A General Equilibrium Model of Portfolio Insurance

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This article examines the effects of portfolio insurance on market and asset price dynamics in a general equilibrium continuous-time model. Portfolio insurers are modeled as expected utility maximizing agents. Martingale methods are employed in solving the individual agents' dynamic consumption-portfolio problems. Comparisons are made between the optimal consumption processes, optimally invested wealth and portfolio strategies of the portfolio insurers and "normal agents." At a general equilibrium level, comparisons across economies reveal that the market volatility and risk premium are decreased and the asset and marketprice levels increased, by the presence of portfolio insurance.

There recently has been a surge in interest amongst both academicians and practitioners in the intertemporal behavior of market volatility (standard deviation of price return).[1] Since the market crash of October

This paper is drawn from Chapter 1 of my doctoral thesis at Carnegie Mellon University. I would like to thank Duane Seppl for useful conversations that led to my Interest in portfolio Insurance and subsequent invaluable discussions. I am also grateful to Chi-Fu Huang (the editor), an anonymous referee for this journal, Franklin Allen (the executive editor), Domenlco Cuoco, Bob Dammon. Rick Green, Burton Hollifield, John Lehoczky, and seminar participants at University of British Columbia. Carnegie Mellon University. Indiana University, INSEAD, University of Iowa, London Business School, University of Michigan, MIT, Northwestern University, University of Pennsylvania (Wharton), University of Warwick, the 1993 IMA Workshop in Mathematical Finance and the 1993 WFA meetings for their helpful comments. I especially thank Steve Shreve for valuable input throughout this research. Financial support from the William Larimer Mellon Foundation is gratefully acknowledged. Address correspondence to Suleyman Basalt. Finance Department, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104-6367.

[1] Although there Is now a large empirical literature on price volatility [Black (1976), French. Schwert and Stambaugh (1987). Merton (1980), Schwert (1989, 1990), Stoll and Whaley (1990)]. thcoretical work [e.g., Basak (1995, §5.4), Hindy (1992)] In this area is mote limited.

The Review of Financial Studies Winter 1995 Vol. 8, No. 4, pp. 1059–1090

1987, when market volatility was very high, much attention has focused on the factors that affect volatility. Of particular interest is the influence of the rate of information flow into the market, various security trading practices, and macroeconomic variables. The efficient markets hypothesis would claim that the volatility of a stocks return is due solely to the rate of arrival of new information about that stock's future payoff. However, the popular press, regulators, and numerous researchers suggest that certain types of dynamic trading strategies tend to increase stock market volatility. In particular, portfolio insurance and program trading have been accused of contributing to the stock market crash by increasing volatility [e.g., the Brady Report (1988)] and having a general destabilizing effect on the market [e.g., Hull (1993, p. 332)].

A portfolio insurer is broadly defined as someone who follows a trading strategy so that at some horizon, he is guaranteed a minimum level of wealth (the "floor"), yet is able to participate in the potential gains of some reference portfolio (e.g., the S&P 500). This definition, which we adopt in this article, agrees with the description of portfolio insurance given in both the academic [Grossman and Vila (1989)] and professional [Luskin (1988, p. 77)] literature.

The primary objective of this paper is to study the effects of portfolio insurance on the market price level, volatility, and risk premium. Our approach is to take a standard environment [Lucas (1978)] in asset pricing theory, one that is well understood and extensively studied, and introduce portfolio insurance into it. To this end we develop a continuous-time consumption-based general equilibrium model of a multiagent pure-exchange economy. The main tool of analysis is the martingale representation technology of Cox and Huang (1989,1991), Karatzas, Lehoczky, and Shreve (1987), and Pliska (1986).

We postulate that trade takes place between two types of traders, "portfolio insurers" and "normal agents." We model the portfolio insurers as expected utility maximizers according to the above definition of portfolio insurance. Our method of modeling the portfolio insurers must assure them of a horizon wealth above the floor. We do this in the most natural way by explicitly applying to their expected utility maximization problem an additional constraint, that the horizon wealth be above the floor.

A comparison of the optimal consumption processes reveals that before the portfolio insurance horizon, portfolio insurers postpone consumption relative to the normal agents, thus enabling them to meet their objectives of above-the-floor wealth. Compared with normal agents, the portfolio insurers value dividend streams paid out after the horizon more highly than those paid out before. As a result, the prehorizon equilibrium price of any asset that pays out posthorizon dividends is higher in the portfolio insurance economy than it is in an economy with no portfolio insurance. Similarly, the market, a claim against all dividends, increases in price in the presence of portfolio insurance. As soon as the portfolio insurance matures and the insurers have met their wealth constraint, their consumption demand increases discontinuously, causing the asset prices to jump down discontinuously.

The need for a price jump in equilibrium leads to one of the technical subtleties involved in our work. Other researchers [Cox and Huang (1989), Merton (1990, chapter 6)] have solved in a continuous-time model the dynamic consumption-portfolio choice problem of constrained agents who are similar to our portfolio insurers. Our analysis demonstrates that to extend such problems to general equilibrium in a pure-exchange model, one needs to expand the usual price space, modeling prices as continuous semimartingales, to allow for a price jump at the time of the constraint.

The main conclusions are that within our model, the equity market volatility and risk premium are decreased by the presence of portfolio insurers. This is contrary to the popular view that portfolio insurance increases volatility. The comparisons are made by deriving explicit analytical expressions for the equilibrium market price, volatility, and risk premium. These expressions are derived under standard environments often studied in finance [constant relative risk aversion (CRRA) preferences]. The intuition for our results is that the portfolio insurers behave as more "risk averse" than the normal agents, in the sense of having a lower demand for risky securities and higher demand for a riskless security. Hence, with portfolio insurers present, to clear the markets the risky securities must become more favorable. Within our setup this is achieved by a decrease in the volatility (riskiness) of the equity market. As the market becomes less risky, we see that its risk premium must simultaneously decrease.

Related literature includes the pure-exchange general equilibrium models of Brennan and Schwartz (1989), Donaldson and Uhlig (1993), and Grossman and Zhou (1994). The main dimension in which these articles differ from ours is that their conclusion is the opposite: they find market volatility to be *increased* by the presence of portfolio insurance. Donaldson and Uhlig (1993) develop a single-period model in which trading decisions are made at only one point in time. They model portfolio insurers by exogenously specifying their demands for a risky and a riskless security. They show the existence of two equilibria and find that increasing the level of portfolio insurance increases the separation between the two equilibrium prices. They interpret this as an increase in volatility. In the continuous-time model of Brennan and Schwartz (1989), the portfolio insurers are also not expected utility maximizers: equilibrium prices are set by the remaining normal agents; the portfolio insurers demand a convex function of the market at the portfolio insurance horizon, which coincides with the final date.

The closest model to ours is Grossman and Zhou's (1994). They expand upon the Brennan and Schwartz work by modeling portfolio insurers as expected utility maximizers. The pertinent modeling difference between their article and ours is that their agents only consume at the portfolio insurance horizon, whereas our agents consume continuously throughout their lifetime. In Section 3, we discuss the implications of this difference. Other papers such as Gennotte and Leland (1990), Grossman (1988), and Jacklin, Kleidon, and Pfleiderer (1992), have looked at the effect of portfolio insurance on market price and volatility in the context of asymmetric information.

The remainder of our article is organized as follows. Section 1 presents the multiagent pure-exchange continuous-time formulation used in this model. Section 2 describes how portfolio insurance is modeled in an expected utility-maximizing framework. Section 3 solves for agents' optimization and for general equilibrium in the presence of portfolio insurers, and Section 4 concludes. The Appendix provides the proofs of all lemmas and propositions.

1. General Formulation

This section describes a continuous-time variation on the Lucas (1978) pureexchange economy. The formulation follows the continuous-time general equilibrium models recently developed by Duffie (1986), Duffie and Huang (1985), Duffie and Zame (1989), Huang (1987), and Karatzas, Lehoczky, and Shreve (1990, 1991). We consider a finite horizon (0, T] economy in which the portfolio insurance horizon is at a time T before T'. There is a single consumption good and all quantities (prices, dividends, etc.) are expressed in units of this good. We let $W = (W_1, \ldots, W_L)^{\top}$ be an L-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\{\mathcal{F}_t;\ t \in [0, T']\}$ be the augmentation by null sets of the filtration generated by W. All the uncertainty in the economy is represented by this L-dimensional Brownian motion.

