Splitting Orders

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A standard presumption of market microstructure models is that competition between risk-neutral market makers inevitably leads to price schedules that leave market makers zero expected profits conditional on the order flow. This article documents an important lack of robustness of this zero-profit result. In particular, we show that if traders can split orders between market makers, then market makers set less-competitive price schedules that earn them* strictly positive *profits and hence raise trading costs. Thus, this article can explain why somebody might willingly make a market for a stock when there are fixed costs to doing so. The analysis extends to a limit order book, which by its nature is split against incoming market orders: equilibrium limit order schedules necessarily yield those agents positive expected profits.

A standard presumption in market microstructure [see, e.g., Easley and O'Hara (1987), Glosten and Milgrom (1985), Kyle (1985)] is that competition between risk-neutral market makers inevitably leads to price schedules that leave market makers zero expected profits

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profits conditional on the order flow. This article shows that this result *does not* hold when traders can split orders between market makers. That is, when traders can split orders, either an equilibrium does not exist, or in equilibrium, market makers set price schedules that earn them *strictly positive profits* and hence lead to greater total trading costs. The analysis extends to a limit order book, which by its nature is split against incoming market orders: equilibrium limit order schedules necessarily yield those agents positive expected profits. Thus, this article can explain why somebody might be willing to make a market for a stock when there are fixed costs to doing so.

We often observe substantial trading of the same stock across exchanges that are open simultaneously [see, e.g., Biais, Hillion, and Spatt (1995) or de Jong, Nijman, and Röell (1994)]. One might think that such intermarket competition would provide better prices for traders. However, this article shows that *because* traders can split orders across exchanges, intermarket competition *widens* spreads and *reduces* trader welfare.

The intuition for our results is simple. Consider an environment where both informed and uninformed agents trade a risky asset with risk-neutral market makers. The presence of informed agents implies that the equilibrium price is increasing in order size. When traders cannot split orders between market makers, trade is perfectly price-elastic, with all trade going to a market maker who offers the best price. Since market makers who do not receive the order earn zero profits, it must be that the market maker who wins the trade also earns zero expected profits. Consequently, when an equilibrium exists,[1] this Bertrand competition between the market makers inevitably leads to zero-expected-profit pricing.

In contrast, when traders facing upward-sloping price schedules can split orders, they equate marginal trading costs across market makers. Hence, a market maker's demand is less price-elastic because when one market maker offers a slightly better price schedule, the other market maker still receives some trade. If market maker A sets a schedule that would earn zero expected profits if matched by market maker B, then market maker B can ensure positive profits by setting a steeper, less-competitive price schedule. This is because the trade of informed agents, who only have profit motivations for trading, is more price-elastic than the trade of agents who also care about portfolio balance. Hence, informed traders reduce their orders by more than liquidity traders in response to a price increase, leading to market-maker profits. As a result, when traders can split orders,

[1] See, for example, Riley (1979) for a discussion of equilibrium existence issues in screening games.

market makers will set steeper schedules, schedules that earn strictly positive expected profits. Indeed, markets may break down because, no matter how steep the schedule of one market maker, the other market maker may have an incentive to set an even steeper one.

There is another implication of our findings. In practice, there are substantial fixed costs to making a market. Such fixed costs are a serious problem in the standard no-splits formulation since market makers earn zero expected profits from each order and then cannot cover their fixed costs. Hence, in the standard no-splits formulation, if there are fixed costs, there can be no entry by a competing market maker in equilibrium. In sharp contrast, when traders can split orders, market makers expect positive profits from handling each order and they can cover their fixed costs.[2] With free entry, there will be just enough market makers that additional entry would render market making unprofitable.

We consider this issue within the classic Kyle (1985) environment. If trade is completely inelastic as in Kyle, then when there are two competing market makers, an equilibrium linear price schedule does not exist when traders can split orders. We then generalize the Kyle environment by replacing the noise traders with agents who have quadratic preferences with ideal share holdings drawn from a normal distribution. We show that these quadratic preferences can be interpreted as the reduced-form preferences of a rational agent with exponential utility who receives a liquidity shock. We characterize equilibrium outcomes and trader welfare for an arbitrary number of market makers. As the number of market makers increases, the effective price schedule faced by traders becomes less steep, and aggregate market-maker profits fall, vanishing as the number of market makers becomes arbitrarily large. For any given fixed cost to making a market, the equilibrium number of market makers who trade a particular stock is thus determined.

We then show that the basic result that market makers must expect positive equilibrium profits when traders can split orders holds in almost any market where there is informed trade, including markets made by agents who compete by setting limit order schedules. It is natural to consider order splitting in any limit order context because the nature of limit orders is such that incoming market orders are split up against the limit book.

We also detail how results are affected when our informational and timing assumptions are altered. In our equilibrium, market makers

[2] So too, in inventory model formulations with risk-averse market makers, market makers' expected utility from each trade is zero as long as interdealer trade is allowed [see Ho and Stoll (1983)], so that they cannot recover the disutility of fixed costs.

correctly infer the total order flow from the volume in their market. Hence, if they also observe total order flow, the positive-profit equilibrium remains. We then consider the possibility that rather than being quote-driven, the exchange is order-driven, with orders being submitted before prices are set. Here, if traders cannot withdraw their orders, markets break down because market makers can set arbitrarily high prices. If traders can withdraw their orders after seeing quotes, then each market maker prices monopolistically over the split orders, extracting all surplus.

Our results have implications for empirical work estimating the price impact of information. First, to account for order splitting when determining informational impact, one must estimate the effective price schedule, taking into account the number of exchanges on which the asset is traded rather than just the price schedule on an individual exchange. Second, one must correct for the fact that the steepness of the schedule is due to imperfect competition, as well as inside information.

1. The Kyle Model

1.1 Inelastic liquidity demand

Consider the environment of Kyle (1985). Let the current value of a claim to an asset be zero. An asset innovation δ, which is normally distributed with zero mean and variance Σ_0, is private information to a risk-neutral insider. Noise traders inelastically demand Q, which is normally distributed with zero mean and variance σ_u^2. Traders can submit their orders to either of two risk-neutral, uninformed agents, A and B. One can interpret the competition between these agents either as competition between market makers on the same exchange or as competition between exchanges. For example, there is nothing preventing traders from splitting orders between dealers on NASDAQ or the LSE. Alternatively, nothing prevents traders from splitting orders between exchanges where the same security is traded, as on the LSE and the Paris Bourse. Unless otherwise noted, we will refer to these agents as market makers.

The timing is as follows:

1. The insider observes the innovation to the asset value, δ.
2. Market makers simultaneously set price schedules, detailing for each level of aggregate order flow in their market, q^i, $i = A, B$, a price.
3. Traders choose how much to trade with each market maker.
4. Payoffs are realized.

1.1.1 No splits. When traders cannot split orders between market makers, in equilibrium, an informed trader chooses an order quantity $q_I(\delta, p_a(\cdot), p_b(\cdot))$ that maximizes expected profits given his information, and both types of traders choose the market maker who will execute their orders most cheaply. Given correct beliefs about the schedule set by the other market maker and the trading strategies of the two trader types, each market maker's price schedule maximizes his expected profits.

In the equilibrium, as is standard when agents cannot split their trade, Bertrand competition demands that the two market makers set identical price schedules, price schedules that break even in expectation, conditional on the order flow. When traders cannot split orders between market makers, the equilibrium solution is that detailed in Kyle (1985):

$$p_a(q^A) = p_b(q^B) = \lambda q^i, \quad i = A, B,$$

where

$$\lambda = \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}.$$

Informed agents trade quantity $\beta\delta$ with one of the two market makers, where

$$\beta = \frac{1}{2\lambda}.$$

1.1.2 Splits. To facilitate the analysis, we suppose that market makers set linear schedules satisfying $p(0) = 0$: $p_a(q^A) = aq^A$, $p_b(q^B) = bq^B$. The price schedules that earn zero expected profits conditional on the order flow are linear, so if these price schedules do not survive as equilibrium schedules when pricing strategies are restricted to be linear, they cannot survive extensions to more general classes of pricing strategies. In fact, we show a stronger result: when traders can split orders, no linear equilibrium price schedule exists.

