Information, Trade, and Derivative Securities

Michael J. Brennan
University of California, Los Angeles,
and London Business School

H. Henry Cao
University of California, Los Angeles

Hellwig's (1980) model is used to analyze the value of improving trading opportunities by more frequent trading in the underlying asset, or by trading in a derivative asset. With multiple trading sessions, uninformed investors behave as rational trend followers, while more informed investors follow a contrarian strategy. As trading becomes continuous, Pareto efficiency is achieved. With trading in an appropriate derivative security, Pareto efficiency may be achieved in only a single round of trading. All derivative claims are then priced on Black and Scholes (1973) principles and, in the absence of further supply shocks, no trading will take place in subsequent trading rounds.

It is common for securities markets to develop initially as periodic call markets and only later to offer the opportunity for continuous trading through the day.[1] Most recently, we have witnessed the development of

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[1] Trading frequency still differs considerably across security markets. Ghana's bourse trades for only 2 days a week and Morocco's for only $1\frac{1}{2}$ hours a day. *Economist*, June 11, 1994.

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after-hours and even round-the-clock trading in some securities, as well as the concomitant introduction of an extensive array of derivative assets. Since the maintenance of markets is costly, it is useful to analyze the magnitude of the benefits conferred by the introduction of new markets, either by the more frequent opening of existing markets, or by the opening of new markets in derivative securities. In this article we address these issues in the context of a noisy rational expectations model of the type of Hellwig (1980), by assuming that knowledge about final asset payoffs emerges gradually over time and then considering the effect of allowing investors to trade more frequently. The welfare gain for an individual investor from more frequent trading is shown to be an increasing function of the absolute value of the difference between the precision of his prior information and the risk tolerance weighted average precision of all investors; it is also (and equivalently) an increasing function of the volume of trade of the investor.

We also demonstrate that, for a given number of market sessions, welfare is maximized when information is released in such a way that the variance of price change between successive market sessions is constant over time. This result provides a social justification for disclosure laws that require firms to disclose material new information immediately, rather than saving the disclosure for the periodic financial statement dates.[2] The model also casts light on trading behavior in markets with disparately informed agents.[3] We show that poorly informed agents tend to be "trend-chasers," purchasing more of the risky security when its price rises and selling when its price falls, while better informed agents act as contrarians, selling on a price rise and buying on a fall.[4] Some authors have suggested that the observed increase in purchases of mutual funds when the market has risen is evidence of irrational "barn door closing behavior" [Patel, Zeckhauser, and Hendricks (1991)] — our model suggests rather that such behavior is rational for poorly informed investors who are the ones most likely to purchase mutual funds.[5]

We also show that, for a given number of market sessions, the total

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[2] Diamond (1985) provides an alternative explanation for why firms wish to release information early.

[3] Previous authors have used the noisy rational expectations model to analyze the effect of public information announcements on the volume of trade. [see, e.g., Grundy and McNicholls (1989), He and Wang (1993), and Kim and Verrecchia (1991a, b)].

[4] This is similar to the result of DeLong et al. (1990) although their result depends on nonrational beliefs on the part of investors, while ours requires only an assumption of information asymmetry.

[5] The behavior of poorly informed investors in our model should be distinguished from that of the positive feedback traders in DeLong et al. (1990) whose demand for the risky security depends on its lagged price change.

volume of trade is maximized when the arrival of new information is smooth, in the sense that the variance of price changes between successive market sessions is constant over time: the volume of trade is proportional to the square root of the number of market openings when the number is large, and approaches infinity as the limit of continuous trading is reached. Thus, the model is able to account for the high volume of trading that is observed in extant securities markets in terms of the portfolio adjustments to new information of investors endowed with information of differential precision:[6] only an infinitesimal level of supply noise is necessary to generate an infinite volume of trading if the market is open continuously and information arrives smoothly.

When the number of market openings increases without limit, the limiting payoff allocation is Pareto efficient,[7] provided that the information flow satisfies a technical smoothness condition. The intuition for this result is that the price process approaches a diffusion, so that the market becomes dynamically complete. In this model in which investors have exponential utility and returns are normally distributed, the Pareto efficient allocation is such that the payoffs of individual investors are quadratic functions of the aggregate payoff — the better informed investors choose trading strategies that yield payoff functions that are concave, while the less well informed synthesize convex payoff functions. The former are analogous to the payoff on a "covered call" strategy, while the latter are similar to a "protective put" strategy. Such strategies have been shown to be optimal for investors whose risk tolerance increases with wealth more or less rapidly than the average, or who have average risk tolerance but are more or less optimistic than the average about the expected return on the security [see Brennan and Solanki (1981) and Leland (1980)]. Our analysis extends the justification of portfolio insurance like payoffs to include differences in the precision of private information when investors have constant (but different) risk tolerance; in interesting contrast to Leland (1980), whether an investor chooses to synthesize a convex or a concave payoff function is independent of the degree of relative optimism, depending only on the precision of the private information signal. The Pareto efficiency of the limiting economy implies that in this economy the prices of derivative claims on the underlying asset satisfy the risk neutral pricing property of the Black and Scholes (1973) model, and are priced using a market consensus volatility.

[6] The total volume of trading depends not only on the aggregate supply noise but also on the dispersion of the precisions of the investors' private information.

[7] See also Duffie and Huang (1985) for an analysis of how multiplicity of trading opportunities can substitue for numbers of securities in achieving Pareto efficiency.

Since the Pareto efficient allocation in this model is a quadratic function of the aggregate payoff on the risky security, it can be achieved by the introduction of a derivative claim whose payoff is equal to the square of the difference between the initial security price and its final payoff. The introduction of this derivative claim has no effect on the equilibrium price of the risky security.[8] However, since the resulting equilibrium is Pareto efficient, the no-trade result of Milgrom and Stokey (1982) applies, and no further trade takes place as new public information about the security payoff becomes available. Thus, in this case, the opening of a single derivative security market makes redundant any market opening after time 0, and is a complete substitute for a continuous market in the underlying risky asset.[9]

This article is organized as follows. In Section 1 the basic single trading period model of Hellwig (1980) is presented for reference. Section 2 analyzes the welfare effects of increasing the number of trading sessions, and discusses trading strategies and the volume of trade. Section 3 shows that as the number of trading sessions increases, the payoff allocation approaches Pareto efficiency, subject to a smoothness condition on the public information flow. Section 4 shows that the Pareto efficient allocation can be achieved with a single round of trading if a particular derivative asset is introduced. Section 5 relaxes the assumption that there is only a single supply shock at time 0 and allows for the effects of supply shocks at each trading session. Section 6 concludes the article.

1. The Basic Single-Period Model

Following Hellwig (1980), the market is assumed to consist of a continuum of investors, each indexed by i where $i \in [0, 1]$. At time 0 each investor is endowed with x_i units of the risky asset and, to avoid unnecessary notation, we assume that individual endowments of the riskless asset are zero; without loss of generality, the riskless interest rate is taken as zero. The risky asset pays off at time 1 an amount $\tilde{u}$, where $\tilde{u}$ is normally distributed with mean $\bar{u}$ and precision h. The per capita supply of risky assets, $\tilde{x}$, is normally distributed with mean 0 and precision p. Before trading at time 0 each investor receives a

[8] The reason for this is that exponential utility prices are the same as if there existed a single representative investor with characteristics independent of the initial wealth distribution [see Breenan and Kraus (1978) and Rubinstein (1974)]. Since the representative investor does not trade, the prices she supports are independent of the available trading opportunities and securities.

[9] If there are additional supply shocks after time 0, then there will still be trade when the derivative security is introduced.

private signal about the risky asset payoff,

$$\tilde{z}_i = \tilde{u} + \tilde{\epsilon}_i, \tag{1}$$

where $\tilde{\epsilon}_i$ is normally and independently distributed with mean 0 and precision s_i.[10] Investor i has a negative exponential utility function defined over time 1 wealth, $U(w_i) = -\exp(-w_i/r_i)$, where the risk tolerance r_i is assumed to be uniformly bounded.

Let P_0 be the price of the risky asset, D_{0i} be the demand of investor i for the risky asset, $\tilde{F}_{0i}$ denote the information set of investor i, and $\tilde{F}_0$ denote the public information set, including the price at time 0. Investor i's time 1 wealth is given by $w_i = P_0 x_i + (\tilde{u} - P_0)\tilde{D}_{0i}$. It is well known that in this setting [Hellwig (1980)] a linear equilibrium exists in which

$$P_0 = K_0^{-1}[(K_0 - s)\tilde{\mu}_0 + s\tilde{u} - \tilde{x}/r], \tag{2}$$

$$\tilde{D}_{0i} = r_i K_{0i}[\tilde{\mu}_{0i} - P_0] = r_i[s_i\tilde{z}_i - s\tilde{u} + \tilde{x}/r - (s_i - s)P_0], \tag{3}$$

where

$$\tilde{\mu}_0 \equiv E(\tilde{u} \mid \tilde{F}_0) = \frac{h\bar{u} + r^2 s^2 p\tilde{q}}{h + r^2 s^2 p},$$

$$\tilde{\mu}_{0i} \equiv E(\tilde{u} \mid \tilde{F}_{0i}) = \frac{(K_0 - s)\tilde{\mu}_0 + s_i\tilde{z}_i}{K_{0i}} = \frac{h\bar{u} + s_i\tilde{z}_i + r^2 s^2 p\tilde{q}}{h + s_i + r^2 s^2 p},$$

$$\tilde{q} = \tilde{u} - \tilde{x}/rs,$$

$$r \equiv \int_0^1 r_i di,\ s \equiv \frac{1}{r}\int_0^1 r_i s_i di,$$

$$K_{0i} \equiv \mathrm{var}^{-1}(\tilde{u} \mid \tilde{F}_{0i}) = h + s_i + r^2 s^2 p,$$

$$K_0 \equiv \int_0^1 \frac{r_i}{r} K_{0i} di = h + s + r^2 s^2 p.$$

Equation (2) expresses the equilibrium price as a precision weighted average of $\tilde{\mu}_0$, the expected payoff conditional on all public information, and $\tilde{u}$, the average value of the private information signals, less a risk premium that depends on the realized per capita asset supply,

[10] Note that we are assuming that investor signal errors are independent. If signal errors are correlated, the equilibrium is qualitatively similar to that described below, except that the coefficients of the linear pricing function in Equation (2) must be derived from the roots of a cubic equation [see Grundy and McNicholls (1989)]; the simplifying assumption made here permits direct calculation of these coefficients.

$\tilde{x}$. The expected utility of investor i conditional on his endowment, but before receipt of the private information signal, is

$$EU_i = \frac{-1}{\sqrt{L_0 K_{0i}}} \exp\left[\frac{-\bar{u}x_i}{r_i} + \left(\frac{1}{h} - \frac{1}{K_0^2 L_0}\right)\frac{x_i^2}{2r_i^2}\right] \tag{4}$$

where

$$L_0 \equiv \mathrm{var}(\tilde{u} - \tilde{P}_0) = K_0^{-2}(K_0 + s + r^{-2}p^{-1}).$$

Note that expected utility is decreasing in s: since the informativeness of the price signal, $\tilde{q}$, is increasing in the (weighted) average signal precision, s, this corresponds to the Hirshleifer (1972) finding that an increase in public information can have adverse welfare consequences by reducing risk sharing opportunities. Investor i's wealth at time 1, w_i, is a linear function of the risky asset payoff, $\bar{u}$, and may be written as

$$w_i(\tilde{u}) = P_0 x_i + r_i(\tilde{u} - P_0)\left[s_i(\tilde{z}_i - P_0) + \frac{h(\bar{u} - P_0)}{1 + r^2 sp}\right]. \tag{5}$$

The marginal rate of substitution for investor i between wealth contingent on $u = u_l$ and $u = u_k$ is given by

$$\begin{aligned} M_{kl}^i &= \frac{\exp\{-K_{0i}(u_k - \mu_{0i})^2/2 - w_i(u_k)/r_i\}}{\exp\{-K_{0i}(u_l - \mu_{0i})^2/2 - w_i(u_l)/r_i\}} \\ &= \exp\left\{-\frac{1}{2}K_{0i}(u_k - u_l)(u_k + u_l - 2P_0)\right\}. \end{aligned} \tag{6}$$

Since the marginal rate of substitution is investor specific through K_{0i}, the precision of the investor's posterior beliefs, this equilibrium in which allocations are constrained to be linear fuctions of the asset payoff is not Pareto efficient.[11]

2. Addtional Market Sessions

In this section we extend the model of the previous section to allow for additional market sessions between time 0 and the asset payoff at time 1. Immediately before each market session there is assumed to be a public signal about the asset payoff, so that the precision of public information about the payoff increases through time. In Section 2.1 we consider the welfare gains from improved risk sharing made possible by the increased trading opportunites. In Section 2.2

[11] The notion of Pareto efficiency that we use here and throughout this article is that of ex post conditional Pareto efficiency; that is, Pareto efficiency conditional on P_0 and realized endowments x_i, where expected utility is evaluated using each investor's posterior beliefs.

we analyze the trading strategies of investors in this multisession setting. Section 2.3 is concerned with the volume of trade. Section 2.4 shows that Pareto efficiency is achieved as the number of trading sessions is increased without limit, provided that the information flow is sufficiently smooth. Section 2.5 derives the stochastic process for the asset price as trading becomes continuous.

