Discrete-Time Valuation of American Options with Stochastic Interest Rates

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We develop an arbitrage-free discrete time model to price American-style claims for which domestic term structure risk, foreign term structure risk, and currency risk are important. This model combines a discrete version of the Heath, Jarrow, and Morton (1992) term structure model with the binomial model of Cox, Ross, and Rubinstein (1979). It converges (weakly) to the continuous time models in Amin and Jarrow (1991, 1992). The general model is "path dependent" and can be implemented with arbitrary volatility functions to value claims with maturity up to five years. The model is illustrated with applications to long-dated American currency warrants and a cross-rate swap from the quanto class.

We develop a discrete time arbitrage-free model to price *American-exercise* type securities that are derivatives of the term structures and currency exchange rate of two countries. Our discrete time model con-

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verges in the limit to the continuous time model in Amin and Jarrow (1991, 1992). Currently; there is no such model in the literature. Effectively, existing models can only value certain European style claims.[1] Our model may also be used to value equity or commodity-linked claims that are also dependent on interest rates, if the foreign term structure volatility is set at zero and an equity or commodity price is substituted for the exchange rate.[2] In this article, we focus on the more interesting general case with multiple term structures.

The need for a computationally tractable model for explicitly combining stochastic interest rates and a stochastic asset price or exchange. rate has recently increased. The offered maturities of exchange-based and over-the-counter market derivatives have lengthened significantly. Equity options (such as long-term equity anticipation securities (LEAPS) and contingent value rights), currency warrants, swaptions, and commodity trust units all trade with maturities of three years and more. At these maturities, interest rate risk is a significant valuation factor. Further, many interesting claims now explicitly depend on both interest rates and the exchange rate or asset price. Examples include "Best of Two Options" with payoffs dependent on the maximum of a bond return and an equity return, cross-currency swaptions, bond warrants, convertible bonds,, callable dual currency bonds, and diff swaps, diff options, and diff futures.[3]

Among these types of claims, quantos are of particular interest. Quantos are contingent claims with a "quantity" or nominal cash flow determined by interest rates or equity values in one currency but paid in another currency. The currency conversion occurs at a *fixed* exchange rate and is often 1:1. A new type of currency or "quanto risk" exists in these claims and must be valued and hedged by new methods.

In recent years, quantos have evolved from exotic to mainstream products. Several banks have quanto portfolios with notional amounts in the billions of dollars. Citing the rapidly growing importance of quantos, Litzenberger (1992) refers to these claims as "fanciful swaps." Specific examples include Nikkei equity index options with the yen Nikkei return paid in dollars and Deutschemark-dollar swaps with

[1] Examples models are Amin and Jarrow (1991), Grabbe (1983), and Nielsen and Saá-

2 Other examples of models with a risky asset and a single term structure include Amin and Jarrow (1992), Bailey and Stulz (1989), Cortazar and Schwartz (1992), Ho, Stapleton, and Subrahmanyan (1993a), Longstaff and Schwartz (1993), Merton (1973), Ramaswamy and Sundaresan (1985), and Ritchken and Sankarasubramanian (1993).

[3] For a recent analysis of the pricing and hedging of diff swaps, see Turnbull (1993).

the absolute rate differential being paid in dollars. The second claim is called a *cross-rate swap*. This claim type was judged "Deal of the Year" by Investment Dealer's Digest [Liebowitz (1991)] in 1991. We provide the first pricing analysis for this class of claims.[4]

The first element of our pricing approach is the specification of domestic term structure dynamics. These dynamics are modeled using a discrete version of Heath, Jarrow, and Morton (1989, 1992), henceforth HJM. Term structure models in the HJM class have recently gained wide popularity because they match the initial term structure by construction and do not require any preference assumptions. The domestic term structure economy is enlarged by the addition of stochastic processes for the exchange rate and the foreign term structure. The form of the foreign term structure is analogous to that of the domestic term structure. Specifically, our model combines the binomial model of Cox, Ross, and Rubinstein (1979), hereafter CRR, and a discrete term structure model in a unified framework. The model is parameterized in terms of the continuous-time limiting covariance matrix that governs the evolution of the state variables. It converges to the continuous-time models in Amin and Jarrow (1991, 1992).

When the volatility functions are permitted to be arbitrary, our discrete model is "path dependent" [i.e., an "up" state followed by a "down" state, as in CRR (1979)] and does not yield the same state as a "down" state followed by an "up" state. We note that path-independent term-structure models of the HJM type are not feasible except when interest rates are assumed to be normally distributed. Conventional techniques such as Amin (1991) or Nelson and Ramaswamy (1990) cannot be used because of the necessity of modeling multiple points on the term structure. Further, option prices cannot in general be represented as solutions to a partial differential equation.[5]

We show that accurate values for options with up to five years in maturity (i.e., to within. 1 or 2 percent of their continuous-time limiting values) can be obtained using our path-dependent model with 10 or 12 time steps chosen appropriately. Therefore, path-dependent models are viable with even multiple state variables. Given the wide popularity of "Asian" and other pathdependent options, it may be possible, to use our path-dependent approach in a broader context. It supplements the work of Amin (1991), Boyle, Evnine, and Gibbs

[4] The first public deal was a Credit Suisse Financial Products-Sallie Mae swap. Invesment Dealer's Digest [Liebowitz (1991)] describes this deal as "one of the hottest new products in the swap market, one which redefined the relationship between interest rates and currency rates to investors."

[5] Even if such a partial differential equation can be derived, the boundary conditions are extremely complicated with three state variables. In fact, no attempts have been made to numerically solve three-state partial differential equations in the finance literature.

and $\sigma_d^2(\cdot)b$, respectively. In each period, there is a single source of uncertainty (or factor), represented by X_d, which influences forward interest rates of all maturities. To permit forward rates of different maturities to vary independently, we can add an additional source of uncertainty in (1). Since we will introduce additional sources of uncertainty for the foreign term structure and the exchange rate, we use a single domestic term structure factor for tractability.

The one-period domestic forward interest rate, $f_d(t, t)$, is the domestic spot of interest, which we denote as $r_d(t)$. Further, the forward rate definition implies that the price at time t of a domestic discount bond of maturity T is given by

$$P_d(t, T) = \exp\left[\sum_{i=\frac{t}{b}}^{\frac{T}{b}-1} f_d(t, ib)b\right].$$

Even though trading occurs only at discrete dates, by convention we assume that interest rates are defined based on continuous compounding. This convention simplifies some of the subsequent algebra.

Substituting Assumption (1) into the definition of $P_d(t, T)$ above yields

$$P_d(t, T) = \exp\left(-\sum_{i=\frac{t}{b}}^{\frac{T}{b}-1}\left\{f_d(0, ib) + \sum_{j=0}^{t-1}[\alpha_d(jb, ib, \cdot)b + \sigma_d(jb, ib, \cdot)X_d((j+1)b)\sqrt{b}]\right\} b\right). \quad (2)$$

As in Harrison and Pliska (1981), our analysis uses a money market account as the numeraire, and we assume that it trades. The money market account, $B_d(t)$, is denominated in dollars and corresponds to the wealth accumulated by an investment strategy under which an investor reinvests the proceeds of an initial $1 investment at the domestic spot interest rate in each subsequent period. Therefore,

$$B_d(t) = \exp\left[\sum_{i=0}^{\frac{t}{b}-1} r_d(ib)b\right] = \exp\left[\sum_{i=0}^{\frac{t}{b}-1} f_d(ib, ib)b\right]. \quad (3)$$

Having defined the domestic term structure economy, we impose a similar structure on the foreign term structure economy. To maintain a correlation structure with the domestic economy, yet permit some independent variation, we require a different source of uncertainty in the foreign term structure. The correlation between this source of uncertainty and the first source of uncertainty, X_d, yields the correlation between the two term structures.

Assumption 2 (Foreign Forward Rate Dynamics). *Foreign forward rates follow:*

$$f_f(t+h, T) - f_f(t, T) = \alpha_f(t, T, \cdot)h + \sigma_f(t, T, \cdot)X_f(t+h)\sqrt{h}, \quad (4)$$

where $X_f(ih)$, $i = 1, 2, \ldots$ is a sequence of independent random variables with mean zero, variance one and the correlation of $X_f(t)$ and $X_d(t)$ is $\rho_{df}(t, \cdot)$. While $\rho_{df}(\cdot)$ and $\sigma_f(\cdot)$ may depend on past and current values of the state variables, this dependence is suppressed for notational simplicity.

The single period mean and variance of the foreign forward rates are $\alpha_f(\cdot)h$ and $\sigma_f^2(\cdot)h$, respectively. Algebraic manipulation shows that the covariance with domestic interest rates is $\rho_{df}\sigma_d(\cdot)\ \sigma_f(\cdot)h$. At this point, we do not restrict these covariance functions further.

As in the domestic case, the foreign spot rate of interest, $r_f(t)$, is equal to $f_f(t, t)$. Denominated in foreign currency, we denote the foreign money market account by $B_f^*(t)$ and the foreign bond prices by $P_f^*(t, T)$. The superscript * denotes the denomination of the price in units of the foreign currency. The evolution of these variables is obtained as in the domestic economy.