Given this strategy space, in the (Perfect-Bayesian) equilibrium, the informed trader maximizes expected profits with his choice of order quantities, $q_I^A(\delta, p_a(\cdot), p_b(\cdot))$ and $q_I^B(\delta, p_a(\cdot), p_b(\cdot))$, in markets A and B, respectively. The noise trader minimizes the expected cost of trading Q with his choice of order quantities, $q_L^A(Q, p_a(\cdot), p_b(\cdot))$ and $q_L^B(Q, p_a(\cdot), p_b(\cdot))$, where $q_L^A(Q, p_a(\cdot), p_b(\cdot)) + q_L^B(Q, p_a(\cdot), p_b(\cdot)) = Q$. The two market makers set linear price schedules, $p_a(q^A) = aq^A$ and $p_b(q^B) = bq^B$, that maximize each market maker's expected profits, given correct beliefs about the schedule set by the other market maker and correct inferences about how each type of trader responds to the price schedules.

Lemma 1. *1. A liquidity trader with demand Q minimizes his trading costs by buying q_L^A from market maker A and $Q - q_L^A$ from B such that*

$$q_L^A = \frac{bQ}{a+b}, \qquad q_L^B = \frac{aQ}{a+b}.$$

2. Linear price schedules that earn zero expected profits conditional on the order flow must satisfy

$$p_a(0) = p_b(0) = 0,$$
$$p_a(q^A) = p_b(q^B),$$
$$a, b > \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}},$$

and

$$(a+b)\sqrt{\Sigma_0} = 2ab\sqrt{\sigma_u^2}.$$

All proofs are in the Appendix.[3]

Observe that the symmetric zero-profit schedule when traders can split orders is twice as steep as the zero-profit schedule when they cannot:

$$p_a(z) = p_b(z) = 2\lambda z.$$

If market makers set this schedule, traders split orders in half so that the effective price schedule for the total order is the zero-profit schedule set when traders cannot split. Put another way, each market maker realizes that he observes only half the order flow and takes that into account by doubling the slope of his price schedule. However, we now argue that for *any* given schedule $p_b(q^B) = bq^B$ set by market maker B, market maker A wants to set a schedule with steeper slope, $a > b$. Since market maker B has similar incentives, an equilibrium with (potentially asymmetric) linear price schedules cannot exist.

Proposition 1. *No linear equilibrium exists when traders can split orders and market makers set linear price schedules.*

The intuition for the nonexistence result is that a price increase in market A always reduces liquidity demand in market A by less than

[3] Note that we do not demonstrate that all zero-profit pricing schedules are linear. It is an open question [see Rochet and Vila (1994)] whether nonlinear equilibria exist in the Kyle model. Rochet and Vila show that nonlinear equilbiria do not exist when the insider observes noise trade. In Section 2, we show that zero-expected-profit price schedules do not generally obtain in equilibrium.

it reduces informed demands.[4] For example, consider informed and uninformed traders who, facing price schedules such that $a = b$, trade q in each market at associated expected price $p_a(q) = p_b(q) = aq$. That is, aggregate liquidity demand is $Q = 2q$, and the informed trader has received signal δ such that his optimal decision is to trade q in each market.

Now suppose that market maker A increases the slope of his pricing function so that $p'_a(q) = a'q$. From Lemma 1, part 1, the liquidity trader shifts some trade to market B, reducing his demand in market A from

$$q_L^A = \frac{Q}{2} = q$$

to

$$q_L^{A\prime} = \frac{aQ}{a' + a} < q.$$

The expected price paid by that liquidity trader in market A increases to

$$p'_a\left(\frac{2aq}{a' + a}\right) = a' q_L^{A\prime} = \frac{2a'aq}{a' + a},$$

which exceeds $p_a(q)$ because $\frac{2a'}{a'+a} > 1$.

The informed trader now trades

$$q_I^{A\prime} = \frac{\delta}{2a'} = \frac{aq}{a'} < q_L^{A\prime},$$

and he expects to pay

$$p'_a(q_I^{A\prime}) = a'\frac{aq}{a'} = aq,$$

which is the same expected price as before.

Market maker A's expected profits are given by

$$\pi_a = a'q^A = -\frac{\delta}{2a'}\left(\delta - \frac{\delta}{2}\right) + a'\left(\frac{2a'aq}{a' + a}\right)^2.$$

It is then straightforward to show that $\frac{d\pi^a}{da'}$ is positive for a' sufficiently close to a. That is, given one market maker's schedule, the other market maker increases profits by setting a strictly steeper schedule. Since the incentives are symmetric, each market maker wants to set a steeper schedule than the other, and no equilibrium exists.

[4] Dennert (1993) shows, however, that with $N > 2$ market makers, linear equilibria exist even when liquidity demand is inelastic.

1.2 Elastic liquidity demand

In the previous section, equilibrium did not exist solely because liquidity demand was completely inelastic. We now show that existence of equilibrium can be recovered by allowing liquidity traders to have demands that are price-sensitive. Accordingly, we generalize the Kyle model by replacing the noise traders with optimizing agents who have quadratic preferences over their portfolio holdings of money and the risky asset:

$$Y - \eta(Q - q_L)^2.$$

Here, $Y = \bar{Y} - q_L p(q_L)$ represents money holdings, $\bar{Y}$ is the agent's initial endowment of money, and Q represents the optimal number of shares in the agent's portfolio. We assume that the agent has no initial endowment of the risky asset. We assume that Q is normally distributed with zero mean and variance σ_u^2. One can interpret Q as portfolio-balancing shock arising from a liquidity need. The greater is η, the less price-elastic is liquidity demand. The standard Kyle model corresponds to $\eta = \infty$. Notice that liquidity traders are now strategic in the sense that higher prices reduce liquidity demand: if prices are zero, then liquidity demand is Q, and if prices are very high, liquidity demand vanishes.

The advantages of this generalization are twofold. First, we can now determine the consequences for liquidity-trader welfare of both informed trade and the strategic interaction between market makers. Second, we can solve for the equilibrium price schedules when traders can split orders and market makers set linear schedules. Because liquidity trade is price-elastic, price increases eventually reduce expected market-maker profits, so that an equilibrium exists even when there is a single market maker.

We now show that one interpretation of these preferences is as the reduced-form objective of agents with exponential preferences.

Proposition 2. *Suppose the liquidity trader has exponential utility with risk-aversion parameter γ and receives an endowment shock $E = -\frac{Q}{2}$. Then, if the effective price schedule is linear with slope λ, the liquidity trader has reduced-form preferences given by $\bar{Y} - q_L p(q_L) - \eta(Q - q_L)^2$, where $\eta = \frac{\gamma \Sigma_0}{8}$.*

1.2.1 No splits. We first detail how equilibrium outcomes are affected by price-elastic liquidity trade when traders cannot split their orders.

Proposition 3. *1. If $\eta > \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, then an equilibrium exists when traders cannot split orders. The equilibrium features a linear price* *schedule with slope*

$$\lambda = \frac{\eta \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}}{\eta - \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}} > \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}.$$

If $\eta \leq \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, *then no linear equilibrium exists.*

2. The liquidity trader's ex ante expected utility is given by $Y - \frac{1}{2}\sqrt{\Sigma_0 \sigma_u^2}$.

The proof parallels the standard Kyle argument. Conjecture that the market makers set price schedules that are linear functions of total order flow. Then an uninformed trader's demand is a linear function of his portfolio balance shock Q, and the informed trader's demand is a linear function of his signal about the asset value δ, so that total demand is normally distributed. Hence, each market maker's conditional expectation of the asset value is a linear function of the total order flow. Bertrand competition then implies that the market maker's schedules earn them zero expected profits, so that their equilibrium schedules are, in fact, linear.

When splitting is infeasible, expected losses to the informed are a function only of λ. The informed trader's problem is identical to that when liquidity demand is inelastic. However, liquidity demand is reduced to $q_L(\eta, \lambda, Q) = \frac{\eta}{\eta+\lambda}Q$, so the equilibrium price schedule is steeper than when liquidity demand is inelastic. The slope of the equilibrium pricing function increases when $\eta < \infty$ because uninformed traders reduce their orders in response to higher prices. If liquidity demand is too price-elastic, that is, if $\eta \leq \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, then a linear equilibrium does not exist, because the uninformed reduce their orders by too much when market makers set steeper price schedules so that market makers can never earn non-negative profits. Thus, markets can break down for the same reason that they can in Glosten and Milgrom (1985). In the exponential preferences interpretation, markets break down if the risk of the asset as captured by its variance weighted by the liquidity trader's coefficient of risk aversion is too small.

1.2.2 Splits. We restrict market makers to linear price schedules and consider first the possibility of two market makers, A and B, with associated linear price schedules, $p_a(q) = aq$ and $p_b(q) = bq$.

The aggregate price schedule faced by a liquidity trader, $p_e(Q)$, is then given by

$$p_e(Q) = \frac{ab}{a+b}Q.$$

This is because the liquidity trader optimally splits his order, trading fraction $\frac{b}{a+b}$ in market A and the remainder in market B.