2.1 Value of additional market sessions

Since the one-period equilibrium is not Pareto efficient, there remain unexploited gains from trade; these could be realized by opening additional (derivative) asset markets, but we consider first the gains from additional openings of the market in the single risky asset, as new information about the final asset payoff becomes publicly available. Specifically, consider a setting in which information about the final payoff $\tilde{u}$ is made available gradually by a series of public signals $\tilde{y}_t = \tilde{u} + \tilde{\eta}_t$ at time $\tau_t = t/T,\ t = 1, \ldots, T-1$, where $\tilde{\eta}_t$ is independently and normally distributed with mean 0 and precision n_t.[12] After each signal the market opens for trading, and at time 1, the return of the risky asset is realized and consumption occurs.[13]

Let P_t denote the price of the risky asset at time τ_t. In a rational expectations equilibrium investors realize that at time τ_t the previous and current prices for the risky security, $\tilde{P}_j,\ j = 0, \ldots, t$, reflect the information held by other traders, and their conjectures about the relation of the price and traders information are self-fulfilling.

Trader i's optimal demand for the risky asset at time τ_t, $\tilde{D}_{ti}$, is required to be $\tilde{F}_{ti}$ measurable, where $\tilde{F}_{ti}$ denotes the subsigma field generated by $\{\tilde{z}_i, \tilde{y}_j, \tilde{P}_j, j = 0, \ldots, t\}$. The subsigma field generated by $\{\tilde{y}_j, \tilde{P}_j, j = 1, \ldots, t\}$ is denoted by $\tilde{F}_t$.

Let a linear conjecture of $\tilde{P}_t$ be written as

$$\tilde{P}_t = \alpha_t \bar{u} + \beta_t \tilde{u} - \gamma_t \tilde{x} + \sum_{j=0}^{t} \theta_{tj} \tilde{y}_j. \tag{7}$$

Then the signal about $\tilde{u}$ derivable from $\tilde{P}_t$ can be written as

$$\tilde{q}_t = \frac{\tilde{P}_t - \alpha_t \bar{u} - \sum_{j=0}^{t} \theta_{tj} \tilde{y}_j}{\beta_t} = \tilde{u} - \frac{\gamma_t}{\beta_t} \tilde{x}. \tag{8}$$

[12] The model can be extended to include new random supply shocks of the risky asset, new private information acquisition, and common errors in their private information. For the ease of exposition, we restrict our model to the present setting. Section 6 considers some of these generalizations.

[13] We set $\eta_T = 0,\ n_T = \infty$ to reflect the assumption that risky asset payoff is realized at time 1.

There are two types of equilibria in this model.[14] In the first type, prices fully reveal private information at the first trading session t, for which $\frac{\gamma_1}{\beta_1} \neq \frac{\gamma_t}{\beta_t}$. For example, if $\frac{\gamma_1}{\beta_1} \neq \frac{\gamma_2}{\beta_2}$ the payoff of the risky asset is revealed at time τ_2 and is equal to

$$\frac{\frac{\beta_1}{\gamma_1}\tilde{q}_1 - \frac{\beta_2}{\gamma_2}\tilde{q}_2}{\frac{\beta_1}{\gamma_1} - \frac{\beta_2}{\gamma_2}}.$$

The price at time τ_2 is equal to the final payoff and individual investor demands are indeterminate. In the second type of equilibrium, the prices only partially reveal the private information; this implies that all the prices contain the same information, so that $\tilde{q}_t = \tilde{q}$ for $t = 0, \dots, T-1$. The fully revealing equilibrium lacks intuitive appeal since it requires that $\beta_2 = 1$ and $\gamma_2 = 0$, and investors' demands depend on neither price nor private information; therefore we focus on the partially revealing equilibrium in the remainder of this article.

The rational expectations partially revealing equilibrium is described in the following theorem. All proofs are in the Appendix.[15]

Theorem 1. *In an economy with T trading sessions, there exists a partially revealing rational expectations equilibrium in which prices, and demands for the risky asset, are given by*

$$\tilde{P}_t = K_t^{-1}[(K_t - s)\tilde{\mu}_t + s\tilde{u} - \tilde{x}/r], \tag{9}$$

$$\tilde{D}_{ti} = r_i K_{ti}[\tilde{\mu}_{ti} - \tilde{P}_t] = r_i[s_i\tilde{z}_i - s\tilde{u} + \tilde{x}/r - (s_i - s)\tilde{P}_t], \tag{10}$$

where

$$\tilde{\mu}_t \equiv E(\tilde{u} \mid \tilde{F}_t) = \frac{h\bar{u} + \sum_{j=0}^{t} n_j\tilde{y}_j + r^2s^2p\tilde{q}}{h + \sum_{j=0}^{t} n_j + r^2s^2p},$$

$$\begin{aligned}\tilde{\mu}_{t_i} &\equiv E(\tilde{u} \mid \tilde{F}_{ti}) = \frac{(K_t - s_i)\tilde{\mu}_t + s_i\tilde{z}_i}{K_{ti}} \\ &= \frac{h\bar{u} + \sum_{j=0}^{t} n_j\tilde{y}_j + s_i\tilde{z}_i + r^2s^2p\tilde{q}}{h + \sum_{j=0}^{t} n_j + s_i + r^2s^2p},\end{aligned}$$

$$\tilde{q} = \tilde{u} - \tilde{x}/rs,$$

[14] The fully revealing equilibrium is also discussed in Grundy and McNichols (1989).

[15] Since there is no public information release in the first trading period, for notational simplicity we define $n_0 = 0$, $\tilde{y}_0 = 0$. This model can be easily extended to the case where there is public information in the first period and all the consIusions derived here hold.

$$K_{ti} \equiv var^{-1}(\tilde{u} \mid \tilde{F}_{ti}) = h + s_i + \sum_{j=0}^{t} n_j + r^2 s^2 p,$$

$$K_t \equiv \int_0^1 \frac{r_i}{r} K_{ti} di = h + s + \sum_{j=0}^{t} n_j + r^2 s^2 p.$$

Notice that in this equilibrium the signal from the price is the same in each period since $\frac{\gamma_t}{\beta_t} = \frac{1}{rs}$ is independent of t, that is, $\tilde{q}_t = \tilde{u} - \tilde{x}/rs = \tilde{q}$. Moreover, since P_0 and $\tilde{P}_t$ jointly reveal $\sum_{j=0}^{t} n_j \tilde{y}_j$, the time τ_t price $\tilde{P}_t$ is a sufficient statistic for all the public information flow up to time τ_t. An interesting feature of the equilibrium is that the allocation of securities at time τ_t is not affected by the anticipation of future public announcements.

In the following lemma the expected utility of investor i, conditional on his endowment but before receipt of the private information signal, EU_i, is expressed as a function of his individual, and the market average, precisions in each of the T market sessions.

Lemma 1.

$$EU_i = \prod_{t=1}^{T-1} \sqrt{\frac{K_{(t-1)i}}{K_{(t-1)i} + n_t(s_i - s)^2/K_t^2}} \times \frac{-1}{\sqrt{L_0 K_{0i}}} \exp\left[\frac{-\bar{u}x_i}{r_i} + \left(\frac{1}{h} - \frac{1}{K_0^2 L_0}\right)\frac{x_i^2}{2r_i^2}\right]. \quad (11)$$

As Kim and Verrecchia (1991a) point out, an investor's expected utility in this type of model can be expressed as the product of the expected utility without any future trade, and a term that corresponds to the utility gain from future trading opportunities. This permits a simple calculation of the monetary value of trading opportunities. Thus define $\gamma_i(T)$ as the (monetary) value of T market sessions for investor i; it is the maximum amount the investor would be willing to pay to have T trading sessions rather than simply one session at time 0. Comparing the expression for expected utility in the single trading session economy of Section 2 with that for the T session economy in Lemma 1, $\gamma_i(T)$ is given by

$$\gamma_i(T) = \frac{r_i}{2} \sum_{t=1}^{T-1} \ln\left(1 + \frac{n_t(s_i - s)^2}{K_t^2 K_{(t-1)i}}\right). \quad (12)$$

Equation (12) implies that the gain from additional market sessions is positive for all investors except those whose private signal precision is equal to the risk tolerance weighted average precision. Contrary to the effect of an increase in public information before trading, which reduces welfare, the effect of increased public information flow after the initial trading session, combined with the additional trading sessions, is to increase welfare for all investors by improving risk sharing opportunites.[16] The magnitude of the welfare gain depends on the rate at which the individual and the market average posterior precisions, K_t and K_{ti}, change between market sessions, as well as the time pattern of the public signal precisions, n_t. This is intuitive, since if, for example, the first public signal were perfectly precise then there would be no trade in subsequent market sessions, and the number of sessions would then be irrelevant. A reasonable conjecture is that the gains from improved risk sharing through additional trading sessions will be maximized when the variance of price changes between sessions is constant. This conjecture is verified in the following lemma. Define $\nabla\tilde{P}_t \equiv \tilde{P}_t - \tilde{P}_{t-1}$. Then

Lemma 2. *$\gamma_i(T)$, the value of T market sessions for investor i, is maximized when n_t, the precision of the public information signals, is given by*

$$n_t = \frac{TK_0}{(T-t)(T-t+1)}. \tag{13}$$

This time path of precisions implies that var$[\nabla P_t]$ is constant. The maximized value of $\gamma_i(T)$ is

$$\gamma_i^*(T) = \frac{r_i}{2}\left[T\ln\left(1+\frac{s_i-s}{TK_0}\right)-\ln\frac{K_{0i}}{K_0}\right]. \tag{14}$$

The lemma implies that welfare is maximized for all investors when information about the asset payoff is released smoothly over time so that the variance of price changes between market sessions is equal. Additional market openings allow traders to take linear positions in a finer partition of the aggregate price change $\tilde{u}-P_0$. As will be shown in Section 3, in a continuous-time economy it is optimal for investors to synthesize by a dynamic trading strategy a payoff that is a quadratic function of the asset payoff; equalizing the variance of price changes between sessions allows traders to minimize the certainty equivalent cost of "hedging errors" given their exponential utility functions.

The lemma suggests that if the firm receives information smoothly then a policy of immediate disclosure is optimal. On the other hand, if information is received by the firm in a lumpy fashion, then the lemma suggests that risk sharing could be improved if the public release of

[16] In this setting there is no endowment effect associated with the increased trading sessions since P_0 remains unchanged.

Table 1
The value of additional market sessions for an investor, as a function of his relative signal precision $(s_i - s)$.

σ_{0i}	$s_i - s$	T: 2	4	8	16	32	64	128	256	512	8192
0.05	300	558	1065	1452	1704	1852	1932	1974	1995	2006	2016
0.06	178	451	809	1051	1195	1275	1317	1339	1350	1355	1361
0.07	104	305	519	650	723	762	782	792	797	800	802
0.08	56	158	256	311	341	356	364	368	370	371	372
0.09	23	45	70	83	90	93	95	96	96	96	97
0.11	−17	55	80	92	98	100	102	103	103	103	103
0.12	−31	238	337	382	404	415	420	423	424	425	425
0.13	−41	575	795	893	939	962	973	978	981	983	984
0.14	−49	1089	1475	1642	1720	1758	1776	1786	1790	1793	1795

σ_{0i} is the standard deviation of the investor's posterior distribution over the asset payoff. T is the number of trading sessions per period. The example is constructed so that the risky asset has an the risky asset, the better informed he is, the more he is willing to pay for near continuous trading. For example, the best informed investor represented in the table would be willing to pay $21 to increase the number of trading sessions from 256 per year, or roughly daily, to 8,192, or roughly 32 times per day; the uninformed investor on the other hand would be willing to pay only $5 for the same increase in trading sessions.

2.2 Trading strategies and price behavior

In order to characterize investors' trading strategies we consider the change in the demand of investor i for the risky asset between successive market sessions:[17] using Equations (9) and (10), we have

$$\tilde{D}_{(t+1)i} - \tilde{D}_{ti} = -r_i(s_i - s)(\tilde{P}_{t+1} - \tilde{P}_t). \tag{15}$$

The following corollaries characterize the relation between price moves and the trades of investors.

Corollary 1.

(i) The trades of the well-informed investors ($s_i > s$) at time τ_t are negatively correlated with the price change at time τ_t.

(ii) The trades of the poorly informed investors ($s_i < s$) in each trading session t are positively correlated with the price change at time τ_t;

(iii) The trades of the poorly informed investors at time τ_t are negatively correlated with those of the well-informed investors at time τ_{t-1}.

(iv) The averagely informed investors ($s_i = s$) will not trade.

(v) Trades exhibit positive autocorrelation for all investors.

Corollary 2. *The price change at time τ_{t+1} is negatively correlated with the price and is positively correlated with the price change at time τ_t, that is,*

$$cov(\tilde{P}_{t+1} - \tilde{P}_t, \tilde{P}_t) < 0, \tag{16}$$

$$cov(\tilde{P}_{t+1} - \tilde{P}_t, \tilde{P}_t - \tilde{P}_{t-1}) > 0. \tag{17}$$

The intuition for these two corollaries is that well-informed investors sell when the price is high and buy when the price is low, since they have more precise information about the risky asset payoff than the average investor. Less-informed investors rely more on the signal revealed by the price, and therefore buy when the price is high and sell when it is low. Corollary 1 shows that positive correlation

[17] Kim and Verrecchia (1991b) analyze trades and trading volume in a two trading session economy with two public signals.

between mutual fund sales and market returns does not necessarily imply irrational "barn-door closing" behavior on the part of the small investors who purchase these funds; rather, it is fully consistent with rational behavior for investors who are less well informed. Warther (1994) reports that security returns are strongly related to contemporaneous cash flows into mutual funds: there is no lagged relation such as would be predicted by trend following behavior, and he is unable to explain the phenomenon in terms of price pressure. However, it is quite consistent with the predictions of Corollary 1 if mutual fund investors are less well informed than average, and therefore place greater weight on new public information signals than the market as a whole.[18] The aggressiveness of traders is proportional to their risk tolerance. In addition, when the proportion of uninformed traders increases (s decreases), informed traders trade more aggressively.

