Dynamic Nonmyopic Portfolio Behavior

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The dynamic nonmyopic portfolio behavior of an investor who trades a risk-free and risky asset is derived for all HARA utility functions and a stochastic risk premium. Conditions are found for when the investor holds more or less than the myopic amount of the risky asset; hedges against or speculates the risk-premium uncertainty; is long or short on the risky asset; and holds more or less of the risky asset at longer horizons. The analytical solutions derived take multiple mathematical forms and include extreme cases in which investors with long but finite horizons can attain nirvana.

In the standard paradigm of portfolio theory, the investor maximizes expected utility, with continuous or periodic revisions of his portfolio within his investment horizon. The purpose of the revisions is to adapt to shifts in wealth, interest rates, and beliefs, and to the shortening of the investor's horizon as time passes.[1]

The investor's *opportunity set* is defined to be the current risk-free rate and his probability beliefs for

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[1] For example, see Fama (1970), Hakansson (1970), and Samuelson (1969) for multiperiod models, and Merton (1969, 1971, 1973) and Breeden (1979) for continuous-time models.

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future asset returns. It has long been recognized that an extraordinary simplification results from making any one of three assumptions: (1) a nonstochastic opportunity set, (2) an opportunity set uncorrelated with asset returns, or (3) logarithmic utility. Each assumption produces a myopic portfolio strategy, so-called because the current portfolio is independent of the distribution of asset returns over revision periods beyond the current one.

Without one of these assumptions, the myopic and nonmyopic portfolios are identical only in the final period of a multiperiod model, or as the remaining horizon approaches zero in a continuous-time model. Longer-horizon myopic and nonmyopic portfolio behavior are potentially very different. Though the first- and second-order conditions for optimal nonmyopic portfolio strategies in continuous-time have been known since Merton (1973), our current knowledge of these differences in behavior is slight. The optimality conditions alone are not highly informative, and the existing analytical solutions for nonmyopic portfolio behavior are few in number and specialized in their assumptions.[2]

In this article, we examine the dynamic nonmyopic behavior of investors in a continuous-time model with *HARA*-class (Hyperbolic absolute risk aversion) utility functions and one risk-free and one risky asset, with a constant risk-free rate and a stochastic risk premium on the risky asset following the simplest mean-reverting diffusion process, the well-known Ornstein-Uhlenbeck process. The analytical solutions for the investment and expected-utility functions are derived for all parameter combinations. The solutions take four different mathematical forms, depending on where they occur in a two-dimensional parameter space. The solutions supply answers to a variety of questions regarding dynamic nonmyopic portfolio behavior: (1) when an investor holds more or less than the myopic amount of the risky asset; (2) whether an investor hedges against or speculates on the risk-premium uncertainty; (3) when an investor is long or short the risky asset; and (4) when an investor holds more or less of the risky asset at long horizons.

The solutions can also be classified as *well-behaved* and *nirvana* solutions. For the nirvana solutions, which occur in certain extreme regions of the parameter space, investors are able to attain nirvana in sufficiently long but finite horizons through optimal and even suboptimal trading strategies, where nirvana is defined as maximum expected utility for utility functions bounded above and infinite expected utility for utility functions unbounded above. In addition, the optimal portfo-

[2] For example, see the Ingersoll (1987) and Samuelson (1988, 1989) models of nonmyopic behavior.

lio strategies for the nirvana solutions require taking infinite positions in the risky asset at the critical horizons at which nirvana is attained, except when myopic portfolio strategies are optimal. Given that no investors show signs of nirvana attainment or take infinite positions, the implication of these properties is either that (1) the parameter combinations producing nirvana solutions do not occur in the real world, or (2) the real world has costs or constraints not found in the model which eliminate the nirvana solutions.

Section 1 describes the basic model. Section 2 outlines the solution method, classifies the solutions by mathematical form and behavior, identifies the regions of the parameter space in which they occur, and describes the behavior of the well-behaved and nirvana solutions. Section 3 supplies answers to the four questions above regarding non-myopic behavior. Section 4 summarizes and concludes.

1. The Model

The investor trades two assets, a risk-free asset and a risky asset, in a frictionless continuous-time market. For simplicity, the risky asset has no cash payout and there is no consumption or labor income during the investment horizon. The investor has *HARA* utility for final wealth, the riskless rate is constant, and the price $P(t)$ of the risky asset follows the diffusion process

$$\frac{dP}{P} = \mu(t)dt + \sigma(t)dZ, \tag{1}$$

where $Z(t)$ is an arithmetic Brownian motion with zero drift rate and unit variance rate. The current drift rate $\mu(t)$ and volatility $\sigma(t)$ of the risky asset are both diffusion processes as well. Define

$$X(t) = \frac{\mu(t) - r}{\sigma(t)} \tag{2}$$

to be the risk premium on the risky asset. The risk premium is assumed to follow the simplest diffusion process with mean-reverting drift, the Ornstein-Uhlenbeck process

$$dX = -\lambda_X(X - \overline{X})dt + \sigma_X dZ_X, \tag{3}$$

where λ_X, σ_X, and $\overline{X}$ are positive constants; $Z_X(t)$ is a second Brownian motion with zero drift rate and unit variance rate; and the correlation between the asset-return and risk-premium processes is given by

$$E\{dZdZ_X\} = \rho_{mX}dt. \tag{4}$$

This risk-premium process can at times take on negative values, an appropriate assumption for a model of investor behavior with time-varying and heterogeneous beliefs.