Define $\lambda(a, b) = \frac{ab}{a+b}$ to be the slope of the aggregate price schedule $p_e(Q)$. The symmetric zero-profit price schedules are now twice as steep as those obtained when agents cannot split orders:

$$a = b = \frac{\eta \left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}}{\eta - \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}}.$$

However, the unique equilibrium linear price schedules feature steeper slopes and positive market-maker profits.

Proposition 4. *If $\eta > (\frac{1}{2}\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, then a unique linear equilibrium exists when there are two market makers and traders can split orders between them. The equilibrium price schedules have slope*

$$a = b = \frac{2\eta}{4^{\frac{1}{3}}\eta^{\frac{2}{3}}\left(\frac{\sigma_u^2}{\Sigma_0}\right)^{\frac{1}{3}} - 1}.$$

Aggregate market-maker profits are $\frac{-\Sigma_0}{2a} + \frac{2a\eta^2\sigma_u^2}{(a+2)^2}$. Aggregate uninformed expected losses are $\frac{2a\eta^2\sigma_u^2}{(a+2)^2}$. Aggregate informed expected profits are $\frac{\Sigma_0}{2a}$. As is the case when traders cannot split orders, when $\eta < \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, an equilibrium does not exist.

The construction of the equilibrium schedule is as follows. First, market maker A conjectures a schedule $p_b(q) = bq$. Given that conjecture, he then chooses his schedule $p_a(q) = aq$ to maximize expected profits, given the optimal responses of both informed and liquidity traders. In an equilibrium, the conjectures of market maker A are fulfilled.

Corollary 4.1. *If $\eta > \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$ so that an equilibrium exists, the slope of the equilibrium aggregate price schedule, $\lambda(a, b) = \frac{a}{2}$, exceeds the slope of the zero-expected-profit aggregate price schedule.*

Because the slope of the aggregate price schedule exceeds that under zero-expected-profit pricing, $E(|\text{aggregate volume}|)$ is less than that under zero-expected-profit pricing, since both aggregate informed demand and aggregate uninformed demand fall.

When traders cannot split orders, a market maker who increases prices obtains zero trade and hence obtains zero profits from doing so. Therefore, competition leads to an equilibrium featuring zero expected profits, since in equilibrium, a market maker is indifferent about whether to raise prices. When traders can split orders, however, when a market maker raises his prices, he still receives some trade. In this case, then, competition does not lead to zero-expected-profit pricing.[5] Market makers earn positive expected profits in equilibrium for the same reason as when liquidity demand was inelastic.

Corollary 4.2. *A price increase reduces informed demand by more than uninformed demand. Given a and b, if both informed and liquidity traders choose order size $q = q_I^A(a, b, \delta) = q_L^A(a, b, Q)$ in market A, then*

$$\frac{\partial q_I^A}{\partial a} = \frac{\partial q_L^A}{\partial a}\left[1 + \frac{\eta b}{a(\eta + b)}\right] < 0.$$

In equilibrium, the market-maker gain from the higher prices and reduced informed demand just offset the costs of the reduction in liquidity trade.

Corollary 4.3. *If an equilibrium exists, increasing the variance of the liquidity shock, σ_u^2, decreases the slope of the equilibrium price schedule, and increasing the variance in the asset payoff Σ_0, increases the slope:*

$$\frac{\partial a}{\partial \sigma_u^2} < 0, \qquad \frac{\partial a}{\partial \Sigma_0} > 0.$$

Surprisingly, the relationship between the price-elasticity of liquidity demand and the aggregate market-maker price schedule is not monotonic. To better focus on the relationship, set $\sigma_u^2 = \Sigma_0 = 1$. Then, in equilibrium, Corollary 4.4 applies.

Corollary 4.4. *The slope of the zero-expected-profit price schedule is given by*

$$\lambda_0 = \frac{\eta}{2\eta - 1}, \qquad \eta > \frac{1}{2}.$$

The liquidity trader's ex ante expected utility, given zero-expected-profit pricing, is $Y - \frac{1}{2}$. The slope of the equilibrium price schedule

[5] In the standard Kyle environment, because liquidity demand is inelastic, a monopolist market maker makes infinite profits. Now, because liquidity demand is elastic, there is an interior solution for the monopolist's profit-maximizing schedule.

Table 1
Equilibrium outcomes as a function of the elasticity of liquidity demand

η	a	$\lambda(a)$	$\frac{\eta}{\eta+\lambda(a)}$	MM π	Liq. Loss	Liq. EU	Inf. π	$E(\lvert\text{vol}\rvert)$	λ_0	$E(\lvert\text{vol}\rvert_0)$
0.5	∞	∞	0	0	0	NA	0	0	∞	0
0.51	76.75	38.38	.013	.0001	.0066	−.503	.0065	.015	25.5	.022
0.75	4.83	2.42	.24	.03	.14	−.57	.10	.25	1.5	.38
1.0	3.40	1.70	.37	.09	.23	−.63	.15	.38	1.0	.56
2.60	2.60	1.30	.67	.38	.58	−.87	.19	.61	0.62	.91
5.0	2.75	1.38	.79	.66	.85	−1.08	.18	.69	0.56	1.02
100	6.02	3.01	.97	2.76	2.84	−2.92	.08	.79	0.5025	1.12
∞	∞	∞	1	∞	∞	$-\infty$	0	.80	0.5	1.12838

Slope of the equilibrium price schedule in each market, a; the equilibrium aggregate price schedule, $\lambda(\cdot)$; the percentage of Q actually traded by liquidity traders (as a function of η) $\frac{\eta}{\eta+\lambda(a)}$; aggregate market-maker profits, MM π; aggregate liquidity-trader losses, Liq. Loss; expected utility of liquidity traders ($Y = 0$), Liq. EU; aggregate insider profits, Inf. π; $E(\lvert\text{aggregate volume}\rvert)$, $E(\lvert\text{vol}\rvert)$; the slope of the zero-profit price schedule, λ_0; and $E(\lvert\text{aggregate volume}\rvert)$ under zero-expected-profit pricing as a function of η when there are two market makers and $\sigma_u^2 = \Sigma_0 = 1$, $E(\lvert\text{vol}\rvert_0)$.

in each market is given by

$$a = \frac{2\eta}{4^{\frac{1}{3}}\eta^{\frac{2}{3}} - 1}.$$

The liquidity trader's ex ante expected utility is

$$E(Y) - \left(\frac{\eta}{4}\right)^{\frac{1}{3}}.$$

Table 1 shows that the slope of the equilibrium pricing function is not monotonic in η. This is because more elastic liquidity trade has two competing effects on the price schedule. The first, illustrated in Proposition 1, is that as liquidity trade becomes more inelastic, there is an incentive for market makers to raise prices to exploit this inelasticity. As η rises, liquidity trade becomes less price-elastic, and the slope a of the equilibrium price schedule increases, becoming unboundedly large as liquidity trade becomes completely price-inelastic. The second effect is that as liquidity trade becomes more price-elastic, market makers may also wish to raise prices because there is less liquidity trade. This is most clearly illustrated by the fact that λ_0, the slope of the zero-profit schedule, declines monotonically in η. As η becomes small, this effect dominates, and a increases. At $\eta = \frac{1}{2}$, liquidity trade and informed trade are equally price-elastic, and markets break down. The minimum value of the price schedule slope a occurs when $\eta = 2.60$.

We now consider how the aggregate price schedule and market-maker profits are affected by the number of market makers. To focus

Table 2
Equilibrium outcomes as a function of the number of competing market makers

N	$a(N)$	$\lambda[a(N)]$	MM π	Ag. MM π	Ag. Liq. Loss	ΔEU	Ag. Inf. π	$E(\|\text{vol}\|)$
1	2.74	2.74	0.10465	0.10465	0.19581	−0.73	0.09116	0.258
2	3.40	1.70	0.04313	0.08626	0.23311	−0.63	0.14685	0.377
3	4.27	1.42	0.02223	0.06669	0.24238	−0.59	0.17569	0.433
5	6.16	1.23	0.00886	0.04433	0.24731	−0.55	0.20280	0.482
10	11.07	1.11	0.00237	0.02366	0.24935	−0.53	0.22569	0.523
100	101.0	1.01	.000025	0.00249	0.24999	−0.503	0.24751	0.560
∞	∞	1	0	0	0.25	−0.5	0.25	0.564

Slope of the equilibrium price schedule in each market, $a(N)$; the equilibrium aggregate price schedule, $\lambda(\cdot)$; individual expected market maker profits, $\pi(N)$, MM π; aggregate expected market maker profits, $\Pi(N) = N\pi(N)$, Ag. MM π; aggregate expected liquidity trader losses, Ag. Liq. Loss; liquidity trader expected utility ($Y = 0$), ΔEU; aggregate expected insider profits, Ag. Inf. π; $E(|\text{aggregate volume}|)$, $E(|\text{vol}|)$; as a function of the number N of competing market makers.

on how the number of market makers affects the equilibrium, we set $\eta = \sigma_u^2 = \Sigma_0 = 1$.

