Pricing Real Assets with Costly Search

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Markets for many real assets are characterized by sequential search followed by bilateral bargaining between matched buyers and sellers. For a category of real assets, the joint, intertemporal valuation problems of buyers, owners, and sellers, and the associated Nash pricing function are solved explicitly. In equilibrium, the average transaction price is a noisy, proportional random walk, and the liquidity premium is positive for matched owners. Depending on the values of the parameters, the liquidity premium can be substantial. In a related problem of optimal development with costly search, the optimal exercise point, cost of development, and value of the undeveloped asset are calculated analytically. With search, development can occur sooner and undeveloped assets have lower market values than the standard solution without search.

Markets for many real assets are decentralized. Common examples are categories of real estate within a metropolitan area, including single and multifamily, housing, commercial and industrial properties, and retail centers. Other examples are types of consumer and producer durables, small businesses, and firms acquired through mergers and takeovers.[1] In all of

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[1] Some decentralized markets have huge trading volumes. For example, from 1989 to 1991 the annual average trading volume for single-family residences in the United States was $395 billion for existing units and $84 billion for new units, as reported in the *Federal Reserve Bulletin,* October 1992. This total was about one-third of the concurrent volume of round lots on the

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these markets, buyers must search for sellers and/or sellers must search for buyers. Once a potential buyer and seller meet, they must bargain to determine a price. If the parties cannot agree on a price, then they must continue to search.. Search is costly because buyers and sellers are mismatched to their assets while searching. The bargaining solution must reflect the options of both the buyer and seller to continue searching, the benefit to both of a match, and the buyer's subsequent costly search as a seller once his match fails. In turn, the benefit from a match must reflect the systematic or common component of the rent or cash flow from the asset. For example, with single-family houses in a metropolitan area the common component could be the local rent on apartments or the implicit rent from owner-occupied houses. A realistic solution for housing must allow the common component of rent to change stochastically though time as conditions in the local rental market change.

Costly search with pairwise bargaining has been studied extensively in steady state.[2] In steady state all factors affecting the matching of potential buyers and sellers, the bargaining between matched agents, and thereby the pricing of real assets are constant through time. This analytically convenient assumption allows careful, detailed study of the relationship between matching, bargaining, and prices. However, it also requires that prices change through time only in response to the changing idiosyncratic component of matches between different buyers and sellers. Most importantly, it precludes stochastic changes through time in aggregate state variables or common factors that affect the cash flows from assets. With housing this precludes unpredictable changes in local employment or income that could affect the common component of the implicit rent on housing services. In option pricing theory such common factors are critical to the pricing of assets.

In this article the pricing process is derived in continuous time for a category of real assets with costly search and bilateral bargaining. Under the assumptions of the model, the expected price or current market value of a representative asset is a continuous, proportional

NYSE and about one-fifth of the concurrent volume on all stock exchanges in the United States. Measured in terms of the stock of assets, decentralized markets for residential real estate are even larger relative to organized stock exchanges.

[2] Initially, the trading price between matched agents was specified as a cooperative Nash bargaining solution. Important examples are Diamond (1982) and Mortensen (1982). Later, the price was determined in the equilibrium of a noncooperative bargaining game with complete information. Important papers include Rubinstein and Wolinsky (1985), Gale (1986, 1987), and Binmore and Herrero (1988a, 1988b). Incomplete information was introduced more recently in Wolinsky (1990) and Samuelson (1992).

random walk with a constant mean and variance per unit of time, while the average transaction price is a proportional random walk with noise. In equilibrium the liquidity premium on an asset is positive and possibly substantially so, depending on the values of the parameters.[3] Also, optimal development of a real asset, conditional on a match, is accelerated by costly search. Simultaneously, the resulting value of the option to develop or, equivalently, the undeveloped asset is depressed relative to the standard solution without search. Finally, some of the comparative statics are surprising. For example, if buyers search more intensively and thereby arrive on average more frequently to inspect the assets of sellers, then these assets become more liquid. In equilibrium these more liquid assets have a higher expected price but the same liquidity premium and the same distribution of selling times. In this model the higher reservation values for more liquid assets and the resulting higher prices offset in equilibrium all effects of more intensive search on expected yields and selling times.

Related work on real assets is limited.[4] In Baldwin and Meyer (1979), firms must search for investments. Opportunities to invest arrive in a Poisson process, and past investments expire in another Poisson process. Because the representative firm has the capacity to invest in only one project at a time, its required rate of return or reservation value for new projects increases in the average arrival rate and decreases in the average departure rate. The pricing problem in continuous time is solved explicitly for a risk-neutral firm with a discount rate equal to zero. In Wheaton (1990) a simple, but innovative model of matching in steady state is applied to owner-occupied housing. There, families can switch between two possible types and three possible states. The types can be interpreted as small versus large families, whereas the states are matched homeowners, unmatched homeowners searching for another house, and matched homeowners seeking to sell their previous house. Comparative statics are calculated for the price in steady state, the fraction of buyers, the expected time to sale, and the effort expended on search. Both articles differ in focus from this article and hence differ in critical details. The first model

[3] Liquidity premia are derived theoretically and documented empirically in Amihud and Mendelson (1986, 1989, 1991). Other theoretical derivations appear in Constantinides (1986) and Vayanos and Vila (1992). These models have costly transactions but no costly search. The choice between searching for transactions and trading in a central market is explored in Pagano (1989).

[4] Without exception the literature on real options has ignored costly search. Applications of option pricing theory to real assets include Brennan and Schwartz (1985), Titman (1985), McDonald and Siegel (1985, 1986), Majd and Pindyck (1987), Paddock, Siegel, and Smith (1988), Clarke and Reed (1988), Dixit (1989), Sick (1989), Capozza and Sick (1991), Williams (1991, 1993), and Bar-Ilan, Sulem, and Zanello (1992).

effectively has both buyers and owners of real assets but no sellers. Without sellers, risky real assets cannot be priced in equilibrium. The second model is effectively static. Without a stochastically evolving state variable, like the common component of rent or implicit rent on real assets, developed and undeveloped property cannot be priced like options in equilibrium and liquidity premia cannot be calculated.

The model is presented in Section 1. By assumption, the cash inflow on each asset is the product of two components. The common component is realized on all assets and observed by all agents, while the residual is specific to the match and observed by the buyer and seller. The common component is a geometric Wiener variate, and the residual is an independent Pareto variate with an independent Poisson jump to zero when the match between an owner and his asset is broken. Also, buyers search independently in the same Poisson process, and all agents agree on the Nash bargaining solution. Under these conditions the unique equilibrium is identified in Section 2. The pricing function is linear homogeneous in the common component. The price per unit of the common component is a constant plus an independent Pareto variate, and the expected price or, equivalently, the current market value of the asset is a geometric Wiener variate. With costly search the expected liquidity premium on each asset is a positive constant. Finally, for each asset the time to sale and fraction of assets for sale have exponential and truncated Beta distributions, respectively.

This solution is then extended in several directions. In Section 3 it is applied to undeveloped assets. With costly search, development can occur at any value of the common component of cash inflows at or above the optimal value at which the owner first accepts an offer to develop his property. Both this optimal point and the resulting value of an undeveloped asset are below the corresponding values without costly search. The cost of development, as determined by the Nash bargaining solution, is increasing and concave in the common component and asymptotic to a linear upper bound. In Section 4 the solution is used to identify necessary and sufficient conditions for the employment of brokers by buyers and sellers. Throughout, the comparative statics are calculated analytically, if possible, and numerically otherwise for plausible values of the parameters. From the numerical computations, for example, brokers are found to be more valuable in markets with persistently low demand versus markets with persistently high demand. In turn, these markets are characterized by higher liquidity premia, higher expected times to sale, and larger fractions of assets for sale. Other extensions are also discussed in Section 4, and the results are summarized in Section 5. All technical details appear in the Appendix.

1. Model

Buyers inspect the assets of sellers, and sellers offer their assets for sale. At time t the market for the representative category of real assets has B_t buyers and S_t sellers, each with one asset for sale. Each buyer inspects the assets of sellers in the same process. During the period from time t to time $t + \Delta t$, the representative buyer b arrives to inspect the asset of some seller with the probability, $\alpha \Delta t + o(\Delta t)$. As usual, the residual $o(\Delta t)$ represents terms of smaller order than At. During the same period buyer b sees no asset with probability $1 - \alpha \Delta t + o(\Delta t)$ and multiple assets with probability $o(\Delta t)$. The numbers of inspections per buyer per period are stationary and independent across both buyers and periods. In this case, buyers inspect assets for sale in independent, identical Poisson processes. The constant Poisson arrival rate per unit of time, $\alpha > 0$, reflects the efficiency of search, as described in Section 4. For analytical convenience, all sampling of specific assets for sale is purely random, with replacement period by period and independent across buyers, Sampling with replacement is an increasingly accurate approximation to the more realistic assumption of sampling without replacement as progressively more assets are offered for sale: $S_t \to \infty$. With these assumptions the representative buyer b visits the representative seller s over the period of length At beginning at time t, with the probability: $(\alpha/S_t)\Delta t + o(\Delta t)$. Ignoring terms of smaller order than Δt, seller s then receives during this period offers from $n_{t+\Delta t}$ buyers with the binomial probabilities:

$$\Pr\{n_{t+\Delta t} = n\} = \binom{B_t}{n}\left(\frac{\alpha\Delta t}{S_t}\right)^n\left(1 - \frac{\alpha\Delta t}{S_t}\right)^{B_t - n},$$

for $n = 0, \ldots, B_t$.

Markets for real assets can have many buyers and sellers. Primary examples are single-family residences, including detached houses and condominiums, and consumer durables, like automobiles — especially in major metropolitan areas. Also, some categories of commercial real estate, small businesses, and producer durables can have reasonably large numbers of buyers and sellers. Accordingly, let the number of buyers and sellers, B_t and S_t, increase without bound, such that the ratio of buyers to sellers, B_t/S_t. converges to a positive constant ρ: $B_t, S_t \to \infty$ with $B_t/S_t \to \rho > 0$. The constant ρ can be interpreted as the ratio of buyers to sellers in a quasi-steady state. This steady state excludes short-term, random variations in the ratio of buyers to sellers, B_t/S_t, around its mean ρ but includes long-term trends that may force the mean ρ away from 1. For example, a metropolitan area may lose jobs and thereby residents for decades, so that ρ

remains below 1 for residential real estate. Another metropolitan area may grow so rapidly that public controls on development constrain the construction of new housing, so that ρ stays above 1. Under these limiting conditions, the above binomial density converges to a Poisson density with the mean $\alpha\rho\Delta\, t$. With progressively shorter and shorter periods, $\Delta\, t \to 0$, this Poisson density converges to

$$\Pr\{n_t = 0\} = 1 - \alpha\rho\Delta t + o(\Delta t), \quad \Pr\{n_t = 1\} = \alpha\rho\Delta t + o(\Delta t). \tag{1}$$

Under the above assumptions, seller s receives, over a short interval of length At beginning at time t, offers from n_t buyers with the Poisson probabilities (1).

