_t}\frac{(L_t - X_t)}{p_t} dM_t^b. \tag{22}$$

But Harrison (1985) shows that the appropriate boundary condition in such models sets the costs of regulation equal to the first derivative of the relevant discounted integral (in our case, L_t); that is,

$$L'(p_b) = -\frac{D(q_t)}{D'(q_t)q_t}\frac{[L(p_b) - X(p_b)]}{p_b}. \tag{23}$$

This condition is in itself quite interesting. $D(q_t)/|D'(q_t)q_t|$ is the price elasticity of demand for the industry's output. If we take $D(q_t)$ to be isoelastic, then we have

$$\frac{L'(p_b)p_b}{L(p_b) - X(p_b)} = \left|\frac{\partial q(p_b)}{\partial p}\frac{p_b}{q_t}\right| = \eta, \tag{24}$$

for some constant η. In the remainder of the article we shall suppose that $D(q)$ is isoelastic.

Figure 3, which is drawn for the special case in which $X(p) = 0$ for all p, illustrates what is going on. By appropriate choice of units, bond value L and aggregate demand q are shown as (locally linearized) functions of price intersecting at C where prices equal p_b. For a given firm, the expected capital loss due to liquidation if the forcing process x_t falls by Δx equals $\beta_2 \equiv L(p_b)CD/CE$. On the other hand, if x_t *rises* by Δx, bondholders gain β_1. Our elasticity condition is equivalent to $\beta_1 = \beta_2$, ensuring that the expected capital gains and losses cancel.

Finally, we require two conditions to determine the trigger prices, p_b and p_e. p_b is chosen by equity-holders who, at that price, are indifferent about leaving the industry or staying in production. The relevant condition is then $V'(p_b) = 0$. The upper trigger price is given by our assumption of free entry and consists of $V(p_e) + L(p_e) = K$, where K is the cost of setting up the firm.

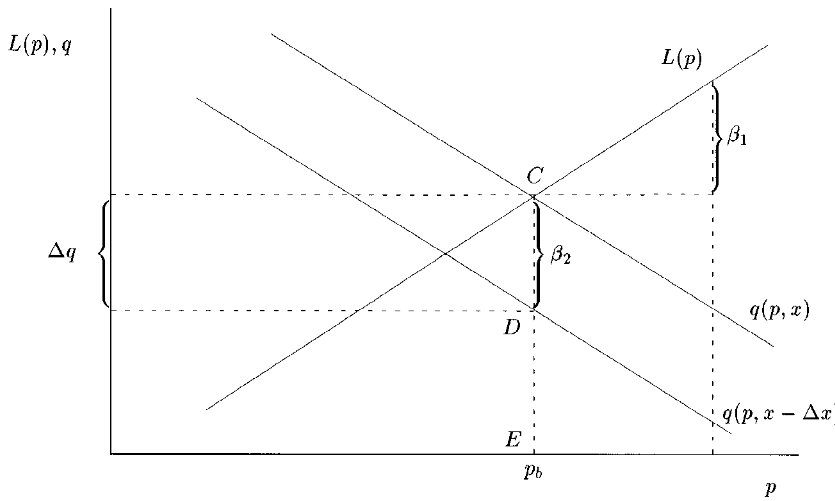

Figure 3
The industry exit condition for bond valuation
Appropriate choice of units allows one to show the bond value $L(p)$ and industry demand $q(p)$ intersecting at the bankruptcy trigger price p_b. The boundary condition for the bonds at p_b balances capital gains and expected losses and corresponds to the equality $\beta_1 = \beta_2$.

2.3 Valuing debt and equity

The following values of debt and equity may then be obtained:

Proposition 5. *In industry equilibrium when the flow of coupon payments is fixed at b, the values of bonds and equity are given by*

$$\begin{aligned} V(p_t) \;=\; & (1-\tau)\left(\frac{p_t}{r-\mu} - \frac{w+b}{r}\right) - \gamma_1(1-\tau)\left(\frac{p_b}{r-\mu} - \frac{w+b}{r}\right) \\ & \times \left[\lambda_2\left(\frac{p_t}{p_b}\right)^{\lambda_1} - \lambda_1\left(\frac{p_e}{p_b}\right)^{\lambda_1}\left(\frac{p_t}{p_e}\right)^{\lambda_2}\right] \\ & - \gamma_1(1-\tau)\frac{p_e}{r-\mu}\left[\left(\frac{p_t}{p_e}\right)^{\lambda_2} - \left(\frac{p_b}{p_e}\right)^{\lambda_2}\left(\frac{p_t}{p_b}\right)^{\lambda_1}\right], \end{aligned} \tag{25}$$