Proposition 5. *Let $\eta = \sigma_u^2 = \Sigma_0 = 1$. 1. With N market makers, the slope of the unique equilibrium pricing function in each market, $a(N)$, is given by the solution to*

$$\frac{1}{4a(N)^2} - \frac{a(N) + N - 2}{[a(N) + N]^3} = 0.$$

2. Aggregate expected market-maker profits fall with the number of market makers and are given by

$$\frac{N a(N)}{[N + a(N)]^2} - \frac{N}{4a(N)}.$$

3. The liquidity trader's ex ante expected utility is given by

$$Y - \frac{a(N)}{N + a(N)}.$$

Table 2 summarizes equilibrium outcomes for different numbers of competing market makers. A profit-maximizing monopolist market maker setting a linear schedule chooses $\lambda = 2.74$ and expects profits of 0.10465. Although the expected price schedule is significantly steeper than when there are two competing market makers, the monopolist market maker's profits are only 21% greater than those that two market makers would expect. This is because the elastic liquidity trade is significantly reduced when there is only one market maker. Indeed, note that liquidity-trading losses are smallest when there is one market maker, simply because they trade very little. Liquidity-trader utility is much lower because their portfolio holdings are far from their ideal points. The addition of a second market maker goes almost halfway to raising the liquidity trader's utility to that obtained in a perfectly competitive economy. The expected volume when there is a monopolist market maker is also about one-half that in a perfectly competitive market.

As the number of competing market makers increases, the slope of the equilibrium pricing function in each market, $a(N)$, becomes arbitrarily steep, but the aggregate linear price schedule $\lambda(a_1, \ldots, a_N)$ converges to the zero-expected-profit aggregate price schedule with slope 1. In turn, as N increases, aggregate expected market-maker profits fall, vanishing as $N \to \infty$. The percentage reduction in market-maker profits is increasing in N. The intuition for this result is familiar. As with oligopolistic competition between firms, the more competing market makers there are, the less order flow any one will handle. This makes it relatively more attractive to offer "better" prices in order to attract more trade because the "cost" of the better prices — the reduced revenues on orders that the market maker would receive in the absence of the better prices — is smaller.

We can now determine the equilibrium number of market makers when there are fixed costs to making a market. In the standard environment where traders cannot split orders, if there are fixed costs, there can be no entry by a competing market maker in equilibrium. These fixed costs imply that in the standard model with fixed costs for making a market, prices will be set by a monopolist market maker. In contrast here, because market makers expect positive profits from handling each order, with free entry to market making, there will be just enough market makers that additional entry would render market making unprofitable.

These results have implications for empirical work involving both estimating components of the bid-ask spread and estimating the price impact of information. First, to account for order splitting between exchanges, when determining the informational impact, one must estimate the aggregate price schedule $\lambda(a(N))$ rather than just the price schedule on an individual exchange, $a(N)$. Second, one must correct for the fact that the steepness of the schedule is due to imperfect competition as well as inside information. Hence, a steep price schedule may reflect not just inside trading but also market-maker profits. In the Kyle model, if $N = 2$, $a(2) = 3.4$, and $\lambda(a(2)) = 1.7$, but the slope of the aggregate price schedule due to insider trading is only 1.

2. The General Argument

Consider the properties of a cononical insider-trading model. Agents trade consumption today for claims to an asset with random consumption payoff tomorrow. Some agents have private information about the value of the claim to the risky asset; others want to trade to rebalance their portfolios. All trade is through one of two risk-neutral, uninformed market makers who set continuous, twice-differentiable, monotone-increasing price schedules detailing the prices at which they are willing to handle each order size. Trade by informed and uninformed agents can be characterized by their associated first-order conditions. The trades of informed agents are more price-elastic than those of agents who trade to rebalance their portfolios because the informed have only profit motives for trade.

Traders can submit their orders to either of two risk-neutral, uninformed market makers, A and B. In this general section we assume that orders are handled individually — as is the practice on most exchanges.

The timing is as follows:

1. The identity of the trader (informed or noise) is determined.
2. Market makers simultaneously set price schedules, detailing a price for each order size.
3. The trader chooses how much to trade with each market maker.
4. Payoffs are realized.

An insider with information that a claim to the asset has value δ trades to maximize

$$max_{q_I^A, q_I^B}\; q_I^A(\delta - P_a(q_I^A)) + q_I^B(\delta - P_b(q_I^B)),$$

where q_I^i is the insider's trade with market maker i, and $i = A, B$.

Suppose that the liquidity trader's reduced-form objective can be written as

$$max_{q_L^A, q_L^B}\; W(q_L^A + q_L^B \mid \phi) - q_L^A P_a(q_L^A) - q_L^B P_b(q_L^B),$$

where $W(\cdot)$ is a strictly concave monotonic function for $q_L^A + q_L^B < \phi$. That is, ϕ represents the optimal portfolio mix; the further away the liquidity trader's holdings are from ϕ, the more he values trading a marginal unit to get closer to ϕ. Both δ and ϕ are drawn from distributions with continuous densities.

2.1 No splits

First, suppose that an agent cannot split his order between the two market makers. Then equilibrium price schedules must provide market makers zero expected profits conditional on the order flow. The only possible equilibrium is for both market makers to set a schedule that breaks even conditional on the order flow, given that one market maker sets that schedule, the other is indifferent about whether to match.

Were one market maker to set a less-competitive schedule, that is, were he to set a schedule on which he expects strictly positive profits for some order flows, the other market maker would have an incentive to offer slightly better prices on those profitable orders and take them himself. Since the effect on informed and liquidity trade of offering a slightly better schedule can be made arbitrarily small, it would be strictly profitable for the other market maker to do this. Therefore, competition inevitably demands that the two market makers set identical price schedules, price schedules that break even in expectation conditional on the order flow. Let $P(q)$ be the (potentially nonlinear) equilibrium price schedule set when agents cannot split orders. We only assume that this zero-profit price schedule is differentiable and monotonically increasing in order flow, and not "too" concave for $q > 0$ [i.e., $p''(q) > -\omega, \omega > 0, \omega$ defined in the proof] and not "too" convex for $q < 0$.[6]

2.2 Splits

Lemma 2. *When traders can split orders, in any potential equilibrium in which market makers earn zero expected profits conditional on the order flow, the market makers must set the same prices, $P_a(q^A) = P_b(q^B)$, and both must receive positive order flow.*

Suppose that $P_a(\cdot)$ and $P_b(\cdot)$ earn zero expected profits. Facing these schedules, a trader splits an order q into q^A and q^B such that $P_a(q^A) = P_b(q^B) = P(q)$ and $q^A + q^B = q$. Since the aggregate price schedule is unchanged, then so are the total order choices of the traders. But we now show that $P_a(\cdot)$ and $P_b(\cdot)$ are not equilibrium price schedules when agents can split orders: given that market maker B sets $P_b(\cdot)$, A can earn strictly positive expected profits by setting a schedule that is slightly steeper than $P_a(\cdot)$.

Proposition 6. *In the canonical model of this section, $P_a(\cdot)$ and $P_b(\cdot)$ are not equilibrium schedules. In any equilibrium, market makers earn strictly positive expected profits.*

The result follows because the demands of the liquidity traders are less elastic than those of the informed. The proof proceeds as follows. Let $P_a(\cdot)$ and $P_b(\cdot)$ be any candidate zero-profit equilibrium price schedules. We than compute market maker A's profits from setting a

[6] Dennert (1993) shows that when there is a two-point distribution for inside information, zero-expected-profit equilibria exist where market makers adopt mixed pricing strategies and the equilibrium price schedule is nondifferentiable.

slightly steeper schedule, $P_a(q) + kq$. The effect of the deviation by A is that liquidity demand in market A is reduced by less than informed demand — the liquidity demand is less price-elastic than informed trade. In the proof, we show that

$$\frac{dq_I^A}{dk} < \frac{dq_L^A}{dk}.$$

We then derive a contradiction by showing that this implies that market maker A must be able to earn positive expected profits.

Note that market maker B also earns positive expected profits when A sets a steeper price schedule because informed trade with B is unaffected by the price schedule set by A, but B receives more liquidity trade than before (the liquidity trader equates marginal trading costs across market makers, substituting some order share toward market maker B).

