Option Pricing with Differential Interest Rates

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The classic option pricing model is generalized to a more realistic, imperfect, dynamically incomplete capital market with different interest rates for borrowing and for lending and a return differential between long and short positions in stock It is found that, in the absence of arbitrage opportunities, the equilibrium price of any contingent claim must lie within an arbitrage-band. The boundaries of an arbitrage-band are computed as solutions to a quasi-linear partial differential equation, and, in general, each end-point of such a band depends on both interest rates for borrowing and for lending. This, in turn, implies that the vector of concurrent equilibrium prices of different contingent claims-even claims that are written on different underlying assets-must lie within a computable arbitrage-oval in the price space.

The classical Black-Scholes option pricing model, as is well known, is based on several ideal assumptions. After the inception of the model some score years ago, many successful efforts were mounted at relaxing most of those assumptions to bring them closer

For their insightful comments and good advice I wish to thank Simon Benninga, Avi Bick, Fischer Black, Wendell Fleming, Mark Garman, Michael Harrison, Steve Heston, Gur Huberman, Jon Ingersoll, Haim Levy, Steve Ross, Mark Rubinstein, and the finance seminar participants at UC Berkeley, The Hebrew University, Tel-Aviv University, and at Yale University. My special thanks go to Darrell Duffie and to the editor, Chi-fu Huang. On the other hand, I am responsible for all possible errors. This work has been supported by a grant from the Krueger Fund. Address correspondence to Yaacov Bergman, School of Business and the Center for Rationality and Interactive Decision Theory, The Hebrew University, Mount Scopus, Jerusalem 91905, Israel.

The Review of Financial Studies Summer 1995 Vol. 8, No. 2, pp. 475–500

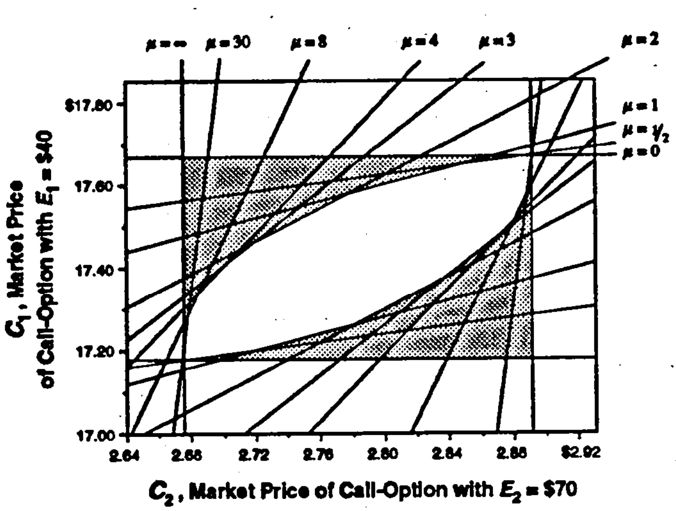

Figure 1
A computed sample of an arbitrage-oval
Any pair of call prices that lies outside the arbitrage-oval represents an arbitrage opportunity. The arbitrage-oval is formed by taking the Intersection of a collection of strips. Each strip is the area outside which a pair of call-option market prices represents an arbitrage opportunity, which is deduced from the arbitrage-band for a μ-money-spread comprising one call-option with exercise price E_1 held long and μ call-options with exercise price E_2 held short.

to agreement with real market environments. However, one of the basic assumptions of the option pricing model remained virtually unchallenged-that of a unique riskless interest rate for borrowing and for lending.

This article generalizes the option pricing model to a more realistic imperfect capital market with different interest rates for borrowing and for lending. It is found that, in the absence of arbitrage opportunities, the equilibrium price of any contingent claim must lie within an *arbitrage-band.* The boundaries of an arbitrage-band are computed as solutions to a quasi-linear partial differential equation, and, in general, each endpoint of such a band depends on *both* interest rates for borrowing and for lending. In turn, arbitrage-bands of portfolios of options put *joint* restrictions on the equilibrium prices of the constituent options. The effects are quite large, A typical result is shown in Figure 1; pairs of call-option prices that fall outside the *arbitrage-oval* represent arbitrage opportunities.

It is also found that, as the borrowing and the lending rates approach a single value, option pricing with differential interest rates approaches that prescribed by the Black&holes option pricing model, which, in turn, implies the robustness of the latter classical model to the introduction of a small borrowing-lending rate differential.

The article is organized as follows. The setting is described in Section 1. It is a dynamically incomplete capital market with differential interest rates for borrowing and for lending, in which contingent claims *cannot* be priced by arbitrage arguments alone. Section 2 derives the arbitrage-bands, within which the equilibrium contingent prices must lie, as solutions to optimization problems. The implications for prices of options are discussed in Section 3. Section 4 pmvides an extension of the model by introducing an imperfection into the underlying asset market in addition to the imperfection in the riskless asset market. Section 5 concludes.

1. The Setting: A Dynamically Incomplete Capital Market

In setting up the framework, we will seek to adhere as closely as possible to the setting of the Black-Scholes option pricing model [Black and Scholes (1973)], introducing only those changes that are necessary to accommodate differential interest rates for borrowing and for lending. The reason that such changes are needed is the following. The Black&holes capital market is perfect and dynamically complete. In that market, the price of a call-option, say, must be a function of time and the concurrent stock price only; it cannot depend on any other variables that may affect the economy lMerton (1977)]. The proof that this is so relies heavily on the fact that the borrowing and the lending rates in the Black&holes market are the same.

What if those rates are not the same? Then it is no longer necessary that the option price be a function of time and concurrent stock price only. Depending on the structure of the economy, either the option price may depend on additional variables or it may not so depend. In the latter case, though, if the markets for the stock and the option are perfect and continually open, then a locally riskless asset with *the same implied borrowing and lending rates* can be synthetically manufactured by a self-financing dynamic trading strategy in the stock and in the option.[1]

[1] It can readily be shown that, if the stock price follows a diffusion with a diffusion coefficient of $\sigma(S_t, t)$ and if the option price, $O(S_t, t)$, depends on time and concurrent stock price only, then the synthetic interest rate is equal to

$$\left(\frac{\partial O(S_t, t)}{\partial t} + 1/2\sigma^2(S_t, t)\frac{\partial^2 O(S_t, t)}{\partial s^2}\right) \Big/ \left(O(S_t, t) - S_t\frac{\partial O(S_t, t)}{\partial s}\right),$$

provided that the derivatives exist and the denominator is not zero.

If, on the other hand, the economy structure is such that the equilibrium option price depends nontrivially on some extra nontraded variables in addition to time and concurrent stock price, then, although the stock and the option markets are perfect and continually open, synthetic locally riskless interest rates cannot be formed by any trading strategy in the stock and in the option. The reason is this. The stock and the option prices share one source of uncertainty, and therefore this source can be dynamically hedged away by an appropriate trading strategy. But the other extra sources of uncertainty in the option price will remain, and there are no other traded assets, whose prices also depend on those extra sources of uncertainty, that could be used to hedge those extra sources away, as well. In other words, in this sort of capital market there are too few long-lived assets relative to the evolution of uncertainty, so that even very frequent trading is not sufficient to complete the market. Such a market is said to be *dynamically incomplete* [Duffie and Huang (1985)].