2.3 Volume of trade

The sources of price volatility and trading volume and their interrelation are topics of enduring interest in financial economics. The numerous empirical studies that have examined the contemporaneous behavior of volume and absolute price changes have found a positive correlation between the two [Karpoff (1987)]. Since the dynamics of price volatility and trading volume can only be studied in a multiple trading session economy, in this section we extend the results of Kim and Verrecchia (1991b) to a T session economy, and present new results on the autocorrelation properties of trading volume and the relation between trading volume and the frequency of trading.

Let $\nabla \tilde{P}_t = P_t - \tilde{P}_{t-1},\ t = 1, \ldots, T-1$ denote the price change at time τ_t. Let trading volume at time τ_t, V_t, be defined as one-half the sum of all purchases and sales,[19] that is,

$$V_t = \frac{1}{2}\int_0^1 |\tilde{D}_{ti} - \tilde{D}_{(t-1)i}|\, di = \frac{1}{2}\int_0^1 r_i |s_i - s|\, di |\nabla \tilde{P}_t|. \tag{18}$$

In our model there is no hedging demand for the risky asset, and all trading is informationally based. As a result we obtain the simple result that the trading volume in each period is proportional to the product of the absolute price change and the dispersion of investors' private information precisions.[20] Expected volume at time τ_t, $E[V_t]$, is

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[18] Capon, Fitzsimons, and Prince (1993) cite evidence of lack of sophistication among mutual fund investors.

[19] Note that D_{Ti} is not well defined since the risky asset becomes riskless at time 1. Nevertheless, we can define $\tilde{D}_{Ti} = r_i(s_i \tilde{z}_i - s\tilde{q} + (s - s_i)\tilde{u})$ to preserve the relation between demands and prices expressed in Equation (10).

[20] A similar result is obtained by Harris and Raviv (1993) in a model in which investors have different

obtained from Equation (18) by noting that $E[\nabla \tilde{P}_t] = 0$ and using an elementary property of the normal distribution:

$$E[V_t] = \sqrt{\frac{\mathrm{var}[\nabla \tilde{P}_t]}{2\pi}} \int_0^1 r_i |s_i - s| \, di.$$

Since

$$\mathrm{var}[\nabla \tilde{P}_t] = \frac{n_t}{K_{t-1} K_t} + \frac{n_t^2}{K_{t-1}^2 K_t^2} \left(s + \frac{1}{r^2 p} \right),$$

it is straightforward to show that the expected volume at time τ_t is increasing in the precision of the public information signal at time τ_t, n_t, and decreasing in the prior average precision, K_{t-1}.

In Corollary 2, we have shown that price changes are negatively serially correlated. Since the correlation coefficient between the absolute values of two normally distributed variables $\tilde{x}$ and $\tilde{y}$ with correlation $r(\tilde{x}, \tilde{y})$ and means equal to zero is given by

$$corr(|\tilde{x}|, |\tilde{y}|) = \frac{2}{\pi - 2} \int_0^{r(\tilde{x}, \tilde{y})} \arcsin t \, dt > 0.$$

The following lemma is immediate.

Lemma 3. *The absolute price change and the volume of trade are positively serially correlated.*

The lemma is consistent with the empirical evidence of Harris (1987) on the autocorrelation of trading volume and squared price changes [see also Harris and Raviv (1993)].

We saw in Lemma 2 that welfare is maximized when the variance of price changes between market sessions is equalized. In the following lemma, we show that the expected volume of trade is also maximized when the variance of price changes between market sessions is equalized.

The total volume of trade in an economy with T trading sessions is given by

$$Volume = \sum_{t=1}^{T} V_t.$$

Using Equation (18) for V_t we get

Lemma 4.

(i) The expected total volume of trade in a T trading session economy

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opinions and so ignore the information in price.

is maximized when the variance of the price change at time τ_t is the same for all $t = 1, \ldots, T-1$.

(ii) When T is large the expected trading volume is proportional to the square root of T, that is,

$$E[Volume] = Q\sqrt{\frac{1}{2\pi}}\sqrt{\frac{K_0 T + s + r^{-2}p^{-1}}{K_0^2}} = Q\sqrt{\frac{T}{2\pi K_0}} + O\left(\sqrt{\frac{1}{T}}\right),$$

where $Q \equiv \int \frac{1}{2} r_i |s_i - s|\, di$ measures information asymmetry and $O(\sqrt{\frac{1}{T}})$ means of order $\sqrt{\frac{1}{T}}$.

The lemma implies that if information arrives smoothly so that price volatility is constant, then the small amount of supply noise or noise trading in the first trading session necessary to prevent the price from being fully revealing can give rise to an unlimited amount of rational trading volume as the number of market sessions increases without limit. The lemma implies that for large T the expected volume per session is proportional to $\sqrt{1/T}$.

3. Convergence to a Continuous-Time Economy

In Section 3.1 we demonstrate formally that as the number of trading sessions increases without limit the wealth allocation converges to the Pareto efficient allocation, provided that the information flow is sufficiently smooth. The intuition for this result is that the multiplication of trading opportunities made possible by a large number of trading sessions makes the market essentially complete with only two securities, the riskless asset and the risky security. We also show that the Pareto efficiency of the limiting economy implies that derivative securities (which are of course redundant) are priced according to Black and Scholes (1973) principles and that the volatility used to price derivative assets is the inverse of the risk tolerance weighted average of the posterior precisions of the individual investors, K_0^{-1}. In Section 3.2 we present a heuristic continuous time model which shows that, despite their different information sets, investors agree on the variation of the price process, and this agreement is of course necessary if they are to agree on the prices of derivative assets.

3.1 Pareto efficiency in the limiting economy

It was shown in Section 1 that the allocation of wealth achieved with a single trading session is not Pareto efficient, and it is straightforward to extend this result to a setting with multiple trading sessions. However, as we have seen, investor welfare is an increasing function of the number of trading sessions, and it is natural to conjecture that Pareto efficiency will be achieved as the number of trading sessions increases without limit. In this section we show that as the number of trading sessions is increased, the equilibrium wealth allocation converges in probability to the Pareto optimal allocation.

As we increase the number of trading sessions we impose the requirement that the public information flow is sufficiently smooth that the variance of the price changes between trading sessions tends to zero. Formally, we assume **Condition A**:

$$\max_t \operatorname{var}[\nabla \tilde{P}_t] = O(\nabla t). \tag{19}$$

The following lemma, which is used in the proof of Theorem 2, gives the limiting vlaue of $\gamma_i(T)$ and the expected utility of the investor as the number of trading sessions approaches infinity.

Lemma 5. *If the information structure satisfies Condition A, then as $T \to \infty$: (i) $\gamma_i(T)$, the value of the additional trading opportunities for investor i converges to*

$$\gamma_i^*(T) = \frac{r_i}{2}[(K_{0i} - K_0)/K_0 - \ln(K_{0i}/K_0)]; \tag{20}$$

(ii) His ex ante utility converges to

$$EU_i^* = \frac{-1}{\sqrt{L_0 K_0}} \exp\left[\frac{-\bar{u}x_i}{r_i} - \frac{s_i - s}{2K_0} + \left(\frac{1}{h} - \frac{1}{K_0^2 L_0}\right)\frac{x_i^2}{2r_i^2}\right]. \tag{21}$$

Thus, as the frequency of trading opportunities increases, the utility of the investor no longer depends on the time path of the public information signal precision as it does in the finite trading session economy as shown in Lemma 1. The following theorem shows that in the limit, each investor's wealth converges in probability to the expost conditional Pareto efficient wealth allocation.

Theorem 2.

(i) The wealth of investor i at period 1 converges in probability to $w_i^e(\tilde{u})$ as $T \to \infty$, where $w_i^e(\tilde{u})$ is given by

$$w_i^e(\tilde{u}) = P_0 x_i + (\tilde{u} - P_0)\tilde{D}_{0i} + \frac{1}{2} r_i (s - s_i)[(\tilde{u} - P_0)^2 - K_0^{-1}], \tag{22}$$

and D_{0i} is as given in Equation (3).

(ii) The allocation $w_i^e(\tilde{u})$ is Pareto optimal, given the period 0 information and initial endowments, and the expected utility of each

investor converges to that yielded by the Pareto efficient allocation $w_i^e(\tilde{u})$, as $T \to \infty$.

Thus, while partially revealing rational expectations models of markets in which there are only two securities and investors have information of different precisions yield equilibria that are Pareto inefficient, the introduction of continuous trading opportunities effectively makes the market complete, yielding a Pareto efficient allocation.[21] We note from Equation (22) that the Pareto optimal payoff for investor i is a quadratic function of the risky asset payoff, $\tilde{u}$, which is concave (convex) if the investor's private information is more (less) precise than that of the risk tolerance weighted average. Moreover, while the payoff depends on the investor's private information signal, $\tilde{z}_i$, through $\tilde{D}_{0i}$, it does not depend on the public information signals, $\tilde{y}_j$.[22]

Lemma 6. *As the limiting economy is Pareto efficient, it follows from the results of Brennan and Kraus (1978) and Rubinstein (1974) that, in the limiting economy, security prices are as if there existed a single representative investor with risk tolerance r, and beliefs $\mathcal{N}(\mu_R, K_0^{-1})$, where $\mu_R = K_0^{-1}[h\bar{u} + (s + r^2 s^2 p)\tilde{u} - rsp\tilde{x}]$.*

Given that prices are supported by a representative investor with constant absolute risk aversion, and that from his perspective the return on the single risky asset is normally distributed with parameters $\mathcal{N}(\mu_R, K_0^{-1})$, it follows from the results of Brennan (1979) that all contingent claims on the risky asset are priced in accord with Black and Scholes (1973) principles, that is, as though the return on the risky asset were distributed normally with mean P_0 and variance K_0^{-1}.[23] In particular, the price of a call option on the underlying risky asset with exercise price E, $C(P_0; E)$ is given by

$$C(P_0; E) = N\left(\frac{P_0 - E}{\sigma}\right) + \sigma\, n\left(\frac{E - P_0}{\sigma}\right), \tag{23}$$

where $n(\cdot)$ and $N(\cdot)$ are the standard normal density and standard normal distribution functions, respectively, and $\sigma = \sqrt{1/K_0}$. Equation (23) shows that in this noisy rational expectations equilibrium with hetero-

[21] Duffie and Huang (1985) have shown that when information is homogeneous a Pareto efficient allocation can be achieved by dynamic trading in a market with a few long-lived securities, provided that the martingale multiplicity of the information structure is not greater than one less than the number of securities. Their martingale multiplicity condition plays the same role as our Condition A.

[22] Laffont (1985) presents an example in which the allocation achievable by a linear rational expectations equilibrium is Pareto dominated by an incentive compatible mechanism in which the allocation is allowed to be nonlinear in the *private* information.

[23] If the gross interest factor is R then the mean is $P_0 R$.

geneous private information precision and continuous trading, not only does the price of the risky asset aggregate the private information about the mean of the asset payoffs, but the prices of derivative assets aggregate the posterior precisions of the individual investors. Note that in this model investors are able to agree on the price of the derivative asset despite having different assessments of the variance of the asset payoff, K_{0i}^{-1}.[24] We shall see in the next section, however, that investors do agree on the instantaneous volatility of the price process.

3.2 A heuristic continuous-time model

To analyze the trading of investors in a continuous-time model, it is convenient to assume that the public information at time τ can be summarized by the sufficient statistic $\tilde{Y}_\tau = \tilde{u} + \tilde{\zeta}_\tau$,[25] where ζ_τ follows the stochastic process

$$d\zeta_\tau = -\frac{n_\tau}{K_\tau - K_0}\zeta_\tau \, d\tau + \frac{\sqrt{n_\tau}}{K_\tau - K_0} dz, \tag{24}$$

where $K_\tau - K_0 = \int_0^\tau n_t \, dt$ and $\int_0^t n_\tau \, d\tau \xrightarrow{t \to 1} \infty$ to ensure that the public information eventually reveals the risky asset payoff $\tilde{u}$ at time 1. Let $\tilde{P}_\tau$ denote the price and $\tilde{D}_{\tau i}$ denote the demand of investor i for the risky asset at time τ. Let $\tilde{F}_{\tau i}$ denote the filtration generated by $\{\tilde{P}_t, \tilde{y}_t, 0 \le t \le \tau, \tilde{z}_i\}$ and $\tilde{F}_\tau$ denote the filtration generated by $\{\tilde{P}_t, \tilde{y}_t, 0 \le t \le \tau\}$. It follows from our previous results that in this economy there exists an equilibrium as described in the following theorem.

Theorem 3. *There exists a partially revealing rational expectations equilibrium in which prices and demands for the risky asset are given by*

$$\tilde{P}_\tau = K_\tau^{-1}[(K_\tau - s)\tilde{\mu}_\tau + s\tilde{u} - \tilde{x}/r], \tag{25}$$

$$\tilde{D}_{\tau i} = r_i K_{\tau i}[\tilde{\mu}_{\tau i} - \tilde{P}_\tau] = r_i[s_i \tilde{z}_i - s\tilde{u} + \tilde{x}/r - (s_i - s)\tilde{P}_\tau], \tag{26}$$

where

$$\tilde{\mu}_{\tau i} \equiv E(\tilde{u} \mid \tilde{F}_{\tau i}) = \frac{h\bar{u} + s_i \tilde{z}_i + (K_\tau - K_0)\tilde{Y}_\tau + r^2 s^2 p \tilde{q}}{K_{\tau i}},$$

$$\tilde{q} = \tilde{u} - \tilde{x}/rs,$$

[24] This result extends also to a multiasset version of the noisy rational expectations economy.