1.2 The spot (currency) exchange rate

Because the spot exchange rate links domestic and foreign money markets, it must reflect the two countries' relative interest rates. In a deterministic world, the covered interest rate parity relation dictates that the rate of return on the exchange rate should be equal to the difference between the domestic and foreign spot interest rates. In an uncertain world, even though the expected rate of return on the currency must be related to this rate difference, it will also exhibit a risk-related premium or discount. Our model reflects this characteristic.

Assumption 3 (Spot Exchange Rate Dynamics) *The exchange rate S(t), denoted in dollars per unit of foreign currency, evolves according to*

$$\ln\frac{S(t+h)}{S(t)} = [\alpha_s(t, \cdot) + r_d(t) - r_f(t)]h + \sigma_s(t, S(t))X_s(t+h)\sqrt{h}, \quad (5)$$

where $\alpha_s(\cdot)$ and $\sigma_s(\cdot)$ are arbitrary drift and variance functions that can depend on current and past values of the state variables and $X_s(ih)$, $i = 1, 2, \ldots$ is a sequence of independent random variables *with mean zero and variance one. Further, the covariance matrix of* $(X_d(t), X_f(t), X_s(t))$ *is given by*

$$\begin{bmatrix} 1 & \rho_{df} & \rho_{ds} \\ \rho_{df} & 1 & \rho_{fs} \\ \rho_{ds} & \rho_{fs} & 1 \end{bmatrix},$$

where r . . . are correlation coefficients that can depend on current and past values of the state variables. As noted in Assumption (2), our notation will suppress this dependence. To ensure complete markets across both term structures, we assume that the distribution of $(X_d(t), X_f(t), X_s(t))$ *has no more states than the total number of domestic and foreign bonds traded at date t - h.*

The single period drift of ln $[S(t+h)/S(t)]$ is given by $[\alpha_s(t,\cdot) + r_d(t) - r_f(t)]h$, and the instantaneous variance is $\sigma_s^2(\cdot)h$. The covariance of the spot exchange rate with the domestic term structure is given by $\rho_{ds}\sigma_d(\cdot)\sigma_s(\cdot)h$, and the covariance with the foreign term structure is given by $\rho_{fs}\sigma_f(\cdot)\sigma_s(\cdot)h$.

2. Risk-Neutral Pricing

Our economy contains four basic securities: the domestic money market account $[B_d(t)]$, domestic discount bonds $[P_d(t, T)]$, the foreign money market account $[B_f^*(t)]$, and foreign discount bonds $[P_f^*(t, T)]$. All other assets can be constructed by appropriately trading in these basic securities. Viewing the valuation exercise from the U.S. perspective, we denominate all of these security prices in dollars. To convert foreign security prices to dollars, one multiplies the foreign currency values by the spot exchange rate. For the foreign currency money market account, let $B_f(t) = B_f^*(t)S(t)$. For the foreign currency bonds, let $P_f(t, T) = P_f^*(t, T)S(t)$. Hence, the dollar values of the foreign bonds $[P_f(t, T)]$ and money market account $[B_f(t)]$ are "domestic" assets. We assume that the domestic investor holds foreign currency only in the form of interest-earning foreign bonds or units of the foreign money market account.

We define the dollar discounted prices of all the assets in the economy by

$$Z_d(t, T) = \frac{P_d(t, T)}{B_d(t)},$$

$$Z_f(t, T) = \frac{P_f(t, T)}{B_d(t)} = \frac{P_f^*(t, T) \times S(t)}{B_d(t)},$$

$$Z_f(t) = \frac{B_f(t)}{B_d(t)} = \frac{B_f^*(t) \times S(t)}{B_d(t)}. \tag{6}$$

These prices are relative dollar prices with the domestic money market account as numeraire. To complete our model specification, we determine the values of the drifts $\alpha_d(\cdot)$, $\alpha_f(\cdot)$, and $\alpha_s(\cdot)$ such that the discounted dollar prices of all securities in our economy are $\tilde{Q}$-martingales.

Lemma 1. *The discounted dollar prices of all securities are $\tilde{Q}$-martingales if and only if for every $T \in [0, h, \ldots, \tau - h]$; $T > t$,*

$$\left[\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}} \alpha_d(t, ih, \cdot)h\right] h = \ln\left[\tilde{E}_t \exp\left(\left[\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}} \sigma_d(t, ih, \cdot)h\right] X_d(t+h)\sqrt{h}\right)\right],$$

$$\left[\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}} \alpha_f(t, ih, \cdot)h\right] = \ln \tilde{E}_t \exp\left[\sigma_s(t, \cdot)X_s(t+h)\sqrt{h} - \left[\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}} \sigma_f(t, ih, \cdot)h\right] X_f(t+h)\sqrt{h}\right] + \alpha_s(t, \cdot)h$$

$$\alpha_s(t, \cdot)h = -\ln[\tilde{E}_t \exp[\sigma_s(t, \cdot)X_s(t+h)\sqrt{h}]], \tag{7}$$

where $\tilde{E}_t$ indicates the time t conditional expectation with respect to $\tilde{Q}$.

Proof. These expressions are obtained by applying the martingale condition to $Z_d(t, T)$, $Z_f(t)$, and $Z_f(t, T)$, respectively, that is,

$$\tilde{E}_t\left[\frac{Z_d(t+h, T)}{Z_d(t, T)}\right] = \tilde{E}_t\left[\frac{Z_f(t+h)}{Z_f(t)}\right] = \tilde{E}_t\left[\frac{Z_f(t+h, T)}{Z_f(t, T)}\right] = 1.$$

We can obtain the necessary result by substituting the stochastic difference equations for the forward interest rates and the exchange rates for the relative bond prices and solving the equation. For details, see the Appendix. ■

An example of a probability distribution for $\tilde{Q}$ illustrates our discrete model. For notational simplicity, we rotate the axes for $(X_d$, X_f,

Example 1
Probability distribution for $\tilde{Q}$ in terms of (Y_1, Y_2, Y_3)

A	$(1, 1, 1)$	$\frac{1}{4}$
B	$(1, -1, -1)$	$\frac{1}{4}$
C	$(-1, 1, -1)$	$\frac{1}{4}$
C	$(-1, 1, -1)$	$\frac{1}{4}$
D	$(-1, -1, 1)$	$\frac{1}{4}$

This example illustrates a probability distribution of three given random variables (Y_1, Y_2, Y_3) over four states such that the random variables each have mean zero and variance one and they are pairwise uncorrelated.

X_s). Let $\bar{\rho}_{fs} = \rho_{fs} - \rho_{df}\rho_{ds}/\sqrt{(1-\rho_{df}^2)}$. Define a sequence of uncorrelated random variables $\{Y_1(t), Y_2(t), Y_3(t)\}$ such that $X_d(t) = Y_1(t)$, $X_f(t) = \rho_{df} Y_1(t) + \sqrt{(1-\rho_{df}^2)} Y_2(t)$, and $X_s(t) = \rho_{ds} Y_1(t) + \bar{\rho}_{fs} Y_2(t) + \sqrt{1-\rho_{ds}^2 - \bar{\rho}_{fs}^2}\, Y_3(t)$. An example of a probability distribution for $\tilde{Q}$ in terms of these (Y_1, Y_2, Y_3) is given in Example 1. Note that the random variables (Y_1, Y_2, Y_3) are uncorrelated and all have mean zero. With this distribution, it is easy to verify that the random variables (X_d, X_f, X_s) satisfy the mean, variance, and covariance requirements in Assumptions (1) to (3).

The variance functions, $\sigma_d(\cdot)$, $\sigma_f(\cdot)$, and $\sigma_s(\cdot)$, and the correlations, ρ_{df}, ρ_{ds}, and ρ_{fs}, constitute the parameter inputs. From Lemma 1, we can determine the drift functions $\alpha_d(\cdot)$, $\alpha_f(\cdot)$, and $\alpha_s(\cdot)$ using the above distribution of $\tilde{Q}$ and generate the full time-state space for any finite number of time periods. Over this state space, contingent claim values are computed as the expected discounted (by B_d) dollar payoff on the claim. American-type option prices are evaluated by the following expression:

$$\sup_{\theta \in \mathcal{T}[0,\tau]} \tilde{E}_0 \left[\frac{C(\theta)}{B_d(\theta)} \right], \tag{8}$$

where $C(\theta)$ is the dollar payoff to the security owner at time θ if she exercises her option early, and $\mathcal{T}[0, \tau]$ is the class of all early exercise strategies (stopping times) in $[0, \tau]$. For an American call option on the foreign currency with strike price K, the payoff function $C(\theta)$ is equal to $\text{Max}[S(\theta) - K, 0]$. Option values in (8) can be computed using backward recursion.