This kind of a dynamically incomplete capital market, unlike the Black-Scholes dynamically complete one, admits meaningful differential interest rates for borrowing and for lending. Clearly, in such a market, the sharp equilibrium option price cannot be determined by pure arbitrage argument,[2] but, as we will see, the existence of differential interest rates can be used to bound the equilibrium option price by pure arbitrage arguments. Our purposes in this article are to show the existence of such bounds, to derive a procedure for computing them, and to investigate their economic implications.

1.1 A formal setting

The following is a simple formalization of the above setting. Agents can borrow by short-selling locally riskless borrowing-bonds with a time t price B_t that obeys

$$dB_t = B_t r_B(t) dt; \qquad t \geq 0, \; B_0 = 1. \tag{1}$$

Agents can also lend by buying locally riskless lending-bonds with a time t price L_t following

$$dL_t = L_t r_L(t) dt; \qquad t \geq 0, \; L_0 = 1. \tag{2}$$

For simplicity, it is assumed that the borrowing rate r_B and the lending rate r_L are constants, but the analysis is identical even when those rates are general functions of time. We also assume that the borrowing rate is greater than the lending rate. The source for the divergence

[2]This is typical. Garman and Ohlson (1981) use a two-date model with transaction costs, in which the equilibrium price of a call-option cannot be determined by arbitrage arguments alone.

of the riskless interest rates for borrowing and for lending can be attributed to the cost of intermediation by a bank, bringing together borrowers and lenders and monitoring borrowers to ensure loan repayment (e.g., detection of debt-covenant breaches), and execution of necessary remedies. (The monitoring costs should be proportional to the term of the loan, and indeed they are, if those costs are remunerated by an interest rate differential.) Clearly, the fact that the bank borrows at the lower lending rate and lends at the higher borrowing rate does not constitute an equilibrium-violating arbitrage opportunity for the bank. The interest rate differential just reflects the normal return to a bank's economic activity, which may even include rent, if competition among banks is not perfect.

To describe the dynamics of the risky assets in a dynamically incomplete market setting, we adapt a simplified version of Merton's (1990, p. 553) diffusion state variable information framework. Let Y_t describe a "nontraded" state variable of the economy, about which agents care and whose dynamics are given by the stochastic differential equation (SDE)

$$dY_t = G(Y_t, t)dt + H(Y_t, t)dq_t; \qquad t \geq 0; \ \ Y_0 \in \Re, \tag{3}$$

where q_t is a Wiener process. It is assumed that agents can also buy or sell a risky asset, stock, in a frictionless and continuously open market. The stock price, S_t, is a regular diffusion that satisfies the SDE

$$dS_t = m(S_t, Y_t, t)dt + \sigma(S_t, t)dZ_t; \qquad t \geq 0; \ \ S_0 \in D, \tag{4}$$

where Z_t is a Wiener process that is not perfectly correlated with q_t, and where D is the domain[3] of real values that are attainable by the diffusion S_t. It is assumed that the functions[4] G, H, m, and σ are such that the solution to the system of SDEs, Equations (3) and (4), exists [see, e.g., Duffie (1992)]. In the specification of the stock price process, we follow Merton (1977, p. 243), who assumes that "*m* can be generated by a stochastic process of a quite general type, while σ is restricted to be at most a function of S_t and t." The purpose of this restriction on σ is to achieve the simplification that allows the bounding of option prices by means of solutions to a partial differential equation (PDE) in time and the stock price as free variables, in analogy to the precise pricing of options by arbitrage in the Black-Scholes theory.[5]

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[3]For example, if the diffusion is a geometric Brownian motion, then $D = \Re_{++}$.

[4]Of these four, only σ will play an explicit role in the sequel. The correlation between the two processes does not play an explicit role either.

[5]Allowing σ to depend on the state variable Y_t as well, while more realistic, would complicate the analysis by driving our model further away from that of Black and Scholes to considerations of

A European contingent claim, option (the two terms will be used interchangeably), which, for simplicity, does not pay dividends, is also traded in a frictionless, continuously open market. The option is a financial contract that pays b(s) dollars if the underlying asset winds up with the price s on the expiration date $T > 0$. It is assumed that the structure of the economy is such that the equilibrium price of this option, denoted O_t^e, besides depending on the stock price and time, also depends nontrivially on the state variable Y_t (or on its path up to time t). As discussed above, this dependence of the option price on the extra state variable leaves the market dynamically incomplete, and this market, in turn, accommodates meaningful divergent interest rates for borrowing and lending. (Of course, if the borrowing and the lending rates are the same, then the option price is a function of the stock price and time.)

At this stage, our modeling strategy is similar to that of Blacks (1972, p. 446) “zero-beta” model, where an imperfection is introduced in the riskless asset market only-borrowing or lending are prohibited-while no restrictions apply to long or short positions in the risky assets. Similarly, we have introduced an asymmetry in the rates of return between long and short positions in the riskless asset but allowed none with respect to long and short positions in the risky assets. This expository simplification is removed in Section 4, where it is shown that the result is a magnification of the effects that we find when friction is present only in the credit market.

In the sequel, we will consider more than one option on the same stock. It will then be implicitly assumed that the market is still dynamically incomplete by reinterpreting Y_t as a state *vector* of a dimension that is equal to or larger than the number of tradable options, so that no option is a redundant asset, and a locally riskless trading strategy cannot be formed.

Remark. Let P_t and C_t denote the prices of a put and a call on our stock with the same exercise price E and the same expiration date T. Then, to preclude arbitrage, the yield to maturity T as of time t, denoted ${}_t y_T$ must be equal to

$$\frac{1}{T-t}\ln\left[\frac{E}{S_t+P_t-C_t}\right];$$

the same yield for borrowing and for lending. This, however, does not preclude differential locally riskless rates for borrowing and for

stochastic volatility of the stock price in addition to the differential interest rates extension, which is our concern in the current article.