[25] In discrete time, the sufficient statistic is given by $\tilde{Y}_{\tau_t} \equiv \sum_{j=0}^t n_j \tilde{y}_j / \sum_{j=0}^t n_j$, so that $\tilde{\zeta}_{\tau_t} \equiv \tilde{Y}_{\tau_t} - \tilde{u} = \sum_{j=0}^t n_j \tilde{\eta}_j / \sum_{j=0}^t n_j$ and $\nabla \tilde{\zeta}_{\tau_t} \equiv \tilde{\zeta}_{\tau_t} - \tilde{\zeta}_{\tau_{t-1}} = -n_t \tilde{\zeta}_{\tau_{t-1}} / \sum_{j=0}^t n_j + n_t \tilde{\eta}_t / \sum_{j=0}^t n_j$.

$$K_{\tau i} \equiv var^{-1}(\tilde{u} \mid \tilde{F}_{\tau i}) = K_\tau + s_i - s,$$

$$K_\tau \equiv \int_0^1 \frac{r_i}{r} K_{\tau i}\, di = h + s + \int_0^\tau n_t\, dt + r^2 s^2 p.$$

The price process conditional on $\tilde{u}$ is obtained by applying Ito's Lemma to Equation (25):

$$d\tilde{P}_\tau = \frac{n_\tau}{K_\tau}(\tilde{u} - \tilde{P}_\tau)d\tau + \frac{\sqrt{n_\tau}}{K_\tau} dz, \tag{27}$$

where dz is the increment to a standard Wiener process.

However, investors have only imperfect information about $\tilde{u}$, and the price process adapted to the filtration $\tilde{F}_{\tau i}$ is obtained by noting that conditional on the filtration $\tilde{F}_{\tau i}$ the public signal follows a stochastic process of the form

$$d\tilde{Y}_\tau = -\frac{n_\tau}{K_\tau - K_0}(\tilde{\mu}_{\tau i} - \tilde{Y}_\tau) + \frac{\sqrt{n_\tau}}{K_\tau - K_0} dz_i. \tag{28}$$

This yields the adapted price process:

$$d\tilde{P}_\tau = \frac{n_\tau}{K_\tau}(\tilde{\mu}_{\tau i} - \tilde{P}_\tau)d\tau + \frac{\sqrt{n_\tau}}{K_\tau} dz_i. \tag{29}$$

Notice that although different investors have different estimates about the drift of the price, they agree on the variation process of the price. This conforms with the no arbitrage requirement in Duffie and Huang (1986) that the variation process should be the same for all individuals even when there is differential information. The variation process of $\tilde{P}$, $[\tilde{P}, \tilde{P}]_\tau$ is given by $P_0^2 + \int_0^\tau \frac{n_t}{K_t^2}\, dt = P_0^2 + \frac{1}{K_0} - \frac{1}{K_\tau}$.

The final wealth for investor i in this continuous-time model is found from Equations (25) and (26):

$$\begin{aligned} w_i(\tilde{u}) &= P_0 x_i + \int_0^1 \tilde{D}(t)\, d\tilde{P}(t) \\ &= P_0 x_i + (\tilde{u} - P_0)\tilde{D}_{0i} + r_i(s - s_i) \int_0^1 (\tilde{P} - P_0)\, d\tilde{P} \\ &= P_0 x_i + (\tilde{u} - P_0)\tilde{D}_{0i} + \frac{r_i}{2}(s - s_i)(\tilde{u} - P_0)^2 \\ &\quad - \frac{r_i}{2}(s - s_i)[\tilde{P} - P_0, \tilde{P} - P_0]_1 \\ &= P_0 x_i + (\tilde{u} - P_0)\tilde{D}_{0i} + \frac{r_i}{2}(s - s_i)[(\tilde{u} - P_0)^2 - K_0^{-1})]. \end{aligned} \tag{30}$$

Comparing Equations (22) and (30), the final wealth of each investor in the continuous-time economy is identical to that in the limiting discrete-time economy; since we saw that the Black and Scholes (1973) option pricing result obtained in the limiting economy, it holds also in this continuous-time economy. This is in contrast to the result of Back (1993) who also analyzes option pricing in a continuous-time economy with differentially informed investors and finds that the Black and Scholes (1973) result does not hold — the major differences between his model and ours is that he assumes that there is net supply noise in the options market and that there is only one informed investor who behaves monopolistically; the results of Holden and Subrahmanyam (1992) suggest that this assumption is critical in preventing the private information from being instantaneously revealed.

4. Pareto Efficiency with a Single Round of Trading and a Quadratic Option

Since in this model with exponential utility and normal payoff distribution, the Pareto efficient allocation of wealth $w_i^e(\tilde{u})$ given in Equation (22) is a quadratic function of the final payoff of the risky asset $\tilde{u}$, the Pareto efficient allocation can be achieved in only a single round of trading if the set of available securities is extended to include a derivative asset with a payoff that is a quadratic function of $\tilde{u}$.[26] In this section we introduce into the market at time 0 a derivative asset whose payoff at time 1 is $C(\tilde{u}) = (\tilde{u} - P_0)^2$.[27] The derivative asset or option is in zero net supply. Theorem 4 establishes that there exists an equilibrium in which the Pareto efficient allocation is achieved in one round of trading, and the price of the risky asset is the same as in the basic single-period model.

Theorem 4. *Let P_C denote the price of the option and D_{C0i} denote the demand for the option by investor i in the first period. Then there exists a rational expectations partially revealing equilibrium in the economy with the option in which*

$$P_0 = K_0^{-1}[h\bar{u} + (s + r^2 s^2 p)\tilde{u} - (r^{-1} + rsp)\tilde{x}], \tag{31}$$

$$\tilde{D}_{0i} = r_i K_{0i}[\tilde{\mu}_{0i} - P_0] = r_i[s_i \tilde{z}_i - s_i \tilde{u} + \tilde{x}/r - (s_i - s)P_0], \tag{32}$$

$$P_C = K_0^{-1}, \tag{33}$$

[26] Nau and McCardle (1991), who have also recognized the role of a quadratic payoff in achieving a Pareto efficient allocation in a model with exponential utility and normal distributions when investors' posterior beliefs have different precisions, point out that such a payoff can be achieved by purchasing "straddles" composed of uniform distributions of call and put options at all exercise prices.

[27] Similar results obtain if the derivative asset has a general quadratic payoff.

$$D_{C0i} = \frac{r_i}{2}(P_C^{-1} - K_{0i}) = \frac{r_i}{2}(s - s_i). \tag{34}$$

It is straightforward to show that the wealth allocation in this economy is the Pareto efficient allocation $w_i^e(\tilde{u})$. In this equilibrium, the less-informed investors ($s_i < s$) take long positions in the option, while the well-informed ($s_i > s$) investors take short positions. Let trading volume in the option be defined as the gross long position,

$$Volume(C) \equiv \frac{1}{2}\int_0^1 |D_{Ci}|\, di.$$

Then we have

$$Volume(C) = Q/2,$$

where

$$Q \equiv \int_0^1 \frac{1}{2} r_i |s_i - s|\, di.$$

The volume of trade in the option is equal to half of the asymmetric information measure Q, and the demand for the option arises because of the differences in the precision of investors' private information, so that if every investor had the same precision of information, the demand for the option would be zero. Thus the Pareto inefficiency in the one trading session economy with only one risky asset is due to the differences in the precision of the private information of investors.

Theorem 4 shows that the introduction of a new option achieves the same goal as continuous market sessions in improving social welfare. However, the continuous trading economy requires an infinite volume of trading as we saw in Section 2.3. It is apparent that if there is any social cost to trading then the introduction of a derivative asset is more efficient than the introduction of continuous trading in promoting investor welfare. The option would be redundant in an infinite trading session economy since it can be replicated by dynamic trading strategies in equilibrium. Nevertheless, if the option is introduced in the first trading session, investors will trade in it and arrive at a Pareto optimal allocation wealth. Then, according to the Milgrom and Stokey (1982) theorem, there will be no further trade as additional public or private information arrives.

While there will be no trade following public information arrivals, it is easy to show that the shadow price of the risky asset at time τ_1 will still be $\tilde{P}_1$. The shadow price of the option at time τ_1 will be changed to $P_{C1} = K_1^{-1} + (\tilde{P}_1 - P_0)^2 = P_{C0} + \{(\tilde{P}_1 - P_0)^2 - E_{\tau 0}[(\tilde{P}_1 - P_0)^2]\}$, where $E_{r0}[\cdot]$ denotes expectations of a representative investor (who has signal precision s and takes no position in the risky asset or the option) at time 0. Thus the shadow price of the option increases with realized price volatility. However, since everybody now wants to hold more of the option, there is no further trade in it.

If new private information arrives after the first round of trading, the final asset payoff will be fully revealed and the volume of trade is not well defined since the risky asset becomes riskless.

5. Further Results for an Economy with Options

In this section we extend our analysis of the economy with quadratic options. In Section 5.1 we consider an economy with two trading sessions and a second supply shock in the second session; the equilibrium is analyzed both with and without a quadratic option. In Section 5.2 we consider the effect on trade of additional private signals prior to the later trading sessions. In Section 5.3 we compare the measure of investors who have no private information with the market maker of the Kyle (1985) model, by analyzing the relation between the trades of the uninformed measure of investors and the price change and comparing it with the relation between price change and order flow in the Kyle model. In Section 5.4 we analyze an economy with only short-term options and consider the dynamic trading strategies that this entails.

5.1 The effect of additional supply shocks in an economy with options

We have seen in Section 4 that there will be no trade after a public signal if a quadratic option is traded in an earlier market session, since the option is sufficient to ensure that the resulting equilibrium is Pareto efficient. However, that result was predicated on there being no further supply shocks. Now we assume that new liquidity traders enter the market in the second trading session with exogenously determined per capita supply $\tilde{x}_1$, which is distributed normally with mean 0 and precision p_1. Then there exists an equilibrium described in the following theorem in which the price in the second session adjusts to reflect the new public signal and the supply shock, but does not reveal any further private information.

Theorem 5.

(i) In an economy with two trading sessions, a single risky asset, and a new supply shock in the second trading session, there exists a partially revealing rational expectations equilibrium in which prices and demands for the risky asset are given by

$$P_0 = K_0^{-1}[(K_0 - s)\tilde{\mu}_0 + s\tilde{u} - \tilde{x}/r], \tag{35}$$

$$\tilde{D}_{0i} = r_i K_{0i}[\tilde{\mu}_{0i} - P_0] = r_i[s_i \tilde{z}_i - s\tilde{u} + \tilde{x}/r - (s_i - s)P_0], \tag{36}$$

$$\tilde{P}_1 = K_1^{-1}[(K_1 - s)\tilde{\mu}_1 + s\tilde{u} - (\tilde{x} + \tilde{x}_1)/r], \tag{37}$$

$$\tilde{D}_{1i} = r_i K_{1i}[\tilde{\mu}_{1i} - \tilde{P}_1] = \tilde{D}_{0i} + r_i(s - s_i)(\tilde{P}_1 - \tilde{P}_0) + \frac{r_i \tilde{x}_1}{r}. \tag{38}$$

(ii) If a quadratic option with payoff $(\tilde{u} - P_0)^2$ *is introduced into the market in the first trading session, then there exists the following equilibrium:*

$$P_0 = K_0^{-1}[(K_0 - s)\tilde{\mu}_0 + s\tilde{u} - \tilde{x}/r], \tag{39}$$

$$\tilde{D}_{0i} = r_i K_{0i}[\tilde{\mu}_{0i} - P_0] = r_i[s_i \tilde{z}_i - s\tilde{u} + \tilde{x}/r - (s_i - s)P_0], \tag{40}$$

$$\tilde{P}_1 = K_1^{-1}[(K_1 - s)\tilde{\mu}_1 + s\tilde{u} - (\tilde{x} + \tilde{x}_1)/r], \tag{41}$$

$$\tilde{D}_{1i} = \tilde{D}_{0i} + \frac{r_i \tilde{x}_1}{r}, \tag{42}$$

$$P_{C0} = K_0^{-1} + [K_1(1 + K_1 r^2 p_1)]^{-1}, \tilde{P}_{C1} = (\tilde{P}_1 - P_0)^2 + K_1^{-1}, \tag{43}$$

$$D_{C0i} = \frac{r_i}{2}\left(\frac{1}{P_{C0} - [K_1(1 + K_1 r^2 p_1)]^{-1}} - K_{0i}\right) = \frac{r_i}{2}(s - s_i), \tag{44}$$

$$D_{C1i} = \frac{r_i}{2}\left(\frac{1}{\tilde{P}_{C1} - (\tilde{P}_1 - P_0)^2} - K_{1i}\right) = \frac{r_i}{2}(s - s_i). \tag{45}$$

The only effect of the second session supply noise is to shift the second period risky asset price by an amount that is proportional to the noise realization.

When there is no option, the trade of investor i in the second market session is given by

$$\nabla \tilde{D}_{1i} \equiv \tilde{D}_{1i} - \tilde{D}_{0i} = \frac{r_i \tilde{x}_1}{r} + r_i(s - s_i)(\tilde{P}_1 - P_0).$$

The first component of the trade, $r_i\tilde{x}/r$, is investor i's share of the increased supply made available by liquidity traders. The second component is the informational trade generated by the differential responses of investors with different prior precisions to the public signal. However, as Equation (42) shows, introduction of the quadratic option eliminates the second motive for trade, so that in equilibrium there are only liquidity trades, and it can be shown that the net result is a decrease in the volume of trading in the risky asset.[28]

[28] In contrast Skinner (1989) finds an adjusted increase of 4 percent in the volume of trading in the underlying security following the introduction of options trading. However, see the discussion following Theorem 6.