$$\begin{aligned} L(p_t) \;=\; & \frac{b}{r} + \eta\gamma_2\left[X(p_b) - \frac{b}{r}\right] \\ & \times \left[\lambda_2\left(\frac{p_t}{p_b}\right)^{\lambda_1} - \lambda_1\left(\frac{p_e}{p_b}\right)^{\lambda_1}\left(\frac{p_t}{p_e}\right)^{\lambda_2}\right], \end{aligned} \tag{26}$$

where

$$\gamma_1 = \left[\lambda_2 - \lambda_1 (p_e/p_b)^{\lambda_1 - \lambda_2}\right]^{-1}, \tag{27}$$

$$\gamma_2 = \left[\lambda_2(\eta - \lambda_1) - \lambda_1(\eta - \lambda_2)(p_e/p_b)^{\lambda_1 - \lambda_2}\right]^{-1}. \tag{28}$$

p_e and p_b are the roots to the free entry condition $V(p_e) + L(p_e) = K$, and the equity-holders' optimality condition for the timing of bankruptcy, $V'(p_b) = 0$. λ_1 and λ_2 are the negative and positive roots, respectively, of the quadratic equation $\lambda(\lambda - 1)\sigma^2/2 + \lambda\mu = r$. η is the constant price elasticity of demand for industry output.

Proposition 5 derives equity and debt values for a given coupon flow b. If equity-holders are cash-constrained, having a maximum of J available to invest, then b will be determined by the equation $L(p_e, b) = K - J$.

We portray the equilibrium industry dynamics in Figure 4. At the entry price p_e, the combined value of bonds and equity, $W(p_e)$, equals the capital cost K. At the lower trigger point p_b, equity value is zero and optimality requires that $V'(p_b) = 0$. Again, we assume for simplicity that $X(p) = 0$ for all p, so that the boundary condition for bonds is $L'(p_b)p_b/L(p_b) = \eta$. This is equivalent to the tangency shown between the W function and the ray starting at $p_b - a$ where $a \equiv \eta/p_b$, since at the bankruptcy price p_b, $V = V' = 0$, and

$$\frac{L'(p_b)p_b}{L(p_b)} = \frac{W'(p_b)p_b}{W(p_b)} = \frac{W(p_b)}{a}\frac{p_b}{W(p_b)} = \eta. \tag{29}$$

In the lower panel of Figure 4, we illustrate the equilibrium that prevails when the elasticity of demand is low, specifically when $\eta \leq 1$. Ceteris paribus, a low elasticity means that only a few firms need to close to regulate the output price at p_b; that is, individual firms face a small chance of bankruptcy. For this reason, bond values, represented by the distance $W - V$ in the figure, do not decrease greatly even when prices approach the bankruptcy trigger.

How do the trigger prices compare with those of an unlevered firm? The latter are shown as the prices at points C and E in the lower panel, where the schedule U gives the value of a pure equity firm. Evidently, leverage has narrowed the distance between the trigger points. The tax deductibility of interest payments encourages entry at a lower price, while the impact of coupon payments on equity-holders' income flow induces a higher price trigger for exit.

Above, we supposed that when p_t hits p_b, each firm has an equal chance of being liquidated. Other approaches to "ordering" liquidation across firms are possible, such as some deterministic ordering of firms based on differences in their wage costs. This would affect