[6]The real function b is assumed to be differentiable a.e. on its domain, D.

lending. It is only required that (i)

$$r_L \leq {}_t y_T \leq r_B,$$

for all t in $[0, T]$, which is the generalization of the put-call-parity in the current market setting; and (ii) that there would not exist similar put-call pairs expiring almost everywhere in $[0, T]$, which is precluded by realistically requiring that the number of contingent claims is always finite. Note that no arbitrage can be generated between the differential locally riskless rates and the yield to maturity T *because reversing the* position in a locally riskless asset means switching from the borrowing to the lending rate, or vice versa.[7]

2. Arbitrage-Bands

We further assume that trade in the stock and bonds can take place continually according to a dynamic *trading strategy,* which in our context is defined to be a three-dimensional process

$$(\alpha_t, \beta_t, \lambda_t), \; t \geq 0$$

that is predictable (determined by information available up to time t). Here, α_t denotes the number of shares of stock held and β_t and λ_t denote, respectively, the number of borrowing and of *lending* bonds held-all at time t. We require that

$$\beta_t \leq 0, \lambda_t \geq 0; \text{ for all } t \geq 0,$$

to ensure that borrowing and lending are carried out at the appropriate rates. At time t, the value of the portfolio held according to a trading strategy is

$$V_t := \alpha_t S_t + \beta_t B_t + \lambda_t L_t; \;\; t \geq 0.$$

Given the contingent claim described above, consider the following optimization problem.

The MIN$^+$ Problem

$$\underset{\{\alpha_t, \beta_t, \lambda_t;\, 0 \leq t \leq T\}}{\text{minimize}} \; V_0$$

subject to:

(i) $V_t = \alpha_t S_t + \beta_t B_t + \lambda_t L_t; \; 0 \leq t \leq T,$

(ii) $dV_t = \alpha_t dS_t + \beta_t dB_t + \lambda_t dL_t; \; 0 \leq t \leq T,$

(iii$^+$) $V_T \geq +b(S_T).$

[7] A fuller discussion of this issue is in the working paper [Bergman (1993)] version of this article.

The MIN^+ optimization problem says the following. Consider the set of trading strategies

$$\{(\alpha_t, \beta_t, \lambda_t), 0 \leq t \leq T\}$$

that obey (i) value dynamics; (ii) self-financing [Merton (1970)]; (iii$^+$) at time T, a trading strategy winds up with a value at least as large as the contractual payoff to the contingent claim. Out of all these eligible trading strategies, MIN^+ selects the one that at time 0 costs the minimum dollar amount.

The solution strategy to MIN^+, denoted

$$(\alpha_t^+, \beta_t^+, \lambda_t^+),$$

will be called a self-financing, minimum-cost, dominating trading strategy, or an SFMCD trading strategy, for shorts.[8] Its time t dollar value will be denoted by V_t^+. In an arbitrage-free economy,

$$O_0^e \leq V_0^+$$

must hold. To see this, suppose that $O_0^e > V_0^+$, then the following is an arbitrage opportunity. Sell the contingent claim short at time 0 and remain short until time T, when the short position is closed. Also, invest according to the strategy, $(\alpha_t^+, \beta_t^+, \lambda_t^+)$, $0 \leq t \leq T$. Arbitrage profits are generated, since the net proceeds at time 0 are $O_0^e - V_0^+ > 0$; at time T they are $V_T^+ - b(S_T) \geq 0$, according to (iii$^+$); and in the interim no external funds are required because the strategy is self-financing.

Let the problem MIN^- be the variant of MIN^+ where constraint (iii$^+$) is replaced by: (iii$^-$) $V_T \geq -b(S_T)$. Denote the strategy that solves MIN^- by $(\alpha_t^-, \beta_t^-, \lambda_t^-)$, and the time t dollar value of the solution strategy by V_t^-, $0 \leq t \leq T$.. By a similar argument, $-V_0^- \leq O_0^e$ must hold to preclude arbitrage. The two inequalities can then be combined to yield $-V_0^- \leq O_0^e \leq V_0^+$, , and,. since time 0 is arbitrary, we have $-V_t^- \leq O_t^e \leq V_t^+$; $0 \leq t \leq T$. The interval $[-V_t^-, V_t^+]$ will be called *the arbitrage-band.* We can now summarize.

Proposition 1. *In a dynamically incomplete market with different interest rates for borrowing and for lending, it is necessary that the*

[8] The adjective dominating signifies that, according to (iii$^+$), the strategy's value dominates the payoff to the option at expiration. An SFMCD trading strategy may be viewed as the dynamic analog of Shaefer's (1982) two-period minimum-cost-dominating-portfolio.

equilibrium price of an option lie in the arbitrage-band, $[-V_t^-, V_t^+]$. *Otherwise, an arbitrage opportunity exists.*

Clearly, in our market with differential interest rates, a necessary condition for a trading strategy to qualify as a ‘solution to either one of the two minimization problems is that the strategy would never entail simultaneous nonzero borrowing- and lending. Simultaneous borrowing and lending should always be offset in order to minimize the initiation cost of a solution strategy.

Remark 1. A contingent claim may itself comprise a portfolio of other options, so that

$$h(s) = \sum_{m=1}^{M} h_m(s).$$

In that case, the arbitrage-band of the portfolio may help establish much narrower arbitrage-bands for component options in the portfolio conditioned on the observed prices of other component options in the portfolio. This is explored in Section 4.

Remark 2. The restriction on the volatility in the stock price to be at most a function of time and concurrent price was not necessary in deriving Proposition 1. It is necessary, however, in characterizing the arbitrage-band by means of a solution to a generalization of the Black-Scholes PDE in time and stock price as free variables. This is done next.

2.4 Characterizing the arbitrage-band by means of solutions to a quasi-hear PDE

The solutions to MIN^+ and to MIN^- are constructively characterized as follows.

Proposition 2. Define the real function $R : \Re \rightarrow \{r_L, r_B\}$ *that maps every nonpositive number to the bon-owing rate and every positive number to the lending rate. Let* $w^+(s, t)$*, a realfunction of two variables, be the solution to the following (quasi-linear) partial differential equation*

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s, t)\frac{\partial^2 w}{\partial s^2} = R\left(w - s\frac{\partial w}{\partial s}\right) \cdot \left[w - s\frac{\partial w}{\partial s}\right];$$

$$(s, t) \in D \times [0, T] \qquad (5)$$

with boundary condition[9]

$$w^+(s,T) = +h(s); \qquad s \in D, \tag{6}$$

where D is the real domain in *which the stock price takes values (usually the positive ray).*

Using the process S_t in Equation (4), define the four stochastic processes

$$\alpha_t^+, l_t^+, \beta_t^+, \lambda_t^+, \qquad t \in [0,T]$$

by

$$\begin{aligned}
\alpha_t^+ &:= \frac{\partial w^+}{\partial s}(S_t, t),\\
l_t^+ &:= w^+(S_t,t) - S_t\frac{\partial w^+}{\partial s}(S_t,t),\\
\beta_t^+ &:= \frac{l_t^+}{B_t},\ \lambda_t^+ := 0 \qquad \text{when } l_t^+ < 0,\\
\beta_t^+ &:= 0,\ \lambda_t^+ := \frac{l_t^+}{L_t} \qquad \text{when } l_t^+ \geq 0.
\end{aligned}$$

Then, (i) the trading strategy, $(\alpha_t^+, \beta_t^+, \lambda_t^+)$, solves MIN^+*; (ii) $w^+(S_t,t)$ is the dollar value of that solution strategy, i.e., $V_t^+ = w^+(S_t,t)$; and (iii) l_t^+ is the dollar amount lent (if positive, borrowed if negative) according to the solution strategy.*

To construct the solution to MIN^-, replace the +h(s) in the PDE's boundary condition with -h(s), and replace all plus-sign superscripts above with minus-sign superscripts.[10]

Proof. In the Appendix. ■

We now have the following corollary, which follows directly from Propositions 1 and 2.