A striking feature of the equilibria described in Theorem 5 is that the price of the underlying risky asset is in no way affected by the introduction of trading in the option. This contrasts with the theoretical results of Detemple and Selden (1991) who find that in an incomplete markets economy with quadratic utility and a particular type of difference of opinion, the introduction of options trading raises the price of the underlying asset. It also conflicts with the bulk of the empirical evidence on the effects of the introduction of option trading, which finds a positive effect on the underlying asset price.[29] However, Figlewski and Webb (1993) provide a possible explanation by showing that one effect of options trading is to relax constraints on short sales which, while assumed away in our model, may be important in practice. In addition, we have ignored any effect of option introduction on incentives to collect information [see Cao (1994)]. Several authors have suggested that options are a more attractive investment for informed investors than the underlying stock,[30] and Easley, O'Hara, and Srinivas (1993) construct a sequential trade model in which informed traders prefer to trade in the options. They, as well as Amin and Lee (1993), find evidence of informed trading in the options market. However, in our model it is the investors with the less precise signals who have the greatest demand for the option, and the demand for the option depends only on the private signal precision and not the signal realization: thus in our model trading in the option conveys no information about the asset payoff. Empirical evidence on lead-lag relations between stock and options markets remains inconclusive, despite earlier claims that options lead stocks [e.g., Manaster and Rendleman (1982)]; our model is consistent with the recent findings of Chan, Chung, and Johnson (1993) who conclude that neither market leads the other.

In the economy with the option, the final wealth of investor i is given by

$$
\begin{aligned}
w_i(\tilde{u}, \tilde{x}_1) \;=\; & P_0 x_i + \tilde{D}_{0i}(\tilde{u} - \tilde{P}_0) + \frac{r_i \tilde{x}_1}{r}(\tilde{u} - P_0) \\
& + \frac{r_i}{2}(s - s_i)[(\tilde{u} - P_0)^2 - P_{C0}],
\end{aligned}
$$

[29] Jennings and Starks (1986) and Skinner (1990) find evidence that the introduction of option trading causes the underlying risky asset to be priced more efficiently. Conrad (1989) reports that the introduction of option trading is accompanied by an increase in the price of the underlying asset and a reduction in its volatility. Similar results are reported by Detemple and Jorion (1990), who also find that the introduction of options trading on an individual stock is associated with an increase in the level of the industry and overall market prices. On the other hand, Lamoureaux (1991) finds that the introduction of options has no effect on the price volatility of the underlying asset.

[30] Skinner (1989) quotes Black (1975), Cox and Rubinstein (1985), and Manaster and Rendleman (1982) to this effect.

and the marginal rate of substitution for individual i across states $(\tilde{u} = u_k, \tilde{x}_1 = x_l)$ and $(\tilde{u} = u_m, \tilde{x}_1 = x_n)$, M^i_{klmn}, is independent of i:

$$M^i_{klmn} = \frac{\exp\{-p_1\tilde{x}_1^2/2\}\exp\{-K_{0i}(u_k - \tilde{\mu}_{0i})^2/2\}\exp\{-w_i(u_k, x_l)/r_i\}}{\exp\{-p_1 x_n^2/2\}\exp\{-K_{0i}(u_l - \tilde{\mu}_{0i})^2/2\}\exp\{-w_i(u_m, x_n)/r_i\}}$$
$$= \frac{\exp\{-K_0(u_k - P_0)^2/2 - p_1 x_l^2/2 - x_l(u_k - P_1)/r\}}{\exp\{-K_0(u_m - P_0)^2/2 - p_1 x_n^2/2 - x_n(u_m - P_1)/r\}}. \tag{46}$$

Thus the option contracts allow the market to achieve Pareto efficient risk sharing with respect to both the asset payoff, $\tilde{u}$, and the second period supply noise, $\tilde{x}_1$. Despite the Pareto efficiency of the equilibrium, we note that the option price in the first trading session does not satisfy the Black and Scholes (1973) pricing principle. The reason for this is that the final wealth of the representative investor is no longer normally distributed on account of the supply shocks. However, conditional on the second-period supply shock, the final wealth is normally distributed, and consequently the option price in the second[31] trading session does satisfy the Black and Scholes (1973) principle.

Comparing Theorem 4 with Theorem 5, it is apparent that the supply noise introduced in the second trading session has no effect on the investors' demands for the option. On the other hand, the price of the option is increasing in the variance of the supply shock in the second session, which increases the variance of the price of the risky asset in the second session.

5.2 Informational trade with new private information

We now extend the model to incorporate private information acquisition after the first trading session. In particular, we assume that before each trading session, $t = 1, \ldots, T$, investor i receives a private signal $\tilde{z}_{ti} = \tilde{u} + \tilde{\eta}_{ti}$ of precision s_{ti}. We also assume that in each trading session, $t = 1, \ldots, T$, there is a new per capita supply shock caused by liquidity traders, $\tilde{x}_t$, which is normally and independently distributed with mean 0 and precision p_t. Public information signals are received as described above. In addition, let $\tilde{F}_{ti}$ denote the subsigma field generated by investor i's information set and $\tilde{F}_t$ denote the subsigma field generated by public information set up to include session t. The following theorem describes the equilibria that obtain with and without the introduction of quadratic options.

[31] More generally, the last.

Theorem 6. *In an economy with T trading sessions, a single risky asset, and, before each trading session, a supply shock, a public information signal, and new private information signals, there exists a partially revealing rational expectations equilibrium in which prices and demands for the risky asset are given by*

$$\tilde{P}_t = K_t^{-1}\left[\left(K_t - \sum_{j=0}^{t} s_j\right)\tilde{\mu}_t + \sum_{j=0}^{t}\{s_j\tilde{u} - \tilde{x}_j/r\}\right], \tag{47}$$

$$\tilde{D}_{ti} = r_i K_{ti}[\tilde{\mu}_{ti} - \tilde{P}_t] = r_i\left[\sum_{j=0}^{t}\{s_{ji}\tilde{z}_{ji} - s_j\tilde{u} + \tilde{x}_j/r - (s_{ji} - s_j)\tilde{P}_t\}\right], \tag{48}$$

$$s_t \equiv \frac{1}{r}\int_0^1 r_i s_{ti}\, di,$$

$$\tilde{\mu}_t \equiv E(\tilde{u} \mid \tilde{F}_t) = \frac{h\bar{u} + \sum_{j=0}^{t}\{n_j\tilde{y}_j + r^2 s_j^2 p_j\tilde{q}_j\}}{h + \sum_{j=0}^{t}\{n_j + r^2 s_j^2 p_j\}},$$

$$\begin{aligned}\tilde{\mu}_{t_i} \equiv E(\tilde{u} \mid \tilde{F}_{ti}) &= \frac{(K_t - \sum_{j=0}^{t} s_j)\tilde{\mu}_t + \sum_{j=0}^{t} s_j\tilde{z}_{ji}}{K_{ti}} \\ &= \frac{h\bar{u} + \sum_{j=0}^{t}\{n_j\tilde{y}_j + s_{ji}\tilde{z}_{ji} + r^2 s_j^2 p_j\tilde{q}_j\}}{h + \sum_{j=0}^{t}\{n_j + s_{ji} + r^2 s_j^2 p_j\}},\end{aligned}$$

$$\tilde{q}_t = \tilde{u} - \tilde{x}_t/rs_t,$$

$$K_{ti} \equiv var^{-1}(\tilde{u} \mid \tilde{F}_{ti}) = h + \sum_{j=0}^{t}\{n_j + s_{ji} + r^2 s_j^2 p_j\},$$

$$K_t \equiv \int_0^1 \frac{r_i}{r} K_{ti}\, di = h + \sum_{j=0}^{t}\{n_j + s_j + r^2 s_j^2 p_j\}.$$

(ii) Suppose that $T-1$ *quadratic options* $\tilde{C}_t$*,* $t = 1, \ldots, T-1$*, with payoffs at time 1 of* $(\tilde{u} - \tilde{P}_t)^2$*, are available for trade in the first trading session. Let* $P_{\tilde{C}_t j}$ *denote the price of* $\tilde{C}_t$ *at trading session* j*, and* $D_{\tilde{C}_t ji}$ *denote the demand of individual* i *for option* $\tilde{C}_t$ *at trading session* j*.*

Then there exists the following equilibrium.[32]

$$\tilde{P}_t = K_t^{-1}\left[\left(K_t - \sum_{j=0}^{t} s_j\right)\tilde{\mu}_t + \sum_{j=0}^{t}\{s_j\tilde{u} - \tilde{x}_j/r\}\right], \tag{49}$$

$$\tilde{D}_{ti} = r_i\left[\sum_{j=0}^{t}\{s_{ji}\tilde{z}_{ji} - s_j\tilde{u} + \tilde{x}_j/r - (s_{ji} - s_j)\tilde{P}_j\}\right], \tag{50}$$

$$P_{\tilde{C}_t 0} = \sum_{j=t+1}^{T}\left[K_{j-1}^{-1} - \left(K_j + s_j + \frac{1}{r^2 p_j}\right)^{-1} + K_j^{-1}\right], \tag{51}$$

$$P_{\tilde{C}_t j} = P_{\tilde{C}_t 0} I_{\{j\le t\}} + [P_{\tilde{C}_j 0} + (\tilde{P}_j - \tilde{P}_t)^2] I_{\{j>t\}},\ ^{33} \tag{52}$$

$$\begin{aligned} D_{\tilde{C}_t j i} &= D_{\tilde{C}_t 0 i} \\ &= \frac{r_i}{2}\left(\frac{1}{P_{\tilde{C}_t 0} + \sum_{j>t}[(K_j + s_j + \frac{1}{r^2 p_j})^{-1} - 2K_j^{-1}]} - K_{ti} - \sum_{j<t} D_{\tilde{C}_j 0 i}\right) \\ &= \frac{r_i}{2}(s_t - s_{ti}). \end{aligned} \tag{53}$$

Without the options, the trade for investor i in the risky asset in session t is given by

$$\begin{aligned} \nabla\tilde{D}_{ti} &\equiv \tilde{D}_{ti} - \tilde{D}_{(t-1)i} \\ &= r_i\left[s_{ti}\tilde{z}_{ti} - s_t\tilde{u} + \frac{\tilde{x}_t}{r} + \sum_{j=0}^{t-1}(s_j - s_{ji})(\tilde{P}_t - \tilde{P}_{t-1}) + (s_t - s_{ti})\tilde{P}_t\right]. \end{aligned} \tag{54}$$

It is clear from this expression that the precision of the private information received prior to t affects the volume of trade in session t. With the options, the trade for investor i is given by

$$\nabla\tilde{D}_{ti} = r_i\left[s_{ti}\tilde{z}_{ti} - s_t\tilde{u} + \frac{\tilde{x}_t}{r} + (s_t - s_{ti})\tilde{P}_t\right]. \tag{55}$$

Thus with the quadratic options, private information causes trading only in the session in which it is received. The intuition is that the options allow investors to trade to a Pareto efficient allocation with respect to their current information, and this eliminates the need for further trading based on that information. However, the net effect of

[32] As the cases before, we define $K_T = \infty$, $p_T = \infty$ for simplicity of notation.

[33] I_A is the indicator function of event A.

the introduction of options on the total volume of trade is indeterminate. Note that, even with the introduction of new liquidity traders and new private information, the options still do not affect the price process for the underlying risky asset. However, the prices of the options are affected by the acquisition of new private information.

5.3 Market depth and a comparison with the Kyle model

Subrahmanyam (1991) has extended the classic Kyle (1985) model of a security market with private information to allow for risk aversion in both the competitive market maker and the risk averse informed traders. If we think of the subset of traders in our model who receive no private information as uninformed market makers, then our model is isomorphic to that of the Subrahmanyam version of the Kyle model, except that informed traders in our model behave competitively rather than strategically, and place limit rather than market orders. Thus denote the subset of investors for whom $s_t = s = 0$ by M for market maker, and define the risk tolerance of the market maker by $r_M = \int_M r_i \, di$. Then in the economy without options, the order flow absorbed by the market maker is given from Equation (54) by

$$\begin{aligned}\nabla \tilde{D}_{Mt} &= r_M \left[s_t(\tilde{P}_t - \tilde{u}) + \tilde{x}_t / r + \sum_{j=0}^{t-1} s_j \nabla P_t \right] \\ &= -\frac{r_M}{1 + r^2 s_t p_t} \left[K_{t-1}(\tilde{P}_t - \tilde{P}_{t-1}) + n_t(\tilde{y}_t - \tilde{P}_t) + \sum_{j=0}^{t-1} s_j \nabla P_t \right].\end{aligned}$$

In the economy with options, on the other hand, the order flow absorbed by the market maker is given by

$$\begin{aligned}\nabla \tilde{D}_{Mt} &= r_M [s_t(\tilde{P}_t - \tilde{u}) + \tilde{x}_t / r] \\ &= -\frac{r_M}{1 + r^2 s_t p_t} [K_{t-1}(\tilde{P}_t - \tilde{P}_{t-1}) + n_t(\tilde{y}_t - \tilde{P}_t)]. \qquad (56)\end{aligned}$$

To facilitate comparison with Kyle type models, we further assume that there are no public announcements, so that $n_t = 0$, and define $\tilde{X}_t \equiv -\nabla \tilde{D}_{Mt}$ as the order flow received by the market maker.[34] Then the price change is related to the order flow as follows: in the market

[34] With this convention a positive order flow corresponds to purchases.

without options

$$\nabla \tilde{P}_t = r_M^{-1} \left[\frac{K_{t-1}}{1 + r^2 s_t p_t} - \sum_{j=0}^{t-1} s_j \right]^{-1} \tilde{X}_t. \tag{57}$$

The corresponding equation for the economy with options is

$$\nabla \tilde{P}_t = r_M^{-1} \left[\frac{K_{t-1}}{1 + r^2 s_t p_t} \right]^{-1} \tilde{X}_t. \tag{58}$$

Following common usage, we define the market depth as $1/\lambda$, the coefficient of $\tilde{X}_t$ in these equations. Then the market depth in the economy with options, $1/\lambda^*$, is given by

$$1/\lambda^* = \frac{r_M K_{t-1}}{1 + r^2 s_t p_t}. \tag{59}$$

The market depth in the economy without options, $1/\lambda^0$, is

$$1/\lambda^0 = 1/\lambda^* - r_M \sum_{j=0}^{t-1} s_j. \tag{60}$$

In the market with options, market depth is increasing in the risk tolerance of the market makers and in the average posterior precision of the market, K_t; it is decreasing in the market risk tolerance, r, and the average precision of the current private signal, s_t, which jointly determine the aggressiveness of informed trading; it is also decreasing in the precision of noise trading in the current session, p_t. These results are consistent with the single trading session model of Subrahmanyam (1991) in which market depth increases with both the risk tolerance of the market maker and the variance of supply noise.