As a final point, we note that the above model is an arbitrage-free model. We have made no assumptions about investor preferences, except the absence of arbitrage opportunities. Our model may be contrasted with other models with interest rate and asset price risk such as Bailey and Stulz (1989) Nielsen and Saa-Requejo (1993), or Turnbull and Milne (1991), which make explicit assumptions about preferences (or equivalently, the marginal rate of substitution).

2.1 Constant elasticity of variance forward rate processes

We now illustrate some volatility functions that can be treated. Motivated by its tractability, its relative empirical performance, and the process' nonnegative rate characteristic,[6] the constant elasticity of variance (CEV) process has received significant attention in the literature. In our framework, the CEV forward rate process is specified as follows:

$$\sigma_k(t, T, \cdot) = \sigma_k f_k(t, T)^\beta \text{ for } k = d, f. \tag{9}$$

Under this parameterization, the shape of the forward rate distribution changes, as β is varied. For β values above zero, forward rates are always nonnegative. Some resultant distributions are the normal ($\beta = 0$), a distribution analogous to the chi square ($\beta = 1/2$), and a distribution analogous to the lognormal ($\beta = 1$).

2.2 Continuous-time limits

As mentioned earlier, our model has been constructed as a discrete-time approximation to the continuous-time models in Amin and Jarrow (1991, 1992). It can be shown that the continuous-time limit of the model (under $\tilde{Q}$) is given by the following system of equations[7]:

$$\begin{aligned} df_d(t, T) &= \alpha_d(t, T, \cdot)dt + \sigma_d(t, T, \cdot)dW_1(t), \\ df_f(t, T) &= \alpha_f(t, T, \cdot)dt + \sigma_f(t, T, \cdot)dW_2(t), \\ d\ln S(t) &= [\alpha_s(t, \cdot) + r_d(t, t, \cdot) - r_f(t, t, \cdot)]dt + \sigma_s(t, \cdot)dW_3(t), \end{aligned} \tag{10}$$

where $(W_1(t), W_2(t), W_3(t))$ is a three-dimensional Brownian motion with instantaneous correlation matrix

$$\begin{bmatrix} 1 & \rho_{df} & \rho_{ds} \\ \rho_{df} & 1 & \rho_{fs} \\ \rho_{ds} & \rho_{fs} & 1 \end{bmatrix}.$$

Further, by taking Taylor series expansions of the right-hand sides in Equations (7) and taking limits as $h \to 0$, we can show that the

[6] For example, Marsh and Rosenfeld (1983), Gibbons and Ramaswamy (1993), Longstaff (1989, 1990), and Chan et al. (1992).

[7] To prove convergence from the discrete-time model to the continuous-time limiting model, the conditions in Equations (2.4)-(2.6), p. 268, in Strook and Varadhan (1979) can be shown to hold. From an economic perspective, these proofs do not add much to the article and are therefore omitted.

continuous-time limiting values of the drift variables $\alpha_d(\cdot)$, $\alpha_f(\cdot)$, and $\alpha_s(\cdot)$ are given by

$$\begin{aligned}
\alpha_d(t, T, \cdot) &= \sigma_d(t, T, \cdot)\int_t^T \sigma_d(t, u, \cdot)du, \\
\alpha_f(t, T, \cdot) &= \sigma_f(t, T, \cdot)\left[\int_t^T \sigma_f(t, u, \cdot)du - \sigma_s(\cdot)\rho_{fs}\right], \\
\alpha_s(t, \cdot) &= -\frac{1}{2}\sigma_s^2(\cdot).
\end{aligned} \tag{11}$$

The domestic rate drift is the same as in HJM, and the spot currency drift is identical to that in the Black-Scholes model. Furthermore, the foreign rate drift is similar to the domestic rate drift but is decremented by the foreign rate-currency covariance. In this respect, the two term structures are not symmetric.

The model's convergence to a continuous-time diffusion limit has two advantages. First, we can use the limiting covariance matrix of the state variablesto estimate model parameters using well-known techniques. Second, HJM and Amin and Jarrow (1991, 1992) (already provide closed-form solutions for certain types of option values. These closed-form solutions can be used to benchmark the accuracy of our discrete model.

3. Path-Dependent Models

Our general model is "path dependent." To highlight this feature, consider Figure 1. This figure depicts the state space for two periods with our sample probability distribution of (Y_1, Y_2, Y_3) in Example 1. The composite state (A, B) in the second period represents the occurrence of state A in the first period followed by the occurrence of state B in the second period. Notice that state (A, B) differs from (B, A). That is, the order of occurrence of the states in each period determines the state at subsequent dates. We illustrate this path dependency for the exchange rate process. Similar path dependencies exist for the forward interest rate processes.

Aggregated from (5), consider the following expression for $S(t)$:

$$\begin{aligned}
\ln\frac{S(t)}{S(0)} = \sum_{i=0}^{\frac{t}{h}-1}\{[\alpha_s(ih, \cdot) + r_d(ih) - r_f(ih)]h \\
+ \sigma_s(ih, \cdot)X_s((i+1)h)\sqrt{h}\}.
\end{aligned} \tag{12}$$

The right-hand side above depends on the order of occurrence of the states at each date. For example, the exchange rate in the second

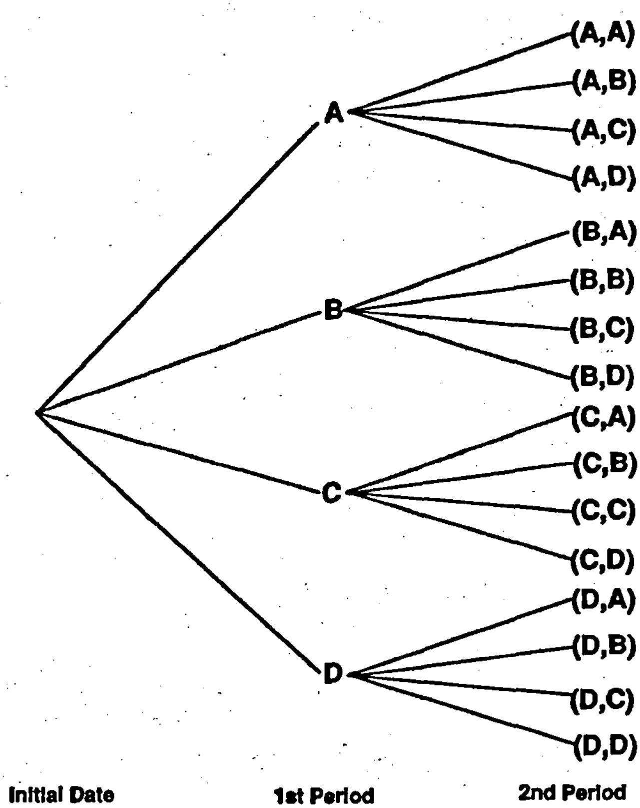

Figure 1
State space for a path-dependent model with four possible states at each date
This figure depicts the evolution of the state space over two periods in a path dependent model in which them are four possible states *(A, B, C* or *D)* at each subsequent date. The tree generating the state space is a path-dependent (or nonrecombining) tree in which state *(A, B)* is not the same as state *(B, A)*. The number of states at any date n is equal to 4^n.

period $(i = 2;\ t = 2h)$ depends on r_d and r_f in periods zero and one $(i = 0.1)$. The exchange rate in state *(A, B)* depends on r_d in state A at date h, but the exchange rate in state *(B, A)* depends on r_d in state *B.* Because of these dependencies, the exchange rates in states *(A, B)* and *(B, A)* cannot be the same; thus, these two states must differ.

Due to variation in the coefficient of X_s, the value of $\sum_{t=0}^{i-1} \sigma_s(th, \cdot) \times X_s((i+1)h)$ depends on the states' order of occurrence and is also path dependent. A final source of path dependency can be due to the drifts $(\alpha_k(\cdot)$ for $k = d, f, s)$, as defined in Lemma 1. If one or more of these drift terms are stochastic, then the cumulative drift also depends on the states' order of occurrence.

In our example, identified path dependencies imply that the number of states at each date n grows at the rate of 4^n. For example, n equal to 40 results in 1.612×10^{12} model states. Current computing technology permits 10 to 12 time step Implementations. On a Pentium DOS personal computer, the 10-step model typically runs in a minute. However, with 12 steps, running times grow to about 15 minutes. We have not attempted to optimize our code for quicker calculations. By reducing the reliance on exponents, cosh functions, etc., it is possible to reduce the running time significantly.

4. A Path-Independent Model

The general model permits us to consider arbitrary volatility and covariance functions. However, certain process parameterizations permit construction of path-independent models. These models are easier to implement and can be implemented with many more time steps. As an example, we now specify such a "path-independent" model. This model also serves to benchmark the American option values computed with the path-dependent model.

A path-independent model is one in which all of the state variables (which uniquely determine both forward rate and exchange rates at each date) can be written as a deterministic function of the cumulative sums of (X_d, X_f, X_s). In a path-independent model, the order in which the different -state values of (X_d, X_f, X_s) occur is irrelevant. Thus, one need not record the entire prior history of the state variables. For example, consider Figure 1. If the values of the state variables in states (A, B) and (B, A) are the same, then these two states are effectively one state. This specification occurs when the state variables are deterministic functions of $(X_d(1) + X_d(2), X_f(1) + X_f(2), X_s(1) + X_s(2))$ only.