1.1 Securities

We assume there are $L + 1$ securities. One is an instantaneously riskless bond in zero net supply; L are risky stocks, each in constant net supply of 1 and paying out a dividend stream at rate $\delta_i(t)$ in [0, T']. We assume that the aggregate dividend process $\delta(t) \equiv \sum_{i=1}^{L} \delta_i(t)$ follows a geometric Brownian motion process (with μ_δ and $\sigma_{\delta j}$ constants):

$$d\delta(t) = \delta(t)\left[\mu_\delta dt + \sum_{j=1}^{L} \sigma_{\delta j} dW_j(t)\right], \quad t \in [0, T']$$

Each security price is modeled as a diffusion process relative to the Brownian filtration, with an $\mathcal{F}_T$-measurable jump at the insurance horizon T. We allow for a jump in anticipation of a price discontinuity required for equilibrium. The bond and risky stock price dynamics are

$$dP_0(t) = P_0(t)\left[r(t)dt + q_0 dA(t)\right], \quad t \in [0, T'],$$

$$dP_i(t) + \delta_i(t)dt = P_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{L} \sigma_{ij}(t) dW_j(t) + q_i dA(t)\right],$$

$$i = 1, \ldots, L, \quad t \in [0, T'].$$

These ex-dividend prices must satisfy $P_i(T') = 0$, $i = 1, \ldots, L$. The interest rate $r(\cdot)$ of the bond, the vector of drifts $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_L(\cdot))^\top$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}$ are (possibly path-dependent) bounded $\mathcal{F}_t$-measurable processes, with $\sigma(\cdot)$ nonsingular.[2] The coefficients $q_0, q_1, \ldots, q_L$ $\mathcal{F}_T$-measurable random variables, and the process $A(t)$ is the right-continuous step function defined by $A(t) \equiv \mathbf{1}_{\{t \geq T\}}$ so that $dA(t)$ is a measure assigning unit mass to time T.[3] The coefficients q_i are related to the time-T price jumps by $q_i = \ln(P_i(T)/P_i(T-))$, where $P_i(T-)$ is the left limit of $P_i(\cdot)$ at T. Since the q_i are predictable at time T, the jumps are not a surprise in the sense that their sizes are revealed immediately before the jumps occur. By "no arbitrage" we must have $q_i = q_0 \equiv q$, for all i; otherwise buying a security with a larger relative jump and short-selling another with a smaller jump just before time T would constitute an arbitrage opportunity.[4]

In our analysis, we use the martingale representation technology, which requires the construction of the following processes, related to the price dynamics [Cox and Huang (1989,1991), Harrison and Kreps

[2] In the constructed equilibrium of Section 3.3, these boundedness and invertibility conditions are indeed satisfied.

[3] The posited stock prices are discontinuous, positive semimartingales, whose bounded variation parts contain an $\mathcal{F}_{T-}$-measurable jump, but whose local martingale parts are continuous. In particular, we are not allowing the stock prices to have discontinuous local martingale parts, as, for example, in the asset pricing models of Back (1991) or of Dothan (1990, chapter 12).

[4] The jump in a security price i is often defined as $\Delta P_i(T) \equiv P_i(T) - P_i(T-)$ and is related to q_i in our setup by $\Delta P_i(T) = P_i(T-)[\exp(q_i) - 1]$. Note that this absolute jump is not necessarily the same for each security, only the relative jump (i.e., jump "return").

(1979), Karatzas, Lehoczky, and Shreve (1987), and Pliska (1986)]. We briefly present the required notions and do not state all regularity conditions.

Given the above prices, market completeness allows us to construct a unique system of Arrow-Debreu securities. Accordingly, we define the *state price density process* as

$$\xi(t) \equiv \frac{1}{P_0(0)} \exp\left\{-\int_0^t r(s)ds - \int_0^t \theta(s)^{\top} dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds - qA(t)\right\},$$

where $\theta(\cdot)$ is the unique L-dimensional $\mathcal{F}_t$-measurable *market price of risk process,* given by $\theta(t) \equiv \sigma(t)^{-1}[\mu(t) - r(t)\mathbf{1}]$, where $\mathbf{1}$ is an L-dimensional vector with every component equal to 1. $\xi(t, \omega)$ is interpreted as the Arrow-Debreu price per unit probability $\mathcal{P}$ of one unit of consumption good in state $\omega \in \Omega$ at time t. Applying Itô's lemma to $\xi(t)$, we can write its dynamics as

$$d\xi(t) = -\xi(t)\left[r(t)dt + \theta(t)^{\top} dW(t) + qdA(t)\right]. \tag{1}$$

By construction, $\xi(\cdot)$ is allowed to jump at T, but the products $\xi(\cdot)P_i(\cdot)$, $i = 0, 1, \ldots, L$ are continuous. If a security price jumps downwards, the state price simultaneously jumps upwards, because agents become less wealthy. Hence, agents derive relatively more satisfaction from an extra unit of consumption in these states, making the Arrow-Debreu state price higher.

The above construction of asset and state prices provides the following well-known relationship between an asset's price and its future dividends, which follows from no arbitrage.

Lemma 1. *The price of asset i is given by*

$$P_i(t) = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right], \quad i = 1, \ldots, L; \quad t \in [0, T']. \tag{2}$$

1.2 Agents' preferences and endowments

We assume there are N agents in the economy. Each agent, n, is endowed at time zero with e_{ni} shares of risky security i. Then, define the initial wealth of agent n by $x_{no} \equiv \sum_{i=1}^{L} P_i(0)e_{ni}$.

For each agent n, we define a nonnegative consumption process, $c_n(t)$, and an L-dimensional portfolio process, $\pi_n(t) = (\pi_{n1}(t), \ldots, \pi_{nL}(t))^{\top}$, where $\pi_{ni}(t)$ denotes the amount (in units of good) invested in the *ith* risky security. Denoting $X_n(t)$ as the wealth of the *nth* agent, $X_n(t) - \pi_n(t)^\top \mathbf{1}$ is the amount invested in the bond. Then, the *nth* agent's wealth process follows

$$\begin{aligned} dX_n(t) &= X_n(t)[r(t)dt + q\,dA(t)] - c_n(t)dt \\ &\quad + \pi_n(t)^\top[\mu(t) - r(t)\mathbf{1}]dt + \pi_n(t)^\top \sigma(t)dW(t). \end{aligned} \tag{3}$$

The wealth process $X_n(\cdot)$ is allowed to jump at T, but the product $\xi(\cdot)X_n(\cdot)$ is continuous. Each agent is assumed to derive utility from consumption at all times, represented by a time-additive state-independent utility function $u_n(c_n(t))$. The function is continuous with continuous first derivatives, strictly increasing, strictly concave, and satisfies $\lim_{c\to\infty} u_n'(c) = 0$ and $\lim_{c\to} u_n'(c) = \infty$. We use a symbol with a caret $(\hat{\ })$ to denote an optimal quantity.

The following lemma provides a result needed in Section 3 to derive explicit expressions for the market and its dynamics. It states that the (optimally invested) wealth at time t equals the cost of the future (optimal) consumption stream.

Lemma 2. *The optimally invested wealth of agent n at time t is given by*

$$\hat{X}_n(t) = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s)\hat{c}_n(s)ds \mid \mathcal{F}_t\right], \quad t \in [0, T']. \tag{4}$$

1.3 Equilibrium

In this article, we characterize security prices by appealing to general equilibrium restrictions. The following definition enforces clearing in the good, risky security, and bond markets.

Definition. *An equilibrium is a collection of* $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$, q *and optimal* $\hat{c}_n(\cdot)$, $\hat{\pi}_n(\cdot)$, *such that*

$$\begin{aligned} &\sum_{n=1}^{N} \hat{c}_n(t) = \delta(t); \quad \sum_{n=1}^{N} \hat{\pi}_{ni}(t) = P_i(t), \quad i = 1, \ldots, L; \\ &\sum_{n=1}^{N} \hat{X}_n(t) = \sum_{i=1}^{L} P_i(t); \quad t \in [0, T']. \end{aligned} \tag{5}$$

1.4 Market drift and volatility

The *equity market,* $X_{em}(t)$, is defined as the aggregate optimally invested wealth in the risky securities. In equilibrium, using the conditions in Equation (5), $X_{em}(t)$ is also equal to both the aggregate optimally invested wealth and the sum of the risky asset prices. Therefore,

$$X_{em}(t) \equiv \sum_{n=1}^{N}\sum_{l=1}^{L}\hat{\pi}_{nl}(t) = \sum_{n=1}^{N}\hat{X}_n(t) = \sum_{l=1}^{L}P_l(t), \quad t \in [0, T].$$

The equilibrium market dynamics can be represented by

$$dX_{em}(t)+\delta(t)dt = X_{em}(t)\left[\mu_{em}(t)dt + \sum_{j=1}^{L}\sigma_{em,j}(t)dW_j(t) + q\,dA(t)\right],$$

where $\mu_{em}(t)$ is the equity market drift, $\|\sigma_{em}(t)\| = \sqrt{\sum_{j=1}^{L}\sigma_{em,j}^2(t)}$ the equity market volatility, and $\mu_{em}(t) - r(t)$ the equity market risk premium. By the definition of the market price of risk process $\theta(t)$, it can be shown that the market risk premium and volatility are related by $\mu_{em}(t) - r(t) = \sigma_{em}(t)^{\top}\theta(t)$.

2. Portfolio Insurance

In this article, we define a portfolio insurer as someone who follows a trading strategy so that at some specified horizon T, he is guaranteed a minimum level of wealth K (the "floor"), yet is able to participate in the potential gains of the value of the equity market.