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[9]Depending on the function $\sigma(\cdot,\cdot)$ the end condition may need extension to another region of the boundary. This is not necessary when the underlying asset's price follows a geometric Brownian motion, for example. Proof of existence and uniqueness of solutions to PDEs like (5) have to be custom made and are beyond the scope of this article. For a treatment of a related nonlinear PDE, see Evans (1979).

[10]In PDE (5), the function $R(\cdot)$ detects the occurence of either borrowing or lending in the solution trading strategy. It's arguement, $w - s\partial w/\partial s$, measures the number of dollars lent; therefore, *$R(\cdot)$ takes on the value of the* appropriate interest rate. Note that, although *$R(\cdot)$ suffers* a jump discontinuity at zero, the right-hand side of PDE (5) is continuous on the equation's domain, so no pasting conditions need be considered.

Corollary 1. The arbitrage-band at time $t \in [0, T]$ of a European option with contractual payoff function h(·), expiration date T, and with an underlying assetpriced at S_t is

$$[-V_t^-, V_t^+] = [-w^-(S_t, t), w^+(S_t, t)].$$

Although,. by construction, the upper bounding trading strategy, $(\alpha_t^+, \beta_t^+, \lambda_t^+)$, is self-financing, and it replicates the payoff to the option at expiration. Nevertheless, the strategy's value, $w^+(S_t, t)$, at any time t is almost never equal to the equilibrium option price O_t^e. This is so because, whereas the former depends only on S_t and t, the latter, recall, depends nontrivially also on the state variable Y_t, by assumption.[11] To preclude arbitrage, as shown in Section 2.2, it is only necessary that $O_t^e \leq w^+(S_t, t)$. However, strict equality-in our market with differential interest rates-cannot be inferred. The reason is the following.

If $O_t^e > w^+(S_t, t)$ at some time t, then arbitrage profits can be captured by selling the option short at t and executing the upper bounding trading strategy $(\alpha_t^+, \beta_t^+, \lambda_t^+)$ throughout $[t, T]$, which costs $w^+(S_t, t)$ dollars to initialize. Therefore, at time t, a net positive amount is received, and at time T, the value of the upper bounding trading strategy is $h(S_t)$, which exactly offsets the payoff on the short option, to form a zero dollar net obligation. This is, clearly, an arbitrage. If, on the other hand, $O_t^e < w^+(S_t, t)$, an attempt to capture arbitrage profits by buying and holding the option and *reversing* the strategy $(\alpha_t^+, \beta_t^+, \lambda_t^+)$ is bound to fail because that reverse strategy, which *is not equal* to $(\alpha_t^-, \beta_t^-, \lambda_t^-)$, is not even self-financing. Deploying the strategy $(\alpha_t^-, \beta_t^-, \lambda_t^-)$ does guarantee $-h(S_T)$ dollars at expiration T, but, unfortunately, this strategy yields at time t only $w^-(S_t, t)$ dollars, which is less than or equal to O_t^e, and therefore arbitrage profits are absent. A similar argument applies to the lower bounding strategy, $(\alpha_t^-, \beta_t^-, \lambda_t^-)$.

3. Properties of Arbitrage-bands

In the limit, when both the borrowing and the lending rates approach a single value, r, then PDE (5) specializes to the *Black-Scholes PDE,*

[11] The equality $O_t^e = w^+(S_t, t)$ can hold at most on a zero Lebesgue measure subset of $[0, T]$. This is still consistent with the assumption that O_t^e depends nontrivially on the state variable Y_t.

which is recorded here for completeness.

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 w}{\partial s^2} = r \cdot \left(w - s\frac{\partial w}{\partial s}\right), \qquad (s,t) \in D \times [0, T]. \quad (7)$$

This implies that the Black-Scholes option pricing model is robust to the introduction of a small borrowing-lending rate differential. For future use, we will denote the solution to the Black&holes PDE (7), with the boundary function $h(\cdot)$ and with constant interest rate r, by $w_{B-S}(s, t; h, r)$ Note that the stock price here (and in the previous propositions) is a quite general diffusion-not necessarily a geometric Brownian motion-and that h is not necessarily that of a call or a put option.

Proposition 3. *(i) Let $h(\cdot)$ be the contractual payoff function of a European contingent claim with expiration date T. If $h(\cdot)$ satisfies*[12] $h(s) - sh'(s) \leq 0,$ *a.e. on D, then, at time* $t \in [0, T]$, *when the underlying asset price is s, the endpoints of the arbitrage-band for that claim are given by*

$$-w^-(s,t) = w_{B-S}(s, t; h, r_L); \qquad w^+(s,t) = w_{B-S}(s, t; h, r_B). \quad (8)$$

If, *on the otherhand,* $h(s) - sh'(s) \geq 0$ *a.e. on D, then r_L and r_B switch roles in Equation (8).*

(ii) *If* $[h(s) - sh'(s)]$ *changes signs on the domain of $h(\cdot)$, then each endpoint of the arbitrage-band depends on both interest rates for borrowing and for lending.*

The proof of this proposition is based on the fact (Bergman (1994)] that a solution function to the Black-Scholes PDE inherits the properties of the boundary function. In particular, if $h(s) - sh'(s) \leq 0$ a.e. on the boundary, then

$$w_{B-S}(s,t) - s\partial w_{B-S}(s,t)/\partial s \leq 0$$

in the PDE's domain. But the latter inequality implies that the solution to the Black-Scholes PDE will also solve PDE (5) with the $R(\cdot)$ switching function picking the borrowing rate everywhere in the domain. The formal proof is omitted.

3.1 The arbitrage-bands for call and put options

Consider a European call-option at time t with exercise price E and expiration date $T(> t)$ on a stock with a concurrent price S_t, when

[12] Except, perhaps, for a set of points in D of measure zero, where the derivative of $h(s)$ may not exist (as is the case for both the call and the put options at $s = E$, the exercise price).