When a buyer acquires an asset, he immediately begins to receive rent or services from the asset. This rent or implicit rent is realized continuously through time until the match between the owner and his asset is broken. If buyer b purchases the asset of seller s, then he receives from time t to time $t + \Delta\, t$ the rent: $x_t y_{bs} \Delta t$. The component x_t is common to all matches, dependent on time t, and known to all agents. The residual y_{bs} is idiosyncratic to the match, constant through time, and observed only by the buyer b and seller s, conditional on a match. The common component x_t evolves continuously though time in a proportional random walk with a constant mean and variance per unit of time. Thus, x_t follows a geometric Wiener process,

$$\frac{dx}{x} = \mu\, dt + \sigma\, dz, \tag{2}$$

with the standard Wiener variate z_t and the constant mean μ and variance σ^2 per unit of time. The residual rents y_{bs} are independently and identically distributed for all possible pairs or matches bs of buyer b to the asset of seller s. The common distribution for the residual rents can reflect the restricted search by the representative category of buyers to its preferred category of assets. By assumption, this distribution is Pareto with a finite mean,

$$\Pr\{y_{bs} \leq y\} = 1 - \left(\frac{\eta}{y}\right)^{\theta}, \tag{3}$$

for all $y \geq \eta > 0$ with $\theta > 1$. The Pareto distribution has a finite mth moment if and only if $m < \theta$. This distribution has been used empirically to measure the size of cities, the returns on securities, and the distribution of income. With (3) the residual rent y_{bs} has the conditional mean: $E[y_{bs} \mid y_{bs} \geq y] = \theta y/(\theta - 1)$ for all $y \geq \eta$. The unconditional mean, $\eta\theta/(\theta - 1)$, is 1 if $\eta = 1 - 1/\theta$.

The match between an owner and his asset is broken by a random event. With residential real estate the event could be a change in the size of the owner's family, his income from employment, or the location of his employment. Examples are marriages, births, deaths, divorces, departures to college, retirement, and new jobs at distant locations. When such an event occurs, the match *bs* disappears and the residual rent y_{bs} drops to zero. Without delay the owner then offers his asset for sale. This event — the termination of a match followed immediately by an offer to sell — occurs during any period of length Δt with the probability, $\beta\Delta t + o(\Delta t)$, satisfying $\beta > 0$. By assumption, the event is independent across both matches and periods. In this case, the length or duration of a match has an exponential distribution with the mean, $1/\beta$. This simple specification is selected mainly for its analytical convenience.[5] In fact, illiquid assets, like single-family houses, are sometimes offered for sale to rebalance portfolios or to buy bigger properties when constraints on downpayments are met.[6] Also, the exponential distribution has no memory, so that the probability of a move does not depend on the duration of the stay.[7]

In each period all buyers are identical before visiting an asset for sale, while all sellers are identical at all times. All agents are risk-neutral with a common discount rate. Without further loss of generality, this discount rate can be interpreted as the riskless rate of interest.[8] Measured per unit of time, the riskless rate is the constant ι. The subsequent problem has a finite solution if and only if this constant satisfies $\iota > \mu$. When buyer *b* inspects the asset of seller *s*, both the buyer and seller observe the residual rent y_{bs}. Next, the buyer and seller negotiate. If there exists a price that both prefer to no transaction, then they agree on the Nash bargaining solution and trade; otherwise they disagree and do not trade. Thereby, buyer *b* purchases the asset of seller *s* if his rent $x_t y_{bS}$ exceeds or equals a lower bound $V(x_t)$. This lower bound is subsequently shown to be the optimal stopping point or reservation value of both the buyer and seller. It is determined endo-

5 Exogenous sales avoid a difficult stopping problem: the optimal values of the continuous state variable x_t at which to offer an illiquid, indivisible asset for sale, subject to proportional transaction costs. To solve this problem, Grossman and Laroque (1990) must make strong assumptions: the homeowner consumes no nondurables and effectively regards his house as a consumer durable and not as a risky asset in his portfolio.

6 In Stein (1993) the latter point is discussed, but the difficult stopping problem with an endogenous pricing process for properties is not solved.

7 Prepayment rates on residential mortgages show a strong seasoning effect during the first few years. A recent example is Schwartz and Torous (1989).

8 The discount rate and the riskless rate do not appear separately in the subsequent valuation equations, (11) through (13). In these equations the constant ι can be Interpreted as either the common discount rate or the riskless rate of interest.

genously and calculated explicitly in the subsequent solution under the restriction that V is continuous on $x_t > 0$. The associated ratio, $v_t = V(x_t)/x_t$, is called the unit reservation value. Finally, if buyer b purchases the asset of seller s, then he pays the price from the subsequent Nash bargaining solution: $P_{bs}(x_t) \equiv P(x_t, y_{bs})$. Without further loss of generality, the seller then pays the transaction cost: $\gamma P_{bs}(x_t)$. The constant, $0 \leq \gamma < 1$, is the proportional transaction cost. After a transaction the seller leaves the market.

Such strong assumptions require some motivation. If agents are risk averse and the capital market is complete, then the subsequent results are changed only slightly. Everywhere in the subsequent solution the expected growth rate μ of the state variable x_t in (2) is replaced by a constant, risk-adjusted, expected growth rate if the following conditions are satisfied. Instantaneous changes in the common component x_t are perfectly locally correlated with concurrent changes in the price of a portfolio of securities that can be traded continuously without transaction costs in a perfectly competitive capital market.[9] Through time the price of this portfolio evolves continuously in a proportional random walk with a constant percentage mean and variance per unit of time. In this case, the portfolio of substitute securities has per unit of time a constant excess expected rate of return per unit of standard deviation **k.** Conditional on this constant, the risk-adjusted expected rate of growth of the state x_t is the constant $\mu - \kappa\sigma$. Again, this risk-adjusted mean would replace the mean μ everywhere in the subsequent analysis. Finally, because matches are either made or broken in completely independent events, the resulting changes in the expected utilities of both buyers and sellers are fully diversifiable. These Poisson jumps can then be valued by the argument in Naik (1991).

Several other assumptions must also be motivated. Risk-neutral buyers and sellers could have different discount rates, reflecting their relative impatience for transactions. These discount rates affect the bargaining powers of buyers and sellers and thereby the allocation of surplus in the Nash bargaining solution but otherwise do not alter the subsequent solution. As previously described, the Nash bargaining solution is supported in models of costly search by noncooperative bargaining under symmetric information. By contrast, symmetric information within the bargaining problem between a matched buyer

[9] It may be possible to generalize this assumption of spanning as follows. The second term in (2) an be decomposed into a systematic component and a residual. The systematic component must be spanned by the stochastic returns on traded securities, whereas the residual is completely diversifiable by construction. In this case, the Brownian residua1 an be written as the limit of a sequence of Poisson variates, and the argument in Naik (1991) can be applied to the Poisson variates.

and seller is more problematic and is motivated mainly by its analytical convenience. Typically, a buyer has private information about his personal preferences for the asset and his urgency to buy, while the seller has private information about the quality of his asset and his urgency to sell. Clearly, a rational agent reveals truthfully his private information in a negotiation only when it improves this personal welfare. However, buyers are typically interviewed by both sellers or their brokers and potential lenders. Brokers of both the buyer and seller act legally as agents of the seller in most jurisdictions, and lenders typically demand detailed information. Although buyers sometimes approach lenders only after negotiating with sellers, sales contracts commonly have financing as a contingency. To protect the buyer, sales contracts commonly have physical inspection of both the asset and the seller's records as another contingency.

Buyers, sellers, and owners have no limit on the length of time during which they can search for a match or hold an asset. In this, case, before buyer b can meet a seller and inspect his asset, he has the expected utility $U_b(x_t)$, conditional only on the current common component of rent x_t. The buyer's current expected utility or indirect value is

$$U_b(x_t) = e^{-\iota\Delta t}\left\{\alpha\Delta t\, E_x\left[\left(\frac{\eta}{v_{t+\Delta t}}\right)^{\theta} E_{x+\Delta x}\left[U_{bs}(x_{t+\Delta t}) - P_{bs}(x_{t+\Delta t}) \mid y_{bs} \geq v_{t+\Delta t}\right]\right] + E_x\left[\left(1-\alpha\Delta t\left(\frac{\eta}{v_{t+\Delta t}}\right)^{\theta}\right) U_b(x_{t+\Delta t})\right]\right\} + o(\Delta t). \quad (4)$$

In (4) the expectation E_x is calculated conditional on the current state, $x_t = x$, whereas the expectation, $E_{x+\Delta x}[\cdot \mid y_{bs} \geq v_{t+\Delta t}]$, is calculated conditional on the subsequent state, $x_{t+\Delta t} = x + \Delta x$, and the event of a transactions, $y_{bs} \geq v_{t+\Delta t} = V(x_{t+\Delta t})/x_{t+\Delta t} = V(x+\Delta x)/(x+\Delta x)$. Also, the residual terms of order At reflect the probability of a buyer's arrival, $\alpha\Delta t + o(\Delta t)$. If buyer b is matched to the asset of seller s at time t, then his expected utility is $U_{bs}(x_t)$. This expected utility satisfies the second recursion:

$$U_{bs}(x_t) = E_x\left[\int_t^{t+\Delta t} e^{-\iota(t'-t)} x_{t'} y_{bs}\, dt'\right] + e^{-\iota\Delta t}\{\beta\Delta t E_x[U_s(x_{t+\Delta t})] + (1-\beta\Delta t)E_x[U_{bs}(x_{t+\Delta t})]\} + o(\Delta t). \quad (5)$$

Finally, for seller s the same asset has the expected utility $U_s(x_t)$:

$$U_s(x_t) = e^{-\iota\Delta t}\left\{\alpha\rho\Delta t\, E_x\left[\left(\frac{\eta}{v_{t+\Delta t}}\right)^{\theta}(1-\gamma)E_{x+\Delta x}\left[P_{bs}(x_{t+\Delta t}) \mid y_{bs} \geq v_{t+\Delta t}\right]\right] + E_x\left[\left(1-\alpha\Delta t\left(\frac{\eta}{v_{t+\Delta t}}\right)^{\theta}\right)U_s(x_{t+\Delta t})\right]\right\} + o(\Delta t). \quad (6)$$

These recursions can be interpreted as follows. In (4) and (6) each agent's current valuation equals his discounted expected value from the next period. Agent b buys: in (4) with the approximate probability $\alpha\Delta t(\eta/v_{t+\Delta t})^{\theta}$, while agent s sells in (6) with the approximate probability $\alpha\rho\Delta t(\eta/v_{t+\Delta t})^{\theta}$. Each probability equals the approximate probability of a match – $\alpha\Delta t$ for a buyer and $\alpha\rho\Delta t$ for a seller - times the probability of a transaction conditional on a match. If a transaction occurs, then buyer b becomes an owner and realizes the net gain: $U_{bs}(x_{t+\Delta t}) - P_{bs}(x_{t+\Delta t})$. Similarly, seller s receives after a transaction the net price: $(1-\gamma)P_{bs}(x_{t+\Delta t})$. Immediately thereafter the seller leaves the market. In (5) a matched owner receives over an interval of time At the rents indicated in the integral and loses his match with the approximate probability $\beta\Delta t$. Without a match he immediately offers his asset for sale and then values it as a seller. With a match his status stays the same and his valuation changes only with the evolution of the state x_t.