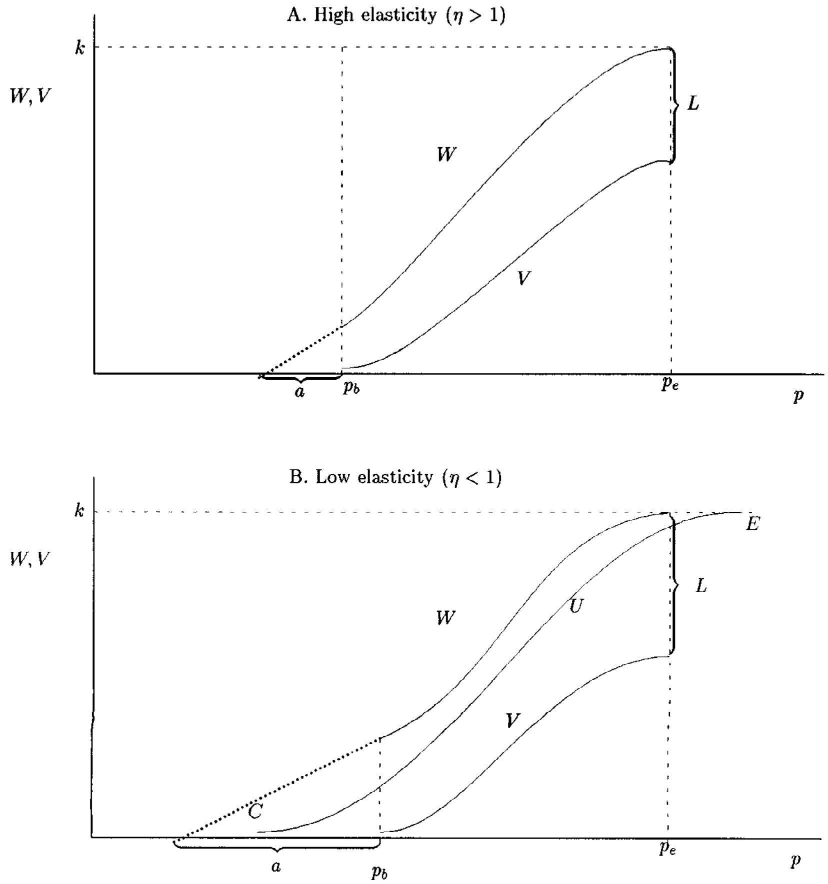

Figure 4
Equilibrium solutions with liquidation
In the industry equilibrium, both the total firm value W and the equity value V smooth-paste at p_e as entry by new firms prevents p_t from rising any further. At p_b, the slope of W is such that the capital gain from an output price rise equals the expected capital loss from a price fall. If the price elasticity η is low, few firms must exit to keep p_t above p_b.

both bond and equity claims, which would become functions of the number of firms left in the industry, q_t, as well as of the output price p_t.

It should be stressed that our assumption of an isoelastic industry demand function is crucial in allowing us to obtain simple closed-form results. Any other specification would mean that the boundary condition to the bond prices would depend on q_t. The solutions would then be functions of two state variables (p_t and q_t) rather than one (p_t). Of course, numerical solutions would still be available in the two-state variable case.

Finally, note that M_t^b and M_t^e, as monotonically increasing and decreasing processes, respectively, must be of finite variation. Equation (21) and a similar equation linking output quantity changes and M_t^e at the upper barrier p_e imply that q_t must also be of finite variation, as claimed prior to Equation (17).

2.4 Equilibrium without cash constraints

How is equilibrium affected if equity-holders are not cash-constrained and hence can freely choose the capital structure at entry? The crucial point to note is that in the equilibrium described in Proposition 5, when p_t hits the closure trigger p_b, debt value (and hence total firm value) of any firm that closes experiences a negative jump.

Suppose that all existing firms adopt a particular coupon flow b. A firm that at entry adopts a marginally lower level of coupon payments enjoys the same tax advantages of leverage but avoids future negative jumps. The reason is that exit by other firms ensures that the output price never falls to a level at which the equity-holders of the low-leverage firm are unwilling to cover operating losses.

Thus, so long as bond values jump at closure, a marginal reduction in leverage (below the level adopted by other firms) discretely increases firm value at entry by eliminating the possibility of bankruptcy. Firms will therefore "undercut" each other at entry, adopting lower and lower levels of leverage. The incentive to undercut disappears only when bond values are continuous at closure. In this case, the firms are in a Nash equilibrium, choosing their level of leverage at entry in a way that is optimal, given the behavior of other firms.

Continuity of bond values at p_b implies the condition $L(p_b) = X(p_b)$. Since most firms do not close at p_b, in the absence of arbitrage possibilities, it must also be true that $L'(p_b) = 0$. Together with the entry condition $L'(p_e) = 0$, these equations uniquely determine the debt value and the parameter b. The equity value and the entry and closure prices are determined by the conditions $V'(p_e) = V(p_b) = V'(p_b) = 0$ and $V(p_e) + L(p_e) = K$.