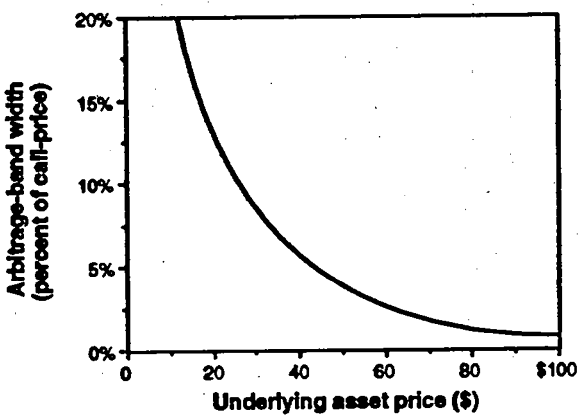

Figure 2
An arbitrage-band of a call-option
The width of an arbitrage-band of a call-option as a percentage of the call's price, when the annual borrowing and lending rates are 8 percent and 9 percent, respectively. The call exercise-price is $60, and its time to expiration is half a year. The underlying asset follows a GBM with an instantaneous standard deviation of 0.3/year.

the concurrent state vector takes the value Y_t. We will denote the equilibrium price of that call-option by

$$C^e(S_t, Y_t, t; E).$$

We will also denote the solution to the Black&holes PDE (7) with a constant interest rate r and with the call-option contractual payoff function, $h_{call}(s) := \max(s - E, 0)$, by $C_{B\text{-}S}(S_t, t; E, r)$. We will use analogous notation for a put-option with contractual payoff $h_{put}(s) :=$ max(E - s, 0). Then the following is a direct corollary of Proposition 3.

Corollary 2. *The arbitrage-bands for call and for put options are given, respectively, by*

$$C_{B\text{-}S}(S_t, t; E, r_L) \leq C^e(S_t, Y_t, t; E) \leq C_{B\text{-}S}(S_t, t; E, r_B),$$
$$P_{B\text{-}S}(S_t, t; E, r_B) \leq P^e(S_t, Y_t, t; E) \leq P_{B\text{-}S}(S_t, t; E, r_L).$$

Proof. By checking that $h_{call}(s)$ and $h_{put}(\cdot)$ satisfy the required conditions of Proposition 3.[13] ■

[13]This is a well-known result, when the stock price follows a GBM, but the corollary is true for quite general diffusion stock price processes.

Figure 2 depicts the arbitrage-band for a call-option as a percentage of the call's price, when the borrowing rate is 9 percent and the lending rate is 8 percent. Clearly, the interest rate differential generates a first-order effect. For example, when the difference between the borrowing and the lending rates is 2 percentage points, as suggested in Brenner and Galai (1986), then the arbitrage-band is as large as 5 percent of the call price at the money (E = $60), and is rapidly increasing the further out of the money the call-option gets.

3.2 Joint arbitrage restrictions on options; the arbitrage oval

Consider a portfolio that is long one call-option with exercise price E_1 and short μ call-options with exercise price E_2. Both calls have the same expiration date T, the exercise prices satisfy $E_2 > E_1$, and the number μ is nonnegative. This portfolio may be viewed as a generalized bullish money-spread (MS), which we will call the p-money-spread. (An ordinary money-spread has $\mu = 1$.) We are interested in the arbitrage-band of the p-money-spread because of its ramifications to joint arbitrage restrictions on the prices of its component call-options. Viewing the p-money-spread as a contingent claim in its own right, its boundary function is given by

$$b_{MS}(s) = \max(s - E_1, 0) - \mu \cdot \max(s - E_2, 0), \qquad s \in D.$$

At time t, when the underlying stock price is S_t, the following must hold to prevent arbitrage

$$MS^e(S_t, Y_t, t; E_1, \mu * E_2) = C^e(S_t, Y_t, t; E_1) - \mu C^e(S_t, Y_t, t; E_2), \quad (9)$$

where the left-hand side denotes' the equilibrium price of the μ-money-spread.

To produce the arbitrage-band for the p-money-spread, we apply Corollary 1 to obtain

$$\begin{aligned} -MS^-(S_t, t; E_1, \mu * E_2) &\le MS^e(S_t, Y_t, t; E_1, \mu * E_2) \\ &\le MS^+(S_t, t; E_1, \mu * E_2), \end{aligned} \quad (10)$$

where

$$MS^+(S_t, t; E_1, \mu * E_2)$$

is the solution to PDE (5) with $+h_{MS}(\cdot)$ as the boundary function, and

$$MS^-(S_t, t; E_1, \mu * E_2)$$

is the solution to the same PDE with $-h_{MS}(\cdot)$ as the boundary function.

Remark For μ-money-spreads with $\mu > E_1/E_2$, the quantity $h_{MS}(s)-sh'_{MS}(s)$ changes signs on the boundary $D \times \{T\}$. It is zero for $s < E_1$; it is negative for $E_1 \le s < E_2$; and it is positive for $E_2 \le s$. Hence, by Proposition 3, the upper and the lower end of the arbitrage-band of such μ-money-spreads depends on both interest rates, for borrowing and for lending.

Combining Equations (9) and (10) we get the following inequality as a necessary condition for no-arbitrage, valid as of time t.

$$\begin{aligned}-MS^{-}(S_t, t; E_1, \mu * E_2) &+ \mu C^{e}(S_t, Y_t, t; E_2)\\ &\le C^{e}(S_t, Y_t, t; E_1) \qquad (11)\\ &\le MS^{+}(S_t, t; E_1, \mu * E_2) + \mu C^{e}(S_t, Y_t, t; E_2).\end{aligned}$$

It is convenient to rewrite double-inequality (11) as

$$-MS_{\mu}^{-} + \mu C_2 \le C_1 \le MS_{\mu}^{+} + \mu C_2.^{14} \qquad (12)$$

Consider the plane of all pairs of market prices-whether equilibrium or not--of the two call-options that comprise the μ-money-spreads. For a fixed underlying asset price s at a fixed time t, double-inequality (12) defines in that plane a linear strip whose twin linear parallel boundaries have a slope of μ. This strip is

$$\mathbf{A}_{\mu} := \{(C_1, C_2) \in \Re^2 : -MS_{\mu}^{-} + \mu C_2 \le C_1 \le MS_{\mu}^{+} + \mu C_2\}.$$

We will refer to it as the $\mathbf{A}_{\mu}$- strip. It is easy to see that $\mathbf{A}_0$ is the arbitrage-band of the E_1 call-option considered in isolation, and define $\mathbf{A}_{\infty}$ to be the arbitrage-band of the E_2 call-option considered in isolation (it can readily be shown that $\mathbf{A}_{\infty} = \lim_{\mu\uparrow\infty} \mathbf{A}_{\mu}$, which justifies the notation). Double-inequality (12) then implies that a pair of call-option prices (C_1, C_2) that lies (as a point) outside any $\mathbf{A}_{\mu}$- strip for positive μ, represents an arbitrage opportunity involving either a long or a short position in the μ-money-spread and the corresponding SFMCD (self-financing, minimum-cost, dominating) trading strategy in the underlying stock and bonds. Proposition 3 implies similarly for μ equals 0 or ∞.

[14] The first double-inequality in corollary 2 above can be used to derive two double-inequalities, one for C_1 and the other for C_2. Then the two inequalities can be used to derive a double-inequality that looks like double-inequality (12). But, of course. it would not contain any new information that is not already in the original double-inequalities. In contrast, (12) contains new information; as will be seen, these inequalities are much tighter. This point is self-evident but was included to preempt such a possible mistake.