Since the opportunity set has three stochastic variables—$\mu(t)$, $\sigma(t)$, and $X(t)$—linked by one equation,

$$\mu(t) = r + \sigma(t)X(t), \tag{5}$$

two stochastic state variables are initially plausible. But one, the risk premium $X(t)$ as defined above, is sufficient. To show this, let $W(t)$ be the investor's current wealth and $y(t)$ be his current monetary investment in the risky asset. The investor's wealth dynamics are

$$dW(t) = r[W(t) - y(t)]dt + y(t)\frac{dP}{P}. \tag{6}$$

Define the normalized return process $R(t)$ by

$$dR(t) = \frac{1}{\sigma(t)}\left(\frac{dP}{P} - rdt\right) = X(t)dt + dZ(t). \tag{7}$$

The normalized return process is clearly the returns on a fully levered monetary investment of $1/\sigma(t)$ in the risky asset. The investor's wealth dynamics can be restated in terms of the normalized return process as

$$dW(t) = rW(t)dt + \theta(t)dR(t), \qquad \theta(t) = y(t)\sigma(t). \tag{8}$$

Thus, the investor's portfolio decision can be viewed as full investment of wealth in the risk-free asset plus a monetary position of $\theta(t)$ in the normalized return process. Given the definition of the normalized return process and the risk-premium dynamics, the current risk premium $X(t)$ supplies all currently available information on current and future investment opportunities in combination with the nonstochastic parameters of the model.

Two parameters of the normalized return process prove to be particularly important in determining the mathematical forms and behavior of the solutions: ρ_{mX} and a derived parameter k_* defined by

$$k_* = \left(\frac{\sigma_X}{\lambda_X}\right)^2 + 2\rho_{mX}\left(\frac{\sigma_X}{\lambda_X}\right). \tag{9}$$

The sign of the cross-sectional correlation ρ_{mX} between the risky asset's return and risk premium determines the sign of the intertemporal correlation in normalized asset returns over discrete time periods: negative (positive) ρ_{mX} produces negative (positive) intertemporal correlation in normalized asset returns. To interpret the derived parameter k_*, consider the variance of the normalized return process over the horizon τ, which can be shown to be[3]

$$var\{R(\tau)\} = \tau + \left(\frac{\sigma_X}{\lambda_X}\right)^2 \left(\tau + \frac{2e^{-\lambda_X \tau}}{\lambda_X} - \frac{e^{-2\lambda_X \tau}}{2\lambda_X} - \frac{3}{2\lambda_X}\right) + \left(\frac{2\rho_{mX}\sigma_X}{\lambda_X}\right)\left(\tau + \frac{e^{-\lambda_X \tau}}{\lambda_X} - \frac{1}{\lambda_X}\right). \quad (10)$$

As the horizon becomes zero or infinite, the marginal and average variances with respect to horizon approach each other and are given by

$$\text{Short run:} \quad \lim_{\tau \to 0} \frac{\partial\, var\{R(\tau)\}}{\partial \tau} = \lim_{\tau \to 0} \left(\frac{1}{\tau}\right) var\{R(\tau)\} = 1 \quad (11)$$

$$\text{Long run:} \quad \lim_{\tau \to \infty} \frac{\partial\, var\{R(\tau)\}}{\partial \tau} = \lim_{\tau \to \infty} \left(\frac{1}{\tau}\right) var\{R(\tau)\} = 1 + k_*.$$

With a nonstochastic risk premium, the long-run marginal and average variances are equal to their short-run values of one. Thus, k_* is the contribution of the stochastic risk premium to the long-run marginal and average variances of the normalized return process. The long-run variances are lower (higher) than their short-run values when k_* is negative (positive). Negative correlation ρ_{mX} between the risky asset's return and risk premium is a necessary though not sufficient condition for the decreasing-variance case $k_* < 0$.

The investor's utility for final wealth $U(W)$ is a member of the well-known two-parameter *HARA* family. For this utility-function class, absolute risk tolerance $T(W)$ is by definition a linear function of wealth, and so can be expressed as

$$T(W) = \frac{U_W}{-U_{WW}} = \begin{cases} \gamma(W - W_*) & \text{for } \gamma \neq 0 \\ T_0 > 0 & \text{for } \gamma = 0 \end{cases} \quad (12)$$

Figure 1 shows the members of the *HARA* class with a common $W_* > 0$, normalized so that the appropriate lower and upper bounds are both zero.

The parameter γ is the sensitivity of absolute risk tolerance to wealth. For *HARA* utility functions with increasing absolute risk-tolerance ($\gamma > 0$), the parameter W_* is a minimum level of wealth, whereas for decreasing absolute risk tolerance ($\gamma < 0$) it is a maximum level of wealth; only the exponential utility function ($\gamma = 0$) has no bounds on wealth. *HARA* utility functions are bounded below and unbounded above for $\gamma > 1$, and are bounded above and unbounded below for

[3] See Kim and Omberg (1993) for proofs of this and various other results below.

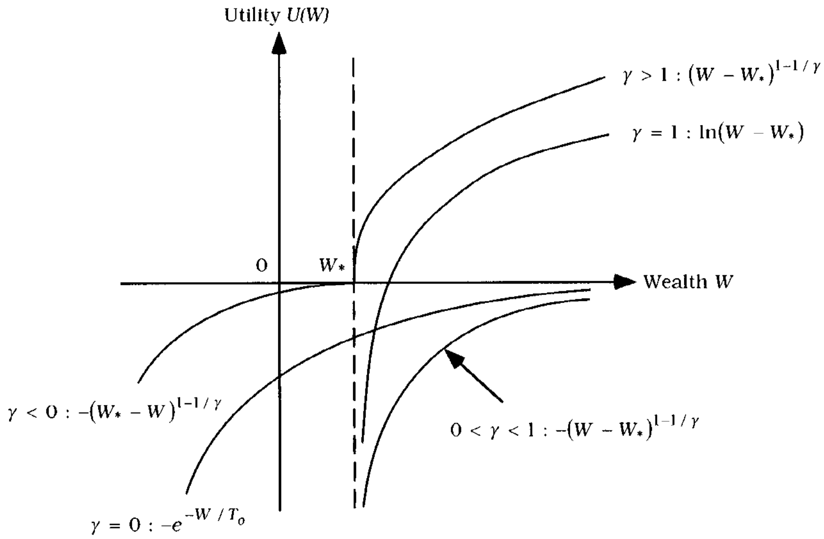

Figure 1
HARA utility functions
For *HARA* utility functions with increasing absolute risk tolerance ($\gamma > 0$), the parameter W_* is a minimum level of wealth, whereas for decreasing absolute risk tolerance ($\gamma < 0$) it is a maximum level of wealth; only the exponential utility function ($\gamma = 0$) has no bounds on wealth. *HARA* utility functions are bounded below and unbounded above for $\gamma > 1$ and bounded above and unbounded below for $\gamma < 1$; only the log utility function ($\gamma = 1$) is unbounded in both directions.