We see from Equation (58) that market depth is less in the economy without options. We state this result as

Theorem 7. *The introduction of options trading in a security leads to an increase in market depth.*

The intuition for the reduced market depth in the economy without options[35] is that in the absence of options less-informed traders (including market makers) tend to be trend followers as the result of prior information disadvantages, as we saw in Corollary 1. This creates a tendency for market makers to trade in the same direction as the price

[35] Fedenia and Grammatikos (1992) find that the bid-ask spread decreases for NYSE stocks following the introduction of options, while the opposite result is found for OTC stocks.

change, the strength of this tendency being directly proportional to the accumulated information disadvantage, $\sum_{j=0}^{t-1} s_j$; if this information disadvantage is sufficiently large market depth may be zero or even negative. In Kyle's (1989) single trading session model, risk averse informed and uninformed investors and noise traders submit demand functions to an auctioneer; whereas in our model, all participants behave competitively. When the numbers of informed and un-informed investors go to infinity, Kyle's (1989) model converges to the Hellwig (1980) model. Therefore a natural conjecture is that if Kyle's (1989) model is extended to many trading sessions then, as the number of informed and un-informed investors goes to infinity, it will converge to our model. Unfortunately, the multiple trading session version of Kyle's (1989) model is extremely difficult to analyze.

5.4 Trading behavior with short-term quadratic options

In section 5.2 it was shown that the introduction of quadratic options would lead to a reduction in the volume of trade based on private information. However, this result is sensitive to the type of options available for trade. We show in this section that when only short-term quadratic options are available, investors have the same ex ante utility although their trading behavior is affected. For simplicity, we discuss the two trading session economy.

Theorem 8. *When options $\tilde{O}_0(\tilde{P}_1) \equiv (\tilde{P}_1 - P_0)^2$, $\tilde{O}_1(\tilde{P}_1, \tilde{u}) \equiv (\tilde{u} - \tilde{P}_1)^2$, are introduced into the economy of section 5.2 with $T = 2$, the prices and demands for the risky assets are the same as in the market without options. Let $P_{\tilde{O}_0}$ be the price of $\tilde{O}_0$ at time 0, $D_{\tilde{O}_0 i}$ be the demand of individual i at time 0. The prices and demands for the options are given by*

$$P_{\tilde{O}_0} = K_0^{-1} - \left[K_1 + s_1 + \frac{1}{r^2 p_1}\right]^{-1},$$

$$P_{\tilde{O}_1} = K_1^{-1},$$

$$D_{\tilde{O}_0 i} = \frac{r_i}{2}(s_0 - s_{0i}),$$

$$D_{\tilde{O}_1 0i} = D_{\tilde{O}_1 1i} = \frac{r_i}{2}[(s_0 - s_{0i}) + (s_1 + s_{1i})].$$

The intuition for this result can be understood as follows. The identity,

$$(\tilde{u} - P_0)^2 = (\tilde{P}_1 - P_0)^2 + 2(\tilde{P}_1 - P_0)(\tilde{u} - \tilde{P}_1) + (\tilde{u} - \tilde{P}_1)^2,$$

implies that the payoff on the holding of $\frac{r_i}{2}(s_0 - s_{0i})$ units of the option $\tilde{C}_0$ of Theorem 6 with payoff $\frac{r_i}{2}(s_0 - s_{0i})(\tilde{u} - P_0)^2$ can be replicated in the economy of this section by the following strategy: purchase $\frac{r_i}{2}(s_0 - s_{0i})$ units of option $\tilde{O}_0$ in the first trading session, and in the second trading session purchase $r_i(s_0 - s_{0i})(\tilde{P}_1 - P_0)$ units of the risky asset and $\frac{r_i}{2}(s_0 - s_{0i})$ units of the option $\tilde{O}_1$. Notice that $r_i(s_0 - s_{0i})(\tilde{P}_1 - P_0)$ units of the risky asset is exactly the amount of informational trades eliminated by the introduction of the options $\tilde{C}_0$, $\tilde{C}_1$. This gives the result that when only the short-term options $\tilde{O}_0$, $\tilde{O}_1$ are available, the volume of trading in the risky security is the same as in the economy without options. However, since the short-term options $\tilde{O}_0$, $\tilde{O}_1$ and the underlying risky asset can replicate the payoff of the long-term option $\tilde{C}_1$, the welfare of investors is the same as in the economy with long-term options.

This example shows that the trading behavior of investors depends crucially upon the nature of the derivative assets available in the market as well as upon differences in the tastes and information of investors.

6. Conclusion

In this article we developed a noisy rational expectations model of the type of Hellwig (1980) in order to consider the effect of allowing investors to trade more frequently when public information about the asset payoff emerges gradually over time. For representative parameter values we find that there are significant welfare gains to both the well informed and the poorly informed from trading more frequently than once a year — only the averagely informed fail to gain. Well-informed traders tend to sell on a price rise, while the relatively poorly informed buy on a rise and, as the limit of continuous trading is reached, the well-informed synthesize by a dynamic strategy a payoff that is a concave function of the risky asset payoff, while the poorly informed synthesize a convex payoff function. This provides a new rationale for portfolio insurance type payoff functions based on differences in the quality of private information. The limiting continuous time economy is shown to achieve a Pareto efficient allocation, provided that a smoothness condition on the arrival of public information is satisfied.

Pareto efficiency may also be achieved in a single round of trading if there exists an option type contract whose payoff is a quadratic function of the payoff on the underlying risky asset; as a result there is no trading in response to new public information in such an economy. This points to the need to give attention to the array of available securities in making predictions about trading volume.

When the market achieves Pareto efficiency, either through continous trading or through trading in options, there exists a representative investor and option contracts are priced in accordance with the martingale principles of Black and Scholes (1973). Despite the differences in the precisions of investors' assessments of the final asset payoff, they agree on the volatility for pricing options.

The foregoing results were developed for an economy with only a single supply shock and set of private information signals at time 0. It is shown that the introduction of new supply shocks and new private information signals prior to each trading session restores trading volume in all trading sessions, even if quadratic options are traded. However, in the economy with options, trading is induced only by the need for each investor to take up the appropriate share of the supply shock.

It is argued that investors who receive no private information may be regarded as Kyle (1985) type market makers in a model in which all participants are risk averse, behave competitively, and submit limit orders. An expression for market depth is derived and compared with that of Subrahmanyam (1991); it is found that the introduction of option contracts leads to an increase in market depth, by eliminating the tendency of the uninformed market makers to trade as trend followers on account of their cumulative information disadvantage. In a subsequent article we extend the results on market making to a competitive risk averse economy in which investors may place market orders as well as limit orders. Further directions for research include extending the model to incorporate many risky assets and endogenizing the acquisition of private information.

Appendix

Throughout the Appendix we set $x_i = 0$ without loss of generality, unless otherwise stated.

Proof of Theorem 1. It is straightforward to verify that in the proposed equilibrium aggregate asset demand is equal to supply in each period. It remains only to show that, given the conjectured equilibrium, the optimal demand for the risky asset by investor i, $\tilde{d}^*_{ti}$ is equal to $\tilde{D}_{ti}$, the conjectured equilibrium demand given in Equation (10). Let $E_{ti}[\cdot]$ denote the expectation with respect to the information set $\tilde{F}_{ti}$, let $\tilde{w}_{ti} \equiv \sum_{j=1}^{t}(\tilde{P}_j - \tilde{P}_{j-1})\tilde{D}_{(j-1)i}$ be the wealth of investor i at time τ_t, and let $V_{it}^{T}(w)$ be the expected utility of trader i at time τ_t in a T trading session economy under the optimal policy. Then at the beginning of the last trading session,

$$V^T_{i(T-1)}(w) = \max_{d_{(T-1)i}} E_{(T-1)i}\left[-\exp\left\{-\left[\frac{w}{r_i} + (\tilde{u} - \tilde{P}_{T-1})\frac{d_{(T-1)i}}{r_i}\right]\right\}\right]. \tag{61}$$

This problem is identical to that in the single-session economy discussed in Section 2, and the optimal asset demand, $\tilde{d}^*_{(T-1)i}$ is given by

$$\begin{aligned}\tilde{d}^*_{(T-1)i} &= r_i K_{(T-1)i}(\tilde{\mu}_{(T-1)i} - \tilde{P}_{T-1}) \\ &= r_i(s_i z_i - s\tilde{u} + \tilde{x}/r - (s_i - s)\tilde{P}_{T-1}).\end{aligned} \tag{62}$$

Comparing Equation (62) with Equation (10), it is apparent that $\tilde{d}^*_{(T-1)i} = \tilde{D}_{(T-1)i}$ and the conjectured equilibrium is confirmed for the final trading session. Then the expected utility of investor i under the optimal policy at the beginning of the final trading session, $V^T_{(T-1)i}(w)$, is

$$\begin{aligned}V^T_{(T-1)i}(w) &= E_{(T-1)i}\left[-\exp\left\{\left[\frac{w}{r_i} + (\tilde{u} - \tilde{P}_{T-1})\frac{\tilde{D}_{(T-1)i}}{r_i}\right]\right\}\right] \\ &= -\exp\left\{-\left[\frac{w}{r_i} + \frac{(\tilde{D}_{(T-1)i}/r_i)^2}{2K_{(T-1)i}}\right]\right\}.\end{aligned}$$

Define $\nabla\tilde{D}_{(T-1)i} \equiv \tilde{D}_{(T-1)i} - \tilde{D}_{(T-2)i}$, the change in demand for the risky asset under the conjectured optimal strategy. The mean and variance of this change in demand are given by

$$E[\nabla\tilde{D}_{(T-1)i}/r_i \mid \tilde{F}_{(T-2)i}] = \frac{n_{T-1}(s - s_i)}{K_{T-1}K_{(T-1)i}}D_{(T-2)i}, \tag{63}$$

$$\mathrm{var}[\nabla\tilde{D}_{(T-1)i} \mid \tilde{F}_{(T-2)i}] = \frac{(s - s_i)^2 n_{T-1}K_{(T-1)i}}{K_{(T-2)i}K^2_{T-1}}. \tag{64}$$

Armed with these expressions, we can calculate the expected utility at time $T-2$ as a function of wealth at $T-2$, $w_{(T-2)i}$, and the number of units of the risky asset purchased, $d_{(T-1)i}$.

$$\begin{aligned}&E_{(T-2)i}[V^T_{(T-1)i}(\tilde{w}_{(T-1)i} \mid w_{T-2} = w, d_{(T-2)i}] \qquad (65)\\ &= E_{(T-2)i}\left[-\exp\left\{-\left[\frac{w}{r_i} + (\tilde{P}_{T-1} - \tilde{P}_{T-2})\frac{d_{(T-2)i}}{r_i}\right.\right.\right. \\ &\qquad\left.\left.\left. + \frac{(\tilde{D}_{(T-1)i}/r_i)^2}{2K_{(T-1)i}}\right]\right\}\right]\end{aligned}$$

$$= -\sqrt{\frac{K_{(T-2)i}K_{T-1}^2}{2\pi(s-s_i)^2 n_{T-1}K_{(T-1)i}}}$$

$$\int \exp\left\{-\left[\frac{w}{r_i} + \frac{\tilde{D}_{(T-1)i}}{r_i}\frac{d_{(T-2)i}/r_i}{s-s_i} + \frac{(\nabla\tilde{D}_{(T-1)i}/r_i + D_{(T-2)i}/r_i)^2}{2K_{(T-1)i}} + \frac{(\nabla\tilde{D}_{(T-1)i} - E_{(T-2)i}[\nabla\tilde{D}_{(T-1)i}])^2}{2\,\mathrm{var}_{(T-2)i}[\nabla\tilde{D}_{(T-1)i}]}\right]\right\} d\nabla\tilde{D}_{(T-1)i}$$

$$= -\sqrt{\frac{K_{(T-2)i}}{K_{(T-2)i} + n_{T-1}((s_i-s)/K_{T-1})^2}}$$

$$\exp\left\{-\left[\frac{w}{r_i} + \frac{[(d_{(T-2)i} - D_{(T-2)i})/r_i]^2}{2(s_i - s + K_{T-2}K_{T-1}/n_{T-1})} \frac{(D_{(T-2)i}/r_i)^2}{2K_{(T-2)i}}\right]\right\}. \tag{66}$$

Maximizing with respect to $d_{(T-2)i}$ at time τ_{T-2} we find $d^*_{(T-2)i} = D_{(T-2)i}$. Therefore, the investor's expected utility at the beginning of session $T-2$ under the optimal policy, $V^T_{(T-2)i}(w)$, is given by

$$V^T_{(T-2)i}(w) = -\sqrt{\frac{K_{(T-2)i}}{K_{(T-2)i} + n_{T-1}((s_i-s)/K_{T-1})^2}}\, V^{T-1}_{(T-2)i}(w). \tag{67}$$

Proceeding recursively, Equation (67) implies that the investor behaves myopically in each trading session, so his demand for the risky security is given by Equation (10). This completes the proof. ■

Proof of Lemma 1. Taking account of the investor's initial endowment, x_i, the ex ante utility of investor i in a one trading session economy is given in Verrecchia (1982):

$$V^1_{0i}(w) = \frac{-1}{\sqrt{L_0K_{0i}}}\exp\left[\frac{-\bar{u}x_i}{r_i} + \left(\frac{1}{h} - \frac{1}{K_0^2L_0}\right)\frac{x_i^2}{2r_i^2}\right].$$