This property, whereby branches in the tree recombine, implies that the number of states in the model is only of the order $(n + 1)^3$, where n is the number of time steps. Even with three state variables, computations of this order are feasible with 60 or more time steps.

To specify a particular path-independent model, first consider Equation (12) for $S(t)$. To generate path independence, the right-hand side of this equation must be a deterministic function of $\sum_{i=0}^{t-1} X_s((i+1)h)$. Unless σ_s is a constant, the term $\sum_{i=0}^{t-1} \sigma_s(ih, \cdot)X_s((i+1)h)\sqrt{h}$ cannot be written as a deterministic function of $\sum_{i=0}^{t-1} X_s((i+1)h)$.

Since an analogous argument holds for the forward interest rate and exchange rate processes, path independence requires $\sigma_k(\cdot)$ to be constants for $k = d, f, s$. By extension, domestic and foreign forward interest rates must have constant variances across all maturities. The term structure model is then equivalent to the Ho and Lee (1986) model.

Another potential source of path dependency is due to the stochastic drift of $\ln\frac{S(t)}{S(0)}$ in (12) (i.e., the term $\sum_{i=0}^{t-1}[r_d(ih)-r_f(ih)]h$, which is clearly path-dependent). In each period, this source of path dependency can be obviated by introducing this drift through the mean of X_s. Since our definition of path independence does not impose any restrictions on the probabilities of the states of X_s, we can eliminate the path-dependent complications that this drift induces by adjustment of the probabilities at each date. Appropriate choice of the probabilities generates the required mean without destroying path independence. Nevertheless, we must still ensure that the potential values of X_s are the same at each date. Only the associated probabilities can differ.

Based on this argument, we modify the stochastic difference equation for $S(t)$ to be

$$\ln\left[\frac{S(t+h)}{S(t)}\right]=\alpha_s(t)h+\sigma_s X_s(t+h)\sqrt{h}, \tag{13}$$

where $X_s(t+h)$ for $t = 0, h, 2h, \ldots$ is a sequence of independent random variables with a mean equal to $\frac{[r_d(t)-r_f(t)]\sqrt{h}}{\sigma_s}$. The covariance matrix of (X_d, X_f, X_s) is defined as before [Assumption (3)]. In Equation (13), the mean of the dispersion term $\sigma_s X_s(t+h)\sqrt{h}$ is equal to $[r_d(t)-r_f(t)]h$. The transfer of the $[r_d(t)-r_f(t)]h$ term from the drift into the mean of X_s facilitates construction of a path-independent model.

Finally, we must also verify that the drift terms $(\alpha_d, \alpha_f$, and $\alpha_s)$ do not make the model path-dependent. With our current volatility function specifications, the continuous-time drifts in Equation (11) are deterministic functions of time. Further, if the step size h is sufficiently small (five or more time steps per year), the discrete time drifts are very close to the continuous-time drifts. Therefore, in practice, the discrete time drifts can be substituted by their continuous-time limiting values without significant loss of accuracy. Since the continuous-time drifts are deterministic functions of time, they cannot induce path dependence. Correspondingly, our particular model is path-independent. With an appropriate probability distribution for $(X_d\ X_f, X_s)$ under Q, the model can be easily implemented.

To summarize, our path-independent model is completely specified by

$$f_d(t+h,T)-f_d(t,T) = \alpha_d(t,T)h+\sigma_d X_d(t+h)\sqrt{h},$$
$$f_f(t+h,T)-f_f(t,T) = \alpha_f(t,T)h+\sigma_f X_f(t+h)\sqrt{h},$$

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = \alpha_s(t)dt + \sigma_s X_s(t+h)\sqrt{h}, \tag{14}$$

where $(X_d(t+h), X_f(t+h), X_s(t+h))$ for $t = 0, h, 2h, \ldots$ are temporally independent random variables, with mean $(0, 0, \frac{[r_d(t)-r_f(t)]\sqrt{h}}{\sigma_s})$ and covariance matrix equal to

$$\begin{bmatrix} 1 & \rho_{df} & \rho_{ds} \\ \rho_{df} & 1 & \rho_{fs} \\ \rho_{ds} & \rho_{fs} & 1 \end{bmatrix}.$$

The drift terms $\alpha_d(\cdots)$, $\alpha_f(\cdots)$ and $\alpha_s(\cdots)$ can be determined from (11).

It is also possible to construct other path-independent variants of the general path-dependent model. For example, if the volatility function of forward rates is of the form

$$\sigma_k(t, T) = \text{Constant} \exp[\lambda_k(T-t)] \text{ for } k = d, f,$$

then a path-independent model can be obtained with suitable transformations.[8] This specification results in mean-reverting spot interest rates and is, essentially, the Vasicek (1977) model with a slight adjustment to match the initial term-structure [see Brenner (1989)].

5. Numerical Implementation

Since the path-independent model is used to benchmark American option values obtained from the path-dependent model, we first provide implementation details for the path-independent model. Next, implementation details are provided for the pathdependent model.

5.1 Numerical implementation of a path-independent model

We are concerned with numerical accuracy and convergence to the continuous-time values. Over many possible distributions (by trial and error), we have found that satisfactory convergence to the continuous-time values is obtained if the three discrete shocks — (Y_1, Y_2, Y_3) defined in Example 1— are assumed to be independent and specified over eight states, $(\pm 1, \pm 1, \pm 1)$. If we assume that the means of Y_1 and Y_2 are zero, then ensuring that $\tilde{E}_t[\sigma_s X_s(t+h)\sqrt{h}] = [r_d(t) - r_f(t)]h$

[8] This model was described in an unpublished earlier draft of this article.

Example 2
Sample probability distribution for $\bar{Q}$ for the path-independent model in terms of (Y_1, Y_2, Y_3)

State	State for (Y_1, Y_2, Y_3)	Probability
A	$(1, 1, 1)$	$\frac{1}{8}(1 + m_3(t))$
B	$(1, 1, -1)$	$\frac{1}{8}(1 - m_3(t))$
C	$(1, -1, 1)$	$\frac{1}{8}(1 + m_3(t))$
D	$(1, -1, -1)$	$\frac{1}{8}(1 - m_3(t))$
E	$(-1, 1, 1)$	$\frac{1}{8}(1 + m_3(t))$
F	$(-1, 1, -1)$	$\frac{1}{8}(1 - m_3(t))$
G	$(-1, -1, 1)$	$\frac{1}{8}(1 + m_3(t))$
H	$(-1, -1, -1)$	$\frac{1}{8}(1 - m_3(t))$

This example illustrates a probability distribution of three given random variables (Y_1, Y_2, Y_3) over eight states such that the random variables have a given mean $(0, 0, m_3(t))$, each has a variance equal to one, and they are pairwise uncorrelated.

requires the following condition:

$$\tilde{E}_t[Y_3(t+h)] = \frac{[r_d(t) - r_f(t)]\sqrt{h}}{\sigma_s\sqrt{1 - \rho_{ds}^2 - \bar{\rho}_{fs}^2}}.$$

If we define $m_3(t) = \tilde{E}_t[Y_3(t+h)]$, then the distribution of (Y_1, Y_2, Y_3) is given in Example 2.[9]

This distribution yields a discrete model with eight possible states evolving from each node, and the number of distinct states at a date n equals $(n + 1)^3$. With 50 time steps, the total number of states (nodes in the multinomial tree), equals 1,625,625. For all reasonable parameter values and 50 or more time steps, this model gives five-year European option prices [benchmarked by the formulae in Amin and Jarrow (1991)] accurate to within a few cents.

Though the general model of Sections 2 and 3 can be parameterized so as to maintain nonnegative interest rates, negative interest rates are possible in the path-independent model.[10] Nevertheless, the path-independent model has two positive characteristics. First, as shown by Dybvig (1989), the empirical fit of this model is quite good for forward rate maturities up to five years. Second, the model is analytically

[9] For the parameter values used in our applications in Section 7, these probabilities are always nonnegative. However, for other parameter values, we may have to truncate the probabilities to ensure that they are in [0, 1]. In the limit, this truncation does not affect the convergence result [see Nelson and Ramaswamy (1990) for a similar construction].

[10] The path-independent model remains consistent with no arbitrage due to the absence of cash.

tractable and yields closed-form solutions for many types of European options [see Amin and Jarrow (1991, 1992) and HJM]. Therefore, its use may be justified in many applications.

5.2 Numerical implementation of the path-dependent model

As noted earlier, if we wish to use arbitrary volatility functions, our general model is pathdependent. However, for option maturities up to five years, path-dependent models with fewer than 12 steps can still yield option values that are accurate to within 1 or 2 percent of their continuous-time limiting values. One method of achieving this level of accuracy is to choose linearly increasing or decreasing step sizes (values of *h*). Empirically, these steps should be chosen such that the ratio of the maximum to minimum step size is between 2 and 3. If uneven step sixes are chosen, then one obtains a representative state space "tree," which is well dispersed at any given date. Since path-dependent models have a large number of states with a relatively small number of time steps, this method can approximate the state space quite well. However, we must prevent separate nodes at a given date (step) from clustering.