Solving for debt and equity values subject to these boundary conditions, we obtain the following proposition.

Proposition 6. *When equity-holders are not cash-constrained and can therefore select b freely at entry, the value of debt is b/r, while total equity value equals that given in Proposition 5. The closure price p_b satisfies the equation $X(p_b) = b/r$.*

This is a strong result. Competitive pressure induces equity-holders entering the industry to reduce leverage until the point at which the face value of the debt, b/r, equals the residual value of the firm's assets available to bondholders at closure, $X(p_b)$. In the special case illustrated in Figure 4 in which the residual value $X(p)$ is zero for all $p > 0$, in the Nash equilibrium leverage would be zero.

3. Optimal Dynamic Releverage

3.1 Basic assumptions

In this section, we reimpose the assumption made in Sections 1 and 2 that the output price is exogenous and behaves as in Equation (2). On the other hand, we relax the assumption made up to this point, that the firm's leverage level is fixed up until the moment of bankruptcy. In so doing, we shall derive a dynamic theory of capital structure. As mentioned in the introduction, our approach to releverage resembles that of Fischer, Heinkel, and Zechner (1989a, 1989b) in that we will derive optimal leverage adjustment policies subject to adjustment costs. Our results differ from theirs because the form of the costs we assume is different.

Consider again the single firm model of Section 1. The following assumptions will combine to yield reasonably tractable pricing expressions.

Assumption 2. *Feasible changes in leverage occur, maximizing the total value of the firm's securities,* $W(p) \equiv V(p) + L(p)$.

Assumption 2 states that any leverage changes that are possible (and some may be ruled out by stakeholders' ex post conflicts of interest and restrictions on side payments) should be carried out so as to maximize the total value of the firm. This is a natural assumption to make since the policy of maximizing W is obviously ex ante efficient and will be followed if it can be enforced ex post.

Assumption 3. *Firms can increase but not reduce their level of leverage.*

Recall that in Proposition 4, we showed that, *in the absence of side payments*, marginal decreases in debt will be opposed by equity-holders, while debt-holders will resist increases in debt of similar maturity if $X(p_b)$ and $X'(p_b)$ are sufficiently small. One may think of Assumption 3 as partially relaxing the hypothesis of Proposition 4 by allowing side payments from equity- to debt-holders but not in the other direction. In these circumstances, if both groups can veto changes in the firm's capital structure, the firm will be able to raise but not lower its leverage level.

If debt-holders are assumed to be relatively small and widely dispersed, whereas equity-holders are represented in negotiations by a single, unified management, then it is quite plausible that side payments from equity- to debt-holders are possible but not vice versa. In effect, hold-out problems by small debt-holders preclude equity-holders from being compensated for changes in capital structure that would maximize firm value.

The last assumption we require concerns the allocation of the benefits of releverage between the two different investor groups. A priori reasoning provides little guidance here, and we shall simply adopt the following reasonable assumption.

Assumption 4. *Equity-holders receive a fraction ζ of the benefits of leverage changes, where $\zeta \in [0, 1]$.*

Finally, note that there is no contradiction between our assumption that feasible changes are carried out so as to maximize firm value, whereas liquidation decisions are made in an inefficient way by equity-holders maximizing the ex post value of their claims. Liquidation involves equity-holders invoking limited liability and washing their hands of the firm at a higher trigger level than that which maximizes total firm value. As with leverage reductions, the problem could be mitigated if debt-holders might be persuaded to make ex post concessions, such as side payments to equity-holders. Supposing that hold-out problems by debt-holders preclude this is consistent with our assumptions.

3.2 Pricing the firm with leverage adjustments

Let us see how the above assumptions combine to yield pricing expressions. For some coupon flow level b, one may think of the total value of the firm as comprising the discounted cash flow with the current level of leverage [i.e., $W(p_t, b)$ as defined in Equations (11) and (12)] *plus* a continuum of options to raise the leverage level by incremental amounts as prices increase. The reason these options will only be exercised gradually as the output price rises is that if the price subsequently falls, the higher the leverage, the greater are the agency costs due to an inappropriately chosen close-down price p_b.