$\gamma < 1$; only the log utility function ($\gamma = 1$) is unbounded in both directions. Simple-power and simple-log utility functions result from $W_* = 0$, and quadratic utility functions and their associated mean-varriance preferences are produced by $\gamma = -1$.

2. The Solutions

Having covered the preliminaries, we now outline the solution method, classify the solutions by mathematical form and behavior, and identify the regions of the parameter space in which they occur.

2.1 The solution method

Let τ be the investor's current horizon, $y_*(W, X, \tau)$ be the optimal monetary investment in the risky asset, and $J(W, X, \tau)$ be the optimal expected utility. The first- and second-order conditions for $y_*(W, X, \tau)$ and $J(W, X, \tau)$ can be derived by specializing Merton's (1973) conditions, or can be independently derived with normal techniques. In either case, the standard dynamic programming conditions result in the optimal investment function,

$$y_*(W, X, \tau) = \left(\frac{J_W}{-J_{WW}}\right)\left(\frac{X(t)}{\sigma(t)}\right) + \left(\frac{J_{WX}}{-J_{WW}}\right)\left(\frac{\rho_{mX}\sigma_X}{\sigma(t)}\right), \quad (13)$$

the partial differential equation,

$$-J_\tau + J_W r W - \left(\frac{1}{2}\right) J_{WW} y_*^2 \sigma^2 - J_X \lambda_X (X - \overline{X}) + \left(\frac{1}{2}\right) J_{XX} \sigma_X^2 = 0, \quad (14)$$

the horizon-end condition,

$$J(W, X, 0) = U(W), \quad (15)$$

and the second-order condition for a maximum $J_{WW} < 0$.

The optimal investment function for the logarithmic case ($\gamma = 1$) is the well-known myopic investment function; for the optimal expected utility function, see Kim and Omberg (1993). For the nonlogarithmic cases, assume trial solutions of the form

$$J(W, X, \tau) = \Phi(X, \tau) U(We^{r\tau}) \quad (16)$$

$$\Phi(X, \tau) = \exp\left[A(\tau) + B(\tau)X + C(\tau)X^2/2\right], \quad (17)$$

$$A(0) = B(0) = C(0) = 0, \quad (18)$$

where $U(W)$ is any positive constant times the utility functions of Figure 1. Solutions of this form automatically satisfy both the horizon-end condition and the second-order condition for a maximum. Substitution of the trial solutions into Equation (13) yields the investment function

$$y_*(W, X, \tau) = T_J(W, \tau)\left(\frac{X(t)}{\sigma(t)} + \frac{\rho_{mX}\sigma_X C(\tau) X + \rho_{mX}\sigma_X B(\tau)}{\sigma(t)}\right), \quad (19)$$

$$T_J(W, \tau) = \frac{J_W}{-J_{WW}} = e^{-r\tau} T(We^{r\tau}) = \begin{cases} \gamma(W - W_* e^{-r\tau}) & \text{for } \gamma \neq 0 \\ T_0 e^{-r\tau} & \text{for } \gamma = 0 \end{cases}$$

Further substitution of the trial solutions into the partial differential of Equation (14) produces a quadratic equation for $X(t)$; its three coefficients must be zero, resulting in the system of first-order nonlinear ordinary differential equations

$$\begin{aligned} \frac{dC}{d\tau} &= cC^2(\tau) + bC(\tau) + a, \quad (20) \\ \frac{dB}{d\tau} &= cB(\tau)C(\tau) + \left(\frac{b}{2}\right) B(\tau) + \lambda_X \overline{X} C(\tau), \end{aligned}$$

$$\frac{dA}{d\tau} = \left(\frac{c}{2}\right) B^2(\tau) + \left(\frac{1}{2}\right) \sigma_X^2 C(\tau) + \lambda_X \overline{X} B(\tau),$$

$$C(0) = B(0) = A(0) = 0,$$

with the parameters

$$\begin{aligned} a &= \gamma - 1, \\ b &= 2[(\gamma - 1)\rho_{mX}\sigma_X - \lambda_X], \\ c &= \sigma_X^2[1 + (\gamma - 1)\rho_{mX}^2]. \end{aligned} \tag{21}$$

Since the first of the differential equations involves only $C(\tau)$, the second $C(\tau)$ and $B(\tau)$, and the third all three functions, the order in which the equations appear above is also the logical order of solution. The first, a Riccati equation with constant coefficients, can be solved by recasting it as the integral equation

$$\int_0^\tau \frac{dC}{cC^2 + bC + a} = \tau. \tag{22}$$

The left-hand side can be determined from standard integral tables, which indicate that the solutions take four mathematical forms, depending on the parameters a, b, and c.[4] Since there are four solutions for the set of three functions $C(\tau)$, $B(\tau)$, and $A(\tau)$, we tabulate them in the Appendix rather than presenting them here.

A key property of the solutions, derivable by phase-plane analysis, is[5]

$$\operatorname{sgn}[C(\tau), C'(\tau), B(\tau), B'(\tau)] = \operatorname{sgn}[\gamma - 1]. \tag{23}$$

Inspection of Equation (19) shows that the traditional myopic investment function results under the appropriate conditions (i.e., when either $\sigma_X = 0$, $\rho_{mX} = 0$, or $\gamma = 1$).

2.2 The mathematical forms of the solutions and their regions

We now classify the solutions by mathematical form, and identify the regions of the parameter space in which they occur. Consider the discriminant

$$q = b^2 - 4ac = 4\lambda_X^2[1 - (\gamma - 1)k_*] \tag{24}$$

[4] For example, see Gradshteyn and Ryzhik (1980, p. 68) for three of the solution forms; the fourth, which we call the polynomial solution, is for the special case $b = c = 0$.

[5] See Kim and Omberg (1993, Appendix D) for the phase-plane analysis.