The results follow under weaker restrictions. The perturbations used in the proofs were global perturbations — the price increase for an order of size q is kq. We need only have considered local price increases on any open interval. Thus, the argument extends immediately unless price schedules are everywhere "too" concave.

Finally, note that similar arguments preclude the existence of a mixed strategy equilibrium because each market maker would want to set a steeper schedule than the zero-expected-profit schedule.

3. Robustness

Here we consider the robustness of our results to informational and timing assumptions. Throughout, we have assumed that schedule setters only observe the order flow that they process. In equilibrium, market makers correctly invert back from their own order flow to determine the equilibrium order flow with each market maker and maximize profits (given the total order flow). This assumption is easy to motivate if there is competition between exchanges and there is a single market maker at each exchange, because a market maker on one exchange cannot set prices conditional on the unobserved order flow in a different market.

But suppose instead that market makers do observe the total order flow so that they can set price schedules that depend on the order flow in each market. Now, another equilibrium exists, one in which each market maker sets the zero-expected-profit price conditional on the total order flow that he will receive in both markets: $p(q^A, q^B) = p(q^A + q^B)$. Given that one market maker sets this schedule, the other market maker cannot possibly earn positive profits, and hence is indifferent about setting that schedule and submitting any other schedule that earns him non-negative expected profits.

However, the equilibrium that we have characterized continues to exist: $p_a(q^A, q^B) = p_a(q^A)$, $p_b(q^A, q^B) = p_b(q^B)$. In our equilibrium (see Section 2), market makers correctly infer the total order flow from that which they expect to receive in their own market, because order flows in the two markets are perfectly correlated. Given that $p_b(q^A, q^B) = p_b(q^B)$, market maker A's profit-maximizing schedule is $p_a(q^A, q^B) = p_a(q^A)$, so that the equilibrium remains even when total order flow is observed.

Standard equilibrium refinements, such as cheap talk between market makers, eliminate all but the most profitable schedule as an equilibrium schedule. Preplay communication [see, e.g., Farrell (1988)] between market makers allows them to select among equilibria, so they will choose the one with the most profitable price schedules. In the case of order splitting across exchanges, the zero-profit equilibrium *cannot* obtain.

We can relate these findings to the multi-exchange framework of Chowdhry and Nanda (1991). In that article informed traders and some uninformed traders can split orders across markets. Prices are assumed set by Bertrand competition in each market (there are multiple market makers in each market), yielding zero-expected-profit pricing. To interpret their results in our context, in our model, zero-expected-profit pricing obtains when agents cannot split orders between the multiple market makers within a market, or when market makers select the zero-profit equilibrium in a situation where both zero-profit and positive-profit market-maker equilibria exist.

One might also contemplate the possibility that only some liquidity traders can split their orders. If the liquidity traders who cannot split can choose where they transact, this leads to a discontinuity in a market maker's order flow around the schedules. If enough liquidity traders cannot split orders, then an upward deviation from the zero-profit price schedules may no longer be profitable because liquidity traders who do not split will not trade with the market maker setting the worse price schedule. However, a slight downward deviation from the zero-profit schedule is now strictly profitable because, with positive probability, the market maker setting the slightly better price receives the strictly profitable trades from any nonsplitting liquidity trader, and the deviation has an arbitrarily small effect on the behavior of traders who split. Indeed, when some, but not all, liquidity traders cannot split their orders, no pure-strategy pricing equilibrium can exist.

Throughout, we have assumed that exchanges are quote-driven: market makers first announce price schedules and then traders submit orders. This seems to be the natural context within which to consider splitting — traders see price schedules and then allocate their trade accordingly. The effect of splitting is very different in an order-driven environment, where first traders submit orders and then market makers set prices. In this instance, market makers set arbitrarily high prices. This is because once a trader submits an order, it cannot be withdrawn, and there is no further competition between market makers to discipline their pricing. Except for agents with completely inelastic demand, no one will submit orders. If liquidity traders have price-elastic demand, and traders can withdraw their orders after seeing the quotes, then the market makers will effectively behave as monopolists facing the split demands.

Only by adding to the game a third stage in which traders can reallocate their orders between the market makers can one recover zero-expected market-maker profits in equilibrium. That is, if traders first *commit* to trading a particular maximum aggregate amount, market makers next post schedules, and finally traders allocate their orders between the market makers, then because market makers observe the maximum total order flow, an equilibrium exists in which they will compete away all profits.

To see this, observe that if market makers set flat schedules $p_a(q) = p_b(q) = P(Q)$, where Q is the total order flow in both markets, then no trader has an incentive to split orders. A market maker setting higher prices receives no trade, and one who sets lower prices makes losses. The third stage has the effect of allowing market makers to price based on total order flow.[7] As in the quote-driven environment, this leads to an equilibrium featuring zero-expected-profit pricing. Unfortunately for small traders, such a market structure does not appear to exist. Perhaps the closest approximation is upstairs trading on the NYSE.

Again, the results for the different order-driven systems reflect differences in the price-elasticity of demand. If traders cannot withdraw their orders after seeing prices, then once the order is submitted, trade is completely price-inelastic, and markets break down. If traders can withdraw orders, then trade is less price-elastic, and market makers earn positive expected profits. Finally, if traders can reallocate their trades after seeing prices, then trade is effectively perfectly price-elastic, and the zero-expected-profit equilibrium obtains.

[7] Note that it is crucial that either traders can precommit not to trade more than their initial submissions, or that market makers can observe the final division of the order between the market makers. Otherwise, traders have an incentive to pad their orders ex post, and markets effectively become quote-driven.

4. Limit Orders

We return to the general environment of Section 2. The only difference is that we now assume that price schedules are set by agents who compete using limit orders. While we assume that there are only two such competing agents, the argument generalizes to any finite number. We will show that the basic result that zero-expected-profit pricing does not obtain when traders can split orders extends naturally to an environment where price schedules are set by limit orders.

As in Section 2, the orders of informed and uniformed traders are handled individually. All trade is through one of two risk-neutral, uninformed agents, A and B, who set continuous, twice-differentiable, monotone-increasing, limit order schedules, $P_A(\cdot)$ and $P_B(\cdot)$, detailing the prices at which they are willing to handle any given order size. Unlike the price schedules in Section 2 that detail the price paid for an order of size q, the limit order schedules here detail the marginal price paid for the qth unit of an order. A trader who submits a market order of size q to market i, where $i = A, B$, receives price $\int_0^q P_i(q)\,dq$ and pays $q\int_0^q P_i(q)\,dq$.

We assume that trade by informed and uninformed agents can be characterized by their associated first-order conditions. The trades of informed agents are more price-elastic than trades of those who trade to rebalance their portfolios, because informed traders have no reason to trade other than profit. The timing is as follows:

1. The identity of the trader (informed or noise) is determined.
2. Market makers simultaneously set limit order schedules.
3. The trader chooses how much to trade with each market maker.
4. Payoffs are realized.

This timing is natural for an environment where there is a limit order book. It is clear that price schedules, in the form of limit orders, are set before market orders are submitted.

We now show that the zero-profit limit order schedule [see Glosten (1994)] cannot be an equilibrium outcome. An agent submitting limit orders can (strictly) increase his expected profits by offering less-competitive prices. The zero-profit limit order schedule can be improved upon because liquidity trade is less price-elastic than informed trade.

Let $P_L(\cdot)$ be the aggregate zero-profit limit order schedule: $P_L(\cdot)$ must generate zero expected profits for any given limit order quantity q, conditional on the fact that it is crossed against all market orders of at least size q. Let the zero-expected-profit limit order schedules be given by $P_A(\cdot)$ and $P_B(\cdot)$. Following Lemma 2, one can show that the (potentially asymmetric) zero-expected-profit limit pricing schedules set by A and B satisfy $P_A(q^A) = P_B(q^B) = P_L(q)$. As a result, traders who submit market orders split their trades such that the price for an order of size q remains $\int_0^q P_L(q)dq$.

We now show that $P_A(\cdot)$ and $P_B(\cdot)$ are not equilibrium schedules: agents submitting limit orders can increase expected profits by submitting less-competitive limit order schedules.

Proposition 7. *In any equilibrium where traders can split orders, the limit order setters must earn strictly positive expected profits.*

The proof mimics that of Proposition 6. Again, the result follows because a price increase causes liquidity traders to reduce their orders by less than informed traders.

5. Conclusion

This article demonstrates that when traders can split orders between risk-neutral market makers, in equilibrium, the market makers must earn strictly positive profits. Competition among risk-neutral market makers or risk-neutral limit order setters leads to zero-expected-profit pricing *only* if agents cannot split orders. We provide a rationale for why somebody might willingly make a market for a stock when there are fixed costs to doing so. We generalize the Kyle environment to allow for rational, price-elastic liquidity traders in order to show that when traders can split orders, market-maker profits decline as the number of market makers increases. Hence, for a given fixed cost to making a market, we can compute the equilibrium number of market makers when there is free entry to market making.