Let $\tilde{W}(p_t, b)$ be the total value of the firm including the option value associated with releverage possibilities, and let $H(p_t, b) \equiv \tilde{W}(p_t, b) - W(p_t, b)$. If $H(p_t, b)$ is a sufficiently smooth function of b, one will always be able to write it as an integral: $H(p_t, b) = \int_b^\infty G(p_t, \xi)\, d\xi$ for some function $G(.\,,\,.)$. This is a useful trick, as one may then interpret $G(p_t, b)$ as the value at p_t of an option to increase the leverage parameter b incrementally, that is, from b to $b + db$.

Assumption 4 implies that the firm is closed down when prices fall to $p_b = p_b(b)$, where $V'(p_b) + \zeta \partial H(p_b, b)/\partial p = 0$. Raising b allows the firm to reap greater tax savings but at the cost of a less-efficient closure decision in the event that prices decline. Since, by assumption,

b cannot decline, it will only change when p_t exceeds its past peak, that is, when

$$p_t = \hat{p}_t \equiv \sup_{0 \le \tau \le t} \{p_\tau\}. \tag{30}$$

The main result of this section may be stated as the following proposition.

Proposition 7. *Under Assumptions 1 through 4, the total value of the firm, $\tilde{W}(p_t, b)(\hat{p}_t))$, is*

$$\tilde{W}(p_t, b(\hat{p}_t)) = W(p_t, b(\hat{p})) + \int_{b(\hat{p}_t)}^{\infty} G(p_t, \xi)\, d\xi, \tag{31}$$

where $\hat{p}_t \equiv \sup_{0\le\tau\le t}(p_\tau)$. Here, $G(p_t, b)$ represents the value of an option to increase leverage incrementally when the current level of coupon payments is b. $G(p_t, b)$ is given by

$$\begin{aligned} G(p_t, b) = \gamma_3 &\left\{ \left[\frac{p_t}{p_x(b)}\right]^{\lambda_2} - \left[\frac{p_b(b)}{p_x(b)}\right]^{\lambda_2} \left[\frac{p_t}{p_b(b)}\right]^{\lambda_1} \right\} \\ &\times \left(\frac{\tau}{r} \left\{ 1 - \left[\frac{p_x(b)}{p_b(b)}\right]^{\lambda_1} \right\} + \frac{b}{r}\frac{\lambda_1}{w+b} \left[\frac{p_x(b)}{p_b}\right]^{\lambda_1} \right. \\ &\left. + \frac{1}{w+b}\left[X'(p_b)p_b - \lambda_1 X(p_b)\right] \left[\frac{p_x(b)}{p_b(b)}\right]^{\lambda_1} \right), \end{aligned} \tag{32}$$

where $\gamma_3 \equiv \{1 - [p_x(b)/p_b(b)]^{\lambda_1-\lambda_2}\}^{-1}$, and $p_x(b)$ and $p_b(b)$ may be found by solving for the root, respectively, of $\partial G(p_x, b)/\partial p = \partial^2 W(p_x, b)/\partial p \partial b$ and $V'(p_b) + \zeta\, \partial H(p_b, b)/\partial p = 0$ for different values of b, where $H(p_t, b) = \int_b^\infty G(p_t, \xi)\, d\xi$.

Thus, the value of the firm, $\tilde{W}(p, b(\hat{p}))$, equals, at any given moment, the value of the firm's cash flows, assuming $b = b(\hat{p})$ (where $\hat{p}$ is the highest price reached up to the present), *plus* the value of a continuum of options to increase leverage incrementally from $b = b(\hat{p})$ to ∞.

Obtaining solutions for $G(p_t, b)$ is relatively straightforward. For a grid of b values, one may solve the boundary conditions to obtain $p_x = p_x(b)$ and $p_b = p_b(b)$. Equation (32) then yields $G(p_t, b)$ for different b.

Figure 5 shows $W_b \equiv \partial W/\partial b$ and $\tilde{W}_b \equiv \partial \tilde{W}/\partial b$ as functions of p. Incremental leverage occurs when $\tilde{W}_b$ smooth-pastes to the horizontal axis as at p_x in the figure. The shaded vertical distance between W_b and $\tilde{W}_b$ is the value of the option exercised at p_x.