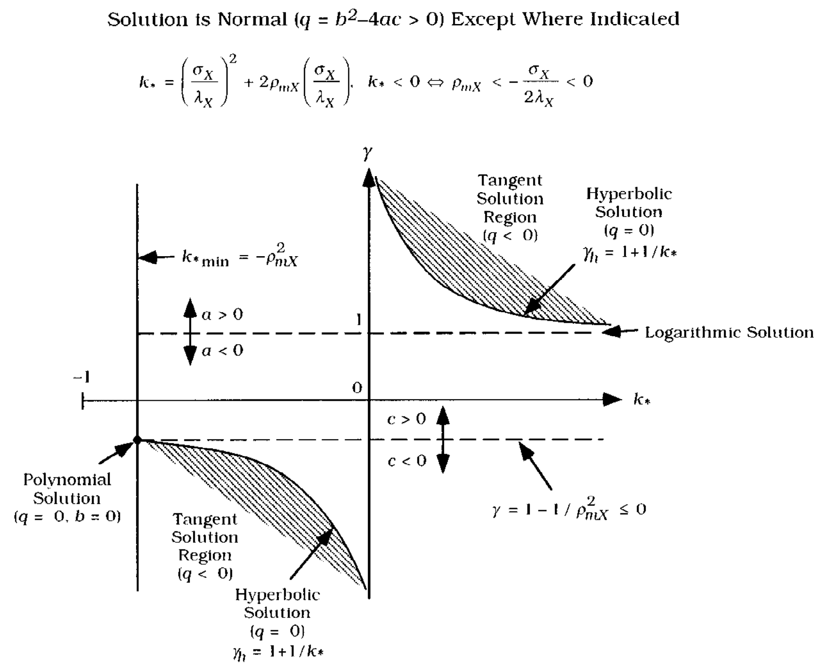

Figure 2
The solution regions
The tangent solution occurs in the extreme upper-right and lower-left regions. The hyperbolic solution occur along the boundaries between the tangent and normal solution regions. The normal solution occurs elsewhere, except for the polynomial solution at one point.

of the quadratic equation

$$\frac{dC}{d\tau} = cC^2(\tau) + bC(\tau) + a = 0. \tag{25}$$

The four solutions are $q > 0$—the **normal solution**; $q = 0$—the **hyperbolic solution** ($b \neq 0$) or the **polynomial solution** ($b = 0$); and $q < 0$—the **tangent solution**. Figure 2 shows where the solutions occur in the two-dimensional parameter space $\{\gamma, k_*\}$.

The *normal solution* is so named because it is the solution everywhere in the parameter space except the extreme regions of the upper-right and lower-left quadrants. The other three solutions are called the *tangent*, *hyperbolic*, and *polynomial solutions* because of the presence of those functions in the solutions. The tangent solution occurs in the extreme regions of the upper-right and lower-left quadrants, and the hyperbolic solution occurs on the two curves separating the normal and tangent solution regions. The polynomial solution oc-

Table 1

Well-behaved regions

Solution form	Additional restrictions
Normal	$a > 0$ or $c > 0$ or $b < 0$
Hyperbolic	$b < 0$
Polynomial	None

Nirvana Regions

Solution form	Additional restrictions
Normal	$a, c < 0$ and $b > 0$ ($\Rightarrow \gamma < 0$)
Hyperbolic	$b > 0$ ($\Rightarrow \gamma < 0$)
Tangent	None

curs only at the left end-point of the lower-left hyperbolic solution curve.

2.3 Well-behaved versus nirvana solutions

The solutions can also be classified as *well-behaved* and *nirvana* solutions. In a *well-behaved solution*, investment in the risky asset $y_*(W, X, \tau)$ and expected utility $J(W, X, \tau)$ are finite whenever their arguments W, X, τ are finite. In addition, for utility functions bounded above ($\gamma < 1$), expected utility $J(W, X, \tau)$ is finitely less than its upper bound whenever its arguments W, X, τ are finite.

A *nirvana solution*, on the other hand, is one for which expected utility $J(W, X, \tau)$ increases so rapidly with the investment horizon that expected utility approaches infinity ($\gamma > 1$ investors) or its upper bound ($\gamma < 1$ investors) at a finite critical horizon τ_c, which depends on the solution's parameters. Except for nirvana solutions with $\rho_{mX} = 0$, where the myopic investment function is optimal, nirvana solutions also have investment functions approaching infinity at the critical horizon. The nirvana solutions are not defined for longer horizons than the critical horizon, understandably so since an investor with a longer horizon can pursue any policy up to the critical horizon and attain nirvana by commencing the optimal policy at that time.

Figure 3 shows the difference between a myopic policy, a well-behaved nonmyopic solution, and a nirvana nonmyopic solution for investors with utility unbounded above ($\gamma > 1$), positive correlation between the risky asset's risk premium and return ($\rho_{mX} > 0$), and zero required wealth ($W_* = 0$), conditions that eliminate horizon effects for the myopic policy, allow both well-behaved and nirvana solutions, and produce positive horizon effects for both.

Whether a solution is well-behaved or a nirvana solution depends on both its mathematical form and additional parameter restrictions, as shown in Table 1.

For nondecreasing absolute risk-tolerance ($\gamma \geq 0$), the regions for

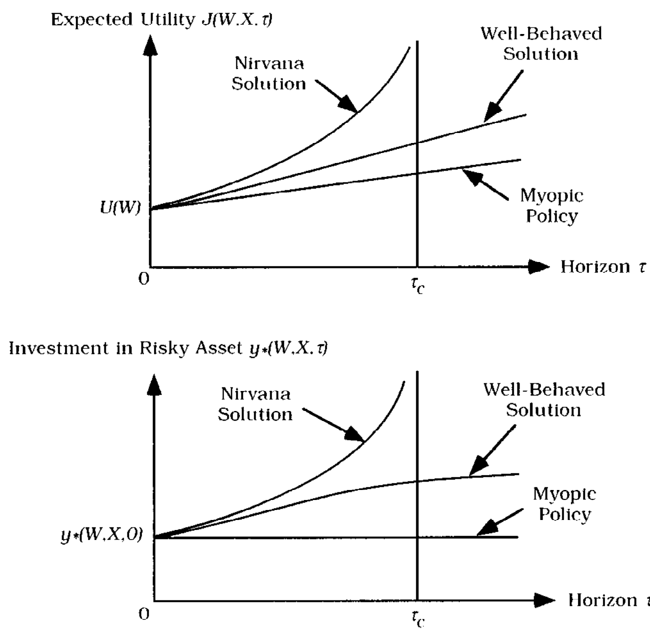

Figure 3
Behavior of myopic, well-behaved, and nirvana solutions for $\gamma > 1$, $W_* = 0$, $\rho_{mX} > 0$
For nirvana solutions, expected utility increases so rapidly with horizon that nirvana is attained at a finite horizon. For $\gamma > 1$, nirvana is infinite expected utility; for $\rho_{mX} \neq 0$, the investment function also explodes at the critical horizon.