Appendix

Proof of Lemma 1. Step 1: A noise trader with demand Q minimizes his trading costs by buying q_L^A from market maker A and $Q - q_L^A$ from B. His optimization problem is

$$min_{q_L^A}\ q_L^A a q_L^A + b(Q - q_L^A)(Q - q_L^A),$$

which has solution

$$q_L^A = \frac{bQ}{a+b}, \quad q_L^B = \frac{aQ}{a+b}.$$

Step 2a: The insider observing δ trades q_L^A with market maker A, where q_I^A solves

$$max_{q_I^A}\ E_Q\left\{q_I^A\left[\delta - a\left(q_I^A + \frac{bQ}{a+b}\right)\right]\right\} = q_I^A(\delta - a q_I^A).$$

By solving the equation above, we find that the informed agent buys claims

$$q_I^A = \frac{\delta}{2a}.$$

Step 2b: Suppose that quantity $q^A = q_I^A + q_L^A = \frac{\delta}{2a} + \frac{bQ}{a+b}$ is observed in market A. Zero-expected-profit pricing conditional on the order flow then implies that

$$\begin{aligned} p_a(q^A) &= aq^A = a\left(\frac{\delta}{2a} + \frac{bQ}{a+b}\right) \\ &= b\left(\frac{\delta}{2a} + \frac{aQ}{a+b}\right) = bq^B = p_b(q^B), \end{aligned}$$

and also that

$$aq^A = E(\delta \mid q^A) = \frac{cov(\delta, q^A)}{var(q^A)}.$$

Solving yields the result.

Both a and b must exceed $\frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$. If b approaches $\frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$ from above, $a \to \infty$.

Proof of Proposition 1. Step 1: Suppose that market maker A believes that market maker B will set schedule $p_b(z) = bz$. Given these beliefs, we will solve for market maker A's optimal linear schedule, $p_a(z) = az$, and show that for any b, a must exceed b.

Step 2: Uninformed demand: From Lemma 1,

$$q_L^A = \frac{bQ}{a+b}, \quad q_L^B = \frac{aQ}{a+b}.$$

Thus, q_L^A is normally distributed with zero mean and variance $\frac{b^2}{(a+b)^2}\sigma_u^2$. This implies that if the uninformed traders have total demand Q, and market makers set price schedules with slopes a and b, uninformed losses to market maker A are

$$\begin{aligned} &-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{bQ}{(a+b)}\left\{\delta - a\left[\frac{\delta}{2a} + \frac{bQ}{(a+b)}\right]\right\} dG(\delta)dH(Q) \\ &\quad = \frac{ab^2}{(a+b)^2}\sigma_u^2. \end{aligned}$$

Step 3: Informed demand: From Lemma 1, an insider observing δ trades $q_I^A = \frac{\delta}{2a}$ in market A. Conditional on trading, the insider's expected profits from trading with market maker A (and hence market maker *A*'s losses) are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta}{2a}\left[\delta - a\left(\frac{\delta}{2a}+\frac{bQ}{a+b}\right)\right]dG(\delta)\,dH(Q) = \frac{1}{4a}\Sigma_0.$$

Step 4: Market maker *A*'s optimization problem is then given by

$$max_a\ \frac{-1}{4a}\Sigma_0 + \frac{ab^2}{(a+b)^2}\sigma_u^2.$$

The first term is market maker *A*'s expected loss to informed traders; the second term is his expected profit from noise traders. The associated first-order conditions are

$$\frac{\Sigma_0}{4a^2} + b^2\sigma_u^2\left[\frac{1}{(a+b)^2} - \frac{2a}{(a+b)^3}\right] = 0.$$

Simplifying and rearranging yields

$$4a^2b^2\sigma_u^2(a-b) = (a+b)^3\Sigma_0.$$

Since both a and b are positive, this implies that $a > b$. Since each market maker always wants to set a steeper price schedule than the competition, no linear equilibrium exists.

Proof of Proposition 2. Let $E = \frac{-Q}{2}$. Then liquidity-trader end-of-period wealth is given by

$$w_1 = \bar{Y} + \left(q_L - \frac{Q}{2}\right)\delta - q_L\lambda(q_L + q_I)$$

Expected end-of-period wealth is

$$E(w_1) = \bar{Y} - \lambda q_L^2,$$

and the variance of end-of-period wealth is

$$var(w_1) = \frac{1}{4}(q-Q)^2\Sigma_0.$$

Liquidity traders' expected utility is then given by

$$EU = E(w_1) = \frac{\gamma}{2}var(w_1) = \bar{Y} - \lambda q_L^2 - \frac{\gamma\Sigma_0}{8}(q_L - Q)^2.$$

Proof of Proposition 3. Step 1a: Conjecture that market makers set schedules that are linear functions of total order flow q: $p(q) = \lambda q$.

Then a liquidity trader receiving shock Q faces the optimization problem:

$$Max_{q_L} E\left[Y - \eta(Q - q_L)^2\right],$$
$$\text{subject to the constraint} \quad p(q_1 + q_L) = \lambda(q_1 + q_L).$$

Solving, the liquidity trader purchases quantity

$$q_L = \frac{\eta Q}{\lambda + \eta}.$$

Since Q is normally distributed with zero mean and variance σ_u^2, liquidity demand is normally distributed with zero mean and variance $(\frac{\eta}{\lambda+\eta})^2\sigma_u^2$.

Step 1b: As in Kyle (1985), informed demand is normally distributed with zero mean and variance $\frac{\Sigma_0}{4\lambda^2}$. Hence, aggregate demand is normally distributed.

Step 1c: Since orders cannot be split, prices must earn the market maker zero expected profits:

$$p(q) = \lambda q = E(\delta \mid q_I + q_L) = \frac{cov(\delta, q)}{var(q)}.$$

Solving yields the result. Note that when $\eta \leq \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}$, the quadratic equation that characterizes the slope of the pricing function yields imaginary roots only, so that no linear equilibrium exists.

Step 2: In equilibrium, the liquidity trader's ex ante utility equals

$$E\left[Y - \lambda q q_L - \frac{\eta}{2}(Q - q_L)^2\right].$$

Substituting in the equilibrium values for q and q_L, taking expectations, and simplifying yields

$$Y - \frac{\lambda\eta}{\eta + \lambda}\sigma_u^2.$$

Substituting the equilibrium vale of λ yields the result.

Proof of Proposition 4. Step 1: Facing a and b, uninformed liquidity traders who receive liquidity shock Q split their orders, trading fraction $\frac{b}{a+b}$ in market A and $\frac{a}{a+b}$ in market B. Therefore, liquidity traders face the same prices in both markets:

$$p(q_L) = aq_L^A = bq_L^B = \left(\frac{ab}{a+b}\right)Q.$$

Hence, the slope of the aggregate price schedule faced by traders is

$$\lambda(a, b) = \left(\frac{ab}{a+b}\right).$$

Since $q_L^A = \frac{\eta}{\lambda(a,b)+\eta}$, substituting for $\lambda(a, b)$ yields

$$q_L^A = \frac{b\eta Q}{ab + \eta(a+b)}.$$

Therefore, uninformed losses in market A are given by

$$E\left[a(q_L^A)^2\right] = \frac{ab^2\eta^2\sigma_u^2}{\left[ab + \eta(a+b)\right]^2}.$$

Step 2: The informed trader's profits in market A depend only on a and again equal $\frac{\Sigma_0}{4a}$.

Step 3: Market maker A's optimization problem is then given by

$$max_a \frac{-\Sigma_0}{4a} + \frac{ab^2\eta^2\sigma_u^2}{\left[ab + \eta(a+b)\right]^2}.$$

Market maker B's optimization problem in turn is given by

$$max_b \frac{-\Sigma_0}{4b} + \frac{ba^2\eta^2\sigma_u^2}{\left[ab + \eta(a+b)\right]^2}.$$

Differentiating with respect to the slope parameters and rearranging yields the following first-order conditions:

$$\frac{\Sigma_0\left[ab + \eta(a+b)\right]^3}{4a^2b^2\eta^2\sigma_u^2} = ab - \eta(b-a),$$

and

$$\frac{\Sigma_0\left[ab + \eta(a+b)\right]^3}{4a^2b^2\eta^2\sigma_u^2} = ab - \eta(a-b),$$

which imply that $a = b$. Substituting for b in either first-order condition and solving the resulting cubic equation,

$$\Sigma_0(a + 2\eta)^3 = 4\eta^2\sigma_u^2a^3,$$

for a yields the result.