Formally, we now have the following testable hypothesis. (It is convenient to denote by $\bar{\mathfrak{R}}_+$ the extended nonnegative real ray including the ideal point ∞.)

Proposition 4. Let (C_1, C_2) be a pair of call-option market prices on the same underlying asset with simultaneous expiration time and with exercise prices E_1 and E_2, respectively. If

$$(C_1, C_2) \in \left(\bigcap_{\mu \in \bar{\mathfrak{R}}_+} \mathbf{A}_\mu \right)^c$$

[i.e., (C_1, C_2) lies outside the intersection of all $\mathbf{A}_\mu$- strips] *then the pair of call prices (C_1, C_2) represents an arbitrage opportunity and therefore should not be observed in equilibrium.*

Proof. If

$$(C_1, C_2) \in \left(\bigcap_{\mu \in \bar{\mathfrak{R}}_+} \mathbf{A}_\mu \right)^c,$$

then

$$(C_1, C_2) \in \bigcup_{\mu \in \bar{\mathfrak{R}}_+} \mathbf{A}_\mu^c,$$

because the two sets are identical. This means that there exists at least one $\mu \in \bar{\mathfrak{R}}_+$ such that $(C_1, C_2) \in \mathbf{A}_\mu^c$, that is, (C_1, C_2) lies outside at least one $\mathbf{A}_\mu$- strip and, therefore, represents an arbitrage opportunity. ∎

It is instructive to discuss Proposition 4 against the backdrop of a numerical solution (closed form solutions are unavailable). The following were selected as input: a stock whose price process is a GBM with σ = 0.3/year and whose current price happens to be \$55; two call-options on that stock with 1 year to expiration and with respective exercise prices E_1 = \$40 and E_2 = \$70; a lending rate of 3.5 percent/year; and a borrowing rate of 5 percent/year (approximating current conditions).

Using a numerical finite-difference method, we solved PDE (5) for p-money-spreads and computed the corresponding $\mathbf{A}_\mu$- strips for μ = 0, ½, 1, 2, 3, 4, 8, 30, ∞ (Actually, many more values of μ were solved for, but those reported suffice to provide the general picture.) In Figure 1 (in the introduction), we drew those $\mathbf{A}_\mu$- strips in the plane of pairs of the call-option market prices. A clear pattern emerges. The intersection of all the $\mathbf{A}_\mu$- strips form a closed oval shape, which we will call the arbitrage-oval its boundary being the envelope curve of the collection of straight line pairs that border the intersected strips.[15]

In terms of Figure 1, Proposition 3 says that pairs of simultaneous call-option prices (C_1, C_2) that fall outside the lightly grayed rectangle, $\mathbf{A}_0 \cap \mathbf{A}_\infty$, represents arbitrage opportunities because either $C_1 \notin \mathbf{A}_0$ or $C_2 \notin \mathbf{A}_\infty$, that is, at least one option price is outside its arbitrage-band. Proposition 4 strengthens this statement considerably by identifying a new arbitrage zone. This is the region inside the rectangle, $\mathbf{A}_0 \cap \mathbf{A}_\infty$ but *outside* the arbitrage-oval, $\bigcap_{\mu \in \bar{\Re}_+} \mathbf{A}_\mu$. For the current numerical solution, the area of the arbitrage-oval is about half the area of the rectangle.

Proposition 4 should be of interest to arbitrageurs, as well. When we say that a pair of simultaneous option prices should not be observed in equilibrium, it does not mean that such prices are not to be observed in the marketplace under any circumstances. In fact, it is the arbitrageurs who, acting in their self-interest, keep a watchful eye for out-of-equilibrium prices that represent arbitrage opportunities. When they identify such prices, they try to capture arbitrage profits by trading appropriately, thus driving the prices back into equilibrium. Although there does not yet exist an accepted theory of out-of-equilibrium price dynamics, it is reasonable to suppose that, due to the activity of the arbitrageurs, the probability is higher for observing market prices that are closer to equilibrium than prices that are further away from it. Therefore, the identification by Proposition 4 of the new arbitrage zone (between the rectangle and the oval) should be even more valuable to arbitrageurs than is suggested by its area only, because out-of-equilibrium pairs of option prices, when they materialize, are more likely to appear in that zone than in regions of equal area lying outside the rectangle.

The reason that the arbitrage-oval, which is computed from the set of μ-money-spreads, is significantly smaller than the intersection of the two respective arbitrage-bands of the two calls is the following. Three factors positively bear upon the width of an arbitrage-band. (i) The difference between the borrowing and lending rates; (ii) the exposure of the minimum cost, dominating trading strategies to these two rates, namely, the dollar amounts borrowed or lent as prescribed by those strategies; and (iii) the length of time until expiration.[16] The

[15]Actually, it is not exactly an oval, because typically this shape has corners on the northeast and southwest edges but, as a mnemonic, oval will do.

[16]In fact, it can be shown that, when the underlying asset follows a geometric Brownian motion, the width of the arbitrage-band of a call-option k approximately the product of (i) the implied borrowing in the SFMCD trading strategy, (ii) the the to expiration, and (iii) the difference between the borrowing and the lending rates.

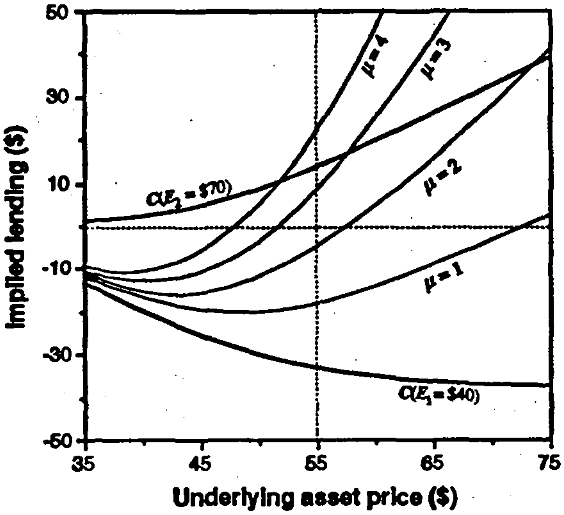

Figure 3
Implied lending in the SFMCD trading strategies
Implied lending in the SFMCD trading strategies of the long call-option (E_1 = $40); of the short call-option (E_2 = $70); and of four μ-money-spreads comprising the two calls, for $\mu = 1, 2, 3, 4$. In the neighborhood of the underlying asset price of $55, the μ-money-spread, with μ equaling about 2, features the closest-to-zero average implied lending, a consequence of the largest average offsetting between borrowing and lending in the corresponding SFMCD trading strategy. This, in turn, gives rise to the narrowest μ-money-spread arbitrage-band, as inferred from Figure 1.

SFMCD trading strategy for a money-spread, because it is minimum-cost, entails *offsetting* the *borrowing* prescribed by the trading strategy that dominates the long call in the money-spread against the *lending* implied by the trading strategy that dominates the *short* call in the same money-spread. As a consequence, the exposure to the interest rates implied in the money-spread is significantly smaller than that implied in either call-option.