which well-behaved and nirvana solutions occur are simple to identify on Figure 2: examining only the upper-half of the diagram where $\gamma \geq 0$, the only nirvana solutions are the tangent solutions in the upper-right quadrant, which require the average and marginal variance of the normalized return process to increase with horizon ($k_* > 0$) and $\gamma > \gamma_b = 1 + 1/k_*$.[6]

Real investors neither attain nirvana nor take infinite positions. The implications of the nirvana solutions are therefore either that (1) the parameter combinations producing nirvana solutions do not occur in the real world, or (2) the real world has costs or constraints not found in the model that eliminate the nirvana solutions.[7] It is interesting

[6] The logarithmic solution is never a nirvana solution. Note however that optimal myopic solutions with $\gamma \neq 1$, $\rho_{mX} = 0$, $\sigma_X > 0$ can be. In addition, myopic solutions can be nirvana solutions when they are suboptimal, though more extreme parameter combinations are required. See Kim and Omberg (1993), which also discusses the intuition of the nirvana solutions.

[7] The introduction of "position costs," specified in such a way that the analytical solutions are preserved, shrinks but does not eliminate the nirvana solution regions [see Kim and Omberg (1993)].

to note that nirvana solutions for decreasing absolute risk tolerance ($\gamma < 0$) are Harrison-Kreps (1979) arbitrage strategies, since initial wealth of zero ($W_0 = 0$) is a feasible starting point and the maximum wealth level $W_* > 0$ is achieved with probability one as the critical horizon is approached. They appear to be the first Harrison-Kreps arbitrage strategies to be identified that are different from the doubling strategies and are derived as solutions of the dynamic programming conditions for optimal trading strategies.[8]

Dybvig and Huang (1988) showed that all arbitrage strategies can be eliminated by a lower bound on wealth. However, the nirvana solutions with increasing absolute risk tolerance ($\gamma > 1 > 0$) are not arbitrage strategies and already have a natural lower bound on wealth $W_* > 0$. By Jensen's inequality they produce infinite expected wealth in a finite period of time. Thus, imposing the condition that admissible trading strategies cannot generate infinite expected wealth in finite horizons would eliminate them without disturbing the well-behaved solutions.

We note a concern of the stochastic optimal control literature that the dynamic programming conditions, also known as the *Hamilton-Jacobi-Bellman (HJB) conditions*, may have more than one solution for a single set of parameters, with some that may not be optimal solutions to the model. For some models, the literature provides *verification theorems* that enable a given *HJB* solution to be checked to ensure it is optimal. However, we are not acquainted with any verification theorem that fits the model above, despite its relative simplicity. For example, the two verification theorems in Fleming and Rishel (1975) are both for models that restrict the final values of the stochastic state variables, which in a portfolio context include investor wealth, to a closed set. But in a portfolio model with utility defined on final wealth, final wealth is not restricted to a closed set and the Fleming-Rishel verification theorems cannot be applied. However, reverse-time, finite-difference solutions to our model, which start with the unique final-value condition $J(W, X, \tau) = U(W)$, show no signs of generating multiple solutions for a single set of parameters, provided horizons are restricted to less than critical horizons for the nirvana solutions, suggesting that any solution that is attainable as the limit of a finite-difference solution is unique.

3. Dynamic Nonmyopic Portfolio Behavior

We now turn to a number of fundamental questions regarding nonmyopic investment behavior: (1) when an investor holds more or less

[8] See Omberg (1989) for doubts that doubling strategies are expected-utility maximizing.

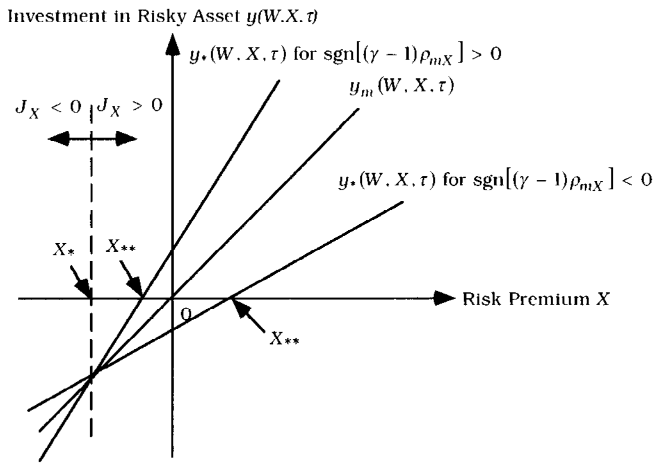

Figure 4
Nonmyopic and myopic investment in risky asset versus risk premium
Investors with $\gamma > 1$ always speculate on the risk-premium uncertainty, while investors with $\gamma < 1$ always hedge against it. However, whether this is implemented as holding more or less than the myopic amount of the risky asset depends on the correlation ρ_{mX} between the risky asset's return and risk premium, and on whether a higher or lower risk premium is preferred, with $J_X > 0$ (preference for a higher risk premium) being the normal case.

than the myopic amount of the risky asset; (2) whether an investor hedges against or speculates on risk-premium uncertainty; (3) when the investor is long or short the risky asset; and (4) whether an investor holds more or less of the risky asset at longer horizons.

From inspection of the nonmyopic investment function, Equation (19), investment in the risky asset $y_*(W, X, \tau)$ is a linear function of the current risk premium $X(t)$ with coefficients that depend on the horizon τ and the current volatility $\sigma(t)$. Figure 4, which shows the relationship for a myopic investor and two hypothetical nonmyopic investors, will serve as a basis for discussion.