These valuation functions must satisfy two sets of boundary conditions. With the geometric Wiener process (2), once the rent $x_t y_{bs}$ reaches zero, it remains there forever. Thus, all utilities must be zero at $x_t = 0$:

$$U_b(0) = 0, \qquad U_{bs}(0) = 0, \qquad U_s(0) = 0. \quad (7)$$

Also, to prevent explosive growth and collapse in prices; the utilities per dollar of implicit rent must be bounded above. In this case, there must exist two positive constants, ζ_b and ζ_s, and a positive, increasing function ζ with the values, $\zeta_{bs} = \zeta(y_{bs})$, such that the utilities satisfy

$$U_b(x_t) \leq \zeta_b x_t, \qquad U_{bs}(x_t) \leq \zeta_{bs} x_t, \qquad U_s(x_t) \leq \zeta_s x_t, \quad (8)$$

as $x_t \to \infty$. These boundary conditions are familiar from many valuation models of contingent claims.

The Nash bargaining solution has the following properties. Because both buyer b and seller s have the same information, they agree to trade if and only if both benefit from the transaction:

$$P_{bs} \leq U_{bs} - U_b, \qquad U_s \leq (1-\gamma)P_{bs}. \tag{9}$$

If a transaction occurs in state x_t then buyer b pays the price $P_{bs}(x_t)$, becomes an owner, and immediately realizes the gain, $U_{bs}(x_t) - U_b(x_t)$ - $P_{bs}(x_t)$. Simultaneously, seller s receives after transaction costs the net price, $(1 - \gamma)P_{bs}(x_t)$, and transfers to the buyer an asset with the value $U_s(x_t)$. With a transaction the total gain is allocated in the following fractions: $1/(1 + \lambda)$ to the buyer and $\lambda/(1 + \lambda)$ to the seller. The constant, $\lambda > 0$, is the seller's relative share of the surplus. This constant can be specified exogenously as a Nash, cooperative bargaining solution. Alternatively, it can be determined as a subgame perfect solution to a noncooperative bargaining problem with complete information, as described in the next section. In either case, the seller's gain, $(1 - \lambda) P_{bs} - U_s$, must equal $\lambda/(1+ \lambda)$ times the total gain, $U_{bs} - U_b - U_s - \gamma P_{bs}$. Equivalently, the seller's gain equals λ times the buyers' gain:

$$(1-\gamma)P_{bs} - U_s = \lambda(U_{bs} - U_b - P_{bs}). \tag{10}$$

This allocation determines the price $P_{bs}(x_t)$ in equilibrium. Given this price, the unit reservation value v_t then follows from either inequality in (9). At this optimal stopping value, the buyer's residual rent y_{bs} just satisfies the inequalities in (9) with the Nash bargaining price in (10).

2 . Equilibrium

In this section the previous problem is solved. The solution includes the utility functions of the representative buyer U_b, matched owner U_{bs}, and seller, U_s, as well as the pricing function P_{bs} and the unit reservation value v_t. A noncooperative bargaining solution to the sellers relative share of the surplus λ is identified. Finally, the expected time to sale of an advertised asset and the expected fraction of assets for sale are determined in equilibrium. Throughout Sections 2 and 3, all subscripts are dropped from the expectation E_x, the unit reservation value v_t, and the common component of rent x_t, and the current values are identified simply as E, v, and x.

The expected utilities of the representative buyer, matched owner, and seller must satisfy three differential equations. These equations are derived as follows. In (4) through (6), divide all terms by $\Delta t > 0$; let $\Delta t \to 0$; and exploit the previously assumed continuity of the unit reservation value v. For buyer b the expected utility U_b, from (4) must satisfy the differential equation:

$$0=\frac{1}{2}\sigma^2x^2U_b''+\mu xU_b'-\iota U_b+\alpha\left(\frac{\eta}{v}\right)^{\theta}E[U_{bs}-U_b-P_{bs}\mid y_{bs}\geq v], \quad (11)$$

for all $x \geq 0$. Again, the expectation E is calculated conditional on the current state x. Similarly, once buyer b is hatched to the asset of seller s, his expected utility U_{bs} from (5) is determined by the valuation equation:

$$0=\frac{1}{2}\sigma^2x^2U_{bs}''+\mu xU_{bs}'-\iota U_{bs}+\beta(U_s-U_{bs})+xy_{bs}, \quad (12)$$

for $x \geq 0$. Finally, for seller s the expected utility U_s from (6) satisfies the equation

$$0=\frac{1}{2}\sigma^2x^2U_s''+\mu xU_s'-\iota U_s+\alpha\rho\left(\frac{\eta}{v}\right)^{\theta}E[(1-\gamma)P_{bs}-U_s\mid y_{bs}\geq v] \quad (13)$$

for $x \geq 0$.

These are familiar valuation equations. Only one term in each equation is new. In (11) the conditional expectation is the expected gain of switching from a buyer to an owner immediately after a purchase. If a transaction occurs, then buyer b acquires the asset of seller s and realizes the net gain $U_{bs}(x) - U_b(x) - P_{bs}(x)$. Conditional on a match, a transaction occurs if and only if the residual rent y_{bs} exceeds or equals the unit reservation value v. This event occurs with the probability $(\eta/v)^{\theta}$, and a match occurs with the approximate probability a per unit of time. In (13) the final terms are similar. If a transaction occurs, then seller s leaves the market with the net gain: $(1-\gamma)P_{bs}(x)-U_s(x)$. Also, a match occurs with the approximate probability $\alpha\rho$ per unit of time, reflecting the ratio of buyers to sellers ρ. In (12) a matched owner receives per unit of time the approximate rents xy_{bs} and loses hi match with the approximate probability β per unit of time. Without a match he immediately offers his asset for sale and realizes the loss of utility: $U_b(x) - U_{bs}(x)$.

The valuation functions of the representative buyer, owner, and seller, as well as the pricing function and the seller's reservation value are determined by (7) through (13). As shown in the Appendix, this problem has a unique solution. The solution is linear homogeneous in the common component of the return on the asset:

$$\begin{gathered}P_{bs}(x)=p_{bs}x, \qquad V(x)=vx,\\ U_b(x)=u_bx, \qquad U_{bs}(x)=u_{bs}x, \qquad U_s(x)=u_sx,\end{gathered} \quad (14)$$

for $x \geq 0$. Here, the coefficients are $p_{bs} \equiv P_{bs}(1)$, $v \equiv V(1)$, $u_b \equiv U_b(1)$, $u_{bs} \equiv U_{bs}(1)$, and $u_s \equiv U_s(1)$. Henceforth, these constants are called *unit values*. The linear homogeneity in (14) follows from the proportionality in the geometric Wiener process (2), the independence between the stochastic increment in (2) and the random variable in (3), the linear homogeneity of the owner's utility U_{bs} in the rent xy_{bs}, and the boundary conditions, (7) and (8). Specifically, the linear homogeneous solution (14) satisfies the system of differential equations, (11) through (13); the boundary conditions, (7) and (8); and the Nash bargaining conditions, (9) and (10). The remaining, nonlinear solutions to the differential equations, (11) through (13), are precluded by the boundary conditions, (7) and (8).

The constants in this solution are identified below, starting with the unit reservation value. As shown in the Appendix, the unit reservation value in (14) is the constant:

$$v = \eta[\max(1, \phi)]^{1/\theta}. \tag{15}$$

In (15) the new constant ϕ is

$$\phi \equiv \frac{\alpha}{(1-\gamma+\lambda)(\theta-1)(\iota-\mu)}\left[(1-\gamma+\lambda\rho) - \frac{\beta\lambda\rho(1-\gamma)}{\beta+\iota-\mu}\right].$$

As before, a matched buyer and seller agree to trade if and only if the residual rent exceeds or equals the unit reservation value: $y_{bs} \geq v$. Thus, a transaction can fail to occur if and only if $v > \eta$ or, equivalently, $\phi > 1$. With $\phi > 1$, the unit reservation value (15) simplifies to $v = \eta\phi^{1/\theta}$. For this case, comparative statics can be calculated analytically when the seller's relative share λ is a constant, defined independently of all other parameters. This occurs with either the Nash cooperative bargaining solution or the noncooperative bargaining Solution with equal numbers of buyers and sellers, $\rho = 1$. If the relative share λ is constant, then the unit reservation value v is increasing in both the average arrival rate of buyers α and the proportional transaction cost paid by sellers γ. Also, it is decreasing in both the average rate at which matches are broken β and the difference between the riskless rate of interest and the expected growth rate of the common component, $\iota - \mu$. Finally, with the Nash cooperative solution the reservation value v, is increasing in the ratio of buyers to sellers ρ.

Using this reservation value, the utilities per unit in (14) can be calculated as shown in the Appendix. These values, u_b, u_{bs}, and u_s, are the constants:

$$u_b = \frac{1-\gamma}{1-\gamma+\lambda}\,\frac{\alpha\eta^{\theta}v^{1-\theta}}{(\theta-1)(\iota-\mu)(\beta+\iota-\mu)}, \tag{16}$$

$$u_{bs} = \frac{\beta u_s + y_{bs}}{\beta + \iota - \mu}, \tag{17}$$

and

$$u_s = \lambda \rho u_b. \tag{18}$$

The constants, (16) through (18), apply respectively to buyer b, an owner with the match bs, and seller s. With (14) through (18), the solution is effectively separated into two components: a common, intertemporal component x in (14) and an idiosyncratic, constant component in (15) through (18).