This is clearly demonstrated in Figure 3, which depicts the implied lending, $(w - S_t \partial w/\partial s)$, in the SFMCD trading strategies of the two call-options and of the &-money-spreads composed of the former with µ = 1, 2, 3, 4; all from the numerical example above. In the neighborhood of the underlying asset price of $55, the p-money-spread, with μ equal to approximately 2, features the closest-to-zero average implied lending that is a result of the largest average offset between borrowing and fending in the corresponding SFMCD trading strategy. This, in turn, generates the narrowest p-money-spread arbitrage-band, as can be inferred from Figure 1.[17]

From this discussion, it is also clear why it was not necessary to consider p-money-spreads with negative *μ's:* Because in such money-spreads both call-options are held long, therefore there is no offsetting of borrowing against lending in the corresponding SFMCD trading strategy, and the width of the arbitrage-band is simply the sum of the arbitrage-bands of the two calls computed separately. In fact, if an $\mathbf{A}_{\mu}$- strip with any negative μ is drawn in Figure 1, then the negatively sloping parallel lines that border it go through the northeast and the southwest comers of the light gray rectangle. Such strips do not add any new restrictions.

The discussion above considered a portfolio of just *two* options of the *call* variety, and therefore Proposition 4 is a statement about the plane of those call-option prices. This analysis can readily be generalized by considering portfolios of several call-options and put-options. The result would be statements, analogous to Proposition 4, about arbitrage-hyperovals in hyperplanes of prices of those options.

The above can also be readily generalized to the case of a portfolio of options written on a set of *different* underlying assets. In particular, the arbitrage-band of a portfolio of such options can be computed using the multidimensional generalization of PDE (5). Consider now *two* call-options, each on a *different* underlying asset and with the same expiration date. Look at the portfolio that is long the first call and short the second; a money-spread of sorts (remember, the underlying assets are different). Repeat now the procedure that was described in Section 2.3 for the money-spreads written on the same underlying asset. This results in a construct similar in all respects to that of Figure 1, including a similar arbitrage-oval and similar empirical implications.

Thus, an observed price of one option on one underlying asset, a call-option on IBM, say, implies restrictions on another contingent claim written on *a different underlying asset,* like *a* call-option on Aetna. The intuition is that the SFMCD trading strategies that determine the arbitrage-band for the spread of the two calls imply, as before, offsetting borrowing against lending even though the underlying assets are different. Thus, interrelationships between unrelated options emerge in our model, where none exist according to the classical option pricing model.

[17] Note also how, in Figure 3, the implied lending in the SFMCD trading strategy of the μ-money-spreads changes signs as a function of the underlying asset price, as it must according to Proposition 3.

4. Adding an Imperfection in the Stock Market

in the foregone analysis, we introduced an imperfection in the riskless asset market; namely, an asymmetry between the rates of return on long and short positions. But we did not introduce a similar asymmetry in the stock market (or in the contingent claim market). We do this now.

A short-seller of stock generally cannot use the proceeds from the short-sale. These remain with the broker (the intermediary), who usually deposits those funds and collects the interest until the short position is closed. This arrangement can be considered as compensation for the broker's continuous monitoring of the short position according to the margin rules, which are designed to prevent contract nonperformance by the short-seller. (Since this institutional arrangement survives competition among brokers, as is the case in the United States, it seems that this monitoring cost is not merely a tent, but rather a real cost that is borne by a broker, even when she trades to her own account.)

It follows that the retention of short-sale proceeds amounts to a rate of return differential between long and short stock positions; if the instantaneous return on a long position in a stock with price S is $+dS/S$, then the instantaneous return on a short position in the same stock can be modeled as $-dS/S - r_L dt$. (The opportunity cost to the short-seller is the lending rate.) With this stock-market imperfection in place, we now characterize the arbitrage-band of a European contingent claim:[8]

Praposition 5. Identical to Proposition 2, except that PDE (5) is replaced by

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 w}{\partial s^2} = R\left(w - s\frac{\partial w}{\partial s}\right)\left[w - s\frac{\partial w}{\partial s}\right] + H\left(\frac{\partial w}{\partial s}\right)\left[\frac{\partial w}{\partial s}s\right] r_L;$$
$$(s,t) \in D \times [0,T], \tag{13}$$

where $H(\cdot)$ is a real function that maps negative numbers to 1 and nonnegative numbers to zero.

[18]Imperfections can be introduced into the contingent-claim market as well. A bid-ask spread will simply add its own size to the arbitrage-band. A rate of return differential between long and short postitions in contingent claims just adds to the effects discussed in Proposition 5.

Proof: Similar to that of Proposition 2 and is therefore omitted. ∎

Remark. Here, as in Proposition 2, the quantity $\partial w/\partial s$ measures the number of shares held in the SFMCD trading strategy. When $\partial w/\partial s <$ 0, a short sale of the underlying stock is indicated by that strategy, which the H(·) function detects. The last term on the LHS of PDE (13) is then activated.

Corollary 3. In the presence of the stock short-sale imperfection, the arbitrage-band of a call-option at time t, when the stock price equals S_t, is given by the interval

$$[\exp(-r_L(T-t))C_{B-S}(S_t, t; E, 0),\ C_{B-S}(S_t, t; E, r_B)].$$

Proof. In the Appendix.

Comparing Corollary 2, which gives the arbitrage-band of a call-option when only the riskless asset-market imperfection is present, to Corollary 3, which gives the call's arbitrage-band when both the riskless and the risky asset-market imperfections are present, reveals that the arbitrage-band is wider in the latter case.[19] Thus, the stock-market imperfection adds to the effect of the riskless asset-market imperfection.

5. Summary

In this article, we have generalized the classic option pricing model to a more realistic, imperfect, dynamically incomplete capital market with different interest rates for borrowing and for lending and a return differential between long and short positions in stock. We find that the Black-Scholes option pricing model is robust to the introduction of such imperfections when these are small. We further find that, in the presence of these imperfections and in the absence of arbitrage opportunities, the equilibrium price of any contingent claim, or of a portfolio of such claims, must lie within an arbitrage-band. The boundaries of an arbitrage-band are computed as solutions to a quasi-linear partial differential equation, and, in general, each endpoint of

[19]The proof generalizes to a much wider class of contingent claims. We conjecture that it is true in general.

such a bind depends on both interest rates, for borrowing and for lending. This, in turn, implies that the vector of concurrent equilibrium prices of different contingent-claims-even claims that are written on different underlying assets-must lie within a computable oval in the price space.