Applying Equation (67) recursively yields the result of the lemma. ■

Proof of Lemma 2. Let $\delta_t \equiv \frac{1}{K_{t-1}} - \frac{1}{K_t}$, then $\sum_{t=1}^{T} \delta_t = \frac{1}{K_0}$. Equation (12) can be simplified to get

$$\gamma_i(T) = \frac{r_i}{2}\left[\sum_{t=1}^{T} \ln(1 + (s_i - s)\delta_t) - \ln \frac{K_{0i}}{K_0}\right].$$

Since $\ln(1+(s_i-s)\delta_t)$ is a concave function, it follows from Jensen's inequality that $\gamma_i(T)$ is maximized by setting δ_t constant. But, direct calculation from Equation (9) gives

$$\mathrm{var}[\nabla \tilde{P}_t] = \delta_t + \delta_t^2\left(s + \frac{1}{r^2 p}\right). \tag{68}$$

This finishes the first part of Lemma 2. Direct calculation gives the second part. ■

Proof of Corollary 1. Follows immediately from Equation (15) and Corollary 2. ■

Proof of Corollary 2. Equation (9) implies that

$$\begin{aligned}\mathrm{cov}(\nabla \tilde{P}_{t+1}, \tilde{P}_t) &= -\frac{n_{t+1} 1 + r^2 sp}{K_t^2 K_{t+1} r^2 p}\,\mathrm{cov}(\nabla \tilde{P}_{t+1}, \nabla \tilde{P}_t)\\ &= \frac{n_{t+1} n_t (1 + r^2 sp)}{K_{t-1} K_t^2 K_{t+1} r^2 p}.\end{aligned} \tag{69}$$

■

Proof of Lemma 4. From Lemma 2, $\mathrm{var}[\nabla \tilde{P}_t] = \delta_t + \delta_t^2(s + \frac{1}{r^2 p})$. Then using the elementary fact that $E[|x|] = \sqrt{2\sigma^2/\pi}$, for a normal distribution with mean 0 and variance σ^2, we have

$$E[\mathit{Volume}] = Q\sqrt{\frac{1}{2\pi}} \sum_{t=1}^{T} \sqrt{\delta_t + \delta_t^2\left(s + \frac{1}{r^2 p}\right)}. \tag{70}$$

Since $\sqrt{\delta_t + \delta_t^2(s + \frac{1}{r^2 p})}$ is a concave function of δ_t, and $\sum_{t=1}^{T} \delta_t = \frac{1}{K_0}$ from the proof of Lemma 2, it follows from Jensen's inequality that total expected volume is maximized when δ_t is constant for all t, and then we have

$$E[\mathit{Volume}] = Q\sqrt{\frac{1}{2\pi}}\sqrt{\frac{T}{K_0} + \frac{1 + r^2 sp}{K_0^2 r^2 p}}. \tag{71}$$

■

Proof of Lemma 5. Let $\gamma_i(T) \equiv \frac{r_i}{2}\sum_{t=1}^{T-1}\ln(1+\frac{n_t(s_i-s)^2}{K_t^2K_{(t-1)i}}) \equiv \sum_{t=1}^{T}\ln(1+\sigma_i(t,T))$. Then, defining $\bar{\delta} \equiv \max_t[\delta_t]$, since $\mathrm{var}[\nabla P_{t+1}] = \delta_{t+1} + \delta_{t+1}^2(s+\frac{1}{r^2p})$, regularity Condition A becomes $\lim_{T\to\infty}\bar{\delta}(T) = 0$.

Now we define

$$\begin{aligned}\bar{\gamma}_i(T) &\equiv \frac{r_i}{2}\sum_{t=1}^{T}\sigma_i(t,T) = \frac{r_i}{2}\sum_{t=1}^{T-1}\frac{n_t(s_i-s)^2}{K_t^2K_{(t-1)i}}\\ &= \frac{r_i}{2}\sum_{t=0}^{T-2}\left(\frac{1}{K_t}-\frac{1}{K_{t+1}}\right)\frac{(s_i-s)^2/K_t}{1+(s_i-s)/K_t}\\ &\quad-\left(\frac{1}{K_t}-\frac{1}{K_{t+1}}\right)^2\frac{(s_i-s)^2}{1+(s_i-s)/K_t}. \end{aligned} \tag{72}$$

■

Obviously the second term converges to 0 and the first term converges to the integral

$$\frac{r_i}{2}\int_0^{\frac{1}{K_0}}\frac{(s_i-s)^2t}{1+(s_i-s)t}\,dt = \frac{r_i}{2}\left[\frac{s_i-s}{K_0}-\ln\left(1+\frac{s_i-s}{K_0}\right)\right].$$

Using the elementary fact that $x/(1+x)\le\ln(1+x)\le x$, we have

$$\begin{aligned}\frac{r_i}{2}\sum_{t=1}^{T-1}\frac{\sigma_i(t,T)}{1+\sigma_i(t,T)} &\le \frac{r_i}{2}\sum_{t=1}^{T-1}\ln(1+\sigma_i(t,T))\\ &= \gamma_i(T)\le\frac{r_i}{2}\sum_{t=1}^{T-1}\sigma_i(t,T)=\bar{\gamma}_i(T).\end{aligned}$$

Since

$$\begin{aligned}\sigma_i(t,T)\equiv\frac{n_t(s_i-s)^2}{K_t^2K_{(t-1)i}} &= \frac{n_t}{K_{t-1}K_t}\frac{(s-s_i)^2}{K_{(t-1)i}}\\ &\le \frac{(s-s_i)^2}{K_{(t-1)i}}\bar{\delta}(T)\le\left[\frac{(s-s_i)^2}{K_{0i}}\right]\bar{\delta}(T),\end{aligned}$$

we get

$$\frac{\bar{\gamma}_i(T)}{1+\bar{\delta}(T)(s-s_i)^2/K_{0i}}\le\gamma_i(T)\le\bar{\gamma}_i(T),$$

which implies that

$$\lim_{T\to\infty}\gamma_i(T)=\lim_{T\to\infty}\bar{\gamma}_i(T)=\frac{r_i}{2}\left[\frac{s_i-s}{K_0}-\ln\left(1+\frac{s_i-s}{K_0}\right)\right].$$

The expected utility at trading session 0 is

$$\lim_{T\to\infty} V_{0i}^{T}(w) = \sqrt{\frac{K_{0i}}{K_0}} \exp\left[\frac{s-s_i}{2K_0}\right] V_{0i}^{1}(w). \tag{73}$$

■

Proof of Theorem 2. The final wealth of investor i is

$$\begin{aligned}
\tilde{w}_i &= \sum_{t=0}^{T-1}(\tilde{P}_{t+1}-\tilde{P}_t)\tilde{D}_{ti} = (\tilde{u}-P_0)\tilde{D}_{0i} + \sum_{t=0}^{T-1}(\tilde{P}_{t+1}-\tilde{P}_t)(\tilde{D}_{ti}-\tilde{D}_{0i}) \\
&= (\tilde{u}-P_0)\tilde{D}_{0i} + r_i(s-s_i)\sum_{t=0}^{T-1}(\tilde{P}_{t+1}-\tilde{P}_t)(\tilde{P}_t-P_0) \\
&= (\tilde{u}-P_0)\tilde{D}_{0i} + \frac{r_i}{2}(s-s_i)[(\tilde{u}-P_0)^2 - \sum_{t=0}^{T-1}(\tilde{P}_{t+1}-\tilde{P}_t)^2] \\
&= (\tilde{u}-P_0)(\tilde{D}_{0i}-r_i\tilde{b}_i(T)) + \frac{r_i}{2}(s-s_i-a_i(T))(\tilde{u}-\tilde{P}_0)^2 \\
&\quad - \frac{r_i(s-s_i)}{2K_0} - r_i\tilde{c}_i(T) \\
&= w_i^e(\tilde{u}) - \frac{r_i}{2}a_i(T)(\tilde{u}-P_0)^2 - r_i\tilde{b}_i(T)(\tilde{u}-P_0) - r_i\tilde{c}_i(T),
\end{aligned}$$

where

$$a_i(T) = (s-s_i)\sum_{t=0}^{T-1}\frac{n_{t+1}^2}{K_t^2K_{t+1}^2},$$

$$\tilde{b}_i(T) = (s-s_i)\sum_{t=1}^{T-1}\left(\frac{1}{K_{t-1}} - \frac{2}{K_t} + \frac{1}{K_{t+1}}\right)\frac{\sum_{j=0}^{t} n_j\tilde{\eta}_j}{K_t},$$

$$\tilde{c}_i(T) = -(s-s_i)\sum_{t=0}^{T-1}\left[\left(\tilde{\eta}_{t+1} - K_t^{-1}\sum_{j=0}^{t} n_j\tilde{\eta}_j\right)K_t^{-1}\sum_{j=0}^{t} n_j\tilde{\eta}_j + \frac{1}{2K_0}\right],$$

and $w_i^e(\tilde{u})$ is given by

$$w_i^e(\tilde{u}) = (\tilde{u}-P_0)\tilde{D}_{0i} + \frac{1}{2}r_i(s-s_i)((\tilde{u}-P_0)^2 - K_0^{-1}). \tag{74}$$

For notational simplicity we use $n_T/K_t = 1$, $\eta_T = 0$ to denote the fact that the risky asset payoff realized at time $\tau_T = 1$. Condition A implies that $\lim_{T\to\infty} a_i(T) = 0$, and that

$$\begin{aligned}
\lim_{T\to\infty} E[\tilde{c}_i(T)] &= \lim_{T\to\infty} -(s-s_i)\left[-\sum_{t=0}^{T-1}\frac{n_{t+1}}{K_{t+1}K_t^2}\sum_{j=0}^{t} n_j + \frac{1}{2K_0}\right] \\
&= (s-s_i)\lim_{T\to\infty}\left[\sum_{t=0}^{T-1}\left(\frac{1}{K_t}-\frac{1}{K_{t+1}}\right)\left(1-\frac{K_0}{K_t}\right)-\frac{1}{2K_0}\right] \\
&= 0.
\end{aligned}$$

The final equality follows since the first term of the previous line is the Riemann representation of the integral $\int_0^{1/K_0}[1-K_0 t]\,dt = \frac{1}{2K_0}$ given Condition A.

Let $\tilde{\eta}$ denote the vector $(\tilde{\eta}_1, \ldots, \tilde{\eta}_{T-1})$. Direct calculation of the expected utility of final wealth conditional on $\tilde{\eta}$ gives

$$\begin{aligned}
E\left[-\exp\left\{\frac{\tilde{w}_i}{r_i}\right\} \mid \tilde{\eta}\right] &= E\left[-\exp\left\{-(\tilde{u}-P_0)\left(\frac{\tilde{D}_{0i}}{r_i}-\tilde{b}_i(T)\right)\right.\right. \\
&\qquad - \frac{s-s_i-a_i(T)}{2}(\tilde{u}-P_0)^2 \\
&\qquad \left.\left. + \frac{s-s_i}{2K_0} + \tilde{c}_i(T)\right\} \mid \tilde{F}_{0i}, \tilde{\eta}\right] \\
&= -\sqrt{\frac{K_{0i}}{2\pi}}\exp\left[\frac{s-s_i}{2K_0}\right] \\
&\quad \times \int \exp\left\{-(\tilde{u}-P_0)\left(\frac{\tilde{D}_{0i}}{r_i}-\tilde{b}_i(T)\right)\right. \\
&\qquad - \frac{s-s_i-a_i(T)}{2}(\tilde{u}-P_0)^2 \\
&\qquad \left. + \tilde{c}_i(T) - \frac{K_{0i}}{2}(\tilde{u}-\tilde{\mu}_{0i})^2\right\} d\tilde{u} \\
&= -\sqrt{\frac{K_{0i}}{2\pi}}\exp\left[\frac{s-s_i}{2K_0}\right] \\
&\quad \times \int \exp\left\{(\tilde{u}-P_0)\tilde{b}_i(T)\right. \\
&\qquad - \frac{K_0-a_i(T)}{2}(\tilde{u}-P_0)^2 + \tilde{c}_i(T)
\end{aligned}$$

$$-\frac{K_{0i}}{2}(\tilde{\mu}_{0i}-P_0)^2\Bigg\}\,d\tilde{u}$$

$$= -\sqrt{\frac{K_{0i}}{K_0 - a_i(T)}}\exp\left[\frac{s-s_i}{2K_0}\right]$$

$$\times \exp\left\{-\frac{\tilde{D}_{0i}^2}{2r_i^2K_{0i}} + \frac{\tilde{b}_i^2(T)}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right\}$$

$$= -\sqrt{\frac{K_{0i}}{K_0 - a_i(T)}}\exp\left[\frac{s-s_i}{2K_0}\right]$$

$$\times \exp\left\{\frac{\tilde{b}_i^2(T)}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right\} V_{0i}^1(0).$$

Taking the expectation over $\tilde{\eta}$ we have

$$V_{0i}^T(0) = -\sqrt{\frac{K_{0i}}{K_0 - a_i(T)}}\exp\left[\frac{s-s_i}{2K_0}\right] \times E\left[\exp\left(\frac{\tilde{b}_i^2(T)}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right)\right] V_{0i}^1(0) \qquad (75)$$

However, from Lemma 5, we have

$$\lim_{T\to\infty} V_{0i}^T(0) = \sqrt{\frac{K_{0i}}{K_0}}\exp\left[\frac{s-s_i}{2K_0}\right] V_{0i}^1(0). \qquad (76)$$

Equating the limit of Equation (75) to Equation (76), and recalling that $\lim_{T\to\infty} a_i(T) = 0$, implies that

$$\lim_{T\to\infty} E\left[\exp\left\{\frac{\tilde{b}_i^2(T)}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right\}\right] = 1.$$