The constants in (15) through (18) are effectively bargaining solutions in steady state. In this steady state the common component of rent x is constant and conveniently normalized at $x = 1$. In (16) the utility of buyer b is the expected unit payoff per period from becoming an owner, $\alpha(\eta/v)^\theta E[u_{bs} - u_b - p_{bs} \mid y_{bs} \geq v]$, discounted as an annuity at the rate, $\iota - \mu$. This result follows from (A15) in the Appendix. The complicated expression in (16) also reflects the solution for the Unit reservation value v, the owner's utility per unit u_{bs}, and the price per unit p_{bs}. In (17) the utility of an owner with the match bs is the unit payoff per period from ownership y_{bs} plus the expected unit payoff from becoming a seller $\beta\, u_s$, discounted as an annuity at the rate, $\beta + \iota - \mu$. This discount rate reflects the average arrival of broken matches β. In (18) the utilities of buyers and sellers are proportional, with a constant of proportionality equal to the ratio of buyers to sellers ρ times the seller's relative share of the surplus λ. When buyers and sellers are equal in number, the two utilities are equal: $\rho = 1 = \lambda$. Finally, the unit reservation value (14) follows from the utilities of the representative buyer and seller, as shown in (14), (16), and (17).

The remaining constant to be determined in equilibrium is the price per dollar of the common component. This unit price p_{bs} is calculated in the Appendix from the Nash bargaining price in (10) and the utilities per unit, (16) through (18). It is

$$p_{bs} = \frac{\psi v + \lambda y_{bs}}{(1 - \gamma + \lambda)(\beta + \iota - \mu)}. \tag{19}$$

in (19) the new constant y is

$$\psi \equiv \lambda(1-\gamma)\min(1, \phi)\frac{\rho(\beta + \iota - \mu) - [\iota + \beta(1 - \lambda\rho) - \mu]}{\lambda\rho(\beta + \iota - \mu) + (1-\gamma)[\iota + \beta(1 - \lambda\rho) - \mu]}.$$

For some values of the parameters, a poorly matched buyer and seller prefer not to trade. If the unit value (15) satisfies $v > \eta$ and thereby $\phi > 1$, then both buyer b and seller s reject all matches bs with

$\eta \leq y_{bs} \leq v$. Otherwise, they reject no matches: $v = \eta$. The former case is not only more plausible than the latter case but also less complicated expositionally than the general case, $v \geq \eta$. For this reason the parameters in (15) are assumed henceforth to satisfy $\phi > 1$. The subsequent results also hold for the more general case, $\phi \geq 1$. with the constant ϕ everywhere replaced by the constant max(1, ϕ).

Typically, average prices are reported in public statistics, If the average residual rent of all successful buyers is $\bar{y}$, then from (19) the average price is $\bar{p}x$, with $\bar{p} \equiv (\psi v + \lambda \bar{y})/[(1-\gamma+\lambda)(\beta+\iota-\mu)]$. As the number of buyers and sellers becomes large, the average residual rent of successful buyers $\bar{y}$ converges almost surely to the conditional mean, $E(y_{bs} \mid y_{bs} \geq v) = \theta v/(\theta - 1)$. In turn, the average unit price $\bar{p}$ converges almost surely to the expected unit price:

$$\bar{p} \rightarrow E(p_{bs} \mid y_{bs} \geq v) = \frac{\psi + \frac{\lambda\theta}{\theta-1}}{(\beta+\iota-\mu)(1-\gamma+\lambda)} v. \qquad (20)$$

The seller's relative share λ is a constant with either the Nash cooperative bargaining solution or the subsequent noncooperative bargaining solution, In either case, the expected unit price on the right side of (20) is a positive constant. In this model, all buyers and sellers are identical and all potential matches are identical ex ante. Conditional on a sale, the representative asset then has the expected price $\bar{p}x$ from (20). For an illiquid asset this expectation is a measure of its current market value to risk-neutral investors. With (9) and (14), it is greater than the seller's valuation $u_s x$ in (18) and less than owners' average valuation $E(u_{bs} \mid y_{bs} \geq v)x$ from (17). Moreover, this measure is consistent with common practice. When searching for a listing agent, homeowners commonly ask prospective brokers: "At what price can I expect to sell my house?" or "What is my house worth?" An honest answer would be an expected price, like (20).

The pricing functions, (19) and (20), have several implications. The average price $\bar{p}x$ implied by (19) is a noisy, proportional random walk. The noise arises from the variation through time in the average residual rent of successful buyers $\bar{y}$. It disappears asymptotically in large markets with many concurrent trades of comparable assets: $\bar{y} \rightarrow \theta v/(\theta - 1)$. In the limit the expected price from (20) follows the same geometric Wiener process (2) as the common component of rent x. Also, illiquid assets have a positive liquidity premium. From (2) and (20), the value of the representative asset grows at the expected rate μ. If the utilities, (16) through (18), and thereby the unit price from (19) are to be finite, then this mean μ must be less than the riskless rate of interest ι. An unmatched seller expects the rate of return μ but receives no rent, $y_{bs} = 0$, and hence earns the negative premium, $\mu - \iota < 0$.

By contrast, a matched owner expects on his asset the total yield: $\mu + E[y_{bs} \mid y_{bs} \geq v]/E[p_{bs} \mid y_{bs} \geq v]$. 'This yield is calculated using the expected unit price in (20) as the measure of current market value. This produces prior to a purchase the positive liquidity premium:

$$\mu - \iota + \frac{E[y_{bs} \mid y_{bs} \geq v]}{E[p_{bs} \mid y_{bs} \geq v]} = \mu - \iota + \frac{\theta(\beta + \iota - \mu)(1 - \gamma + \lambda)}{(\theta - 1)b + \lambda\theta} > 0. \quad (21)$$

This premium must be positive because the matched owner must be compensated for his expected loss after his match is broken. The magnitude of the premium must compensate owners for both the time spent searching for matches and the proportional cost of transactions.

The expected time to sale for the representative asset is determined in equilibrium. For the asset of seller *s*, the remaining time on the market t_s^m is distributed exponentially under the assumptions above (1). With this distribution the expected time to sale is the inverse of the Poisson arrival rate calculated from (1), (3), and (15):

$$E(t_s^m) = \frac{\phi}{\alpha\rho}. \quad (22)$$

As the number of assets offered for sale increases, $S_t \rightarrow \infty$, the average time to sale converges almost surely to (22). If $\rho = 1$, so that $\lambda = 1$, then the expected time to sale (22) is independent of the average arrival rate of buyers α, increasing in the proportional transaction cost γ, and decreasing in both the average arrival rate of broken matches β and the excess rate of return $\iota - \mu$.

The expected fraction of assets on the market is also determined 'in equilibrium. Under the above assumptions, the expected fraction of all assets that are offered for sale at any one time is equal to the expected fraction of time that any one asset is offered for sale. Let t_s^n be the remaining time that asset *s* is off the market or, equivalently, the interval to its next time on the market. From the assumptions in Section 1, the time not on the market t_s^n has an exponential distribution with the mean, $1/\beta$. This distribution is independent of the above exponential distribution for the time on the market t_s^m. As shown in the Appendix, the share of time that an asset spends on the market, $m_s \equiv t_s^m/(t_s^m + t_s^n)$, has a truncated β -distribution of the second kind with the mean[10]:

$$E(m_s) = \frac{\beta\phi}{\alpha\rho - \beta\phi}\left[\frac{\alpha\rho}{\alpha\rho - \beta\phi}\ln\left(\frac{\alpha\rho}{\beta\phi}\right) - 1\right]. \quad (23)$$

[10] A Beta distribution of the second kind is related to the F-distribution. See Kendall and Stuart (1977) p. 163.

This mean is the expected fraction of time spent on the market by any one asset. Under the above assumptions, it equals the expected fraction of all assets on the market at any one time. In larger markets with more buyers and sellers, the average time of each asset on the market and thereby the average fraction of all assets on the market converges almost surely to (23).

The seller's share of the surplus from a match depends on the Nash bargaining solution in (10). As previously mentioned, the Nash bargaining solution can be viewed as either axiomatic or the equilibrium of a noncooperative bargaining game with symmetric information. This equilibrium can be derived as the limit in continuous time of a sequence of unique, perfect equilibria to bargaining games in discrete time with costly delay. Given the linear homogeneity in (14), the surplus per dollar of the state x_t can be divided as in the steady-state solution of Rubinstein and Wolinsky (1985) and Binmore and Herrero (1988a, 1988b).[11] In this noncooperative bargaining solution, the seller's relative share of the surplus λ is

$$\lambda = \frac{\alpha\rho + \iota\phi}{\alpha + \iota\phi}. \tag{24}$$

In (24) the constant riskless rate of interest per unit of time ι is the common discount rate of buyers and sellers, and the constants, α/ϕ and $\alpha\rho/\phi$, are the Poisson arrival rates in equilibrium of transactions for buyers and sellers, respectively. Given equal numbers of buyers and sellers, $\rho = 1$, a matched buyer and seller share the surplus equally, $\lambda = 1$. With larger ratios of buyers to sellers ρ, transactions occur more rapidly on average for sellers relative to buyers, so that sellers can bargain successfully for larger relative shares λ in (24).

With the noncooperative bargaining solution (24), the comparative statics can be complicated, especially for the expected pricing coefficient (19). Consequently, all comparative statics are computed numerically for plausible values of the parameters. In Table 1 buyers and sellers are equal in number, $\rho = 1$, so that the surplus is always shared equally, $\lambda = 1$. In Table 2 demand is either low, $\rho = 0.8$, or high, $\rho = 1.2$, so that the seller's relative'share λ is affected by the values of the parameters in (24). The parameters in the base case are intended to represent owner-occupied residences. For residences the large average arrival rates α in the table include searches by buyers through multiple listing services without subsequent site visits. Most of the comparative statics are consistent with conventional wisdom. For example, more turnover of owner-occupied housing, as reflected

[11] For example, see Corollary 1 in Binmore and Herrero (1988a).