In practice, to achieve this objective, portfolio insurers follow various dynamic trading strategies. These are often prescribed asset allocation rules that tell the portfolio insurer what risky assets/bond mix to hold at each point in time.[5] Currently, the two best-known allocation rules are the "synthetic put approach" of Rubinstein and Leland (1981) and the "constant proportion portfolio insurance" (CPPI) of Black and Jones (1987) and Perold (1986). The synthetic put approach is based on the option pricing theory and relies on the ability of portfolio insurers to dynamically replicate Black&holes (1973) put options. The CPPI involves a trading rule that keeps a constant multiple of the "cushion" — the difference between wealth and a specified floor — in the risky assets up to a borrowing limit. The optimality of these trading rules is questionable. In this paper, instead of restricting the portfolio insurers to possibly suboptimal strategies, we model portfolio insurers as expected utility maximizers, and simultaneously derive their optimal trading strategies.

[5] These strategies typically involve the transfer of funds to the risky assets as their prices go up and away from the risky assets as their prices go down ("trend-chasing" behavior), which is often argued to increase market volatility. However, Basalt (1993, chapter 1, §6.2) finds no clear-cut relationship between the effect of a portfolio insurer on the market volatility and the extent of his trend-chasing behavior.

In accordance with the above description of portfolio insurance, in Section 3 we formulate a portfolio insurer's (*mth* agent's) consumption-portfolio problem by adding to his original [0, T'] optimization problem the additional constraint $X_m(T) \geq K$ on the time T ($< T'$) wealth.

Grossman and Vila (1989) analyze portfolio insurance behavior in this way in a one-period, partialequilibrium framework. One possible criticism of this formulation is that when the portfolio insurer's horizon wealth hits the floor, his marginal indirect utility of wealth suddenly jumps to infinity. It might seem more reasonable for an investor to become gradually, rather than suddenly, unhappy as his wealth approaches the floor. Such an alternative formulation of portfolio insurance [Basak (1993)] leads to equilibria with qualitative properties similar to this formulation.

We let the portfolio insurance horizon, T come before the end of period T because the portfolio insurers require $X_m(T) \geq K$. Since all end-of-period security prices $P_i(T')$ are zero, this wealth condition could not be attained in equilibrium if $T = T$. An alternative might be to define the portfolio insurers as agents who have a floor on a lump sum consumption rather than their wealth at time $T = T'$. However, this formulation is less appealing both from an economic and a modeling point of view. Economically, investors in portfolio insurance do not consume all their wealth at the portfolio insurance horizon; rather, they use the wealth to finance future consumption. This alternative formulation's modeling problem is that the prices at the portfolio insurance horizon would, be given by the exogenous liquidating dividends. Furthermore, in order to attain equilibrium, the aggregate time T' dividend, an exogenous quantity, would need to be high enough to satisfy all the portfolio insurers' floors.

3. An Equilibrium Model of Portfolio Insurance

To capture the effects of portfolio insurance, we study an economy of N agents, the first M being portfolio insurers and the remaining N - M "normal" agents. All agents have the same utility function of consumption $u(c(t))$, $t \in [0, T']$. We first solve each agent's optimization problem and identify how a portfolio insurer differs from a normal agent in his dynamic consumption and time T wealth choices. We then analyze the implications of this portfolio insurance behavior on the equilibrium prices and dynamics in the multiagent economy.

3.1 Individual agents' preferences and optimization problems

The traditional route to solving an agent's consumption-portfolio problem is to use stochastic dynamic programming [Merton (1969, 1971)]. Here, we use instead the martingale representation approach of Cox and Huang (1989, 1991), and Karatzas, Lehoczky, and Shreve (1987), a powerful alternative that converts the dynamic optimization problem of the agent into a static variational problem. The normal agents $n = M + 1, N$ solve the following optimization problem:

$$\max_{c_n(\cdot)} E\left[\int_0^T u(c_n(t))dt\right] \text{ subject to } E\left[\int_0^T \xi(t)c_n(t)dt\right] \leq \xi(0)x_{no}$$

For the portfolio insurers, in order to incorporate the portfolio insurance constraint on time T wealth, we break the budget constraint into two constraints, one for $[0, T)$ and one for [T, T']. The portfolio insurers $m = 1, \ldots, M$ solve the following problem:

$$\max_{(c_m(\cdot), X_m(T-))} E\left[\int_0^T u(c_m(t))dt\right]$$

$$\text{subject to} \quad E\left[\int_0^T \xi(t)c_m(t)dt + \xi(T-)X_m(T-)\right] \leq \xi(0)x_{mo},$$

$$E\left[\int_T^{T'} \xi(t)c_m(t)dt \mid \mathcal{F}_T\right] \leq \xi(T-)X_m(T-), \quad \text{almost surely,}$$

$$X_m(T-) \geq K \text{ almost surely.}$$

Recall that the wealth is allowed to have an $\mathcal{F}_T$-measurable jump at T; $X_m(T-)$ denotes the left limit of $X_m(\cdot)$ at T. We apply the constraint to the left limit to preserve the standard convention of right continuity of the price and wealth processes. We will refer to this left limit at T as "time T." For an optimal solution to be possible, we need to impose the condition that $\xi(0)x_{m0} \geq KE[\xi(T-)]$. The following lemma characterizes the optimal solutions to the above static problems, assuming a solution exists.

Lemma 3. *Assume there exist (nonnegatiye) constants y_n, y_{m1} and (nonnegative) $\mathcal{F}_T$-measurable random variables y_{m2} satisfying*

$$E\left[\int_0^T \xi(t)I(y_n\xi(t))dt\right] = \xi(0)x_{no}, \tag{6}$$

$$E\left[\int_0^T \xi(t)I(y_{m1}\xi(t))dt + \xi(T-)\right.$$
$$\left.\max \times \left\{K, \frac{1}{\xi(T-)}E\left[\int_T^{T'} \xi(t)I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right]\right\}\right] = \xi(0)x_{mo}, \quad (7)$$

$$E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right]$$
$$= \xi(T-)\max\left\{K, \frac{y_{m2}}{y_{m1}}\frac{1}{\xi(T-)}E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right]\right\}, \quad (8)$$

where $I(\cdot)$ is the inverse of $u'(\cdot)$. Then, under mild regularity conditions, the optimal consumption and time T wealths of the N agents are

$$\hat{c}_n(t) = I(y_n\xi(t)), \quad t \in [0, T'], \quad n = M+1, \ldots, N, \quad (9)$$
$$\hat{c}_m(t) = I(y_{m1}\xi(t)), \quad t \in [0, T), \quad m = 1, \ldots, M, \quad (10)$$
$$\hat{c}_m(t) = I(y_{m2}\xi(t)), \quad t \in [T, T'], \quad m = 1, \ldots, M, \quad (11)$$

$$\hat{X}_n(T-) = \frac{1}{\xi(T-)}E\left[\int_T^{T'} \xi(t)I(y_n\xi(t))dt \mid \mathcal{F}_T\right],$$
$$n = M+1, \ldots, N, \quad (12)$$
$$\hat{X}_m(T-) = \max\left\{K, \frac{1}{\xi(T-)}E\left[\int_T^{T'} \xi(t)I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right]\right\}$$
$$m = 1, \ldots, M. \quad (13)$$

We have the following properties: (i) if $x_{m0} = x_{n0}$ then $y_{m1} \geq y_n$; and (ii) when m's portfolio insurance constraint is binding $y_{m1} \geq y_{m2}$, and when not binding, $y_{m1} = y_{m2}$.

The optimal consumption processes are given by the inverse marginal utility evaluated at the properly normalized state price process, where the normalization factors (y_n, y_{m1}, y_{m2}) are the "weights" associated with each agent (the Lagrange multipliers ensuring that each agent's budget constraints hold with equality at the optimal solution). It is these weights that capture the portfolio insurance behavior, as summarized by the properties in Lemma 3.

Comparing a portfolio insurer and a normal agent with identical initial wealths $(x_{m0} = x_{n0})$, property (i) and Equations (9) and (10) reveal that $\hat{c}_m(t)$ is less than $\hat{c}_n(t)$ in [0, *T*). Relative to the normal agent, the portfolio insurer postpones consumption in [0, *T*) in striving to keep his horizon wealth above the floor. Once he has met this wealth objective, he no longer postpones consumption. Hence, his consumption may jump upwards at time *T*. Property (ii) shows that for continuous $\xi(t)$, when portfolio insurer m's constraint is binding, his consumption jumps up at *T*; when his constraint is not binding, his consumption is continuous. The normal agent *n's* consumption is continuous if $\xi(t)$ is continuous. Since both types of agents have identical preferences after time *T*, the agent with the higher time *T* wealth will consume more after *T*.

Since a portfolio insurer is characterized by certain tastes over horizon wealth, we next compare his time *T* wealth with that of a normal agent. Equation (13) can be rearranged as

$$\hat{X}_m(T-) = \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t) I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right] + \max\left\{K - \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t) I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right], 0\right\}.$$

Comparing with Equation (12), we see that a portfolio insurer's horizon wealth is equal to the horizon wealth of a fictitious normal agent with weight y_{ml}, plus the payoff of a put option with exercise price *K* on the fictitious normal agent's horizon wealth. The option ensures that the portfolio insurer's horizon wealth remains above *K*. This put option interpretation of such a constrained strategy was first made by Cox and Huang (1989). This behavior is very similar to the trading strategy of the synthetic put approach to portfolio insurance [Rubinstein and Leland (198l)].