In path-independent models, a small number of time steps yields clustered states and a sparse tree. Given the same number of time steps and appropriate step size choices, the tree representing the path-dependent model state space is relatively unclustered. For example, with three time steps, a CRR-type binomial model will use only four distinct states to approximate the, state space. In a three-time step, path-dependent model, there are eight distinct states. Given an equivalent number of time steps and appropriate choices of step sizes, the pathdependent model approximates the density function of an underlying state variable much more accurately than its path independent analogue does.

In implementing path-dependent models, it is important to use the exact values of the drifts $[\alpha_k(\cdots), k = d, f, s,$ given by Equations (7)]. For example, use of the continuous-time limiting values of these drifts results in five-year maturity option values that are mispriced by up to 10 percent. To understand why this happens, note that the discrete-time drifts ensure that

$$\tilde{E}_0\left[\frac{P_k(t,T)}{B_d(t)}\right] = P_k(0,T) \text{ for } k = d, f \text{ and } t > 0,$$

(i.e., the martingale condition implies that the expectation of the discounted — by the domestic interest rate — bond price at any future date is equal to the current bond price). Since the next periods currency value is equivalent to the value of a foreign discount bond maturing at the end of this-period, this condition holds for the exchange rate as well. Essentially, these martingale conditions determine expected future bond prices in both the discrete-time and continuous-time models. Hence, the discrete-time model approximates the first moments of the continuous-time model exactly. Only the higher order moments are not approximated exactly. Therefore, the error in the discrete-time model is only of second order. Since an inaccurate first moment definition is the largest source of model pricing error, eliminating this error reduces the overall pricing error significantly.[11]

A final concern is the number of different maturity forward rates that are needed at each node (to represent the current term structure and to compute various security prices). An acceptable level of accuracy is obtained as follows: use forward rates with maturities approximately three months apart up to the option's maturity and, thereafter, forward rates with maturities two or three years apart. Intermediate maturity forward rates at each tree node are obtained by linearly interpolating between available maturities.

To illustrate the accuracy of a discrete-time path-dependent model, we use two types of models to generate value benchmarks: exact continuous-time European option models and the path-independent American option model. We will assume that the volatilities of both the domestic and foreign term structures are constant across time and maturity and that exchange rate volatility is constant. For European currency options, closed-form valuation models are available in Amin and Jarrow (1991). For European discount bond options, closed-form valuation models are provided by HJM. The associated path-independent American option model values are calculated by the method described in Section 5.1.

Panel A in Table 1 reports prices computed with 6 through 12 time steps for different maturity European and American puts to sell 10,000 yen with a strike price equal to the initial forward currency price and a strike price equal to 95 percent of the initial forward price.[12] The probability distribution in Example 1 is used to generate the state space, and the ratio of the maximum to minimum steps sixes used to compute option values is 2.0. The accurate values are determined by

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[11] We can also envision more accurate second or higher moment approximations. Although these approximations will undoubtedly lead to even more accurate values, their complexity leads us to leave this task to subsequent research.

[12] **For conciseness, we report only the results for puts since we study only puts in more depth in subsequent sections.**

the closed-form formula in Amin and Jarrow (1991) for the European option and from the path-independent model with 60 time steps for the American option.[13]

In reviewing the warrant pricing errors, we find that a very large component of the error is driven by the necessity to interpolate between different maturity forward rates. Since we store only about 20 forward rates while computing option prices, when the initial forward rate curves are not flat (or linear in maturity) as assumed in Table 1, interpolation necessarily introduces additional errors. If the initial forward rates are linear in maturity, the option values are much more accurate than those reported in panel A of Table 1.

Since the literature has not yet addressed path-dependent HJM model implementations with only interest rate risk, we provide analogous results for an option on a pure discount bond. These results are indicative of the accuracy level of path-dependent HJM models or other single state path-dependent models. Panel B in Table 1 reports different maturity put option prices computed with 6 through 12 time steps for both European and American puts on zero coupon bonds. Each option gives the owner the right to sell a 10-year maturity zero coupon bond anytime before maturity at a strike price equal to the forward price of the bond at the initial date.[14] Once again, the "Accurate Values" column reports the results obtained from the closed-form solution (provided by HJM) in the European case and from the path-independent implementation with 60 time steps in the American case [see Section (5.111. Model errors are less than 1.5 percent of the option value for all maturities.

Pricing error comparisons across panels A and B show that the currency option pricing errors are usually larger than the bond option pricing errors. Nevertheless, the maximum error for a five-year currency option is still less than 1.5 percent of the option value (and this maximum error level is maintained for options with other strike prices). Therefore, the results in this subsection demonstrate the viability of implementing path-dependent models for options with maturities up to five years.

Given the wide popularity of path-dependent options in many markets, the path-dependent procedure has wide applicability. We have also benchmarked our model against the Asian option prices reported in Ritchken, Sankarasubramanian, and Vijh (1992). These authors value equity Asian options with a single state variable. In many

[13] All currency and forward rate data are for April 13, 1988, the issue date of the Citicorp Yen Warrant analyzed in Section 7. Benchmark parameter estimates are presented in Section 6.

[14] For parameters, the U.S. April 13, 1988, swap yields and the U.S. interest rate volatility based on January 9, 1987, to November 5, 1990, daily data (as discussed in Section 7) are used.

Table 1
European and American put prices from the path-dependent model with different numbers of time steps

A: European and American puts on 10,000 Japanese yen

Option type	Maturity in years	Option prices — Number of time steps: 6	7	8	9	10	11	12	Accurate value
		Strike price = forward price							
European	1	3.73	3.83	3.74	3.81	3.75	3.79	3.75	3.73
American	1	4.34	4.35	4.34	4.34	4.35	4.33	4.34	4.31
European	2	5.10	5.16	5.12	5.14	5.12	5.13	5.12	5.07
American	2	6.99	6.94	7.00	6.93	6.99	6.94	6.98	6.93
European	3	6.04	6.03	6.06	6.04	6.06	6.05	6.06	5.99
American	3	9.77	9.74	9.81	9.80	9.85	9.86	9.85	9.87
European	4	6.75	6.73	6.76	6.75	6.75	6.76	6.77	6.71
American	4	13.39	13.39	1339	13.39	13.39	13.39	13.39	13.39
European	5	7.30	7.32	7.31	7.33	7.34	7.36	7.37	7.32
American	5	17.61	17.61	17.61	17.61	17.61	17.61	17.61	17.61
		Strike price = 0.95 × forward price							
European	1	2.16	2.05	2.14	2.05	2.12	206	2.10	2.08
American	1	2.36	2.31	2.37	2.30	2.36	2.30	2.35	2.32
European	2	3.39	3.42	3.40	3.41	3.40	3.40	3.40	3.35
American	2	4.32	4.44	4.36	4.44	4.38	4.43	439	4.38
European	3	4.32	4.37	4.34	4.36	4.34	4.35	4.35	4.29
American	3	6.69	6.75	6.73	6.73	6.73	6.72	6.72	6.68
European	4	5.09	5.08	5.09	5.09	5.11	5.11	5.11	5.05
American	4	9.29	9.29	9.41	9.41	9.47	9.46	9.48	9.45
European	5	5.72	5.70	5.73	5.74	5.75	5.76	5.76	5.70
American	5	12.78	12.78	12.78	12.78	12.78	12.78	12.78	12.84

This table reports the prices of different maturity and strike price American and European put warrants as the the number of time steps used to compute these prices using a discrete path-dependent model with stochastic domestic and foreign interest rates and a stochastic exchange rate. The accurate values in the last column are computed using dosed form formulae In the European case [Amin and Jarrow (1991)] and using a path-independent model with 60 time steps in the American case.

The model (volatility functions) corresponds to constant correlations and

$$\sigma_k(t, T, \cdot) = \sigma_k \text{ for } k = d, f;$$
$$\sigma_s(t, \cdot) = \sigma_s,$$

where σ_k := standard deviation of domestic forward rates (for $k = d$) or foreign forward rates (for $k = f$) at he exchange rate (for $k = s$). All interest rate and exchange rate data are for April 13, 1988. and the parameter values are σ_d = 0.01481, σ_f := 0.01535, ρ_{df} := -0.0821 σ_s := 0.1236, ρ_{ds} := -0.013, ρ_{fs} = 0.0628, initial exchange rate (S(0)) = 0.0079101 $/yen. ρ. denotes the correlations.

Table 1
Continued.