Table 1
Expected price per dollar of common rent, expected liquidity premium, expected time to sale, and expected fraction of assets on the market

Parameter values for base case; medium demand

$\alpha = 10{,}000$	$\beta = .10$	$\gamma = .10$	$\eta = .95$
$\theta = 20$	$\iota = .08$	$\mu = .02$	$\rho = 1.0$

α average arrival rate of buyers			β average rate of broken matches		
5,000	**10,000**	**15,000**	**.05**	**.10**	**.15**
6.880	7.123	7.269	9.383	7.123	5.734
.1572	.1572	.1572	.1058	.1572	.2085
.6175	.6175	.6175	.6883	.6175	.5804
.1295	.1295	.1295	.0887	.1295	.1596

γ proportional transaction cost			η minimum residual rent		
.05	**.10**	**.15**	**.90**	**.95**	**1.0**
7.0169	7.123	7.232	6.748	7.123	7.498
.1604	.1572	.1541	.1572	.1572	.1572
.6101	.6175	.6253	.6175	.6175	.6175
.1285	.1295	.1305	.1295	.1295	.1295

θ exponent of matching probability			ι riskless rate of interest		
10	**20**	**30**	**.04**	**.08**	**.12**
12.35	7.123	5.997	11.52	7.123	5.160
.1605	.1572	.1561	.1208	.1572	.1935
1.304	.6175	.4046	1.593	.6175	.0983
.2013	.1295	.0988	.2245	.1295	.4017

μ expected growth rate of rent			ρ ratio of buyers to sellers		
.00	**.02**	**.04**	**0.8**	**1.0**	**1.2**
5.986	7.123	8.792	5.269	7.123	8.867
.1754	.1572	.1390	.2343	.1572	.1142
.4848	.6175	.8706	.8044	.6175	.4968
.1111	.1295	.1596	.1523	.1295	.1129

In each vertical quadruplet, the top number is the expected price per dollar of the common component of rent (20). The second number is the expected liquidity premium (21). The third number is the expected time to sale (22). The bottom number is the expected fraction of assets on the market (23). The column heading is the value of the indicated parameter, and the values of all other parameters are indicated in the base case. The base case with medium demand has an equal number of buyers and sellers: $\rho = 1$. In the base case the residual rent has a mean of 1. The unit of time is one year. The parameters are defined in more detail below Table 2.

in a higher average rate of broken matches β, reduces the expected transaction price of owner-occupied housing (20), raises the required liquidity premium (21), reduces the expected time to sale (22), and raises the expected fraction of houses on the market (23). Similarly, higher demand for housing, as reflected in a higher ratio of buyers to sellers ρ, raises the expected price (20) but reduces the liquidity premium (21), the expected time to sale (22), and the expected fraction of assets for sale.

Some of the comparative statics are surprising. All of these surprising results reflect the subtle interaction in equilibrium between

Table 2
Expected price per dollar of common rent, expected liquidity premium, expected time to sale, expected fraction of assets on the market, and seller's fraction of the surplus

Parameter values for base case: low demand

$\alpha = 10{,}000$ $\beta = .10$ $\gamma = .10$ $\eta = .95$
$\theta = 20$ $\iota = .08$ $\mu = .02$ $\rho = 0.8$

α average arrival rate of buyers

5,000	10,000	15,000
5.089	5.269	5.377
.2343	.2343	.2343
.8044	.8044	.8044
.1523	.1523	.1523
.4475	.4475	.4475

β average rate of broken matches

.05	.10	.15
7.135	5.269	4.178
.1583	.2343	.3101
.8839	.8044	.7627
.1047	.1523	.1871
.4477	.4475	.4473

γ proportional transaction cost

.05	.10	.15
5.136	5.269	5.409
.2417	.2343	.2268
.7969	.8044	.8123
.1514	.1523	.1532
.4474	.4475	.4475

η minimum residual rent

.90	.95	1.0
4.991	5.269	5.546
.2343	.2343	.2343
.8044	.8044	.8044
.1523	.1523	.1523
.4475	.4475	.4475

θ exponent of matching probability

10	20	30
9.308	5.269	4.408
.2337	.2343	.2345
1.694	.8044	.5274
.2321	.1523	.1173
.4504	.4475	.4465

ι riskless rate of interest

.04	.08	.12
8.204	5.269	3.893
.1783	.2343	.2895
2.119	.8044	.5177
.2605	.1523	.1159
.4483	.4475	.4474

μ expected growth rate of rent

.00	.02	.04
4.471	5.269	6.421
.2625	.2343	.2057
.6279	.8044	1.143
.1309	.1523	.1869
.4468	.4475	.4486

ρ ratio of buyers to sellers

1.0	1.2	1.4
3.431	5.269	7.123
.3930	.2343	.1572
1.126	.8044	.6175
.1854	.1523	.1295
.3829	.4475	.5000

In each vertical quintuplet, the top four numbers match the entries in the previous table. The bottom number is the seller's fraction of the surplus, $\lambda/(1+\lambda)$, caculated from (24). This base case with low demand has fewer buyers than sellers $\rho = 0.8$. The subsequent base case with high demand has more buyers than sellers: $\rho = 1.2$. In all other details Tables 1 and 2 are identical.

parameters and the expected price (20). For example, assets with higher average arrival rates of potential buyers a are more liquid. Surprisingly, these more liquid assets do not have lower liquidity premia (21). Instead, these assets have higher unit reservation values (15) and thereby higher expected unit prices (20). Moreover, the higher prices exactly offset the more rapid arrival of buyers, so that the extra liquidity alters in equilibrium neither the liquidity premium (21), the expected time to sale (22), nor the expected fraction of assets for sale (23). For the same reason, assets with lower proportional transactions

Table 2
Continued.

parameter values for base case: high demand

$\alpha = 10,000$ $\beta = .10$ $\gamma = .10$ $\eta = .95$
$\theta = 20$ $\iota = .08$ $\mu = .02$ $\rho = 1.2$

α average arrival rate of buyers			β average rate of broken matches		
5,000	**10,000**	**15,000**	**.05**	**.10**	**.15**
8.565	**8.867**	**9.048**	11.39	8.867	7.266
.1142	.1142	.1142	.0764	.1142	.1518
.4968	.4968	.4968	.5607	.4968	.4633
.1129	.1129	.1129	.7722	.1129	.1393
.5436	.5436	.5436	.5433	.5436	.5437

γ proportional transaction costs			η minimum residual rent		
.05	**.10**	**.15**	**.90**	**.95**	**1.0**
8.825	8.867	8.910	8.400	**8.867**	**9.334**
.1149	.1142	**.1135**	.1142	.1142	**.1142**
.4896	.4968	**.5044**	.4968	.4968	.4968
.1118	.1129	.1140	.1129	.1129	.1129
.5436	.5436	.5435	.5436	.5436	.5436

θ exponent of matching probability			ι riskless rate of interest		
10	**20**	**30**	**.04**	**.08**	**.12**
15.20	8.867	7.494	14.92	8.867	6.304
.1185	.1142	.1128	.1972	.1142	.1399
1.050	.4968	.3253	1.257	.4968	.0856
.1782	.1129	.0854	.1972	.1129	.3262
.5416	.5436	.5442	.5431	.5436	.5436

μ expected growth rate of rent			ρ ratio of buyers to sellers		
.00	**.02**	**.04**	**1.0**	**1.2**	**1.4**
7.378	**8.867**	**11.10**	**7.123**	**8.867**	**13.24**
.1269	.1142	.1015	.1572	.1142	.0784
.3920	.4968	.1394	.6175	.4968	.5746
.0967	.1129	.6955	.1295	.1129	.1238
.5440	.5436	.5429	.5000	.5436	.5791

α —average arrival rate of buyers, equal to the probability per unit of time that the representative buyer inspects the asset of some seller.
β —average rate of broken matches, equal to the probability per unit of time that the match between the representative owner and his asset terminates.
γ —proportional transaction cost, equal to the total transaction cost divided by the sales price.
η —minimum rent, equal to the lower support of the Pareto distribution (3) for the residual rent.
θ —exponent of matching probability, equal to the exponent in the Pareto distribution (3).
ι—riskless rate of interest, equal to the discount rate per unit of time for each buyer and seller.
μ —expected growth rate of rent, equal to the proportional drift in the diffusion process (2).
ρ —ratio of buyers to sellers, equal to the current number of buyers divided by the current number of sellers.

costs γ have higher, not lower liquidity premia. Again, the unit reservation values (15) are sufficiently higher in equilibrium, so that higher transaction costs are reflected in higher prices (20), not higher liquidity premia (21). Also, increments in the proportional transaction cost γ decrease the net expected price $(1-\gamma)\pi x$ more with higher ratios of buyers to sellers ρ. This is consistent with the result in Section 4 that brokers are more valuable in markets with persistently low demand, $\rho < 1$, than in markets with persistently high demand, $\rho > 1$.

3. Development

Developers and owners of undeveloped assets must also search for matches. Given the previous results, this search can be modeled very simply, as follows. Potential developers appear at the property of the representative owner with the Poisson arrival rate δ. The constant, $\delta > 0$, replaces the product $\alpha\rho$ in the Poisson density (1). If a developer arrives at time t, then he proposes to transform the undeveloped asset with the current market value $V(x)$ into a developed asset with the current market price $\pi\, x$. For the undeveloped asset the market value $V(x)$ in this section is distinct from and should not be confused with the reservation value from the previous sections. For the developed asset the constant, $\pi > 0$, is the current market price per dollar of the common component x. This pricing function requires a perfectly competitive market for developers. Each developer must regard both the cost of development and the expected price of a developed asset as independent of his decision to develop.[12] Without further loss of generality, all development can then be assumed to be instantaneous, once it is started.[13] Also, all development is assumed to occur at a fixed density, such as the legal maximum.[14] Under these assumptions, if the developed asset is immediately offered for sale, then its market price per dollar of common rent is $\pi = u_s$, the seller's unit utility in (18). Given a fixed density, the cost of development to the developer is specified exogenously as the constant, $\kappa > 0$.[15] If the owner of the undeveloped asset receives an offer to develop at time t, then he can immediately accept the offer with any probability q satisfying $0 \le q \le 1$. Otherwise, the owner rejects the offer.

Once an owner and developer meet, they must bargain to determine the price of development. The price or cost of development $C(x)$ determines the allocation between the owner and developer of

[12] This can occur with atomistic developers and atomistic owners of undeveloped real estate at the expanding boundary of a monocentric city, as in Capozza and Sick (1991). Downward sloping demand curves for developed assets appear in Pindyck (1988), Dixit (1989), Bar-Ilan, Sulem, and Zanello (1992), and Williams (1993). Developers can be imperfectly competitive in Williams

[13] Because all agents are price takers, time to build is easily introduced. The price $\pi\, x$ must be replaced by the present value of the expected future price upon completion of construction. This avoids the complications in Bar-Ilan, Sulem, and Zanello (1992).

[14] The optimal density of development is explored in Clarke and Reed (1988) and Williams (1991).