There is a horizon-dependent value of the risk premium $X_*(\tau) = -B(\tau)/C(\tau)$ at which nonmyopic and myopic investment in the risky asset are equal. (For simplicity, Figure 4 shows the same $X_*(\tau)$ for both nonmyopic investors, but it is in general investor specific.) From Equation (23), $X_*(\tau)$ is always negative; further, it can be shown that the marginal utility $J_X(W, X, \tau)$ of the risk premium is positive (negative) at higher (lower) risk premia than $X_*(\tau)$:

$$X_*(\tau) = -\frac{B(\tau)}{C(\tau)} < 0, \;\; \mathrm{sgn}[J_X(W, X, \tau)] = \mathrm{sgn}[X(t) - X_*(\tau)]. \quad (26)$$

The linear relationship between $y_*(W, X, \tau)$ and $X(t)$ has a positive slope, except for one of the nirvana solutions.[9] The slope is greater than (less than) that of the myopic investment function $y_m(W, X, \tau)$ for sgn$[(\gamma - 1)\rho_{mX}]$ positive (negative).

We first explain why investors prefer higher risk premia for $X(t) > X_*(\tau)$, but lower risk premia when $X(t) < X_*(\tau) < 0$. Since investors can be either short or long the risky asset, it is the magnitude of the risk premium rather than its sign that determines its *current* attractiveness. But the nonmyopic investor also considers future risk premia and an asymmetry in the dynamics of the risk-premium process: the reversion to a positive normal risk premium $\overline{X}$. With negative risk premia, the expected movement to $\overline{X}$ passes through zero, where positions in the risky asset are least attractive, whereas with positive risk premia it does not. Thus, the investor normally (i.e., when $X(t) > X_*(\tau) < 0$) has a preference for higher risk premia. However, with sufficiently low risk premia still lower values are preferred because they increase the time spent away from the region around zero risk premia.

We now have sufficient information to consider substantive issues of behavior. Define *hedging against risk-premium uncertainty* to be a departure from myopic behavior that tends to increase (decrease) wealth when the risk premium becomes less (more) favorable. Similarly, define *speculating on risk-premium uncertainty* as a departure that tends to increase (decrease) wealth when the risk premium becomes more (less) favorable.

For simplicity, consider the region $X > X_*$ where $J_X > 0$, that is, higher risk premia are preferred. If $\rho_{mX} < 0$, holding more of the risky asset is hedging, since it tends to combine higher wealth with lower risk-premium states and vice versa. Thus, inspection of Figure 4 shows that $\gamma < 1$ investors are hedging by holding more of the risky asset than the myopic amount, while $\gamma > 1$ investors are speculating by holding less.

Consideration of all other cases ($\rho_{mX} > 0$ and/or $X < X_*$) yields the same result: $\gamma < 1$ investors always hedge against risk-premium uncertainty, while $\gamma > 1$ investors always speculate on it. Whether they hold more or less than the myopic amount of the risky asset is simply an implementation detail that depends on the sign of the correlation between the risky asset's return and risk premium, and their current preference for higher or lower risk premia. Table 2 summarizes the results for whether the investor holds more or less than the myopic amount of the risky asset (**M/L**) and hedges or speculates (**H/S**).

The results are helpful in understanding why logarithmic investors

[9] This is the tangent solution at longer horizons with $\gamma > 1$ and $\rho_{mX} < 0$. See Kim and Omberg (1993) for the intuition of this strange case.

Table 2
Summary of Results for M/L and H/S

$\gamma - 1$	ρ_{mX}	$X - X_*$	**M/L**	**H/S**
+	+	+	M	S
+	−	+	L	S
+	+	−	L	S
+	−	−	M	S
−	+	+	L	H
−	−	+	M	H
−	+	−	M	H
−	−	−	L	H

($\gamma = 1$) are myopic: they lie on the boundary between investors who always hedge risk premium uncertainty ($\gamma < 1$) and investors who always speculate on it ($\gamma > 1$), and therefore do neither.

The nearest parallel to these results in the literature is found in Breeden (1984), who considered an investor who, in addition to holding the risk-free asset and the market portfolio, takes positions in a hedging portfolio consisting of futures contracts. Breeden effectively assumed $W_* = 0$ by explicitly ignoring relative wealth effects, and referred to speculating as "reverse hedging." However, with $W_* \neq 0$, as in our model, γ cannot be equated with either absolute or relative risk tolerance; it is simply the sensitivity of absolute risk tolerance to wealth. Thus, one cannot say that investors who are more risk tolerant than log investors speculate while investors who are less risk tolerant hedge; the effects are more subtle.[10]

Next consider whether investors are long or short the risky asset. Figure 4 shows that there is an $X_{**}(\tau) > X_*(\tau)$ at which the position in the risky asset is zero, with the investor being long (short) at higher (lower) risk premia. The critical risk premium and its sign are given by

$$\begin{aligned} X_{**}(\tau) &= \frac{-\rho_{mX}\sigma_X B(\tau)}{1 + \rho_{mX}\sigma_X C(\tau)} > X_*(\tau), \\ \operatorname{sgn}[X_{**}(\tau)] &= -\operatorname{sgn}[(\gamma - 1)\rho_{mX}]. \end{aligned} \tag{27}$$

Thus, it is possible for some investors to be long the risky asset while others with the same beliefs, but different preferences, are short.

[10] Note that all *HARA*-class investors are *unconditional* risk-premium hedgers and speculators: whether they hedge, speculate, or do neither depends entirely on their preferences (i.e., is independent of their current wealth, their current horizon, and the current risk premium). See Kim and Omberg (1993) for a reasoned conjecture that unconditional behavior does not extend beyond the *HARA* class.