Step 4: Checking the second-order conditions implies that for the first-order conditions to characterize an optimum, a, $b > 0$. This implies that for the linear equilibrium to exist,

$$\eta > \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}.$$

Proof of Corollary 4.1. Recall from Proposition 3 that the slope of the zero-expected-profit schedule is given by

$$\lambda = \frac{\eta \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}}{\eta - \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}}. \tag{1}$$

The slope of the aggregate price schedule $\lambda(a, b) = 2a$ given in Proposition 4 can be rewritten as

$$\lambda(a, b) = \frac{\eta\left[\frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}\right]^{\frac{2}{3}}}{\eta - \left[\frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}}\right]^{\frac{2}{3}}}.$$

Substituting the condition for an equilibrium to exist,

$$\eta > \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_u^2}\right)^{\frac{1}{2}} > 0,$$

into $\lambda(a, b)$ and λ, we see that $\lambda(a, b)$ exceeds the slope λ of the zero-expected-profit schedule.

Proof of Corollary 4.2. Recall that $q_I^A = \frac{\delta}{2a}$ and $q_L^A = \frac{b\eta Q}{ab+\eta(a+b)}$. Suppose that $q_I^A = q_L^A = q$.

Now, for the same-sized order, look at the reaction of each type of trader to a price increase in market a. That is, evaluate $\frac{dq_I^A}{da}$ and $\frac{dq_I^A}{da}$ at $q_I^A = q_L^A = q$.

Differentiating, $\frac{dq_I^A}{da} = -\frac{\delta}{2a^2}$ and $\frac{dq_L^A}{da} = \frac{-b\eta Q(b+\eta)}{[ab+\eta(a+b)]^2}$.

Using $q = \frac{\delta}{2a}$ to substitute for δ in $\frac{dq_I^A}{da}$ and $q = \frac{b\eta Q}{ab+\eta(a+b)}$ to substitute for Q in $\frac{dq_L^A}{da}$ and rearranging yields the result.

Proof of Proposition 5. Step 1: The proof is nearly identical to that of Proposition 4. Let the slope of the pricing function in market i be a_i, where $i = 1, \ldots, N$. Informed trader profits which are unaffected by the price schedules in the other markets, in market i, are given by $\frac{1}{4a_i}$.

A liquidity trader who demands q solves

$$min_{q_L^1,\ldots,q_L^N} \sum_{i=1}^{N} a_i (q_L^i)^2, \qquad \text{such that} \quad \sum_{i=1}^{N} q_L^i = q.$$

Solving,

$$q_L^i = q \left[\frac{\prod_{j\neq 1}^{N} a_j}{\sum_{k=1}^{N} \left(\prod_{j\neq k}^{N} a_j \right)} \right].$$

This implies that the slope of the aggregate price schedule, $\lambda(a_1, \ldots, a_N)$, is given by

$$\lambda(a_1, \ldots, a_N) = \frac{\prod_{j=1}^{N} a_j}{\sum_{k=1}^{N} \left(\prod_{j\neq k}^{N} a_j \right)}.$$

Uninformed losses in market i equal

$$E \left\{ a_i \left[q_l^i \frac{1}{\lambda(a_1, \ldots, a_N) + 1} \right]^2 \right\}.$$

Substituting for $\lambda(a_1, \ldots, a_N)$ and q_L^i yields market makers i's optimization problem:

$$max_{a_i} \frac{-1}{4a_i} + \frac{a_i \prod_{j\neq i}^{N} a_j^2}{\left[\sum_{k=1}^{N} \left(\prod_{j\neq k}^{N} a_j \right) + \prod_{j=1}^{N} a_j \right]^2}.$$

Differentiating with respect to a_i for each $i = 1, \ldots, N$ and solving yields the N first-order conditions. Following the argument of the proof to Proposition 4, one can show that the solution must be symmetric: $a_1 = a_2 = \ldots a_N$. Substituting $a_i = a$, $i = 1, \ldots, N$, into the first-order conditions and solving the resulting cubic equation then yields the result.

Step 2: Differentiating the expression for profits with respect to N, the number of market makers, yields the result.

Step 3: The derivation of the welfare result is analogous to that for Proposition 3.

Proof of Lemma 2. The cost-minimizing way to trade a given total order flow $q = q^A + q^B$ does not depend on whether the trader is informed. If both market makers receive positive order flow, the cost-minimizing way to trade is given by the solution to

$$min_{q^A} \left(P_q(q^A) q^A + P_b(q^B) q^B \right),$$

which (since $q^B = q - q^A$) has first-order condition

$$P_a(q^A) + q^A P_a'(q^A) = P_b(q^B) + q^B P_b'(q^B).$$

If both market makers receive positive order flow, then for both to earn zero expected profits, that is, $E\{q^i[p_i(q^i) - \delta] \mid q^i\} = 0$, $i = A, B$, they must charge the same price, $P_a(q^A) = P_b(q^B)$, since $E(\delta \mid q^A)$ must equal $E(\delta \mid q^B)$.

Now suppose that only one market maker receives positive order flow (without loss of generality, let it be market maker A). But then $P_a(q^A) < P_b(0)$, where $P_a(\cdot) = P(\cdot)$. But then market maker A cannot be maximizing profits, for by setting a schedule

$$\hat{P}_a(q) = \begin{cases} P_b(0) - \epsilon, \quad \epsilon \text{ arbitrarily small} & q < P^{-1}(P_b(0) - \epsilon) \\ P_a(0) & q \geq P^{-1}(P_b(0) - \epsilon), \end{cases}$$

market maker A can earn strictly positive profits, a contradiction.

Proof of Proposition 6. Since orders are handled independently, it is without loss of generality to focus on buy orders $q > 0$. Let market maker B set price schedule $P_b(q)$ and let A deviate by setting

$$P_a^d(q) = P_a(q) + kq, \qquad k > 0.$$

Recall that the insider's problem is

$$max_{q_I^A, q_I^B} \; q_I^A \left[\delta - P_a^d(q_I^A)\right] + q_I^B \left[\delta - P_b(q_I^B)\right].$$

The first-order condition determining the insider's trade with market maker B,

$$\delta - P_b(q_I^B) - q_I^B \frac{dP_b(q_I^B)}{dq_I^B} = 0,$$

is not a function of the price schedule set by market maker A, $P_a^d(q_I^A)$. The first-order condition for the insider's trade with market maker A is given by

$$\delta - P_a(q_I^A) - q_I^A \frac{dP_a(q_I^A)}{dq_I^A} = 2q_I^A k.$$

The second-order conditions for a maximum are given by

$$2\frac{dP_a(q)}{dq} + q\frac{d^2P_a(q)}{dq^2} > 0,$$

and

$$2\frac{dP_b(q)}{dq} + q\frac{d^2P_b(q)}{dq^2} > 0.$$

Note that the second-order conditions place a limit on the concavity of the price schedules.

The liquidity trader's problem is

$$max_{q_L^A, q_L^B} W(q_L^A + q_L^B \mid \phi) - q_L^A P_a^d(q_L^A) - q_L^B P_b(q_L^B).$$

The associated first-order conditions are given by

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^B} - P_b(q_L^B) - q_L^B \frac{dP_b(q_L^B)}{dq_L^B} = 0,$$

and

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^A} - P_a(q_L^A) - q_L^A \frac{dP_a(q_L^A)}{dq_L^A} = 2q_L^A k.$$

Consider now liquidity and informed traders who would submit the same order when facing $P_a(\cdot)$ and $P_b(\cdot)$. The effects of an increase in the slope of market maker A's price schedule on their orders are given by (at $k = 0$)

$$\frac{dq_I^A}{dk} = \frac{-2q_I^A}{2P_a' + q_I^A P_a''} < 0,$$

$$\frac{dq_I^B}{dk} = 0,$$

$$\frac{dq_L^A}{dk} = \frac{2q_L^A(W'' - 2P_b' - q_L^B P_b'')}{(W'' - 2P_a' - q_L^A P_a'')(W'' - 2P_b' - q_L^B P_b'') - (W'')^2} < 0,$$

and

$$\frac{dq_L^B}{dk} = \frac{-2q_L^A W''}{(W'' - 2P_a' - q_L^A P_a'')(W'' - 2P_b' - q_L^B P_b'') - (W'')^2} > 0,$$

where we drop the arguments of the functions, and the primes refer to the associated derivatives.