[15] The implications of capacity constraints among developers are studied in Williams (1993).

the total profit or capital gain from development, $\pi x - \kappa - V(x)$. With the cost $C(x)$ the owner has the capital gain, $\pi x - C(x) - V(x)$, and the developer earns the profit, $C(x) - \kappa$. Again, the Nash cooperative bargaining solution is used, and the owner is assumed to receive the fraction ω of the total gain, while the developer is allocated the residual fraction, $1 - \omega$. The constant, $0 < \omega < 1$, replaces the seller's share, $\lambda/(1 + \lambda)$, in the previous analysis. This Nash bargaining solution determines the cost of development:

$$C(x) = \omega\kappa + (1 - \omega)[\pi x - V(x)], \tag{25}$$

for all $x \geq 0$.

This simple specification of the cost of development is consistent with equilibrium under the following conditions. Suppose that each owner of an undeveloped asset currently accepts a developer's offer with the optimal probability $Q(x)$. Also, represent by $R(x)$ the current ratio of developers' aggregate capacity to units of undeveloped assets. Given the Poisson matching of developers and owners, each developer currently earns per unit of time on each unit of capacity the expected operating revenue: $\delta Q(x)[C(x) - \kappa]/R(x)$. Finally, assume that to maintain his capacity each developer incurs per unit of time the constant cost $K(x)$. If developers can adjust capacity either up or down without additional cost, then the relative number of developers $R(x)$ must satisfy in equilibrium the relationship: $R(x) = \delta Q(x)[C(x) - \kappa]/K(x)$. Otherwise, developers would adjust their collective capacity. In this case; the constant arrival rate of developers δ is consistent with competitive entry and exit.

With these simplifying assumptions, the value of the undeveloped asset must satisfy a familiar differential equation:

$$0 = \max_{0 \leq q \leq 1} \left\{ \frac{1}{2}\sigma^2 x^2 V'' + \mu x V' - \iota V + \delta\omega q(\pi x - \kappa - V) \right\}, \tag{26}$$

for all $x \geq 0$. For some finite constant, $\zeta > 0$, this differential equation must satisfy the usual boundary conditions:

$$V(0) = 0, \qquad V(x) \leq \zeta x, \tag{27}$$

as $x \to \infty$. This problem differs in only one detail from a standard option pricing problem without costly search. In (26) the last term is the expected benefit from immediate development. It equals the probability per unit of time of a developer's offer δ times the owner's probability of acceptance q times the expected benefit conditional on immediate development. With the Nash bargaining solution (25), this conditional benefit is the expected value of the developed property

π x minus the cost of development κ minus the foregone value of the undeveloped property $V(x)$.

As shown in the Appendix, this problem has a unique solution. The owner of the undeveloped asset optimally accepts an offer to develop with the probability:

$$Q(x) = \begin{cases} 0 & 0 \leq x < x^* \\ 1 & x^* \leq x < \infty. \end{cases} \tag{28}$$

At all values of the common component of rent on the developed asset x below the optimal switching point x^*, the owner rejects the offer to develop. Otherwise, he always accepts the offer. The optimal switching point in (28) is the constant:

$$x^* \equiv \frac{\kappa}{\pi} \frac{1 + \frac{\xi_1}{\xi_0 - \xi_1} \frac{\delta\omega}{\delta\omega + \iota}}{1 - \frac{1-\xi_1}{\xi_0 - \xi_1} \frac{\delta\omega}{\delta\omega + \iota - \mu}}. \tag{29}$$

The constant (29) depends on two new parameters:

$$\xi_0 \equiv \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\iota - \mu}{\sigma^2}}$$

and

$$\xi_1 \equiv \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\iota + \delta\omega}{\sigma^2}}.$$

These constants satisfy the inequalities: $\xi_0 > 1$ and $\xi_1 < 0$. Given the optimal development policy in (28) and (29), the undeveloped asset has the value:

$$V(x) = \begin{cases} (\pi x^* - \kappa)\left(\frac{x}{x^*}\right)^{\xi_0} & 0 \leq x < x^* \\ \delta\omega\left(\frac{\pi}{\delta\omega + \iota - \mu} x - \frac{\kappa}{\delta\omega + \iota}\right) & \\ \quad + \left(\frac{\iota - \mu}{\delta\omega + \iota - \mu} \pi x^* - \frac{\iota\kappa}{\delta\omega + \iota}\right)\left(\frac{x}{x^*}\right)^{\xi_1} & x^* \leq x < \infty. \end{cases} \tag{30}$$

This solution is shown in Figure 1. As illustrated, the valuation function V is everywhere increasing and strictly convex in the common

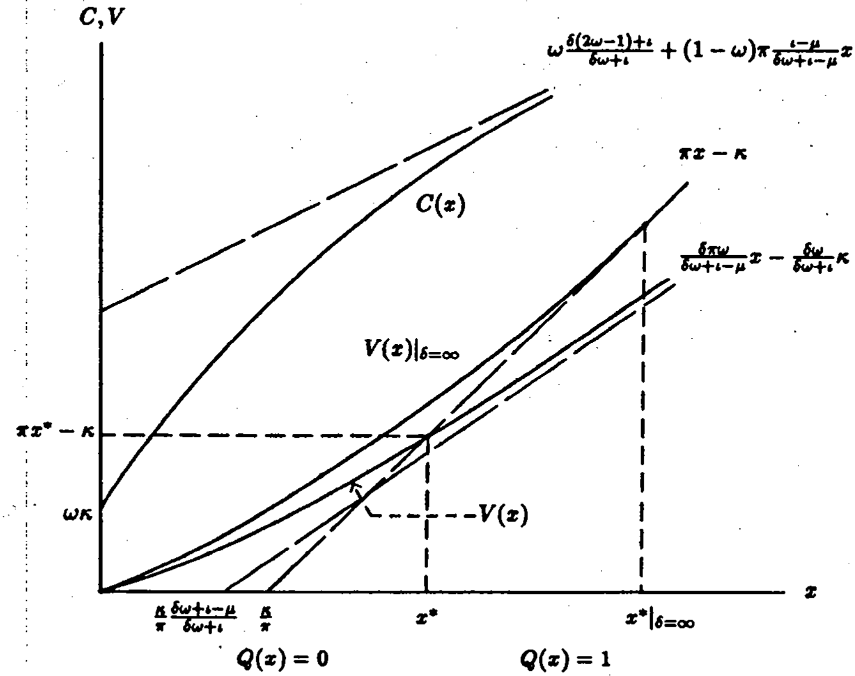

Figure 1
Cost of development and value of an undeveloped asset
The cost of development $C(x)$ and the value of an undeveloped asset $V(x)$ depend on the current common component of rent x from a developed asset. Developers arrive at the Poisson rate δ and offer to transform the undeveloped asset into a developed asset at the price or cost $C(x)$. For a developer the cost is κ. The owner always accepts the current offer if and only if the expected current price of the developed asset minus the cost of development exceeds or equals the current value of the undeveloped asset: $Q(x) = 1$ if and only if $x \geq V(x)$. This occurs if and only if the common component exceeds or equals the optimal switching point. $x \geq x^*$. Otherwise, the owner always rejects the offer, $Q(x) = 0$. As developers arrive on average more frequently, $\delta \to \infty$, the solution converges to the standard option price without costly search: $V(x) \to V(x)|_{\delta=\infty}$ and $x^* \to x^*|_{\delta=\infty}$.

component of rent x. It crosses the exercise value, max $\{0, \pi x - \kappa\}$, at the optimal switching point (29). That is, from the maximand in (26) switching optimally occurs when the value of the undeveloped asset equals the value of the developed asset minus the cost of development, $V(x^*) = \pi x^* - \kappa$. At all lower values of the common component, $0 \leq x < x^*$, the value of the undeveloped asset exceeds the value of the developed asset minus the cost of develop ment, and the owner optimally rejects in (28) all offers of development. At higher values of x, the reverse relationship holds and the owner. accepts all offers to develop. Also, as $x \to \infty$, the valuation function V approaches asymptotically the unexercised value,

$\max\{0, \frac{\delta\pi\omega}{\delta\omega+\iota-\mu}x - \frac{\delta\kappa\omega}{\delta\omega+\iota}\}$. Both the switching point x^* and the valuation function V are increasing in the arrival rate of developers δ, the owners share of the surplus from development ω, and the variance σ^2 of the growth rate in the common component of rent x. In turn, these properties of the valuation function (30) determine through the Nash bargaining solution the properties of the development cost (25). As illustrated in Figure 1, the cost function C is increasing, strictly concave, and asymptotic to its linear upper bound. It is also decreasing in the above parameters, δ, ω, and σ^2.

This solution is related to the standard option pricing result without costly search. As potential developers arrive increasingly more frequently, $\delta \to \infty$, the optimal switching point (29) converges to the optimal exercise point without costly search:

$$x^* \to \frac{\kappa}{\pi}\frac{\xi_0}{\xi_0 - 1}. \tag{31}$$

Simultaneously, the value of the undeveloped property (30) converges to the corresponding value without costly search:

$$V(x) \to (\pi x^* - \kappa)\left(\frac{x}{x^*}\right)^{\xi_0}, \tag{32}$$

for $0 \leq x \leq x^*$. Alternatively, as developers arrive less frequently, $\delta \to 0$, the underdeveloped asset becomes worthless and switching occurs as soon as the option to develop comes into the money: $V(x) \to 0$ and $x^* \to \kappa/\pi$. The optimal switching point x^* in (29) is increasing in the Poisson arrival rate of developers δ, bounded below by κ/π and bounded above by (32).

4. Extensions

Brokerage

Buyers and sellers now have a choice. Each buyer b can search for an asset either with or without a broker. Similarly, each seller s can offer his asset for sale with a broker and then pay the proportional transaction cost γ upon a sale or, alternatively, sell without a broker and thereby pay only a minimal closing cost. For notational convenience, the latter cost is set equal to zero. For both buyers and sellers, brokers can improve the efficiency of search.[16] Buyers arrive more slowly without brokers α_0 than with brokers α_1: $0 < \alpha_0 < \alpha_1 < 1$. The resulting expected utilities per unit from the previous section are u_{bi}

[16] Additional properties of a steady state are developed in Yinger (1981).

and u_{si} for buyer b and seller s. The index i identifies the presence of a broker; the representative buyer or seller has no broker if $i = 0$ and a broker if $i = 1$. In the previous equilibrium buyer b searches with a broker if and only if $u_{b0} \leq u_{b1}$. Similarly, seller s searches with a broker if and only if $u_{s0} \leq u_{s1}$. because buyers and sellers have proportional utilities in (18), buyer b optimally uses a broker if and only if seller s optimally employs a broker. This result also holds if brokers improve the efficiency of search only when both the buyer and seller have a broker. The latter assumption may be more realistic for one important category of assets. With single-family houses, only brokers have access to the local multiple listing service. in this case, a broker for a buyer (seller) may search more efficiently than a nonbroker only for a seller (buyer) with a broker.