B: European and American put prices on a 10-year discount bond with face value $100

Option type	Maturity in years	Option prices: Number of time steps 6	7	8	9	10	11	12	Accurate value
Strike price = forward price									
European	1	2.44	2.43	2.44	2.42	2.43	2.42	2.43	2.42
American	1	2.45	2.44	2.45	2.43	2.44	2.43	2.44	2.43
European	2	3.23	3.19	3.21	3.18	3.21	3.17	3.20	3.18
American	2	3.35	3.30	3.33	3.29	3.33	3.29	3.32	3.30
European	3	3.60	3.53	3.58	3.53	3.57	3.53	3.55	3.53
American	3	3.94	3.81	3.90	3.82	3.88	3.83	3.87	3.84
European	4	3.74	3.66	3.72	3.66	3.70	3.66	3.69	3.67
American	4	4.32	4.18	4.28	4.20	4.26	4.21	4.24	4.22
European	5	3.74	3.65	3.72	3.66	3.70	3.67	3.69	3.67
American	5	4.57	4.47	4.54	4.48	4.52	4.50	4.51	4.50

This table reports the prices of different maturity and strike price American and European puts to purchase a 10-year discount bond as a function of the number of time steps used to compute these prices using a discrete path-dependent HJM model of interest rates. The accurate values in the last column are computed using closed form formulate in the European case [HJM] and using a path-independent model with 60 time steps in the American case.

The model (volatility function) corresponds to

$$\sigma_d(t, T, \cdot) = \sigma_d$$

where σ_d = standard deviation of domestic forward rates. Interest rate data are for April 13,1988. Parameter values: $\sigma_d = 0.01481$.

cases, our model prices are much more accurate than those reported in Table 1. As discussed earlier, the need to interpolate between different maturity forward rates adds significantly to our pricing errors in Table 1.

6. Data and Estimation

To calibrate the effect of interest rate risk on various claims, we first estimate some benchmark parameter values under a specific parameterization. The proportional volatility of the exchange rate (σ_s) and the absolute volatility of domestic and foreign forward rates [(σ_d) and (σ_f), respectively] across time and maturity are assumed to be constant.[15]

[15] There is a need for additional research to fit the CEV processes [i.e., Equation (9)] that we use for pricing. Except in the normal case estimated here, closed form expressions for the forward rate distribution over *finite* intervals are not available. Further, maximum likelihood parameter

Our database, supplied by Citicorp, covers international money market rates and currency prices for Germany, Japan, and the United States from January 9, 1987, through November 5, 1990. It consists of daily observations on spot currency prices and bid-ask averages of one-month and one-year interbank Eurocurrency rates. From the Eurocurrency rates, one-year maturity continuously compounded zero coupon rates are calculated.

To identify interest rate process parameters, we discretize the continuous-time processes given in (10). For ease of estimation, we estimate the parameters with discount bond yields rather than forward rates. The unexpected changes in the exchange rate and the domestic and foreign continuously compounded one-year yields are defined to be $\epsilon(t) = [\epsilon_d(t), \epsilon_f(t), \epsilon_s(t)]'$. Furthermore, if h_t is the time between observations,

$$\begin{aligned}
\epsilon_k(t) &= [y_k(t, t+1) - y_k(t-h_t, t-h_t+1) - \mu_k(t-h_t, t-h_t+1)h_t], \\
&\qquad k = d, f, \\
\epsilon_s(t) &= [\ln S(t) - \ln S(t-h_t) - (y_d(t-h_t, t) - y_f(t-h_t, t))h_t - \alpha_s h_t],
\end{aligned}$$

where $y_k(t, t+T)$; $k = d, f$ is the T-period continuously compounded yield in country k (d or f) at date t and $\mu_k(t-h_t, t-h_t+1)$ is the expected drift of the one-year continuously compounded yield over $[t-h_t, t]$. Based on our procedure for computing forward rates on each date, $y_k(t-h_t, t)$ is equated to the one-month discount yield (the shortest maturity for which a yield is available).

If $\sigma_d(\cdot)$ and $\sigma_f(\cdot)$ are constant across time and maturity, it can be shown that $\epsilon(t)$ is joint normally distributed with a zero mean vector and covariance matrix:

$$h_t\Sigma(t) = h_t \begin{bmatrix} \sigma_d^2 & \rho_{df}\sigma_d\sigma_f & \rho_{ds}\sigma_d\sigma_s \\ & \sigma_f^2 & \rho_{fs}\sigma_f\sigma_s \\ & & \sigma_s^2 \end{bmatrix}. \tag{15}$$

Due to weekends and holidays, the time between observations varies. Therefore, the dating interval, h, is subscripted by t. Parameter estimates are now computed by maximizing the loglikelihood function of $\epsilon(t)$ assuming that successive (daily) observations are independent.

Note that our assumptions on the joint distribution of forward rates and the exchange rate yield joint normality of ϵ_t over *finite* intervals.

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estimates based on conventional approximations will be biased [see Lo (1988)]. In principal, we can estimate the parameters consistently using General Method of Moments using covariance restrictions. Since our focus is on model building rather than estimation, this exercise is not attempted.

Table 2
Forward rate and currency price process parameter estimates

Currency	Means			Standard deviations			Correlations		
	μ_d	μ_f	α_s	σ_d	σ_f	σ_s	ρ_{df}	ρ_{ds}	ρ_{fs}
Yen	0.005	0.010	-0.055	0.0148	0.0153	0.1236	0.082	-0.013	0.063
	(0.27)	(0.38)	(0.46)	(15.65)	(6.32)	(25.95)	(1.79)	(0.050)	(2.11)
D-Mark	0.005	0.010	-0.066	0.0148	0.0113	0.1180	0.295	−0.023	0.045
	(0.67)	(2.02)	(1.20)	(15.65)	(26.96)	(29.35)	(7.27)	(0.02)	(1.27)

Changes in the one-year maturity discount rate and the exchange rate are modeled. Bollerslev and Wooldridge (1988) quasi-maximum likelihood T-statistics are presented below the associated parameter estimates in parenthesis. Daily data were used, January 9, 1989, to November 5, 1990, and were supplied by Citicorp.

$y_k(t, t+T)$ is the T-period continuously compounded yield in country k (d or f) at date t. $S(t)$ is dollar spot price of one unit of foreign currency at time t.

Let h_t be the time between observations. Relative to date t, define the time t error vectors for the three processes:

$$
\begin{aligned}
\varepsilon_k(t) &= y_k(t, t+1) - y_k(t-h_t, t-h_t+1) - \mu_k h_t, \qquad k = d \text{ or } f,\\
\varepsilon_s(t) &= \log S(t) - \log S(t-h_t) - [y_d(t-h_t, t) - y_f(t-h_t, t)]h_t - \alpha_s h_t,\\
\varepsilon(t) &= [\varepsilon_d(t), \varepsilon_f(t), \varepsilon_s(t)]' \sim N(0, h_t\Sigma)\\
\Sigma &= \begin{bmatrix} \sigma_d^2 & \rho_{df}\sigma_d\sigma_f & \rho_{ds}\sigma_d\sigma_s \\ \rho_{df}\sigma_d\sigma_f & \sigma_f^2 & \rho_{fs}\sigma_f\sigma_s \\ \rho_{ds}\sigma_d\sigma_s & \rho_{fs}\sigma_f\sigma_s & \sigma_s^2 \end{bmatrix}
\end{aligned}
$$

μ_k is the expected drift of the One-year continuously compounded yield in country k (d or f) at date t. a_s is the expected drift of the proportional exchange rate change. σ_k is the proportional volatility of the exchange rate ($k = s$) and the absolute volatility of the country k forward rate (d or f). ρ_{jk} represents the correlations between currency (j or $k = s$), domestic rate (j or $k = d$), and foreign rate (j or $k = f$).

As $\sup_t h_t \to 0$, the parameter estimates are consistent and not subject to the type of biases studied by Lo (1988).

Table 2 presents the parameter estimates and their t-statistics based on the Quasi-maximum Likelihood approach of Bollerslev and Wooldridge (1988). All the drift terms, α_d, α_f, and α_s, are insignificant except for the drift of Deutschemark interest rates. We observe that the U.S. forward rate volatility, $\sigma_d = 1.48$ percent, was lower than the Japanese forward rate volatility, $\sigma f = 1.53$ percent, and higher than that for Germany, $\sigma_f = 1.11$ percent. The estimated currency standard derivations are 11.8 percent for the Deutschemark and 12.36 percent for the yen.

The domestic rate-foreign rate correlations are positive, 0.082 for Japan and 0.295 for Germany. Currency-domestic rate correlations are small and negative, -0.023 for Germany and -0.013 for Japan. The correlations between foreign rates and currency values are positive and larger in absolute value than the domestic rate-currency correlations, 0.045 for Germany and 0.063 for Japan.

7. Currency Warrants and Cross-Rate Swaps

To highlight the effect of stochastic interest rates on currency-linked claims, we consider two applications: an AMEX Citibank currency put warrant and a Sallie Mae-Credit Suisse Financial Products (CSFP) cross-rate swap. This cross-rate swap is the first public deal of its kind. We focus on the additional risks that interest rates introduce into warrant valuation and the additional risks that the "quanto currency feature" adds to the cross-rate swap. As the benchmark, we use our estimated forward rate and currency process parameters in Table 2. In our comparative analysis, we vary a single parameter, while holding all other parameters constant at their estimated levels.