Comparing Figure 5 with Figure 2, one may see how the option

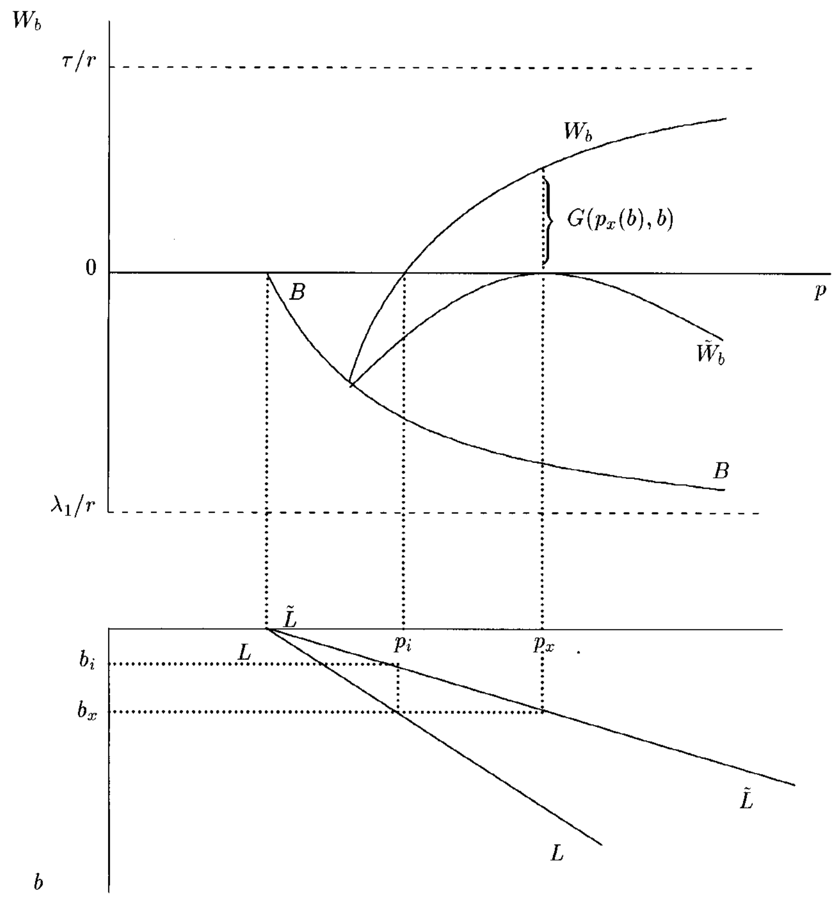

Figure 5
Optimal leverage choice with leverage options
Optimal marginal increases in the coupon flow occur when the change in firm value, $\tilde{W}_b$, equals zero. At this point, the value of the exercised leverage option G exactly offsets the increase in the "constant-coupon firm value" W_b. For a given entry price, optimal leverage is lower when there are options to increase the coupon in the future.

to increase leverage in the future shifts the optimal leverage schedule to the right, as claimed in Section 1. Leverage level b_x is now only chosen when the price goes to p_x and not at the lower price p_i, which applies when there is no option to wait. For any initial price, say p_i, the schedule $\tilde{L}\tilde{L}$ gives the amount of leverage chosen when the firm is set up, b_i, and shows how bonds will be increased incrementally as and when p exceeds p_i.

Note that the assumption made in this section that the output price process is exogenously given (i.e., that there is no reflecting barrier on prices due to free entry by competitor firms) is not innocuous in this context. A reflecting barrier would reduce the option values associated with leverage increments that would have been implemented at prices above the barrier.

The model of this section is closely linked to the analysis of irreversible investment by Pindyck (1988). In Pindyck's case, incremental amounts of real capital are installed by the firm as the output price rises, but irreversibility means that if prices fall, the capital stock does not change. At any given moment, the current capital stock depends upon the maximum output price so far attained.

In our model, as in Pindyck's, the total value of the firm equals the value of the future cash flows, assuming no change in the control variable (capital stock or coupon flow b, respectively) plus a continuum of option values associated with marginal changes in the control.

4. Conclusion

This article has priced corporate debt in an industry equilibrium with entry and exit of firms. For general assumptions about the residual, post-bankruptcy value of the firm, we derive the values of debt and equity. We show how, when equity holders are cash-constrained at entry, the demand elasticity for industry output crucially affects the default premium on the bonds. If equity-holders are not constrained and can freely choose the initial level of leverage, competitive pressure leads firms to reduce leverage until the face value equals the residual value of the firm's assets at closure, that is, until the bonds are riskless.