In Figure 1, we plot a portfolio insurer's and a normal agent's wealth against the level of the equity market at the portfolio insurance horizon *T*. This figure is for identically endowed power utility agents in an economy of one normal agent and one portfolio insurer. We see that, as stated in the definition, the portfolio insurer achieves a minimum level of wealth *K*, yet is able to participate in the gains of the reference portfolio $X_{em}(T-)$.

To gain further insight into the nature of the portfolio insurer's horizon wealth, in Figure 2 we plot the (estimated) probability density of the equilibrium horizon wealths of the normal agent and the portfolio insurer of Figure 1 for a given set of exogenous parameters and

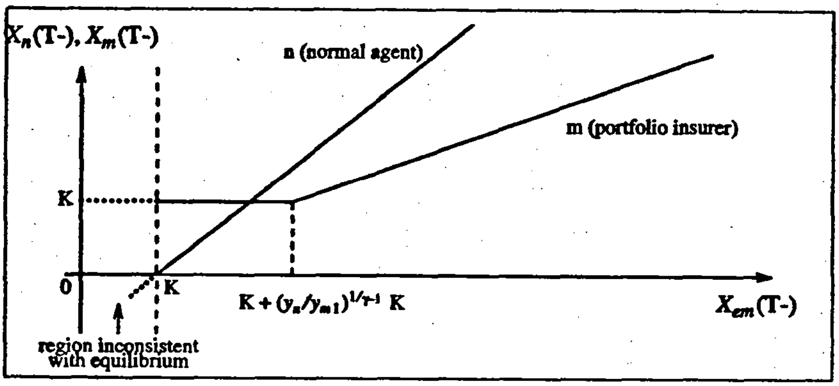

Figure 1
Agents' horizon wealth in the portfolio insurance economy
Agents have power utility of consumption, $u(c) = c^\gamma/\gamma,\ \gamma < 1$. The solid lines are the agent's time T wealths (Lemma 3), $X_n(T-)$, $X_m(T-)$, plotted against the time T level of the equity market of a two-agent economy. $X_{em}(T-)$. The dotted lines are extrapolations into the region inconsistent with equilibrium. K denotes the portfolio insurer's floor, and y_n, y_{m1} denote the Lagrange multipliers of the two agents, as described in the text.

two different values of the floor.[6] For the portfolio insurer, there is a probability mass buildup at the floor and the density loses its left tail. The higher the floor, the more pronounced the buildup.

3.2 Equilibrium conditions

Here we characterize equilibrium in a heterogeneous multiagent economy, and discuss the effect of the presence of portfolio insurance on the equilibrium state prices.

Proposition 1. *If the equilibrium conditions of Equation (5) are satisfied for some process $\xi(t)$, then $\xi(t)$ satisfies*

$$\delta(t) = \sum_{m=1}^{M} I(y_{m1}\xi(t)) + \sum_{m=M+1}^{N} I(y_n\xi(t)) \quad t \in [0, T), \tag{14}$$

[6] The exogenous parameters of the aggregate dividend process(μ_δ = 12 percent, $\|\sigma_\delta\|$ = 20 percent) were chosen so that in a normal agent log-utility (γ = 0) environment in equilibrium μ_{em} = 12 percent, $\|\sigma_{em}\|$ = 20 percent, and r = 8 percent. Then we computed two equilibria: for K = 50, and for K = 75. Taking the initial aggregate dividend as 100, for K = 50, each agent's initial wealth in equilibrium was 102.2 units of consumption, and for K = 75 it was 112.9. We simulated random realizations of the aggregate dividend process at T, computed the equilibrium time T wealths for each realization, and nonparametrically estimated the probability density function using a Gaussian kernel.

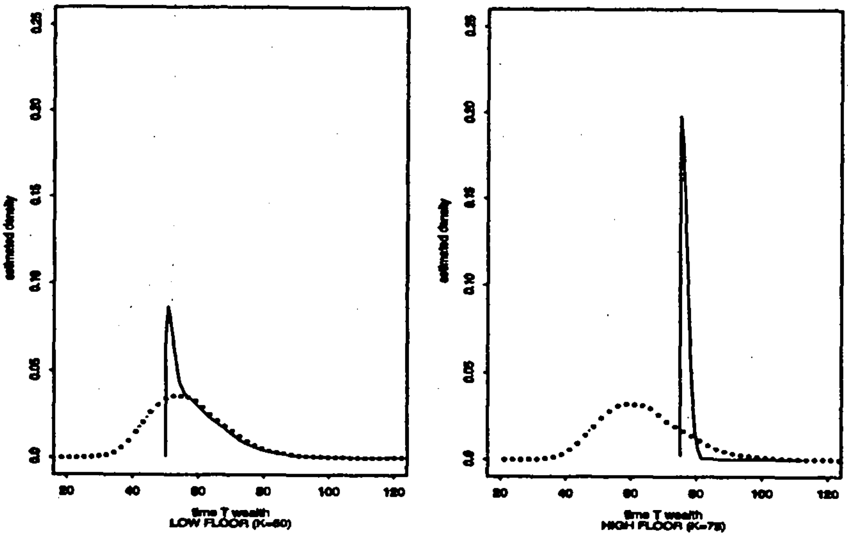

Figure 2
Probability density of agents' horizon wealths in the portfolio insurance economy
The solid line is for the portfolio insurer; the dotted line is for the normal agent of a two-agent economy. The details are provided in footnote 6.

$$\delta(t) = \sum_{m=1}^{M} I(y_{m2}\xi(t)) + \sum_{m=M+1}^{N} I(y_n\xi(t)) \quad t \in [T, T'], \qquad (15)$$

where y_n, y_{m1} and y_{m2} are defined by Equations (6) through (8). If there exists $\xi(t)$ satisfying Equations (14) and (15) then the equilibrium conditions are satisfied by the associated optimal policies.

Proposition 1 simply requires clearing in the consumption good market at all times. The existence of equilibrium is established by showing the existence of the weights (y_n, y_{m1}, y_{m2}) solving Equations (6) through (8). We do not establish existence in the general case, but we do show that equilibrium exists in the special case of Section 3.3.

The equilibrium interest rate process $r(t)$, market price of risk process $\theta(t)$, and jump parameter $q = \ln(\xi(T-)/\xi(T))$ can be derived from Equations (14) and (15). Whenever at least one of the y_{m2} is greater than y_{m1} (one of the portfolio insurers' wealth constraints is binding), there is a discontinuous (upwards) jump in the state price density at T. The intuition is as follows: If ξ were continuous, the normal agents' consumption demands would be continuous, whereas the demands of the portfolio insurers with binding constraints would jump upwards, giving rise to a discontinuous aggregate consumption demand. This is not possible, since the aggregate supply $\delta(t)$ in this type of model is continuous. Hence, the price of consumption, ξ, jumps up at T to counteract the upward jump in aggregate consumption demand. This in turn leads to discontinuous (downwards) jumps in the asset prices. It should be pointed out, however, that the jump is a consequence of the pure-exchange nature of the economy and of the assumption of one portfolio insurance horizon at the fixed date T, neither of which may be applicable to the real economy.

3.3 Asset and market prices, market volatility, and risk premium

It is complicated, at this level of generality, to analyze the asset and market prices in a heterogeneous multiagent economy containing more than one constrained portfolio insurer. This is because the time T wealth constraint must be applied to each portfolio insurer's individual optimization problem, and the constraints of differently endowed agents will not all be binding in the same states. Here we analyze the special case of log-utility agents, $u(c) = \log(c)$, and show that explicit solutions can be derived despite this complication. The agents may be heterogeneously endowed. We compare the portfolio insurance economy with a benchmark "normal agent economy" in which $M = 0$, and quantities are denoted by a superscript $^{(0)}$.

We first consider the equilibrium state price dynamics. From Equations (14) and (15) we have for log utility

$$\xi(t) = \left(\sum_{m=1}^{M} 1/y_{m1} + \sum_{n=M+1}^{N} 1/y_n\right)/\delta(t), \quad t \in [0, T),$$

$$\xi(t) = \left(\sum_{m=1}^{M} 1/y_{m2} + \sum_{n=M+1}^{N} 1/y_n\right)/\delta(t), \quad t \in [T, T'],$$

with the individual agents' weights appearing only in a multiplying factor. Since ξ follows the multiplicative dynamics in Equation (1), we conclude that $r(t)$ and $\theta(t)$ are unaffected by individual agents' weights. Hence, they are unaffected by portfolio insurance, since it is these weights that capture the portfolio insurance effects. This is true more generally for power and negative exponential utilities. Lemma 4 also shows this explicitly for log utility.

Lemma 4. *Assume $u(c) = \log(c)$ for both the portfolio insurance economy ($M > 0$) and the normal agent economy ($M = 0$). Then the equilibrium interest rate and marketprice of risk are constants, at* all $t \in [0, T']$, given by

$$r = \mu_\delta - \|\sigma_\delta\|^2 \quad \text{and} \quad \theta_j = \sigma_{\delta j}, \quad j = 1, \ldots, L.$$

In the normal agent economy with no constrained agents present, no price jump occurs (i.e., $q^{(0)} = 0$). In the portfolio insurance economy, however, where constrained agents are present, a nonzero jump may occur at T in the state prices, asset prices, and wealth. The equilibrium jump size parameter is

$$q = \ln\left(\left(\sum_{m=1}^{M} 1/y_{m1} + \sum_{n=M+1}^{N} 1/y_n\right) \Big/ \left(\sum_{m=1}^{M} 1/y_{m2} + \sum_{n=M+1}^{N} 1/y_n\right)\right) \leq 0.$$

An explicit expression for this jump parameter q in terms of exogenous quantities may not be derived in a heterogeneous-agent economy. For log-utility it is not necessary (as shown in the Appendix) to compute the jump in order to derive the market expressions; this is why we specialize to the case of log.