7.1 Currency warrants

Currency warrants began trading on the American Stock Exchange (AMEX) in mid-1987. Generally a retail market product, these currency options are notable for having relatively long maturities, three or five years.[16] Some issued warrants and their associated attributes are listed in Table 3. Predominantly, issued currency warrants have been puts written on the Deutschemark and yen.

Figure 2 contains plots of Deutschemark (Dm) and yen values from January 1987 to November 1990. The (put) warrant issues and the associated strike prices are also plotted. Generally, we see that all of the warrants were out of the money relative to the spot price when issued. They were also well out of the money relative to the forward price.

As the focus of our warrant analysis, we consider Citicorp yen warrants that were issued on April 13, 1988, with a strike price of 132.9 yen, $.0075245, and maturity of April 15, 1993. The first panel of Figure 3 presents the dollar- and yen-based swap yield curves on April 13, 1988. It is apparent that yen rates were 3 to 4 percent below U.S. rates. Based on the parameter estimates presented in panel B of Table 2, the April 13, 1988, spot currency price, $.0079101, and the associated swap yield curve levels, this American warrant has a value of $2.02. This value can be decomposed into a European warrant component of $1.25 and an early exercise premium of $0.77. If we ignore interest rate risk by setting all the interest rate volatility parameters to zero, the American warrant value is equal to only $1.77.

The calculated American warrant value falls below the initial offering price of $3.375. This type of underpricing has been documented in a more rudimentary manner by Rogalski and Seward (1991). Using a Black-Scholes framework and assuming that both domestic and

[16] The over-the-counter long-term currency option market is very large, with individual deals in the hundreds of millions of dollars. However, obtaining the associated data is quite difficult.

Table 3
American Stock Exchange foreign currency warrants

Issuer	Listing date	Yen strike price	Yen warrants $ Strike price × 100	Expiration date	Offering price	Public distribution, millions
AT& T Credit Corp.	7/09/87	158.25	0.6319	7/01/92	53.500	3.0
Citicorp	6/24/87	152.50	0.6557	7/01/92	$4.125	2.0
Citicorp	4/13/88	132.90	0.7524	4/15/93	$3.375	2:o
Ford Motor	6/19/87	152.20	0.6570	7/01/92	$4.375	3.0
Fold Motor	1/21/88	134.00	0.7463	2/01/93		2.5
Ford Motor	7/15/88	139.78	0.7154	7/15/91	$3.250	2.0
Gen. Electric	6/11/87	149.70	0.6680	6/15/92	$4.500	2.0
Student Loan Marketing Assoc.	6/22/87	152.20	0.6570	7/15/92	$4.375	2.0
Student Loan Marketing Assoc.	3/03/88		0.7357	3/01/93	$3.375	2.0
Xerox Credit	6/30/87	134.13	0.6487	7/01/92	$3.875	2.0

Issuer	Listing date	DM strike price	Deutschemark warrants $ Strike price	Expiration date	offering price	Public distribution, millions
Citicorp	7/10/87	DM 1.9320	0.5176	7/15/92	$4.125	2.0
Emerson Electric Co.	6/26/87	DM 1.9180	0.5214	7/01/92	54.625	2.0
Gen. Electric Credit Corp	6/25/87	DM 1.9120	0.5230	7/01/92	$4.750	2.0
JP Morgan & Co.	6/24/88	DM 1.9040	0.5252	7/01/91	$2.500	2.5
student Loan Marketing Assoc.	6/26/87	DM 1.9200	0.5208	7/15/92	$4.625	2.0

Listed American Stock Exchange currency warrants, as of November 1989. AU warrants have the American exercise feature and on exercise pay the following proceeds: $50 − (50 × $ spot price / $ strike price).

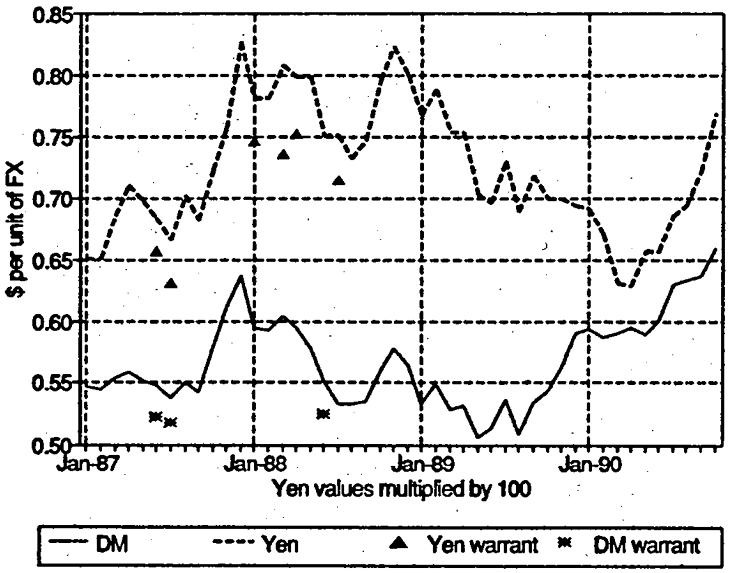

Figure 2
Deutschemark and yen exchange rates and warrant exercise prices
Currency warrants are issued and traded on the American Stock Exchange. Triangles and asterisks indicate offering dates and exercise price levels for the yen and Deutschemark warrants, respectively. All indicates warrants were foreign currency puts with the following exercise proceeds: $50 - (50 × spot price / $ strike price).

foreign interest rates are deterministic Rogalski and Seward compute implied currency volatilities from short-term currency options and use them to price longer-term warrants. They find that the market prices of the warrants are on average more than double the estimated values.

Effectively, short-term option implied volatilities measure only exchange rate volatility. Since short-term option values are relatively insensitive to interest rate risk, the short-term option implied volatility fails to account for interest rate risk, which significantly affects long-term option values. Using these volatilities results in long-term option values that are much too low. Relative to the findings of Rogalski and Seward (1991), incorporation of interest rate risk in our valuation improves on their undervaluation. Based on the historical parameter estimates and the single warrant analyzed, warrant undervaluation is reduced from 91 percent to 67 percent.

There are two potential reasons for the remaining warrant underpricing. First, we have made a particular distributional assumption. Second, we base our parameter estimates on the 1987-90 sample period. There is no assurance that the estimates based on this sample period reflect expectations on April 13, 1988. Given our pricing model,

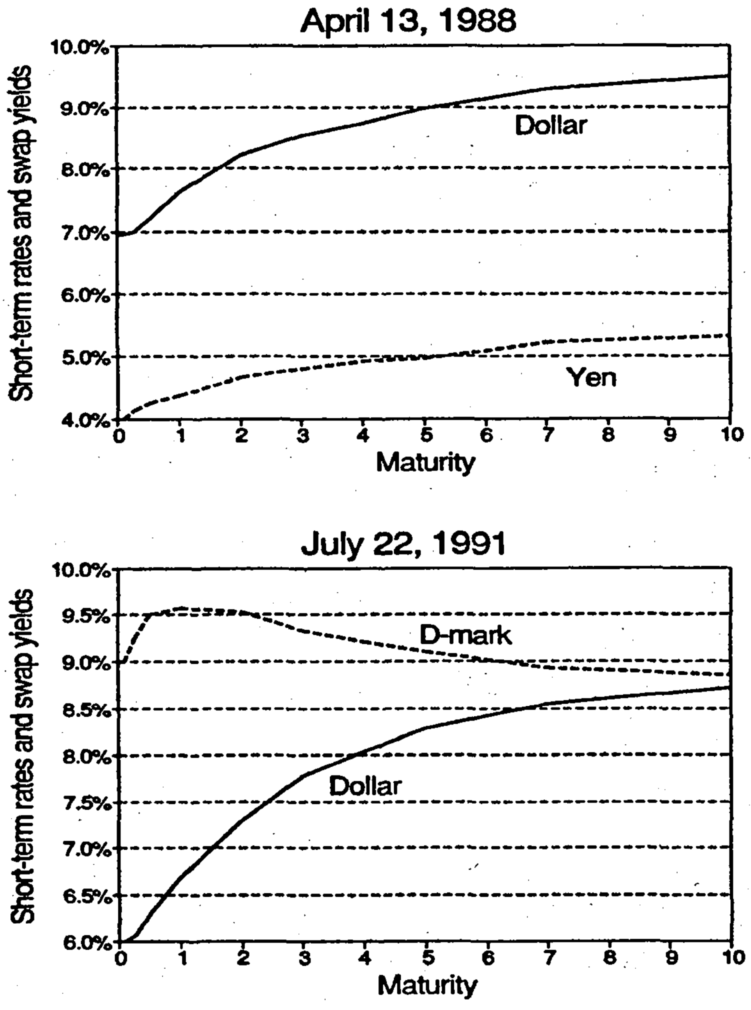

Figure 3
Eurocurrency rates and swap yields
Citibank 1, 3, 6, and 12-month semiannual simple interest rates and 2, 3, 4, 5, 7, and 10-year semiannual bond-equivalent swap yields, which prevailed on the Citicorp yen warrant issue date. April 13, 1988, and on the Sallie Mae Deutschemark/dollar cross-rate swap issue date, July 22, 1991. Rates and yields are bid-ask averages.

we can investigate these potential explanations. Particularly, we conduct a comparative statics pricing analysis over the model parameters and the functional form of the forward rate volatility function. Our analysis has two parts: European warrant values and early exercise values.