Much of our analysis is conducted under the assumption that the firm's leverage level is fixed once and for all when the firm is initially set up. Adopting this assumption, we obtain simple, intuitive expressions for the value of debt that depend explicitly on the price elasticity of demand for total industry output.

However, we think it is important to explore the implications of relaxing the "fixed leverage" assumption by allowing firms to increase their borrowing as their profitability improves. The degree to which a firm takes on debt when it is initially set up may depend significantly on the option it has to issue more bonds in the future. The observation that firms appear relatively under-leveraged given apparent bankruptcy costs could reflect the fact that the option to increase debt levels induces them to take on relatively little debt to start with.

The last section, therefore, develops a dynamic theory of optimal leverage. As we show, the value of the firm may be written as the value of its cash flow assuming a fixed level of leverage, plus that of a continuum of options to increase borrowing. These options are exercised progressively each time the price of the firm's output exceeds its previous peak.

As a final point, we have supposed throughout our analysis that the firm's debt is of infinite maturity. Various assumptions would allow one to price finite maturity bonds within our framework. For example, one may suppose that the firm is funded overwhelmingly with perpetuals and equity but that it issues a small amount of finite maturity debt. In this case, the trigger levels for entry, exit, etc., would be stationary, and the short debt may easily be priced using Fourier or other methods. Leland and Toft (1996) have considered other assumptions that yield stationary overall capital structure and trigger prices and hence facilitate pricing of finite maturity bonds in models similar to ours.

Appendix

Proof of Proposition 1. First, consider the equity value. The general solution to Equations (5) and (6) are

$$V(p) = A_0 + A_1 p + A_2 p^{\lambda_1} + A_3 p^{\lambda_2}, \tag{33}$$
$$L(p) = C_0 + C_1 p + C_2 p^{\lambda_1} + C_3 p^{\lambda_2}. \tag{34}$$

Take derivatives and substitute in Equations (5) and (6). Equating coefficients on like terms yields $A_0 = -(1-\tau)(w+b)/r$, $A_1 = (1-\tau)/(r-\mu)$, $C_0 = b/r$, and $C_1 = 0$, while λ_1 and λ_2 are the negative and positive roots to $\lambda(\lambda-1)\sigma^2/2 + \lambda\mu = r$. Since $\lambda_2 > 0$, A_3 and C_3 must be zero if $V(p)$ and $L(p)$ are to approach the unlimited liability values as $p \to \infty$. A_2 and C_2 are then easily obtained using $V(p_b) = 0$ and $L(p_b) = X(p_b)$.

Proof of Proposition 2. Let $\partial W(p_e, b)/\partial b = 0$ and then substitute for p_b using the expression in Equation (11) and rearrange.

Proof of Proposition 3. We obtain $\hat{p}_b^*$ by solving $\partial W^*(\hat{p}_b^*)/\partial p = 0$. Taking derivatives and rearranging yields the desired expression.

Proof of Proposition 4. Buying back \$1 of debt reduces the coupon flow b by

$$\frac{b}{L(p_t)} = \frac{b}{\left\{\frac{b}{r}[1-(p_t/p_b)^{\lambda_1}] + (p_t/p_b)^{\lambda_1} X(p_b)\right\}}. \tag{35}$$

Now, since

$$\begin{aligned}\frac{\partial V(p_t)}{\partial b} &= \frac{(1-\tau)}{r}\left[1-\left(\frac{p_t}{p_b}\right)^{\lambda_1}\right] \\ &\quad - \left\{\frac{p_b(1-\tau)}{(r-\mu)(w+b)} - \frac{\lambda_1(1-\tau)}{w+b}\left[\frac{p_b}{r-\mu} - \frac{w+b}{r}\right]\right\} \\ &\quad \times \left(\frac{p_t}{p_b}\right)^{\lambda_1} \qquad (36) \\ &= \frac{(1-\tau)}{r}\left[1-\left(\frac{p_t}{p_b}\right)^{\lambda_1}\right], \qquad (37)\end{aligned}$$

one may easily show that change in equity value $(\partial V(p_t)/\partial b)b/L(p_t) < (1-\tau)$. Since this is less than \$1, equity-holders will never wish to carry out such a transaction.