Conditions under which brokers are used in equilibrium can now be identified. Let λ_i represent the seller's relative share (24) with $i = 0, 1$. Again, select values of the parameters such that some matches do not produce transactions: $v > h$. In this case, both buyers and sellers use brokers if and only if the ratio of arrival rates, α_1/α_0, satisfies the inequality:

$$\frac{\alpha_1}{\alpha_0} \geq \frac{1-\gamma+\lambda_1}{(1-\gamma)^{\theta}(1+\lambda_0)}\left[1-\gamma\frac{\beta(1-\lambda_1\rho)+\iota-\mu}{\beta+(\iota-\mu)(1+\lambda_0\rho)}\right]^{\theta-1}, \qquad (33)$$

calculated from (15) and (16). This inequality is necessary and sufficient for brokerage in the previous equilibrium. In other words, buyers and sellers employ brokers if and only if brokers improve sufficiently the efficiency of search. If $\rho = 1$ so that $\lambda_0 = \lambda_1 = 1$, then the lower bound on the right side of (33) increases in both the arrival rate of broken matches β and the proportional transaction cost γ and decreases in the excess return $\iota - \mu$. Also, with the parameter values in the lower right comer of Table 1, the lower bound in (33) increases in the ratio of buyers to sellers ρ. In this sense, brokers are more valuable, other things equal, in markets with longer matches, lower transaction costs, higher interest rates, lower growth rates of rent, and persistently lower ratios of buyers to sellers.

Alternatively, brokers can improve the efficiency of search, not by increasing the frequency of inspections, but by decreasing the probability of a mismatch, conditional on an inspection. Typically, brokers screen assets for their clients. With single-family houses, for example, brokers for buyers can select sites to be visited based on characteristics of design, construction, maintenance, and neighborhoods. This observable heterogeneity can be introduced very simply into the previous model, as follows. Suppose that brokers increase for the buyers the lower bound in the matching distribution (2) from η_0 to η_1. In this second case, both buyers and sellers use brokers if and only if the ratio of lower bounds, η_1/η_0, satisfies the inequality:

$$\frac{\eta_1}{\eta_0} \geq \left[\frac{1-\gamma+\lambda_1}{(1-\gamma)(1+\lambda_0)}\right]^{1/\theta} \left[\frac{\frac{1-\gamma+\lambda_1\rho}{1-\gamma}(\beta+\iota-\mu)-\beta\rho\lambda_1}{(1+\lambda_0\rho)(\beta+\iota-\mu)-\beta\lambda_0\rho}\right]^{1-1/\theta}, \tag{34}$$

also calculated from (15) and (16). Again, this inequality is necessary and sufficient for brokerage in the previous equilibrium. If $\rho = 1$ so that $\lambda_0 = \lambda_1 = 1$, then the lower bound on the right side of (34) also increases in both β and γ and decreases in $\iota - \mu$. Again, with the parameter values in the lower right comer of Table 1, the lower bound in (34) increases in the ratio ρ. Thus, the comparative statics from (33) and (34) are identical.

Heterogeneous assets

Previously, the representative category of assets was homogeneous in the following sense: before an inspection the residual rent of buyers y_{bs} was independently and identically distributed across all assets within the category. In fact, buyers or their brokers can observe some attributes of assets before an inspection. Again, with single-family houses these observable attributes can include characteristics of design, construction, maintenance, and neighborhoods. This observable heterogeneity can be introduced into the previous model as follows. Suppose that all observable attributes of the representative class of assets can be clustered into J categories: $j = 1, \ldots, J$. Also, suppose that each buyer restricts his search to one category of assets: In other words, each buyer initially chooses his preferred category of assets over which to search, based on some unmodeled heterogeneity across buyers. Categories are distinguished only by the lower bound in the Pareto distribution (3) for residual rent. For all assets in category j, this lower bound is η_j. Without further loss of generality, these lower bounds can be arranged in increasing order: $\eta_1 < \eta_2 < \cdots < \eta_J$. For example, poorly designed, constructed, and maintained houses in bad neighborhoods can be clustered in lower categories j with smaller lower bounds η_j. Otherwise, all assets in the representative class are indistinguishable by potential buyers before an inspection.

With this modification the results in the previous section are changed only slightly. For category j, the constant η is replaced everywhere by the constant η_j. This generates the utilities, u_{bj}, u_{bsj}, u_{sj}, the reservation value v_j, and the price p_{bsj}. In (15) the reservation value v_j is linear in η_j, and from (19) the average price $\bar{p}_j$ is linear in

v_j. As a result, the lower bound η can be interpreted as the average minimum residual rent across all assets, and v can be interpreted as the resulting average reservation value from (15). If assets for sale are drawn randomly from the population of all assets, then the average unit price, $\bar{p} \equiv \sum_j \bar{p}_j$, is again approximately (20) in large samples. This single change in the lower bound η does not affect the expected liquidity premium (21), the expected time to sale (22), or the expected fraction of assets on the market (23).

Hot and cold markets

In Section 1 the ratio of buyers to sellers was assumed to be constant through time. The constant ratio, $r_t \equiv B_t/S_t = \rho$, greatly simplified the analysis. With this-strong assumption all functions are linear homogeneous in (14). In this case, the problem effectively separates into two, analytically tractable parts: an intertemporal pricing problem, as reflected in the linear functions (14), and a steady-state bargaining problem, as reflected in the constant coefficients, (15) through (19). The noncooperative bargaining solution to the seller's relative share of the unit surplus λ can then be identified in (24). More realistically, the ratio of buyers to sellers r_t changes stochastically over time t as conditions change in the market. Moreover, movements in the ratio r_t may be correlated with changes in the current common component of rent x_t. In at least one special case, the pricing problem can be solved explicitly. Suppose that the ratio r_t depends only on the current common component of rent, $r_t = R(x_t)$. Assume that the rental function R has only two steps, $R(x_t) = \rho_0$ for $0 \leq x_t < \xi$ and $R(x_t) = \rho_1$ for $\xi \leq x_t < \infty$. The constants in this function satisfy $0 < \xi < \infty$ and $0 < \rho_0 < 1 < \rho_1 < \infty$. In other words, the market can be either cold or hot. Demand is low, $r_t = \rho_0 < 1$, in a cold market, $x_t < \xi$, and high, $r_t = \rho_1 > 1$, in a hot market, $x_t \geq \xi$. In this case, the previous problem can be solved explicitly if the seller's relative share of the surplus λ is constant, as in the cooperative Nash solution.

Under these assumptions the pricing function is no longer linear homogeneous. To see this, consider the expected pricing function $E(P)$ that replaces the expected pricing function from (20). Also, consider two other expected pricing functions: $E(P_m)$ for $m = 0, 1$. The first function holds in a perpetually cold market, $m = 0$, with low demand, $r_t = \rho_0$, whereas the second holds in a perpetually hot market, $m = 1$, with high demand, $r_t = \rho_1$. By the argument above (19), these two expected pricing functions are linear homogeneous in the common component x_t. The expected pricing function E(P) is then increasing everywhere, strictly convex in a cold market and strictly concave in a hot market. It is bounded below by the function in a cold market $E(P_o)$ and above by the function in a hot market $E(P_1)$.

Finally, the function $E(P)$ is tangent to $E(P_o)$ at $x_t = 0$ and asymptotic to $E(P_1)$ as $x_t \rightarrow \infty$. As a result, both the expected growth rate of prices and the variance of the growth rate of prices are highest during intermediate states x_t. These occur during transitions between hot and cold markets.

5. Summary

Many real assets are traded in decentralized markets. In these markets buyers, sellers, or their brokers must search for matches. Once a potential buyer and seller meet, they must bargain to determine a price. This produces three related problems of valuation, one each for the representative. buyer, owner, and seller. The three differential equations must be solved simultaneously with the. Nash pricing function. In this article the problem is solved analytically under the following conditions. The rent realized on each asset is the product of a common component and a residual that is specific to the match between the buyer and seller. The common component of rent is assumed to follow a geometric Wiener process, whereas the residual rent is an independent Pareto variate with an independent Poisson jump to zero when the match between the owner and his asset is broken. All buyers search independently in the same Poisson process, and all matched buyers and sellers agree on the Nash bargaining solution. Finally, all agents are risk neutral with identical discount rates, or the capital market is complete. In equilibrium the unique pricing function is shown to be linear homogeneous in the common component of cash flows, and both the pricing coefficient and the liquidity premium are identified. The expected time to sale and expected fraction of assets for sale are also derived in equilibrium. Next, a problem of optimal development with costly search is solved. Both the optimal decision to develop and the resulting value of an undeveloped asset are calculated explicitly. Several extensions are also discussed.

This article has three main results. First, under a reasonably plausible set of conditions, the average transaction price for a category of illiquid assets is a noisy, proportional random walk. The noise decreases with increasingly larger markets, and the process approaches a proportional random walk. Second, under the same conditions the expected liquidity premium can be calculated explicitly. The liquidity premium is positive, and its comparative statics are consistent with conventional wisdom—with some surprises. The premium is positive because a matched owner must be compensated for the time spent searching for a match and the transaction cost incurred conditional on a match. Finally, if the owner of an undeveloped asset must search for a developer, then he alters his optimal exercise policy for his option to develop. On average he develops sooner as developers arrive on average less frequently. As the average arrival rate of developers decreases, so does the market value of his undeveloped property or, equivalently, his option to develop.

Many extensions of this model are possible. In this article the liquidity premium can be calculated explicitly only because the portfolio problem of the representative investor is not considered. instead, the owner is assumed to offer his illiquid asset for sale when his match is broken by an exogenous event, like death, divorce, departure of children to school, or a new job in a dint location. in fact, an owner may sell his illiquid asset to rebalance his portfolio. Consider Grossman and Laroque (1990). There, the representative individual can trade both risky and riskless securities without transaction costs and a consumer durable (e.g., his house) subject to proportional transaction costs. The securities are perfectly divisible, but the durable is completely indivisible. The prices of the risky securities are geometric Brownian motions, and the flow of services from the real asset is proportional to its price. Also, the utility of consumption is isoelastic, and no nondurables can be consumed. Under these assumptions the optimal trading points for the durable are characterized analytically and computed numerically.’ Numerical solutions may also be possible with nondurable consumption, a risky durable good, and potentially binding constraints on down payments, as in Stein (1993). Similarly, it should be possible to add costly search. This would endogenize the times at which owners offer their assets for sale, Also, it would introduce issues of portfolio balance with financial constraints.