3.3.1 Market and asset prices. The following proposition presents expressions for the market and asset prices, in a normal agent and a portfolio insurance economy.

Proposition 2. *Assume $u(c) = \log(c)$. Then equilibrium exists in the normal agent and portfolio insurance economies. The equilibrium market and asset prices in a normal agent economy are*

$$X_{em}^{(0)}(t) = (T' - t)\delta(t), \quad t \in [0, T'], \tag{16}$$

$$P_i^{(0)}(t) = \delta(t) E\left[\int_t^{T'} \frac{\delta_i(s)}{\delta(s)} ds \mid \mathcal{F}_t\right],$$

$$t \in [0, T'], \quad i = 1, \ldots, L, \tag{17}$$

and before the portfolio insurance horizon in a portfolio insurance economy are

$$X_{em}(t) = X_{em}^{(0)}(t) + \delta(t) E\left[\frac{1}{\delta(T)} \sum_{m=1}^{M} \max\left\{K - \frac{X_{em}^{(0)}(T)}{y_{m1}}, 0\right\} \mid \mathcal{F}_t\right], \tag{18}$$

$$= X_{em}^{(0)}(t) - (T' - T)\delta(t) \sum_{m=1}^{M} \frac{N(-d_{m1})}{y_{m1}}$$

$$+ K \exp\{-r(T-t)\} \sum_{m=1}^{M} N(-d_{m2}),$$

$$P_i(t) = P_i^{(0)}(t) + \delta(t) E\left[\frac{1}{\delta(T)} \frac{P_i^{(0)}(T)}{X_{em}^{(0)}(T)} \times \sum_{m=1}^{M} \max\left\{K - \frac{X_{em}^{(0)}(T)}{y_{m1}}, 0\right\} \mid \mathcal{F}_t\right], \quad (19)$$

where

$$d_{m1} \equiv \frac{\ln\left((T' - T)\delta(t)/y_{m1}K\right) + \left(r + \frac{1}{2}\|\sigma_\delta\|^2\right)(T-t)}{\|\sigma_\delta\|\sqrt{T-t}},$$

$$d_{m2} \equiv d_{m1} - \|\sigma_\delta\|\sqrt{T-t},$$

$N(\cdot)$ is the distribution function for a standard normal random variable, and r is as given in Lemma 4. Consequently, (i) $X_{em}(t) \geq X_{em}^{(0)}(t)$; (ii) $P_i(t) \geq P_i^{(0)}(t)$, $t \in [0, T)$. After the portfolio insurance horizon, the prices in both types of economy are identical.

The portfolio insurance market price is equal to the normal agent market price plus the value (adjusted for dividends) of M Black-Scholes put options, with exercise price K and maturity T, on a fraction of the normal agent market. Each put option is associated with one individual portfolio insurer, An implication is that before the portfolio insurance horizon, the market level in a portfolio insurance economy is always higher than in a normal agent economy. This arises from the portfolio insurers' different valuation of dividend streams. The portfolio insurance constraint is on time T wealth, which equals the present value of consumption after T. Hence, consuming after time T not only provides the portfolio insurer with utility, but also helps him meet his time T wealth constraint. Consuming before time T, however, hinders him from meeting his constraint. As a result, relative to current consumption, the portfolio insurers value prehorizon dividends the same, but posthorizon dividends more highly, than do the normal agents. Thus the prehorizon value of the equity market, a claim against all posthorizon dividends, is higher in the presence of portfolio insurance.

Proposition 2 also allows us to address how individual asset prices are affected by portfolio insurance. As with the equity market, each prehorizon asset price is higher in the presence of portfolio insurance, for the reason discussed above. We can rearrange Equation (19) as

$$\begin{aligned} P_i(t) &= P_i^{(0)}(t) + \delta(t) E\left[P_i^{(0)}(T) \mid \mathcal{F}_t\right] \\ &\quad \times E\left[\frac{1}{\delta(T)} \sum_{m=1}^{M} \max\left\{\frac{K}{X_{em}^{(0)}(T)} - \frac{1}{y_{m1}}, 0\right\} \mid \mathcal{F}_t\right] \\ &\quad + \delta(t) \mathrm{cov}_t \left(P_i^{(0)}(T), \frac{1}{\delta(T)} \sum_{m=1}^{M} \max\left\{\frac{K}{X_{em}^{(0)}(T)} - \frac{1}{y_{m1}}, 0\right\}\right). \end{aligned}$$

According to the second term, the effect of portfolio insurance is more pronounced for an asset whose expected time T price is high; that is, the present value of its posthorizon dividends is expected to be high. Such assets are favorable to portfolio insurers. Since the third term is the conditional covariance of $P_i^{(0)}(T)$ with a decreasing function of $X_{em}^{(0)}(T)$, the effect of portfolio insurance is more pronounced for any asset whose horizon price covaries highly negatively with the (normal agent economy) horizon market value. A portfolio insurer favors an asset whose price is high in states where the time T market is low and he is less likely to meet his time T wealth constraint.

3.3.2 Market volatility and risk premium. Proposition 3 presents the market dynamics in the two types of economy.

Proposition 3. *Assume $u(c) = \log(c)$. Then before the portfolio insurance horizon, the equilibrium market volatility and risk premium in the normal agent economy are*

$$\begin{aligned} \|\sigma_{em}^{(0)}(t)\| &= \|\sigma_\delta\|, \\ \mu_{em}^{(0)}(t) - r &= \|\sigma_\delta\|^2. \end{aligned}$$

In the portfolio insurance economy they are

$$\|\sigma_{em}(t)\| = \left(1 - \frac{K \exp\{-r(T-t)\} \sum_{m=1}^{M} \mathcal{N}(-d_{m2})}{X_{em}(t)}\right) \|\sigma_\delta\|,$$

$$\mu_{em}(t) - r = \left(1 - \frac{K \exp\{-r(T-t)\} \sum_{m=1}^{M} \mathcal{N}(-d_{m2})}{X_{em}(t)}\right) \|\sigma_\delta\|^2,$$

where $\mathcal{N}(d_{m2})$, $X_{em}(t)$ are as given in Proposition 2, and r is as given in Lemma 4. Consequently, (i) $\|\sigma_{em}^{(0)}(t)\| \leq \|\sigma_{em}(t)\|$; (ii) $\mu_{em}(t) \leq \mu_{em}^{(0)}(t)$; that is, the presence of portfolio insurance decreases the market *volatility and risk premium. After the portfolio insurance horizon, the presence of portfolio insurance does not affect the equilibrium.*

In the normal agent economy, the market volatility and risk premium are constants. In the portfolio insurance economy, the market risk premium and volatility are stochastically time-varying. The volatility and risk premium are driven by the level of the market and by the total additional amount of wealth invested in the bond to synthetically create the Black-Scholes put option. It is straightforward to show that in a given portfolio insurance economy, $\mu_{em}(t) - r$ and $\|\sigma_{em}(t)\|$ are both increasing in $X_{em}(t)$. This shows that the effect of portfolio insurance in decreasing the volatility and risk premium is less pronounced in "good" states where the market is high, and the portfolio insurers have to strive less to meet their constraint.

In this heterogeneous-agent economy, to investigate the effect of varying the portfolio insurance floor, K, the number of portfolio insurers or the initial "wealthiness" of the portfolio insurers is complicated by the presence of the various agents' weights. Varying any of these parameters leads to indirect wealth effects (through the y's of all the agents), adding to the richness of this type of heterogeneous-agent model.[7]

To provide intuition for the most important comparisons, the decrease in market volatility and risk premium in the presence of porfolio insurers, we consider the optimal portfolio strategies of these log-utility agents.