Appendix

Proof of Proposition 2. We first confine attention to solutions to MIN$^+$ in the class of path-independent trading strategies. A trading strategy in that class can be written as follows.

$$\begin{array}{lll} \alpha_t = a(S_t, t), \ t \geq 0 & & a : D \times [0, \infty) \rightarrow \mathfrak{R}, \\ \beta_t = b(S_t, t), \ t \geq 0 & \text{where} & b : D \times [0, \infty) \rightarrow (-\infty, 0] \\ \lambda_t = c(S_t, t), \ t \geq 0 & & c : D \times [0, \infty) \rightarrow [0, \infty). \end{array}$$

The value process (6) then takes the form $V_t = w(S_t, t)$, where $w : D \times [0, \infty) \rightarrow \mathfrak{R}$ is defined by

$$w(s, t) := a(s, t)s + b(s, t)B_t + c(s, t)L_t. \tag{A1}$$

We also require that the functions a, b, and c be twice continuously differentiable in the first argument and once in the second, on $D \times [0, \infty)$. It follows that the same is true of w. Then, using Equation (4) and Itò's lemma:

$$dw(S_t, t) = \left(\frac{\partial w(S_t, t)}{\partial t} + \mu(S_t, t)\frac{\partial w(S_t, t)}{\partial s} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 w(S_t, t)}{\partial s^2}\right) dt$$
$$+ \sigma(S_t, t)\frac{\partial w(S_t, t)}{\partial s} dZ_t.$$

On the other hand, using Equations (1) and (2), the self-financing constraint (ii) of MIN$^+$ takes the form:

$$dw(S_t, t) = \left[\alpha_t \mu(S_t, t) + \beta_t r_B B_t + \lambda_t r_L L_t\right] dt + \alpha_t \sigma(S_t, t) dZ_t.$$

In order for the last two equalities to hold, the instantaneous standard deviations must be identical; that is,

$$\alpha_t = \partial w(S_t, t)/\partial s,$$

or necessarily,

$$a(s, t) = \frac{\partial w}{\partial s}(s, t). \tag{A2}$$

The drifts must also be equal:

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 w}{\partial s^2} = \beta_t r_B B_t + \lambda_t r_L L_t,$$

or necessarily,

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s, t)\frac{\partial^2 w}{\partial s^2} = b(s, t) r_B B_t + c(s, t) r_L L_t. \tag{A3}$$

Clearly, a trading strategy that involves simultaneous borrowing and lending, when the respective interest rates differ, cannot minimize the initial costs. Planning to offset borrowing against lending will cost less initially. It follows that, at any time $t > 0$ the following must hold:

$$\text{either} \quad \beta_t \leq 0 \text{ and } \lambda_t = 0, \tag{A4}$$

$$\text{or} \quad \beta_t = 0 \text{ and } \lambda_t \geq 0. \tag{A5}$$

Suppose now that, at a given time t and for a given underlying asset's price level, $S_t = s$, Equation (A4) holds. Then, from Equations (A1) and (A2),

$$b(s, t) = \frac{\left(w - s\frac{\partial w}{\partial s}\right)}{B_t} \leq 0, \qquad c(s, t) = 0, \tag{A6}$$

and substituting Equation (A6) into Equation (A3) yields

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s, t)\frac{\partial^2 w}{\partial s^2} = r_B\left(w - s\frac{\partial w}{\partial s}\right),$$

$$\text{whenever} \quad \left(w - s\frac{\partial w}{\partial s}\right) \leq 0. \tag{A7}$$

Suppose, on the other hand, that at a given time t and for a given underlying asset's price level, $S_t = s$, Equation (A5) holds. Then, following a similar calculation results in

$$b(s, t) = 0, \qquad c(s, t) = \frac{\left(w - s\frac{\partial w}{\partial s}\right)}{L_t} \geq 0, \tag{A8}$$

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s, t)\frac{\partial^2 w}{\partial s^2} = r_L\left(w - s\frac{\partial w}{\partial s}\right),$$

$$\text{whenever} \quad \left(w - s\frac{\partial w}{\partial s}\right) \geq 0. \tag{A9}$$

Using the definition of $R(\cdot)$, Equations (A7) and (A9) are spliced into PDE (5). We have thus proved that a necessary condition for the self-financing constraint in MIN^+ to hold is that the value of the optimal strategy should satisfy PDE (5). Reversing the steps in the proof shows that the condition is also sufficient.

It can also be shown that a uniform increase in the boundary function $h(\cdot)$ on its domain D is a necessary and sufficient condition for a uniform increase in the solution to PDE (5). It follows that, to minimize in MIN^+, it is both necessary and sufficient that in constraint (iii$^+$), equality holds. Assembling both conditions together we have that the solution to PDE (5) with boundary condition (6) is the value of the optimal strategy of MIN^+ within 'the class of path-independent trading strategies; Equations (A2), (A6), and (A8) specify the optimal trading strategy in terms of the solution value function.

In the context of a unique interest rate for borrowing and lending, Harrison and Pliska (1981) have shown that the trading strategy that solves MIN^+ in the path-independent strategies set also solves the same optimization problem within the more general class of (adapted) path-dependent strategies. Following their line of argument, it can be shown that our optimal solution also remains optimal in that extended class.

The construction of the solution to MIN^- is similar to that of MIN^+. ∎

Proof of Corollary 3. The proof of the right-hand inequality is similar to that of Proposition 3: Since the contractual payoff to a call-option, $h_{\text{call}}(s) := \max(s - E, 0)$, satisfies $h'_{\text{call}}(s) \geq 0$ and $h_{\text{call}}(s) - s h'_{\text{call}}(s) \leq 0$, $s \in D$, it follows that the solution $C_{B-S}(s, t; E, r_B)$ to the Black-Scholes PDE (7) with $h_{call}(\cdot)$ as end-condition and with interest rate r_B, satisfies $\partial C_{B-S}/\partial s \geq 0$, and $C_{B-S} - s\partial C_{B-S}/\partial s \leq 0$, throughout the domain of that PDE. It then follows that $C_{B-S}(s, t; E, r_B)$ also solves PDE (7) with the same $h_{call}(\cdot)$ as the end-condition. Hence, by Proposition 5, $C_{B-S}(s, t; E, r_B)$ is the upper end of a calloption's arbitrage-band in a market with differential interest rates and with differential long-short stock-returns.

Following a similar line of reasoning, it can be shown that the solution to PDE (7) with $-h_{call}(\cdot)$ as end-condition also solves

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2(s, t)\frac{\partial^2 w}{\partial s^2} - r_L w = 0, \qquad (s, t) \in D \times [0, T], \qquad \text{(A10)}$$

with the same end-condition, $-h_{call}(\cdot)$. Using the expectation representation of solutions to parabolic PDEs [e.g., Friedman (1975)], the solution to Equation (A10) is given by

$$e^{-r_L(T-t)}\mathrm{E}_{\xi(t)=s}[-b(\xi(T)],$$

where

$$d\xi = \xi(\tau)\cdot\sigma(\xi(\tau),\tau)dW(\tau).$$

Evaluating the expectation results in

$$-\exp[-r_L(T-t)]C_{B-S}(s,t;E,0),$$

whose negative is the lower-end of the arbitrage-band. ■

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