However, since $\exp(\cdot)$ is convex Jensen's inequality implies that

$$1 = \lim_{T\to\infty} E\left[\exp\frac{\tilde{b}_i(T)^2}{2[K_0 - a_i(T)]} + \tilde{c}_i(T))\right] \geq \lim_{T\to\infty} E[\exp(\tilde{c}_i(T))]$$
$$\geq \lim_{T\to\infty}\exp(E[\tilde{c}_i(T)]) = \exp(\lim_{T\to\infty} E[\tilde{c}_i(T)]) = 1,$$

which implies that $\tilde{c}_i(T)$ converges to 0 in probability. To show that $\tilde{b}_i(T)$ also converges to zero in probability note that Jensen's inequality also implies that

$$
\begin{aligned}
1 &= \lim_{T\to\infty} E\left[\exp\left(\frac{[\tilde{b}_i(T)]^2}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right]\right. \\
&\geq \lim_{T\to\infty} \exp\left(E\left[\frac{\tilde{b}_i(T)^2}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right]\right) \\
&= \exp\left(\lim_{T\to\infty} E\left[\frac{\tilde{b}_i(T)^2}{2[K_0 - a_i(T)]} + \tilde{c}_i(T)\right]\right) \\
&= \exp\left(\lim_{T\to\infty} E\left[\frac{\tilde{b}_i(T)^2}{2[K_0 - a_i(T)]}\right]\right) \geq 1.
\end{aligned}
$$

Therefore, $\lim_{T\to\infty} E[\tilde{b}_i(T)^2/2[K_0 - a_i(T)]] = 0$, which implies that $\tilde{b}_i(T)$ converges to 0 in probability. In summary, we have shown that the wealth of investor i converges to $\tilde{w}_i^e(\tilde{u})$ as $T \to \infty$. To show that the limiting wealth allocation $w_i^e(\tilde{u})$ is Pareto optimal, we need to show that the marginal rate of substitution between state $\tilde{u} = u_k$, u_l is the same for all investors. The marginal rate of substitution between state k and l for investor i is

$$
\begin{aligned}
M_{kl}^i &= \frac{\exp\{-K_{0i}(u_k - \tilde{\mu}_{0i})^2/2 - w_i^e(u_k)/r_i\}}{\exp\{-K_{0i}(u_l - \tilde{\mu}_{0i})^2/2 - w_i^e(u_l)/r_i\}} \\
&= \exp\{K_0(u_l - u_k)(u_l + 2P_0 + 2u_k)\},
\end{aligned}
$$

which is independent of i. ■

Proof of Theorem 4. Substituting for the investor's terminal wealth in the utility function, using the conjectured equilibrium price of the option, and taking expectations, investor i's portfolio problem may be written as

$$
\begin{aligned}
\max_{d_{C0i}, d_{0i}} EU_i \equiv & -\sqrt{\frac{1}{1 + \frac{2d_{C0i}}{r_i K_{0i}}}} \\
& \times \exp\left\{\frac{[\frac{d_{0i}}{r_i} - K_{0i}(\mu_{0i} - P_0)]^2}{K_{0i} + \frac{2d_{C0i}}{r_i}} + \frac{d_{C0i}}{K_0 r_i} - \frac{K_{0i}}{2}(\mu_{0i} - P_0)^2\right\},
\end{aligned}
$$

where d_{C0i} and d_{0i} are the number of units of the option and the risky asset purchased by the investor. The optimal portfolio is given by

$$
d_{0i}^* = r_i K_{0i}(\mu_{0i} - P_0),\ d_{C0i}^* = r_i(s - s_i)/2.
$$

The optimal portfolio is identical to the equilibrium portfolio of the theorem: market clearing for the risky asset follows from our previous results and is immediate for the option. ■

Proof of Theorem 5. See proof of Theorem 6. ■

Proof of Theorem 6.

(i) For simplicity of exposition, we provide the proof for the case of two trading sessions: the extension to a multiple-period model is straightforward. It is immediate that the conjectured price functions are consistent with market clearing given the conjectured demand functions. It remains to show that the conjectured demand function is optimal. Let $V_{1i}(w)$ be the derived utility of individual i in the final trading session:

$$V_{1i}(\tilde{w}) \equiv \max_{d_{1i}} E\left[-\exp\left\{\frac{-1}{r_i}\left[\tilde{w} + (\tilde{u} - P_1)\, d_{1i}\right]\right\} \mid \tilde{F}_{1i}\right], \qquad (77)$$

where $\tilde{w} = (\tilde{P}_1 - P_0)D_{0i}$. It follows from our previous results that

$$V_{1i}(\tilde{w}) = -\exp\left\{\frac{-1}{r_i}\left[(\tilde{P}_1 - P_0)\tilde{D}_{0i} + \frac{r_i K_{1i}}{2}(\tilde{\mu}_{1i} - \tilde{P}_1)^2\right]\right\}, \qquad (78)$$

$$\tilde{D}_{1i} = r_i K_{1i}(\tilde{\mu}_{1i} - \tilde{P}_1) = r_i \sum_{j=0}^{1}\{s_{ji}z_{ji} - s_j\tilde{u} + \tilde{x}_j/r - (s_{ji} - s_j)\tilde{P}_1\}. \quad (79)$$

Define $A_i \equiv \tilde{\mu}_{1i} - \tilde{P}_1$, $B \equiv \tilde{P}_1 - \tilde{P}_0$, $G_i \equiv (K_{1i}E[A_i \mid F_{0i}],\ d_{0i}/r_i)'$, $I_i \equiv E[(A_i, B)' \mid F_{0i}]$, $\Sigma_i \equiv \text{var}[(A_i, B) \mid F_{0i}]$, and

$$\Phi_i \equiv \frac{1}{|\Sigma_i|}\begin{pmatrix} \Sigma_{22i} + K_{1i}|\Sigma_i| & -\Sigma_{21i} \\ -\Sigma_{12i} & \Sigma_{11i} \end{pmatrix}.$$

Then the objective function for the first trading session may be written as

$$E[V_{1i}(\tilde{w} \mid F_{0i}] = -\sqrt{\frac{1}{1 + K_{1i}\Sigma_{11i}}}\exp\{G_i'\Phi_i^{-1}G_i/2 - G_i'I_i\}. \qquad (80)$$

The first-order condition is

$$G_i'\Phi_i^{-1}\frac{dG_i}{dd_{0i}} = I'\frac{dG_i}{dd_{0i}}.$$

Simplifying we obtain the optimal first-session demand for the risky asset:

$$d_{0i}^* = r_i\frac{(1 + K_{1i}\Sigma_{11i})E[B \mid F_{0i}] - K_{1i}\Sigma_{12i}E[A_i \mid F_{0i}]}{\Sigma_{22i} + K_{1i}|\Sigma_i|};$$

where

$$E[A_i \mid F_{0i}] = \frac{K_0}{K_1}[\mu_{0i} - P_0],\ E[B \mid F_{0i}] = \frac{K_1 - K_0}{K_1}[\mu_{0i} - P_0],$$

$$\Sigma_{22i} = \frac{1}{K_1^2}\left[\frac{(K_1 - K_0)(K_{1i} + s_1 - s_{1i})}{K_{0i}} + s_1 + 1/r^2 p_1\right],$$

$$K_{1i}|\Sigma_i| = \frac{1}{K_1^2}\left[\frac{(1 + r^2 s_1 p_1)(K_{1i} - K_{0i})}{K_{0i} r^2 p_1} + \frac{(s_{1i} - s_1)(K_1 - K_0)}{K_{0i}}\right],$$

$$1 + K_{1i}\Sigma_{11i} = \frac{K_{1i}}{K_1^2 K_{0i}}[K_0^2 + K_{0i}(K_1 - K_0 + s_1 + 1/r^2 p_1)],$$

$$\Sigma_{12i} = \frac{1}{K_1^2}\left[\frac{(K_1 - K_0)(K_0 - K_{0i})}{K_{0i}} - s_1 - 1/r^2 p)\right].$$

Simple algebra yields

$$d_{0i}^* = r_i K_{0i}[\tilde{\mu}_{0i} - P_0],$$

which is identical to the conjectured demand, D_{0i}.

(ii) Follows from Theorem 8 and the following discussion which shows the equivalence of the options analyzed in Theorems 6 and 8. ■

Proof of Theorem 8. With the options $\tilde{O}_0 \equiv (\tilde{P}_1 - P_0)^2$, $\tilde{O}_1 \equiv (\tilde{u} - \tilde{P}_1)^2$, the price for $\tilde{O}_1$ can be determined in the second trading session and is given by $P_{\tilde{O}_1} = K_1^{-1}$. Since $P_{\tilde{O}_1}$ is a nonrandom, it follows that the price for $\tilde{O}_1$ in the first period is also the same.

The following expression for the derived utility at the last trading session is obtained by substituting the equilibrium portfolio demands into the objective function stated in the proof of Theorem 4:

$$V_{1i}(w) = -\sqrt{\frac{K_{1i}}{K_{1i} + 2D_{\tilde{O}_1 i}/r_i}} \exp\left[-\frac{w}{r_i} + \frac{D_{\tilde{O}_1 i}}{r_i K_1} - \frac{K_{1i}}{2}(\tilde{\mu}_{1i} - \tilde{P}_1)^2\right],$$

where $w = (P_1 - P_0)D_{0i} + D_{\tilde{O}_0 i}((P_1 - P_0)^2 - P_{\tilde{O}_0})$. Dropping the nonrandom terms in $V_{i1}(w)$, investor i's optimization problem in the first trading session is to maximize the expected value of $\tilde{U}_{i1}(d_{0i}, d_{\tilde{O}_0 i})$ by the portfolio choice d_{0i}, $d_{\tilde{O}_0 i}$, where

$$\tilde{U}_{1i}(d_{0i}, d_{\tilde{O}_0 i}) \equiv \exp\left[-\frac{(P_1 - P_0)d_{0i} + d_{\tilde{O}_0 i}((P_1 - P_0)^2 - P_{\tilde{O}_0})}{r_i} - \frac{K_{1i}}{2}(\tilde{\mu}_{1i} - \tilde{P}_1)^2\right]. \tag{81}$$

Defining $A_i \equiv \mu_{1i} - \tilde{P}_1$, $B \equiv \tilde{P}_1 - P_0$, $G_i \equiv (K_{1i}E[A_i \mid F_{0i}], D_{0i}/r_i)'$, $I_i \equiv E[(A_i, B)' \mid F_{0i}]$, $\Sigma_i \equiv \text{var}[(A_i, B) \mid F_{0i}]$, and taking the expectation in Equation (81), we get

$$E[\tilde{U}_{1i}(d_{0i}, d_{\tilde{O}_0 i}) \mid F_{0i}] = -\sqrt{\frac{1}{1 + K_{1i}\Sigma_{11i} + 2D_{\tilde{O}_0 i}(\Sigma_{22i} + K_{1i}|\Sigma_i|)/r_i}} \times \exp\left\{\frac{\tilde{P}_{\tilde{O}_0} D_{\tilde{O}_0 i}}{r_i} + G_i'\Phi_i^{-1}G_i/2 - G_i'I_i\right\}, \tag{82}$$

where

$$\Phi_i = \frac{1}{|\Sigma_i|}\begin{pmatrix} \Sigma_{22i} + K_{1i}|\Sigma_i| & -\Sigma_{21i} \\ -\Sigma_{12i} & \Sigma_{11i} + 2D_{\tilde{O}_0 i}|\Sigma_i| \end{pmatrix}.$$

The first-order condition for the optimal holding of the options, $d^*_{\tilde{O}_0 i}$, is given by

$$\frac{\Sigma_{22i} + K_{1i}|\Sigma_i|}{1 + K_{1i}\Sigma_{11i} + 2d^*_{\tilde{O}_0 i}(\Sigma_{22i} + K_{1i}|\Sigma_i|)/r_i} = P_{\tilde{O}_0},$$

which simplifies to

$$\left[\frac{1}{K_0} - \frac{1}{K_1 + s_1 + 1/r^2 p_1}\right]^{-1} + s_{0i} - s_0 + 2d^*_{\tilde{O}_0 i}/r_i = P^{-1}_{\tilde{O}_0}.$$

Equating the aggregate demand for the option to zero yields the equilibrium price of the option

$$P_{\tilde{O}_0} = \frac{1}{K_0} - \frac{1}{K_1 + s_1 + 1/r^2 p_1},$$

and the equilibrium holding of the option, $d_{O_0 i}$, is obtained by substituting the expression for the price into the previous equation:

$$d^*_{\tilde{O}_0 i} = \frac{r_i}{2}(s_0 - s_{0i}) \equiv D^*_{\tilde{O}_0 i}.$$

The first-order condition for the equilibrium holding of the risky asset, d_{0i}, is

$$G_i' \Phi_i^{-1} \frac{dG_i}{dd_{0i}} = I' \frac{dG_i}{dd_{0i}},$$

which may be written as

$$\begin{aligned}(2d_{\tilde{O}_0 i} E[B \mid F_{0i}] + d_{0i}/r_i)(\Sigma_{22i} + K_{1i}|\Sigma_i|) + K_{1i}\Sigma_{12i}E[A_i \mid F_{0i}] \\ = (1 + K_{1i}\Sigma_{11i} + 2d_{\tilde{O}_0 i}\Sigma_{22i} + 2d_{\tilde{O}_0 i}K_{1i}|\Sigma_i|)E[B \mid F_{0i}],\end{aligned}$$

and simplifies to

$$d_{0i}^* = r_i \frac{(1 + K_{1i}\Sigma_{11i})E[B \mid F_{0i}] - K_{1i}\Sigma_{12i}E[A_i \mid F_{0i}]}{\Sigma_{22i} + K_{1i}|\Sigma_i|}$$

This is the same as the expression for d_{0i}^* in the proof of Theorem 6, which implies that the equilibrium demand for the risky asset is the same with or without the option. ∎

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