Models of intertemporal pricing with costly search may also have interesting empirical implications. Markets for many real assets, including single-family houses, have two puzzling properties: rates of return are positively correlated with previous trading volumes, and price/earnings multiples are procyclical. Neither empirical observation is consistent with the equilibria of standard financial models. Indeed, if operating returns are procyclical and business cycles are common knowledge, then price/earnings multiples should be countercyclical, rather than procyclical. However, a model of pricing with costly search may produce these two properties. Imagine that investors first search across local markets or categories of assets, acquire and process local data, and only then buy assets. In equilibrium investors may then have poorly diversified portfolios with binding financial constraints. Also, information about markets may spread only slowly through time, and past prices may affect current demands.

Appendix

Characterization of the pricing function and reservation value
From (10) the pricing function P_{bs} must satisfy

$$P_{bs} = \frac{U_s + \lambda(U_{bs} - U_b)}{1 - \gamma + \lambda}, \tag{A1}$$

for $x \geq 0$. Also, with (10) the two inequalities in (9) are equivalent. The pricing function (A1) can then be eliminated from these inequalities to yield the equivalent condition:

$$U_{bs} - U_b - \frac{1}{1-\gamma} U_s \geq 0, \tag{A2}$$

for $x \geq 0$. Given the Nash pricing function (A1), this inequality is necessary and sufficient for a transaction to occur. The reservation pricing rule — trade if and only if $y_{bs} \geq v$ — is an optimal stopping rule if and only if the left side of (A2) is decreasing in y_{bs}. In this case, the reservation value v is the smallest value y_{bs} satisfying (A2):

$$0 = \left[U_{bs} - U_b - \frac{U_s}{1-\gamma} \right] \Bigg|_{y_{bs}=v}, \tag{A3}$$

for $x \geq 0$. Subsequently, problem (4) through (10) is solved, ignoring the required monotonicity in (A2). The resulting solution is then shown to have this monotonicity.

Derivation of (14) through (19)
The system of differential equations, (11) through (13), is solved, subject to the boundary conditions, (7) and (8), in several steps, as follows. First, define the function $F \equiv \lambda\rho U_b - U_s$. Equations (11) and (13) have the weighted difference

$$0 = \frac{1}{2}\sigma^2 x^2 F'' + \mu x F' - \iota F. \tag{A4}$$

Also, the boundary conditions, (7) and (8), have the weighted differences:

$$F(0) = 0, \qquad F(x) \leq (\lambda\rho\zeta_b - \zeta_s)x. \tag{A5}$$

The differential equation (A4) has two real roots. One root is defined below (29) as the constant, $\xi_0 > 1$. The other root is identical to the second constant below (29), $\xi_1 < 0$, with one modification: $\delta\omega$ is replaced by $\alpha\lambda\rho/(1 + \lambda)$. Given the boundary conditions in (A5), the coefficients associated with these roots must be zero. Thus, (A4) and

(A5) have the unique solution: $F(x) = 0$. In other words, the system, (7), (8), (11), and (13), must uniquely satisfy the condition:

$$U_s = \lambda\rho U_b. \tag{A6}$$

With (A6) the system of differential equations simplifies to (11) and (12). The homogeneous equation associated with (12) has $xy_{bs} = 0$. As a result, any solution to the homogeneous system associated with (11) and (12) does not depend on y_{bs}. Using (A1) and (A6), this homogeneous system can be written as

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma^2 x^2 U_b'' + \mu x U_b' - \iota U_b \\ & + \frac{\alpha}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}\left[(1-\gamma)U_{bs} - (1-\gamma+\lambda\rho)U_b\right], \end{aligned} \tag{A7}$$

$$0 = \frac{1}{2}\sigma^2 x^2 U_{bs}'' + \mu x U_{bs}' - \iota U_{bs} + \beta(\lambda\rho U_b - U_{bs}). \tag{A8}$$

Now, define the function $G \equiv U_b - kU_{bs}$. From (A7) and (A8), calculate the weighted difference:

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma^2 x^2 G'' + \mu x G' - \left[\iota + \beta\lambda\rho k + \alpha\frac{1-\gamma+\lambda\rho}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}\right] G \\ & + \left\{\alpha\frac{1-\gamma}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta} + \left[\beta - \alpha\frac{1-\gamma+\lambda\rho}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}\right]k - \beta\lambda\rho k^2\right\} U_{bs}. \end{aligned} \tag{A9}$$

Also, the boundary conditions from (A5) are

$$G(0) = 0, \qquad G(x) \leq (\zeta_b - k\zeta_{bs})x. \tag{A10}$$

The quadratic in (A9) has one positive root,

$$\begin{aligned} k = {} & \frac{1}{2\lambda\rho}\sqrt{\left[1 - \frac{\alpha}{\beta}\frac{1-\gamma+\lambda\rho}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}\right]^2 + 4\frac{\alpha}{\beta}\frac{1-\gamma}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}} \\ & + \frac{1}{2\lambda\rho}\left[1 - \frac{\alpha}{\beta}\frac{1-\gamma+\rho\lambda}{1-\gamma+\lambda}\left(\frac{\eta}{v}\right)^{\theta}\right]. \end{aligned} \tag{A11}$$

With (A11) the argument above (A6) produces for (A9) and (A10) the unique solution: $G(x) = 0$. In other words, with the boundary conditions, (7) and (8), and the constant (A11), the homogeneous system, (A7) and (A8), must uniquely satisfy

$$U_b = kU_{bs}. \tag{A12}$$

The unique homogeneous solution to (11) through (13) can now be identified. Inserting (A12) into the remaining homogeneous equation (A8) yields

$$0 = \frac{1}{2}\sigma^2 x^2 U''_{bs} + \mu x U'_{bs} - [\iota + \beta(1 - \lambda\rho k)]U_{bs}. \tag{A13}$$

The parameter (All) satisfies $k < 1/\lambda\rho$. In this case, the argument above (A6) produces for (A13) and the associated boundary conditions in (7) and (8) the unique solution: $U_{bs}(x) = 0$. Combining this homogeneous solution for (13) with the previous results, (A6) and (A12), yields for the system, (7), (8), and (11) through (13), the unique homogeneous solution:

$$U_b = U_{bs} = U_s = 0. \tag{A14}$$

To complete the argument, it is sufficient to show that (14) is a particular solution to the system, (7), (8), and (11) through (13). Clearly, (14) satisfies both boundary conditions. Inserting (14) into (11), (12), and (A6) then produces the unit values:

$$u_b = \frac{\alpha}{\iota - \mu}\left(\frac{\eta}{v}\right)^{\theta} E[u_{bs} - u_b - p_{bs} \mid y_{bs} \geq v], \tag{A15}$$

(17), and (18). The associated unit price from (A1) is

$$p_{bs} = \frac{u_s + \lambda(u_{bs} - u_b)}{1 - \gamma + \lambda}. \tag{A16}$$

Also, with (14), (17), and (18), the unit value v in (13) satisfies

$$v = \left[\frac{1 - \gamma + \lambda\rho}{1 - \gamma}(\beta + \iota - \mu) - \beta\lambda\rho\right] u_b. \tag{A17}$$

In turn, (17), (18), (A16), and (A17) imply that

$$\begin{aligned} &E[u_{bs} - u_b - p_{bs} \mid y_{bs} \geq v] \\ &\quad = \frac{1-\gamma}{1-\gamma+\lambda} E\left[\frac{\beta\lambda\rho u_b + y_{bs}}{\beta + \iota - \mu} - \frac{1-\gamma+\lambda\rho}{1-\gamma} u_b \mid y_{bs} \geq v\right] \\ &\quad = \frac{1-\gamma}{1-\gamma+\lambda} \frac{v}{(\theta - 1)(\beta + \iota - \mu)}. \end{aligned} \tag{A18}$$

To derive (16), insert (A18) into (A15) and rearrange terms. For (15) insert (16) into (A17). Finally, (19) follows from (16) through (18) and (A16). This solution satisfies the monotonicity condition in (A2).

Derivation of the mean (23)

With $m_s = t_s^m/(t_s^m + t_s^n)$, the time from purchase to the next initial advertisement t_s^n satisfies $t_s^n = t_s^m(1 - m_s)/m_s$. In this case, the conditional density of the fraction m_s is $\Pr\{m_s = m \mid t_s^m = t\} = \Pr\{t_s^m = \frac{1-m}{m} t\}\frac{t}{m^2} = \beta\frac{t}{m^2}\exp(-\beta\frac{1-m}{m}t)$. Thus, its density is the mixture:

$$\begin{aligned}\Pr\{m_s = m\} &= \frac{\alpha\beta\rho}{am^2}\int_0^\infty t\exp\left[-\left(\beta\frac{1-m}{m} + \frac{\alpha\rho}{\phi}\right)t\right]dt \\ &= \frac{\alpha\beta\rho}{[\alpha\rho m + \beta\phi(1-m)]^2},\end{aligned} \tag{A19}$$

for $0 \le m \le 1$.

Derivation of (28) through (30)

Because the maximand in (26) is linear in q, the optimal control $Q(x)$ is bang-bang in (28). With the probability, $q = Q(x)$, the differential equation (26) has the general solution:

$$V(x) = \begin{cases} \nu_0 x^{\xi_0} & 0 \le x < x^* \\ \nu_1 x^{\xi_1} + \delta\omega\left(\frac{\pi}{\delta\omega+\iota-\mu}x - \frac{\kappa}{\delta\omega+\iota}\right) & x^* \le x < \infty. \end{cases} \tag{A20}$$

The remaining roots are eliminated by the boundary conditions (27). In (A20) the constants, v_o and v_1, are uniquely determined by the continuous differentiability of V at the switching point, $x = x^*$. These constants are

$$\nu_1 = \delta\omega\left(\frac{1-\xi_1}{\xi_0-\xi_1}\frac{\pi x^*}{\delta\omega+\iota-\mu} + \frac{\xi_1}{\xi_0-\xi_1}\frac{\kappa}{\delta\omega+\iota}\right)x^{*^{-\xi_0}} \tag{A21}$$

and

$$\nu_1 = -\delta\omega\left(\frac{1-\xi_0}{\xi_1-\xi_0}\frac{\pi x^*}{\delta\omega+\iota-\mu} + \frac{\xi_0}{\xi_1-\xi_1}\frac{\kappa}{\delta\omega+\iota}\right)x^{*^{-\xi_1}}. \tag{A22}$$

Finally, the switching point (29) uniquely satisfies the condition:

$$V(x^*) = \pi x^* - \kappa. \tag{A23}$$

Collectively, (A20) through (A23) determine the switching point (29). This switching point, combined with (A20) through (A22), yields the valuation function (30).

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