Finally, we turn to horizon effects, assuming increasing absolute risk tolerance ($\gamma > 0$), nonlogarithmic utility ($\gamma \neq 1$), a nonnegative required level of wealth ($W_* \geq 0$) with a positive risk premium, and the investor holding a long position in the risky asset. First, consider the horizon effects for a myopic investor whose investment function is

$$y_m(W, X, \tau) = \gamma \left(W - W_* e^{-r\tau} \right) \frac{X(t)}{\sigma(t)}. \tag{28}$$

The myopic investor's horizon effect, abstracting from any drift in $X(t)$ or $\sigma(t)$, is given by

$$\frac{\partial y_m}{\partial \tau} = \gamma r W_* e^{-r\tau} \frac{X(t)}{\sigma(t)}. \tag{29}$$

Thus, there are no myopic horizon effects for $r = 0$ or $W_* = 0$. With $r > 0$ and $W_* > 0$, there is a positive horizon effect (i.e., the myopic investor holds more of the risky asset at longer horizons). Presumably this results from the investment strategy requiring less investment in the risk-free asset to assure the minimum level of wealth W_* at the horizon's end when the horizon is longer. For convenience, we will refer to this as the *insurance horizon effect.*

For a nonmyopic investor, an additional horizon effect is produced by holding more or less than the myopic amount of the risky asset. Since the investor holds the myopic amount at the horizon's end, holding more than the myopic amount at longer horizons reinforces the positive insurance horizon effect, while holding less opposes it. We term this second influence the *H/S horizon effect*, since its ultimate source is the nonmyopic investor's desire to hedge against or speculate on the risk premium uncertainty by departing from the myopic position in the risky asset.

From the nonmyopic investment function, Equation (19), the total horizon effect is given by

$$\frac{\partial y_*(W, X, \tau)}{\partial \tau} = \frac{\partial T_J(W, \tau)}{\partial \tau} f(X, \tau) + T_J(W, \tau) \frac{\partial f(X, \tau)}{\partial \tau}, \tag{30}$$

where $f(X, \tau)$ is the investment in the risky asset per unit of risk tolerance $T_J(W, \tau)$. The first term on the right-hand side is the insurance effect, the second the H/S effect.

For $r = 0$ or $W_* = 0$, the insurance effect vanishes, and the investor's holding of the risky asset increases with horizon when he holds more than the myopic amount and vice versa.[11] For $r > 0$, $W_* > 0$, the insurance and H/S effects are both present, with the insurance effect being positive and the H/S effect having the same sign as $(\gamma - 1)\rho_{mX}$.

Consequently, when $(\gamma - 1)\rho_{mX} > 0$, the net horizon effect is always positive, in agreement with the conventional wisdom that long-horizon investors should hold more of the risky asset than their short-horizon equivalents. But for $(\gamma - 1)\rho_{mX} < 0$, the insurance effect is positive and the H/S effect is negative, with the net horizon effect depending on which is larger. The magnitude of the insurance effect is independent of wealth, but the magnitude of the H/S effect is proportional to absolute risk tolerance $T_J = \gamma(W - W_* e^{-r\tau})$, so that it increases with the investor's level of wealth. It follows that for lower-wealth investors the insurance effect dominates and the net horizon effect is positive, consistent with the conventional wisdom, with the reverse being true for higher-wealth investors.

4. Summary and Conclusions

The results above indicate that even a minimal two-asset model of nonmyopic behavior, simple enough to have an analytical solution, produces investment strategies of considerable complexity and variety. Some investors hold more of the risky asset than the myopic amount, while others hold less; some hedge against risk-premium uncertainty, while others speculate; some hold less of the risky asset as their horizon decreases, while others hold more; some investors can be long the risky asset, while others with the same beliefs are short. There are, however, some limited generalizations: within the class of *HARA* investors, all nonlogarithmic investors are always risk-premium hedgers or always risk-premium speculators, depending on their preferences; but whether they implement this behavior by holding more or less of the risky asset depends on other details, such as the correlation between the risky asset's risk premium and return and the current risk premium.

The variety and complexity of nonmyopic investment behavior should not be surprising. When myopic conditions are removed, investors think and act strategically, fully exploiting the statistical relationships they perceive between current and future opportunities, and

[11] These horizon effects are in agreement with the ones inferred by Samuelson (1988) from a series of discrete-time examples with $r = 0$, $W_* = 0$.

in a way highly adapted to their particular preferences. It should be expected that more fully featured models, such as ones with multiple risky assets and a stochastic riskless rate, will produce an even richer set of investment behaviors.

Appendix

The four solutions for $C(\tau)$, $B(\tau)$, and $A(\tau)$

The normal solution. The condition for normal solutions is

$$q = b^2 - 4ac = 4\lambda_X^2 \left[1 - (\gamma - 1)k_*\right] > 0$$

$$\eta = \sqrt{q}$$

$$\begin{aligned}
C(\tau) &= \frac{2(\gamma-1)\left(1 - e^{-\eta\tau}\right)}{2\eta - (b+\eta)\left(1 - e^{-\eta\tau}\right)}, \\
B(\tau) &= \frac{4(\gamma-1)\lambda_X \overline{X}\left(1 - e^{-\eta\tau/2}\right)^2}{\eta\left[2\eta - (b+\eta)\left(1 - e^{-\eta\tau}\right)\right]}, \\
A(\tau) &= (\gamma - 1)\left(\frac{2\lambda_X^2\overline{X}^2}{\eta^2} + \frac{\sigma_X^2}{\eta - b}\right)\tau \\
&\quad + \frac{4(\gamma-1)\lambda_X^2\overline{X}^2\left[(2b+\eta)e^{-\eta\tau} - 4be^{-\eta\tau/2} + 2b - \eta\right]}{\eta^3\left[2\eta - (b+\eta)\left(1 - e^{-\eta\tau}\right)\right]} \\
&\quad + \frac{2(\gamma-1)\sigma_X^2}{\eta^2 - b^2}\ln\left|\frac{2\eta - (b+\eta)\left(1 - e^{-\eta\tau}\right)}{2\eta}\right|.
\end{aligned}$$