Proposition 4. *Assume r and q are constants. Before the portfolio insurance horizon, the fractions of wealth $\hat{\phi}(t)$ optimally invested in the risky assets by a normal agent (n) and a portfolio insurer (m) with log utility of consumption are*

$$\hat{\phi}_n(t) = \left[\sigma(t)^{\top}\right]^{-1} \sigma(t)^{-1} (\mu(t) - r\mathbf{1}), \tag{20}$$

$$\hat{\phi}_m(t) = \left[1 - \frac{K \exp\{-r(T-t)\}\mathcal{N}(-d_{m2})}{\hat{X}_m(t)}\right] \times \left[\sigma(t)^{\top}\right]^{-1} \sigma(t)^{-1} (\mu(t) - r\mathbf{1}), \tag{21}$$

[7] Comparing the volatility of the individual asset returns across economies is a hard exercise since we an only consistently assume that either the $\delta_i(t)$ or $\delta(t)$ follows geometric Brownian motion. Hence, the analysis of the conditional expectations in the price formula leads to multiple terms whose signs cannot be compared unambiguously across economies.

where

$$d_{m2} \equiv \frac{\ln\left((T' - T)/y_{m1}\xi(t)K\right) + \left(r - \frac{\|\theta\|^2}{2}\right)(T - t)}{\|\theta\|\sqrt{T - t}}.$$

Hence, $\hat{\phi}_{mi}(t) \leq \hat{\phi}_{ni}(t), \quad t \in [0, T), \quad i = 1, \ldots, L.$

At the same prices, a portfolio insurer demands less in the risky assets and more in the bond than a normal agent; in a sense he behaves as more risk-averse. This is because the distinguishing feature of a portfolio insurer's strategy, a synthetic put option, consists of a short position in risky securities and a long position in a bond. When we introduce portfolio insurers into a normal agent economy, there will be less aggregate demand for the risky assets and more for the bond, so to clear the markets, the prices must adjust to make the risky assets more favorable. The notion of favorableness here is given by the demand functions of Equations (20) and (21), which are driven by two parameters: the volatility matrix σ *(t)* and the market price of risk $\sigma(t)^{-1}(\mu(t) - r\mathbf{1}) \equiv \theta(t)$. In our setup (CRRA preferences), the market price of risk is unchanged across economies, so the only way to achieve "favorable" risky assets is for the volatility of some risky assets (and hence the market) to decrease. Since the market volatility is decreased, for the market price of risk to remain unchanged the market risk premium must also decrease.

The reader may ask about other price adjustments that generally make a risky asset more attractive relative to a bond: decreasing the bonds interest rate, or decreasing the risky asset's current price. However, these factors are summarized by Equations (20) and (21) from which, without needing to concern ourselves further, we conclude that volatility must decrease.

As we mentioned, Grossman and Zhou (1994) find that market volatility increases in the presence of portfolio insurance. Their portfolio insurers (like ours) differ from their normal agents in their preferences for horizon wealth, and thus affect the equilibrium state prices at the horizon. Their agents derive utility from horizon wealth and not from prehorizon consumption. Thus the prehorizon-state prices are determined from agents' expectations about the horizon-state prices, and so are also affected by portfolio insurance. This is in contrast to our model, where the prehorizon-state prices are determined by market clearing in the continuous consumption flow, and so are not affected by portfolio insurance. One other difference is that their portfolio insurers are more risk-seeking (with CRRA parameter of less than 1) over their horizon wealth above the floor than are the normal agents (with log utility or CRRA of 1). They find the market price of risk to be increased in "bad" states and decreased in "good" states by the presence of portfolio insurance. Since their market price of risk is changed by portfolio insurance, extending our intuition would suggest that the presence of portfolio insurance now affects the market volatility by two means: by the introduction of agents with different risk preferences (as in our case), and by the effect of portfolio insurance on the market price of risk. In their numerical analysis, for a given set of exogenous parameters, they show that the net effect is an increase in market volatility. However, intuition would suggest that there might also be cases in which volatility would decrease.

A related issue is the robustness of our conclusions to other utility functions. No matter what the choice of utility function, the addition of more "risk averse" portfolio insurers requires that to achieve clearing, risky assets become more favorable relative to the bond. However, CRRA preferences are special in that the market price of risk remains unaffected by portfolio insurance. For other utility functions where the market price of risk may be affected, more risk aversion in the economy may not always imply lower market volatility. A further question is how our results might extend to a production economy in which the aggregate consumption is chosen endogenously rather than given exogenously. This could affect the discontinuities occurring at time T, and might change some of the conclusions in this paper.

Remark (extensions and ramifications)

(i) In this heterogeneous-agent economy, we could easily allow portfolio insurers to have differing floors, K_m, and differing portfolio insurance horizons, T_m. Then, in each of the M put options in the above expressions, K and T would be replaced by their respective K_m and T_m. The prices could now exhibit M possible jumps, one at each horizon. T_m.[8] Market volatility and risk premium would still be decreased by the presence of portfolio insurance.

(ii) Further extension to portfolio insurers with repeated portfolio insurance horizons would lead to a nesting of put options on put options. In this case, the effect on market and asset prices is the same, but we do not obtain explicit formulae for the market dynamics.

(iii) In the special case of an economy populated by identical portfolio insurers $(M = N, x_{m0} = x_0/N)$, we may extend some of our conclusions to more general utility functions. In this economy, the market and asset price comparisons of Proposition 2 hold true for

[8] As pointed out to me by Chi-fu Huang, with a continuum of portfolio insurers having a continuum of horizons, the jumps might be smoothed out. Similarly, to eliminate discontinuities one might consider horizons which are continuously receding as time unfolds and never reached.

any time-additive utility function, in particular, the increase in these prices compared with an all-normal-agent economy. In addition, the volatility and risk premium comparisons of Proposition 3 extend more generally to power utility of consumption. [See Basak (1993) for the details.]

4. Summary and Conclusion

Recently, attention has focused on market volatility as a measure of market instability, particularly since high volatility was observed during the stock market crash of 1987. This paper focuses on the effects of portfolio insurance trading strategies on market dynamics. We construct a consumption-based general equilibrium model of a continuous-time economy that captures the effects of portfolio insurance. These effects are represented by explicit formulae for the quantities of interest derived under common assumptions made in finance.

One major result is that the market price before the portfolio insurance horizon is increased by the presence of portfolio insurance. At the portfolio insurance horizon, the price jumps down discontinuously. An important point is that because of the effective discontinuity of portfolio insurers' consumption preferences and the pure-exchange nature of the model, the equilibrium market, security, and state prices may all jump discontinuously at the horizon.

Our main conclusion is that the market volatility and risk premium before the portfolio insurance horizon are decreased in the presence of portfolio insurance. The intuition is that the portfolio insurers, in striving to meet their objectives, demand less than the normal agents in risky assets and more in a bond. The portfolio insurers behave like the normal agents, but purchase an additional synthetic put option consisting of a long position in a bond and a short position in risky assets. Since the supply of each asset is unchanged across economies, to achieve equilibrium in the presence of portfolio insurers, the risky assets must become more favorable and the riskless asset less favorable. With CRRA preferences over continuous consumption, the interest rate and market price of risk are unaffected by portfolio insurance. Hence, we find that the only way to make the equity market more favorable is to decrease its volatility, and that the risk premium must decrease simultaneously. Our result is in sharp contrast to the popular belief that portfolio insurance increases market volatility. The striking point about our conclusions is that this popular belief breaks down even in one of the most standard, best understood setups in finance [Lucas (1978) and CRRA preferences].

Appendix

Proof of Lemma 1. Define $\bar{P}_i(t)$ as the current price plus value of accumulated dividends:

$$\bar{P}_i(t) \equiv P_i(t) + \frac{1}{\xi(t)} \int_0^t \xi(s)\delta_i(s)ds, \quad t \in [0, T'], \quad i = 1, \ldots, L.$$

For all $t \neq T$, $dA(t) = 0$, and Itô's lemma implies $d(\xi(t)\bar{P}_i(t)) = \xi(t)P_i(t)(\sigma_i(t) - \theta(t)^{\top})dW(t)$, $\forall t \neq T$, where $\sigma_i(t)$ denotes the *ith* row of $\sigma(t)$. This result and $P_i(T') = 0$ together imply that

$$\begin{aligned}\bar{P}_i(t) &= \frac{1}{\xi(t)} E[\xi(T')\bar{P}_i(T') \mid \mathcal{F}_t] \\ &= \frac{1}{\xi(t)} E\left[\int_0^{T'} \xi(s)\delta_i(s)ds \mid \mathcal{F}_t\right], \quad t \in [T, T'],\end{aligned}$$

giving the desired result for $t \in [T, T']$. Similarly we may obtain

$$\begin{aligned}\bar{P}_i(t) &= \frac{1}{\xi(t)} E[\xi(T-)\bar{P}_i(T-) \mid \mathcal{F}_t] \\ &= \frac{1}{\xi(t)} E[\xi(T)\bar{P}_i(T) \mid \mathcal{F}_t] \\ &= \frac{1}{\xi(t)} E[\xi(T')\bar{P}_i(T') \mid \mathcal{F}_t], \quad t \in [0, T),\end{aligned}$$

where for the second equality we use $\exp(q) = P_i(T)/P_i(T-) = \xi(T-)/\xi(T)$, giving the result for $t \in [0, T)$. ■

Proof of Lemma 2. See, for example, Lemma 2.4, Cox and Huang (1989). ■

Proof of Lemma 3. A normal agent's optimal consumption, given by Equations (6) and (9), is derived in, for example, Karatzas, Lehoczky, and Shreve (1987), henceforth KLS. Then Equation (12) is derived from Equation (4). To solve the portfolio insurer's problem we divide it into two parts. First define

$$V(x; \mathcal{F}_T) \equiv \max_{c_m(t);\, t \in [T, T']} E\left[\int_T^{T'} u(c_m(t))dt \mid \mathcal{F}_T\right]$$

$$\text{subject to} \quad E\left[\int_T^{T'} \xi(t)c_m(t)dt \mid \mathcal{F}_T\right] \leq x\xi(T-),$$

with solution (KLS) $c_m^x(t) = I(y_{m2}^x \xi(t)),\ t \in [T, T']$, where y_{m2}^x exists and uniquely solves

$$E\left[\int_T^{T'} \xi(t) I(y_{m2}^x \xi(t)) dt \mid \mathcal{F}_T\right] = \xi(T-)x.$$