The impact on the value of existing bonds if the firm issues more debt is

$$\begin{aligned}\frac{b}{r}\frac{\partial}{\partial b}\left[1-\left(\frac{p_t}{p_b}\right)^{\lambda_1}\right] &= \frac{\lambda_1}{w+b}\left(\frac{p_t}{p_b}\right)^{\lambda_1} \\ &\quad \times \left\{\frac{b}{r} + \frac{1}{\lambda_1}\left[X'(p_b)p_b - \lambda_1 X(p_b)\right]\right\}. \qquad (38)\end{aligned}$$

The right-hand side is negative if $X(p_b)$ and $X'(p_b)$ are small, and, in this case, bondholders will never favor increases in leverage.

Proof of Proposition 5. The proof is very similar to that of Proposition 1. Take derivatives of the general solutions given in Equations (34) and (35), substitute in Equations (5) and (6), and equate coefficients on like terms. The boundary conditions for the equity value form two linear systems in the coefficients A_2 and A_3, and C_2 and C_3, respectively. Solving these two systems by inverting a pair of simple, two-by-two matrices yields the required results.

Proof of Proposition 6. Solving for debt and equity values in an industry equilibrium for a given b/r when b is large enough to imply that debt is risky debt, one obtains the expressions in Proposition 5. If debt is risky, debt values drop discretely if the firm in question goes bankrupt. If all firms adopt a particular leverage b, for an individual firm, total firm value at entry is increased if it adopts a leverage level slightly below b, as, in this case, it will never experience bankruptcy.

Firms will therefore have an incentive to undercut each other's leverage level until there is no jump in bond values at bankruptcy. This occurs when $b/r = X(p_b)$, that is, when the debt principal equals the residual value at closure. Since, in industry equilibrium, only a fraction of firms actually close at p_b, (and bondholders of firms that do close are indifferent about closure and continuation) the latter price acts as a reflecting barrier. Hence, the boundary condition $L'(p_b) = 0$ must hold. Solving for the debt and equity values with the other boundary conditions, $V(p_e) + b/r = K$, $L(p_b) = b/r$, $V'(p_e) = 0$, $V'(p_b) = 0$, yields the results in the proposition.

Proof of Proposition 7. As the difference between two asset prices, H may be thought of as the value of a security. Since holders of this security only receive capital gains income, H must satisfy the equation

$$rH(p_t, b) = \mu p_t \frac{\partial H(p_t, b)}{\partial p} + \frac{\sigma^2}{2} p_t^2 \frac{\partial^2 H(p_t, b)}{\partial p^2}. \tag{39}$$

Taking derivatives with respect to b, we obtain

$$rG(p_t, b) = \mu p_t \frac{\partial G(p_t, b)}{\partial p} + \frac{\sigma^2}{2} p_t^2 \frac{\partial^2 G(p_t, b)}{\partial p^2}. \tag{40}$$

The general solution to this equation is

$$G(p, b) = C_1(b) \left(\frac{p}{p_b}\right)^{\lambda_1} + C_2(b) \left(\frac{p}{p_x}\right)^{\lambda_2}. \tag{41}$$

p_x is the price at which it is optimal for the firm to increase its leverage incrementally and is a function of the current leverage level b. At p_x, it must be the case that $\partial \tilde{W}(p_x(b), b)/\partial b = 0$. But, as one may easily show, this implies that

$$G(p_x(b), b) = \frac{\partial W(p_x(b), b)}{\partial b}. \tag{42}$$

Also, when $p_t = p_b(b)$ and the current level of leverage is b, the firm will be liquidated, so options to releverage must case to have value. Therefore,

$$H(p_b(b), b) = 0 \;\forall\; b. \tag{43}$$

If we apply the boundary condition,

$$G(p_b(b), b) = 0 \;\forall\; b, \tag{44}$$

and let $G(p, b) = 0$ for all pairs (p, b) where $p < p_b(b)$, then

$$\int_b^\infty G(p_b, \xi)\, d\xi = H(p_b(b), b) = 0 \;\forall\; b, \tag{45}$$

as required. Lastly, the optimal exercise price $p_x(b)$ may be obtained from a smooth-pasting condition:

$$\frac{\partial G(p_x, b)}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\partial W(p, b)}{\partial b}\right)\bigg|_{p=p_x}. \tag{46}$$

Solving this yields p_x as a function of b.

To complete the derivation of G, take derivatives of the general solution of the differential equation, substitute in the boundary conditions, and determine the free parameters in the usual way.

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