The condition for well-behaved normal solutions is

$$a > 0 \text{ or } c > 0 \text{ or } b < 0.$$

The condition for nirvana normal solutions and the critical horizon is

$$a, c < 0 \text{ and } b > 0 \left(\Rightarrow \gamma < 1 - 1/\rho_{mX}^2 \le 0\right),\ \tau \in [0, \tau_c),$$

$$\tau_c = \frac{1}{\eta}\ln\left(\frac{b+\eta}{b-\eta}\right).$$

4.0.1 The hyperbolic solution. The condition for hyperbolic solutions is

$$q = b^2 - 4ac = 4\lambda_X^2[1 - (\gamma - 1)k_*] = 0 \text{ with } b \neq 0$$

$$
\begin{aligned}
C(\tau) &= \frac{-1}{c\left(\tau - \frac{2}{b}\right)} - \frac{b}{2c}, \\
B(\tau) &= -\frac{\lambda_X \overline{X}}{bc\left(\tau - \frac{2}{b}\right)} - \frac{b\lambda_X \overline{X}\left(\tau + \frac{2}{b}\right)}{4c}, \\
A(\tau) &= -\frac{b\sigma_X^2 \tau}{4c} + \frac{\lambda_X^2 \overline{X}^2 b^2 \left(\tau - \frac{8}{b}\right)\tau^3}{96c\left(\tau - \frac{2}{b}\right)} - \frac{\sigma_X^2 \ln\left[\frac{1}{2}\left|b\left(\tau - \frac{2}{b}\right)\right|\right]}{2c}.
\end{aligned}
$$

The condition for well-behaved hyperbolic solutions is

$$b < 0.$$

The condition for nirvana hyperbolic solutions and the critical horizon is

$$b > 0, \quad \tau \in [0, \tau_c), \quad \tau_c = 2/b.$$

4.0.2 The polynomial solution. The condition for polynomial solutions is

$$q = b^2 - 4ac = 4\lambda_X^2[1 - (\gamma - 1)k_*] = 0 \text{ with } b = 0$$

$$\left(\Rightarrow c = 0 \Rightarrow \gamma = 1 - 1/\rho_{mX}^2 \leq 0\right)$$

$$
\begin{aligned}
C(\tau) &= (\gamma - 1)\tau \\
B(\tau) &= \left(\frac{\gamma - 1}{2}\right)\lambda_X \overline{X}\tau^2 \\
A(\tau) &= \left(\frac{\gamma - 1}{4}\right)\sigma_X^2 \tau^2 + \left(\frac{\gamma - 1}{6}\right)\lambda_X^2 \overline{X}^2 \tau^3.
\end{aligned}
$$

All polynomial solutions are well behaved.

4.0.3 The tangent solution. The condition for tangent solutions is

$$q = b^2 - 4ac = 4\lambda_X^2[1 - (\gamma - 1)k_*] < 0$$

$$
\begin{aligned}
\eta &= \sqrt{-q}, \\
\omega &= \frac{\eta}{2}, \qquad \varphi = \tan^{-1}\left(\frac{b}{\eta}\right),
\end{aligned}
$$

$$
\begin{aligned}
C(\tau) &= \frac{\eta}{2c}\tan[\omega\tau + \varphi] - \frac{b}{2c}, \\
B(\tau) &= \frac{\lambda_X \overline{X}}{c}\{-1 - \tan[\varphi]\tan[\omega\tau + \varphi] + \sec[\varphi]\sec[\omega\tau + \varphi]\},
\end{aligned}
$$

$$\begin{aligned}
A(\tau) \;=\; & \frac{2\lambda_X^2 \overline{X}^2 b\sqrt{b^2+\eta^2}}{c\eta^3}\{\sec[\varphi] - \sec[\omega\tau+\varphi]\} \\
& + \frac{\lambda_X^2 \overline{X}^2 (2b^2+\eta^2)}{c\eta^3}\{\tan[\omega\tau+\varphi] - \tan[\varphi]\} \\
& - \frac{2(b^2+\eta^2)\lambda_X^2 \overline{X}^2 + b\eta^2\sigma_X^2}{4c\eta^2}\tau - \frac{\sigma_X^2}{2c}\ln[\sec[\varphi]\cos[\omega\tau+\varphi]].
\end{aligned}$$

All tangent solutions are nirvana solutions, thus

$$\tau \in [0, \tau_c), \qquad \tau_c = \frac{\pi}{\eta} - \frac{2}{\eta}\operatorname{Tan}^{-1}\left(\frac{b}{\eta}\right).$$

References

Breeden, D., 1979, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics*, 7, 265–296.

Breeden, D., 1984, "Futures Markets and Commodity Options: Hedging and Optimality in Incomplete Markets," *Journal of Economic Theory*, 32, 275–300.

Dybvig, P., and C. Huang, 1988, "Non-Negative Wealth, Absence of Arbitrage, and Feasible Consumption Plans," *Review of Financial Studies*, 1, 377–402.

Fama, E., 1970, "Multiperiod Consumption-Investment Decisions," *American Economic Review*, 60, 163–174.

Fleming, W. H., and R. W. Rishel, 1975, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York.

Gradshteyn, I. S., and I. M. Ryzhik, 1980, *Table of Integrals, Series, and Products* (4th ed.), Academic Press, New York.

Hakansson, N., 1970, "Optimal Investment and Consumption Strategies under Risk for a Class of Utility Functions," *Econometrica*, 38, 587–607.

Harrison, J. M., and D. Kreps, 1979, Martingales and Arbitrage in Multi-Period Securities Markets, *Journal of Economic Theory*, 20, 381–408.

Ingersoll, J., Jr., 1987, *Theory of Financial Decision Making*, Rowman & Littlefield, Totowa, N.J.

Kim, T., and E. Omberg, 1993, "Dynamic Non-Myopic Portfolio Behavior," Working paper, San Diego State University.

Merton, R., 1969, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, 51, 247–257.

Merton, R., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3, 373–413.

Merton, R., 1973, "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41, 867–887.

Omberg, E., 1989, "The Expected Utility of the Doubling Strategy," *Journal of Finance*, 44, 515–524.

Samuelson, P., 1969, "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–246.

Samuelson, P., 1988, "Long-Run Risk Tolerance When Equity Returns Are Mean Regressing: Pseudoparadoxes and Vindication of 'Business Man's Risk," *Yale Symposium in Honor of James Tobin.*

Samuelson, P., 1989, "The Judgment of Economic Science on Rational Portfolio Management: Indexing, Timing, and Long-Horizon Effects," *Journal of Portfolio Management*, Fall, 4–12.