Furthermore, the indirect utility function $V(x; \mathcal{F}_T)$ is strictly increasing, strictly concave, and satisfies $\lim_{x\to\infty} V'(x; \mathcal{F}_T) = 0$, $\lim_{x\to 0} V'(x; \mathcal{F}_T) = \infty$, under the assumption $E[\int_0^T |u(I(y\xi(t)))| dt] < \infty$, $\forall y \in (0, \infty)$. We define the inverse of $V'(\cdot; \mathcal{F}_T)$ by $I_v(y; \mathcal{F}_T)$. To embed the wealth constraint [Bardhan (1992)] into an effective utility function, we define

$$V_{eff}(z; \mathcal{F}_T) \equiv \begin{cases} V(z+K; \mathcal{F}_T) - V(K; \mathcal{F}_T) & z \geq 0 \\ -\infty & z < 0 \end{cases},$$

$$I_{v_{eff}}(b; \mathcal{F}_T) \equiv \begin{cases} I_v(b; \mathcal{F}_T) - K & 0 < b \leq V'(K; \mathcal{F}_T) \\ 0 & b > V'(K; \mathcal{F}_T) \end{cases}.$$

Then the portfolio insurer solves

$$\max_{c_m(t);\, t\in[0,T),\, z} E\left[\int_0^T u(c_m(t))dt + V_{eff}(z; \mathcal{F}_T)\right] + E\left[V(K; \mathcal{F}_T)\right]$$

$$\text{subject to} \quad E\left[\int_0^T \xi(t) c_m(t) dt + \xi(T-)z\right] \leq \xi(0)x_{m0} - E\left[\xi(T-)K\right],$$

which has the unique solution (KLS) $\hat{c}_m(t) = I(y_{m1}\xi(t)),\ t \in [0, T)$, $z = I_{v_{eff}}(y_{m1}\xi(T-))$ implying $\hat{X}_m(T-) = I_{v_{eff}}(y_{m1}\xi(T-)) + K$, where y_{m1} is the unique number satisfying

$$E\left[\int_0^T \xi(t) I(y_{m1}\xi(t)) dt + \xi(T-) I_{v_{eff}}(y_{m1}\xi(T-))\right]$$
$$= \xi(0)x_{m0} - E\left[\xi(T-)K\right], \tag{22}$$

assuming that $\xi(0)x_{m0} \geq E\left[\xi(T-)K\right]$.

Substituting this $\hat{X}_m(T-)$ into the first problem, we obtain $\hat{c}_m(t) = I(y_{m2}\xi(t))$, $t \in [T, T']$, where y_{m2} uniquely solves

$$E\left[\int_T^{T'} \xi(t) I(y_{m2}\xi(t)) dt \mid \mathcal{F}_T\right] = \xi(T-)\hat{X}_m(T-)$$
$$= \xi(T-)\left[I_{v_{eff}}(y_{m1}\xi(T-); \mathcal{F}_T)\right] + K. \tag{23}$$

To prove property (ii), note first that, if we take any (fictitious) normal agent with "weight" y, his budget constraint in [T, T'] combined with his optimal consumption imply

$$E\left[\int_T^{T'} \xi(t)I(y\xi(t))dt \mid \mathcal{F}_T\right] = \xi(T-)I_v(y\xi(T-);\mathcal{F}_T), \tag{24}$$

which must then hold for all nonnegative numbers y. For a portfolio insurer, when $\hat{X}_m(T-) > K$, $I_{v_{eff}}(y_{m1}\xi(T-);\mathcal{F}_T) > 0$, $0 < y_{m1}\xi(T-) \leq V'(K;\mathcal{F}_T)$, and $I_{v_{eff}}(y_{m1}\xi(T-);\mathcal{F}_T) = I_v(y_{m1}\xi(T-);\mathcal{F}_T) - K$, so we have that y_{m2} solves

$$E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right] = \xi(T-)I_v(y_{m1}\xi(T-);\mathcal{F}_T).$$

Hence by Equation (24) $y_{m1} = y_{m2}$. When $\hat{X}_m(T-) = K$, $I_{v_{eff}}(y_{m1}\xi(T-);\mathcal{F}_T) \geq I_v(y_{m1}\xi(T-);\mathcal{F}_T) - K$, so we have

$$E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right] \geq \xi(T-)I_v(y_{m1}\xi(T-);\mathcal{F}_T).$$

Since the right-hand side is strictly decreasing in y_{m1}, we conclude from Equation (24) that $y_{m1} \geq y_{m2}$.

To derive Equations (7), (8), and (13) from the above we can write

$$\begin{aligned} I_{v_{eff}}(y\xi(T-);\mathcal{F}_T) &= \max\{0, I_v(y\xi(T-);\mathcal{F}_T) - K\} \\ &= \max\left\{0, E\left[\int_T^{T'} \frac{\xi(t)}{\xi(T-)} I(y\xi(t))dt \mid \mathcal{F}_T\right] - K\right\} \\ &\quad \forall y \in (0,\infty), \end{aligned}$$

implying Equation (22) can be written as Equation (7). Equation (23) immediately implies Equation (13) and using property (ii) we can rewrite it as Equation (8) by expressing $\hat{X}_m(T-)$ as the maximum term as follows: When $\hat{X}_m(T-) \geq K$,

$$\begin{aligned} \hat{X}_m(T-) &= \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right] \\ &= \frac{y_{m2}}{y_{m1}} \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t)I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right] \geq K. \end{aligned}$$

When $\hat{X}_m(T-) = K$,

$$\hat{X}_m(T-) = K = \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t) I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right]$$
$$\geq \frac{y_{m2}}{y_{m1}} \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t) I(y_{m2}\xi(t))dt \mid \mathcal{F}_T\right].$$

To prove property (i), Equation (7) can be arranged as

$$E\left[\int_0^{T'} \xi(t) I(y_{m1}\xi(t))dt\right]$$
$$+ E\left[\xi(T-) \max\left\{K - \frac{1}{\xi(T-)} E\left[\int_T^{T'} \xi(t) I(y_{m1}\xi(t))dt \mid \mathcal{F}_T\right], 0\right\}\right]$$
$$= \xi(0)x_{mo}.$$

This equation is similar to Equation (6), but has an extra nonnegative term on the left-hand side. Hence if $x_{m0} = x_{n0}$,
is decreasing in its argument. ∎

Proof of Proposition 1. This is a variation on Karatzas, Lehoczky, and Shreve (1990) to include nonredundant positive net supply securities. To prove the former statement, $\hat{c}_m(t)$ and $\hat{c}_n(t)$ given by Equations (9) through (11) together with Equation (5) imply Equations (14) and (15). To prove the second statement, $\hat{c}_m(t)$ and $\hat{c}_n(t)$ given by Equations (9) through (11) together with Equations (14) and (15) imply the first equality in Equation (5). Using Lemmas 1 and 2, and the first equality in (5) we obtain the last equality in Equation (5) as follows:

$$\sum_{n=1}^{N} \hat{X}_n(t) = \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s) \sum_{n=1}^{N} \hat{c}_n(s)ds \mid \mathcal{F}_t\right]$$
$$= \frac{1}{\xi(t)} E\left[\int_t^{T'} \xi(s) \sum_{l=1}^{L} \delta_l(s)ds \mid \mathcal{F}_t\right] = \sum_{l=1}^{L} P_l(t)$$

Applying Itô's lemma to $\xi(t)\hat{X}_n(t)$, using Equation (3) and summing over all agents, we obtain

$$d\left(\xi(t)\sum_{n=1}^{N}\hat{X}_n(t)\right) = \xi(t)\left[\sum_{n=1}^{N}\hat{\pi}_n(t)^{\mathsf{T}}\sigma(t) - \sum_{n=1}^{N}\hat{X}_n(t)\theta(t)^{\mathsf{T}}\right]dW(t) - \xi(t)\sum_{n=1}^{N}\hat{c}_n(t)dt,$$

which using Equations (6) and (8) can be written as

$$d\left(\xi(t)\sum_{i=1}^{L}P_i(t)\right) = \xi(t)\left[\sum_{n=1}^{N}\hat{\pi}_n(t)^{\mathsf{T}}\sigma(t) - \sum_{i=1}^{L}P_i(t)\theta(t)^{\mathsf{T}}\right]dW(t) - \xi(t)\sum_{i=1}^{L}\delta(t)dt.$$

Applying Itô's lemma to $\xi(t)P_i(t)$ and summing over all risky securities i we obtain

$$d\left(\xi(t)\sum_{i=1}^{L}P_i(t)\right) = \xi(t)\left[\sum_{i=1}^{L}P_i(t)\sigma_i(t) - \sum_{i=1}^{L}P_i(t)\theta(t)^{\mathsf{T}}\right]dW(t) - \xi(t)\sum_{i=1}^{L}\delta(t)dt.$$