Since $W(\cdot)$ is strictly concave, at $q_L^A = q_I^A$, $\frac{dq_I^A}{dk} < \frac{dq_L^A}{dk}$. Intuitively, a liquidity trader has a reason other than profit maximization to trade, so that when A sets a less-competitive price schedule, a liquidity trader who would trade what an insider would trade if $k = 0$ reduces his order with A by less than the insider does. To see this, rewrite $\frac{dq_L^A}{dk}$ in terms of $\frac{dq_I^A}{dk}$:

$$\frac{dq_L^A}{dk} = \frac{dq_I^A}{dk}\left[\frac{W'' - 2P_b' - q_L^B P_b''}{(W'' - 2P_b' - q_L^B P_b'') + \frac{W''(2P_b' + q_L^B P_b'')}{2P_a' + q_L^A P_a''}}\right].$$

That is, $\frac{dq_L^A}{dk}$ equals $\frac{dq_I^A}{dk}$ times a number less than one.

The more concave is $W(\cdot)$, the less price-elastic is liquidity trade with market maker A, and hence the less by which liquidity trade with A falls when A increases price.[8]

We now show that the profits A expects from informed and liquidity traders who would trade the same arbitrary quantity $q > 0$ if A were to set $P_a(\cdot)$ are increased by a marginal price increase. Since informed trade is monotone in δ, and liquidity trade is monotone in ϕ, one can invert from q and determine the associated δ and ϕ. If A were to set schedule $P_a(\cdot)$, then by assumption, his expected profits conditional on order flow q would be zero:

$$P_a(q) = E(\delta \mid q) = Pr(\text{informed} \mid q)\delta^{-1}(q) + Pr(\text{liquidity} \mid q)E_0(\delta),$$

where $\delta^{-1}(q)$ is the asset value associated with an inside trade of q, and $E_0(\delta)$ is the ex ante expected value of a claim to the risky asset. Letting $\rho(q^A) = Pr(\text{informed} \mid q^A)$, we can write A's profits from these traders as

$$\rho(q^A)\left\{q_I^A\left[P_a(q_I^A) + kq_I^A - \delta^{-1}(q^A)\right]\right\} + \left[1 - \rho(q^A)\right]\left\{q_L^A\left[P_a(q_L^A) + kq_L^A\right]\right\} = 0,$$

evaluated at $k = 0$. Differentiating A's profits from these agents with respect to k and evaluating at $k = 0$ yields

$$\rho(q^A)\left\{\frac{dq_I^A}{dk}\left[P_a(q_I^A) - \delta^{-1}(q^A) + q_I^A P_a'(q_I^A)\right] + (q_I^A)^2\right\} + [1 - \rho(q^A)]\left\{\frac{dq_L^A}{dk}\left[P_a(q_L^A) + P_a'(q_L^A)q_L^A\right] + (q_L^A)^2\right\}.$$

From the informed's first-order conditions, $[P_a(q_I^A) - \delta^{-1}(q^A) + q_I^A P_a'(q_I^A)] = 0$, so this simplifies to

$$\frac{dq_L^A}{dk}\delta(q^A)[1 - \rho(q^A)] + (q^A)^2 > \frac{dq_I^A}{dk}\delta(q^A)[1 - \rho(q^A)] + (q^A)^2,$$

[8] The standard noise trader formulation obtains when $lim_{q\to\phi} W''(q) = -\infty$.

where the above inequality follows from $|\frac{dq_L^A}{dk}| < |\frac{dq_I^A}{dk}|$. Substituting for $\frac{dq_I^A}{dk}$ yields

$$[1-\rho(q^A)]\left[\frac{-2q_I^A\delta(q^A)}{2P_a'+q_I^AP_a''}\right]+(q^A)^2=\frac{(q^A)^3P_a''}{2P_a'+q_I^AP_a''}.$$

This last expression follows from substituting in the market maker's premised zero-profit condition and the informed trader's first-order condition. This expression is non-negative if the price schedule is weakly convex, and because of the strict inequality obtained when substituting $\frac{dq_I^A}{dk}$ for $\frac{dq_L^A}{dk}$, the positive deviation increases profits provided that P_a is not "too" concave.

Since q was arbitrary, it must be that market maker A expects strictly positive profits when he sets a schedule that is steeper than the zero-expected-profit schedule. Hence, the zero-profit schedule cannot be an equilibrium schedule. Finally, it should be noted that the argument was global in nature. The argument extends immediately unless $P_a(\cdot)$ is globally "too" concave for $q > 0$: A can obtain positive profits by raising price slightly on an open interval where $P_a(\cdot)$ is less concave.

Proof of Proposition 7. Since orders are handled independently, it is without loss of generality to focus on buy orders $q > 0$. Let agent B set a limit-order schedule $P_B(q)$, and let A set schedule

$$P_A^d(q)=P_a(q)+kq, \qquad k>0.$$

The insider's orders $q_I^j(\delta),\ \ j=A,B$ are given by the solution to

$$P_j(q_I^j(\delta))=\delta.$$

The liquidity trader's reduced-form objective is

$$max_{q_L^A,q_L^B}\ W(q_L^A+q_L^B\mid\phi)-\int_0^{q_L^A}[P_A(q)+kq]dq-\int_0^{q_L^B}P_B(q)dq.$$

The associated first-order conditions are given by

$$\frac{dW(q_L^A+q_L^B\mid\phi)}{dq_L^B}=P_B(q_L^B),$$

and

$$\frac{dW(q_L^A+q_L^B\mid\phi)}{dq_L^A}=P_A(q_L^A)+kq_L^A.$$

Consider now liquidity and informed traders who would submit the same order quantity to A if A were to set $P_a(\cdot)$. The effects of marginal increase in market maker A's price schedule on their orders are given by

$$\frac{dq_I^A}{dk} = \frac{-q_I^A}{P_a'} < 0,$$

$$\frac{dq_I^B}{dk} = 0,$$

$$\frac{dq_L^A}{dk} = \frac{q_L^A(W'' - P_B')}{(W'' - P_A')(W'' - P_B') - (W'')^2} < 0,$$

and

$$\frac{dq_L^B}{dk} = \frac{q_L^A W''}{(W'' - P_A')(W'' - P_B') - (W'')^2} > 0,$$

where we drop the arguments of the functions, and the primes refer to the associated derivatives.

Since $W(\cdot)$ is strictly concave, at $q_L^A = q_I^A$, $\frac{dq_I^A}{dk} < \frac{dq_L^A}{dk}$. To see this, write $\frac{dq_L^A}{dk}$ as a function of $\frac{dq_I^A}{dk}$:

$$\frac{dq_L^A}{dk} = \frac{dq_I^A}{dk}\left(\frac{W'' - P_B'}{W'' - P_B' + \frac{W'' P_B'}{P_A'}}\right)$$

That is, $\frac{dq_L^A}{dk}$ equals $\frac{dq_I^A}{dk}$ times a number less than one. The argument now follows directly from the argument when market makers set average price schedules, liquidity trade is elastic, and traders can split orders.

References

Biais, B., P. Hillion, and C. Spatt, 1995, "An Empirical Analysis of the Limit Order Book and the Order Flow in the Paris Bourse," forthcoming in *Journal of Finance*.

Chowdhry, B., and V. Nanda, 1991, "Multimarket Trading and Market Liquidity," *Review of Financial Studies*, 4, 483–511.

de Jong, F., T. Nijman, and A. Röell, 1994, "A Comparison of the Cost of Trading French Shares on the Paris Bourse and on SEAQ," International Center Discussion Paper, Tilburg University.

Denner, J., 1993, "Price Competition between Market Makers," *Review of Economic Studies*, 60, 735–51.

Easley, D., and M. O'Hara, 1987, "Prices, Trade Size and Information in Securities Markets," *Journal of Financial Economics*, 19, 69–90.

Farrell, J., 1988, "Meaning and Credibility in Cheap-Talk Games," in M. Dempster (ed.), *Mathematical Models in Economics*, Oxford University Press, Oxford.

Glosten, L. R., 1994, " Is the Electronic Open Limit Order Book Inevitable?," *Journal of Finance*, 49, 1127–61.

Glosten, L. R., and P. Milgrom, 1985, "Bid, Ask, and Transaction Prices in a Broker Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14, 71–100.

Ho, T., and H. Stoll, 1983, "The Dynamics of Dealer Markets Under Competition," *Journal of Finance*, 38, 1053–74.

Kyle, A., 1985, "Continuous Auctions and Insider Trading," *Econometrica*, 53, 1315–35.

Riley, J., 1979, "Informational Equilibrium," *Econometrica*, 47, 331–59.

Rochet, J. C, and J. L. Vila, 1994, "Insider Trading Without Normality," *Review of Economic Studies*, 61, 131–52.