Equating the coefficients of the $dW_j(t)$ terms in the above two dynamics of $\xi(t)\sum_{i=1}^{L}P_i(t)$ yields $P(t)^{\mathsf{T}}\sigma(t) = \sum_{n=1}^{N}\hat{\pi}_n(t)^{\mathsf{T}}\sigma(t)$, which using the nonsingularity of $\sigma(t)$ implies the second equality of Equation (5). ∎

Proof of Lemma 4. Applying Itô's lemma to the $\xi(t)$ expressions yields the desired results. ∎

Proof of Proposition 2. Since the y's are only determined up to a multiplicative constant from Equations (6) through (8), we can set $\left(\sum_{m=1}^{M}1/y_{m1} + \sum_{n=M+1}^{N}1/y_n\right) = 1$ in the expression for ξ in $[0, T)$. Then, substituting this expression for ξ into Equations (9) through (13), the agents' equilibrium consumptions in $[0, T)$ and time T wealths are $c_n(t) = \delta(t)/y_n$, $t \in [0, T)$, $n = 1, \ldots, M$, $c_m(t) = \delta(t)/y_{m1}$, $t \in [0, T)$, $m = M+1, \ldots, N$, $X_n(T-) = (T' - T)\delta(T)/y_n$, $n = 1, \ldots, M$, $X_m(T-) = \max\left\{K, (T' - T)\delta(T)/y_{m1}\right\}$, $m = M+1, \ldots, N$ where the

ys satisfy Equations (6) through (8); that is,

$$\frac{T'}{y_n} = \frac{x_{n0}}{\delta(0)}, \tag{25}$$

$$E\left[\frac{T}{y_{m1}} + \frac{1}{\delta(T)} \max\left\{K, \frac{1}{y_{m1}}(T' - T)\delta(T)\right\}\right] = \frac{x_{m0}}{\delta(0)}, \tag{26}$$

$$\frac{1}{y_{m2}}(T' - T) = \frac{1}{\delta(T)} \max\left\{K, \frac{1}{y_{m1}}(T' - T)\delta(T)\right\}. \tag{27}$$

First we establish the existence of these y's

decreasing in y_{m1} and the right-hand side is independent of y_{m1}. For $y_{m1} = 0$, the left-hand side is greater than the right-hand side, and for $y_{m1} \to \infty$ the right-hand side is greater. For a given y_{m1}, the same argument follows for y_{m2} in Equation (27), and again for y_n in Equation (25). Hence, a solution exists.

Using Lemma 2 and summing over all agents, we obtain the market value as Equation (16) in the normal agent economy, and in the portfolio insurance economy as

$$\begin{aligned} X_{em}(t) &= (T - t)\delta(t) + \left(\sum_{n=M+1}^{N} \frac{1}{y_n}\right)(T' - T)\delta(t) \\ &\quad + \sum_{m=1}^{M} \delta(t) E\left[\frac{1}{\delta(T)} \max\left\{K, \frac{1}{y_{m1}}(T' - T)\delta(T)\right\} \mid \mathcal{F}_t\right] \\ &= (T - t)\delta(t) + (T' - T)\delta(t) \\ &\quad + \sum_{m=1}^{M} \delta(t) E\left[\frac{1}{\delta(T)} \max\left\{K - \frac{1}{y_{m1}}(T' - T)\delta(T), 0\right\} \mid \mathcal{F}_t\right], \\ &\quad t \in [0, T), \end{aligned}$$

that is, Equation (18). Substituting back for the equilibrium $\xi(t) = 1/\delta(t)$, $t \in [0, T)$, each term in the summation can be written as

$$\frac{1}{\xi(t)} E\left[\xi(T-) \max\left\{K - \frac{1}{y_{m1}}(T' - T)\delta(T), 0\right\} \mid \mathcal{F}_t\right].$$

We define the unique *equivalent martingale measure* $\tilde{\mathcal{P}}$, as $\tilde{\mathcal{P}}(A) \equiv E[z(T')1_A]$, $A \in \mathcal{F}_{T'}$, where

$$z(t) \equiv \exp\left\{-\int_0^t \theta(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\} = \xi(t)P_0(t).$$

Changing the measure to the equivalent measure $\tilde{\mathcal{P}}$, the conditional expectation becomes

$$\frac{P_0(t)}{P_0(T-)}\tilde{E}\left[\max\left\{K-\frac{1}{y_{m1}}(T'-T)\delta(T),0\right\}\mid\mathcal{F}_t\right].$$

Since $\delta(t)$ follows geometric Brownian motion, this can be integrated out explicitly, yielding $(T'-T)\delta(t)\mathcal{N}(-d_{m1})/y_{m1}+K\exp\{-r(T-t)\}\mathcal{N}(-d_{m2})$, where d_{m1} and d_{m2} are as given in the proposition. Hence, we have the second equation for the market price in a portfolio insurance economy.

$P_t^{(0)}(t)$ is immediate from Lemma 1, since $\xi(t)=1/\delta(t)$, $t\in[0,T']$ in the normal agent economy. In the portfolio insurance economy, Equation (27) implies

$$\sum_{m=1}^{M}\frac{1}{y_{m2}}+\sum_{n=M+1}^{N}\frac{1}{y_n}=\frac{1}{(T'-T)\delta(T)}\sum_{m=1}^{M}\max\left\{K,\frac{1}{y_{m1}}(T'-T)\delta(T)\right\}+\sum_{n=M+1}^{N}\frac{1}{y_n}.$$

So,

$$\xi(t)=1/\delta(t),\quad t\in[0,T)$$

$$\xi(t)=\frac{(T'-T)\delta(T)}{\delta(t)\{\sum_{m=1}^{M}\max\{K,\frac{1}{y_{m1}}(T'-T)\delta(T)\}+\sum_{n=M+1}^{N}\frac{1}{y_n}(T'-T)\delta(T)\}},$$
$$t\in[T,T'].$$

Substituting into Lemma 1, rearranging, and substituting in $X_{em}^{(0)}(t)$ yields Equation (19). Then properties (i) and (ii) are immediate from Equations (16) through (19). ■

Proof of Proposition 3. Applying Itô's lemma to Equations (16) and (18) yields the expressions for $\|\sigma_{em}^{(0)}(t)\|$, $\|\sigma_{em}(t)\|$, $\mu_{em}^{(0)}(t)$, $\mu_{em}(t)$. Then properties (i) and (ii) are immediate. ■

Proof of Proposition 4. For any wealth $\hat{X}(t)$ satisfying the wealth dynamics in Equation (3), applying Itô's lemma to the product $\xi(t)\hat{X}(t)$ gives

$$d(\xi(t)\hat{X}(t))+\xi(t)\hat{c}(t)dt=\xi(t)\left[\hat{\pi}(t)^{\top}\sigma(t)-\hat{X}(t)\theta(t)^{\top}\right]dW(t).\quad(28)$$

Using Lemma 2 and the optimal consumptions from Equations (6) through (8), integrating the conditional expectations in a similar manner to the proof of Proposition 3, we obtain the optimal wealths:

$$\hat{X}_n(t) = \frac{(T'-t)}{y_n\xi(t)},$$
$$\hat{X}_m(t) = \frac{(T'-t)}{y_{m1}\xi(t)} - \frac{(T'-T)}{y_{m1}\xi(t)}\mathcal{N}(-d_{m1}) + K\exp\{-r(T-t)\}\mathcal{N}(-d_{m2}),$$

where

$$d_{m1} \equiv \frac{\ln\left((T'-T)/y_{m1}\xi(t)K\right) + \left(r+\frac{\|\theta\|^2}{2}\right)(T-t)}{\|\theta\|\sqrt{T-t}},$$
$$d_{m2} \equiv d_{m1} - \|\theta\|\sqrt{T-t}.$$

Applying Itô's lemma to the products $(t)\hat{X}_n(t)$ and $\xi(t)\hat{X}_m(t)$, we get

$$d\left(\xi(t)\hat{X}_n(t)\right) = -\frac{1}{y_n}dt$$
$$d\left(\xi(t)\hat{X}_m(t)\right) = -\xi(t)K\exp\{-r(T-t)\}\mathcal{N}(-d_{m2})\theta(t)^{\top}dW(t) + dt \text{ terms.}$$

Equating these coefficients of $dW(t)$ with those of Equation (32) yields the optimal portfolio strategies $\hat{\pi}_n(t)$ and $\hat{\pi}_m(t)$, which upon dividing by $\hat{X}_n(t)$ or $\hat{X}_m(t)$ leads to Equations (25) and (26).

To show that $\hat{\phi}_{mi}(t) \le \hat{\phi}_{ni}(t)$ it is sufficient to show that $0 \le [1 - \frac{K\exp\{-r(T-t)\}\mathcal{N}(-d_{m2})}{\hat{X}_m(t)}] \le 1$. The second inequality is obvious, since both numerator and denominator of the second term in the square brackets are nonnegative. Using the above expressions for $\hat{X}_m(t)$, we rearrange to give

$$\left[1 - \frac{K\exp\{-r(T-t)\}\mathcal{N}(-d_{m2})}{\hat{X}_m(t)}\right] = \frac{1}{\hat{X}_m(t)}\left[\frac{(T'-T)}{y_{m1}\xi(t)} + \frac{(T'-T)}{y_{m1}\xi(t)}(1-\mathcal{N}(-d_{m1}))\right] \ge 0. \quad \blacksquare$$

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