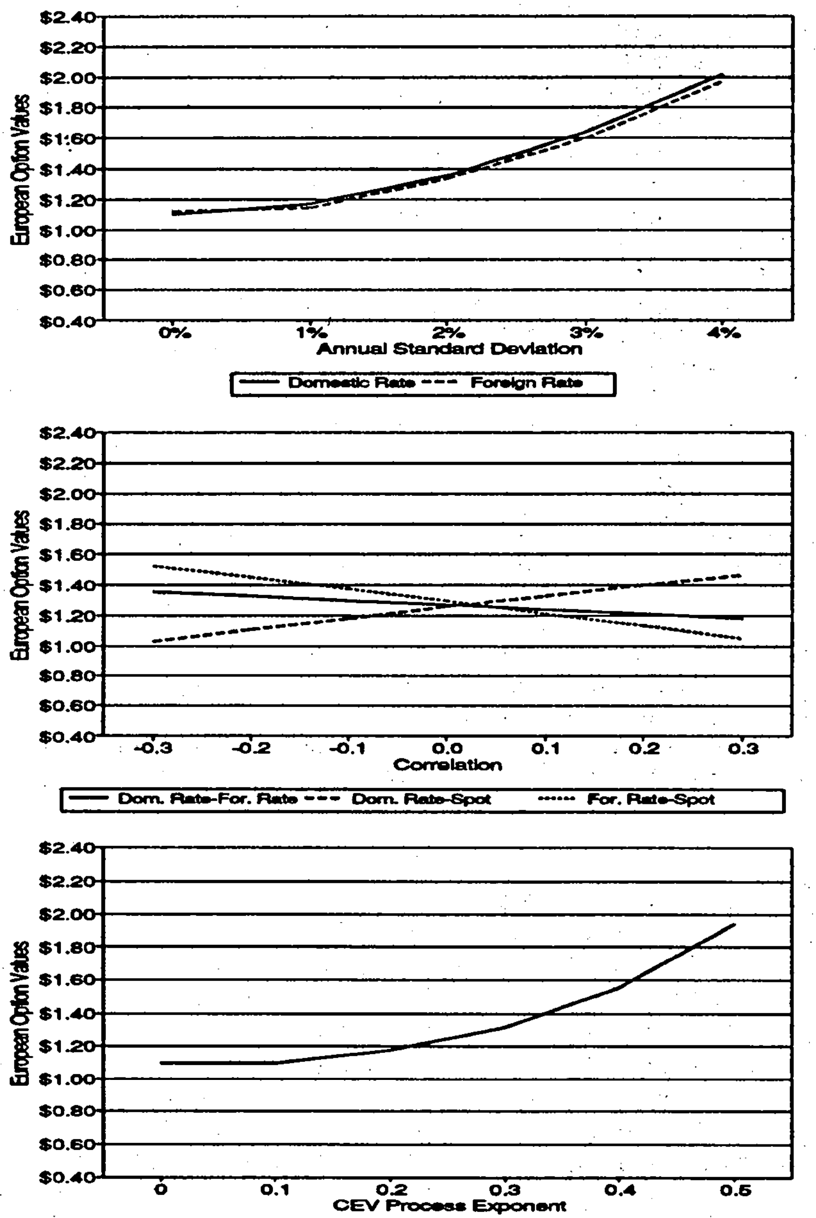

Figure 4
AMEX five-year maturity yen warrant—European option values
Comparative statics analysis of a Citicorp put currency warrant's April 13, 1988, offering value. We vary one parameter, while holding all others at the following benchmark values: $S(0) = 0.0079101$, $\sigma_d = 1.48$ percent, $\sigma_f = 1.53$ percent, $\sigma_s = 12.36$ percent, $\rho_{df} = 0.082$, $\rho_{ds} = -0.013$, and $\rho_{fs} = 0.063$. Strike price — $K_{\text{km}} = 0.0075245$. The CEV process exponent, β, equals zero for the first two graphs. The warrant exercise proceeds are $50 — (50 × $ spot price / $ strike price).

7.1.1 European warrant values. Figure 4 presents the comparative statics results for the European warrant component of the Citicorp April 15, 1993, maturity yen put. The pricing effects of changing rate standard deviations, correlations, and forward rate standard deviation exponents (β) are presented in the first, second, and third panels, respectively. To facilitate comparison, all graphs are set to the same scale. The benchmark parameter values are listed at the figure. bottom.

We vary σ_d and σ_f from zero to 4 percent. Domestic and foreign rate standard deviation increases result in similar European component value changes, about \$0.87. The correlations range from -0.3 to +0.3. Varying the rate-spot price correlations generates price changes of similar magnitude but opposite sign (-\$0.47 for the foreign rate and \$0.44 for the domestic rate). Lowering the foreign rate-spot currency correlation or raising the domestic rate-spot currency correlation adds to the warrant value. Raising the domestic rate-foreign rate correlation yields a smaller negative pricing effect, –\$0.18.

If we consider the influence of parameter changes on forward currency price volatility, then interpretation of the above valuation effects is straightforward. Higher total forward currency price volatility raises warrant values. Higher domestic and foreign interest rate standard deviations and higher domestic rate-spot currency price correlation add to forward price variability. Lower correlations for both the domestic rate-foreign rate and the foreign rate-currency price pairs and lower β 's reduce forward price volatility. The signs of the European component pricing effects follow directly.

To analyze the effect of changing the CEV parameter β, we must also vary the parameters σ_d and σ_f. Otherwise, the (CEV-weighting of the standard deviations by small forward rates dramatically lowers the terminal exchange rate's total volatility. Choosing among many possible adjustments, we scale the rate standard deviations so that the instantaneous standard deviations of five-year maturity discount bond prices are equal across specifications of β. This convention also ensures that the instantaneous volatility of the five-year forward exchange rate corresponding to the option maturity is unaltered by changes in β. This adjustment seems consistent with estimation of the forward rate parameters using data sampled at short intervals.

To study the effect of the CEV parameter β, we change both the domestic and foreign parameters together. Changing β from zero to one-half roughly doubles the European warrant value component. This model pricing sensitivity is a plausible rationale for the Citibank warrant being significantly above our model price calculated earlier.

The intuition for the large valuation effects due to changes is β is as follows. When $\beta > 0$, σ_d is multiplied by $f_d(t, T)^\beta$ to yield the volitility of $f_d(t, T)$. Therefore, an increase (decrease) in the forward rate is accompanied by an increase (decrease) in the volatility, whereas the volatility remains constant even as the level of forward rates changes when $\beta = 0$. Since the effect at high levels of forward rates dominates, a model with $\beta > 0$ yields higher interest rate volatility over long time intervals than does a model in which $\beta = 0$, even though the volatility over very short time intervals is identical. If the parameters are estimated from frequently observed data, the models (with $\beta = 0$ and $\beta > 0$) will yield virtually identical instantaneous volatilities. However, the volatility over long time intervals will be larger in the model with $\beta > 0$. Since an increase in the interest rate volatility serves to increase the warrant value, our result documents this intuition.

7.1.2 Early exercise premium. Figure 5 plots the early exercise premium (American option value minus the European option value) as the parameter values are varied.[17] The first panel indicates that the early exercise premium is not very sensitive to changes in interest rate volatility. However, the second panel shows that the correlations yield significant valuation effects. Interestingly, the effect of correlation changes on early exercise premia are the opposite of those for European warrant values. Changing the domestic rate-currency correlation from -0.3 to +0.3 results in a decrease in the early exercise premium of roughly the same absolute magnitude as the increase in the European warrant value. Similarly, a foreign rate-currency correlation increase from -0.3 to +0.3 increases the early exercise premium and lowers the European option value. Taken together, these results indicate that the composite value of the American warrant is not significantly affected by currency-related correlation changes.

A change in the correlation between domestic and foreign interest rates (ρ_{df}) affects early exercise values by a much smaller amount than changes in the Other correlations. Raising ρ_{df} lowers the warrant early exercise premium (just as it does European warrant values). However, both the early exercise premium and the European warrant value effects are of the same sign. Therefore, the American warrant is more sensitive to this correlation than to the currency-related correlations.

The intuition behind the early exercise value results is as follows. For the put currency warrants, the early exercise privilege is valuable when the option is in-the-money (spot price low), yield (the foreign interest rate) is low, and the cost of funds (the domestic interest rate) is high. The likelihood of attaining such states is increased by posi-

[17] For an analysis of early exercise of foreign currency options with constant interest rates, see Bodurtha and Courtadon (1987).

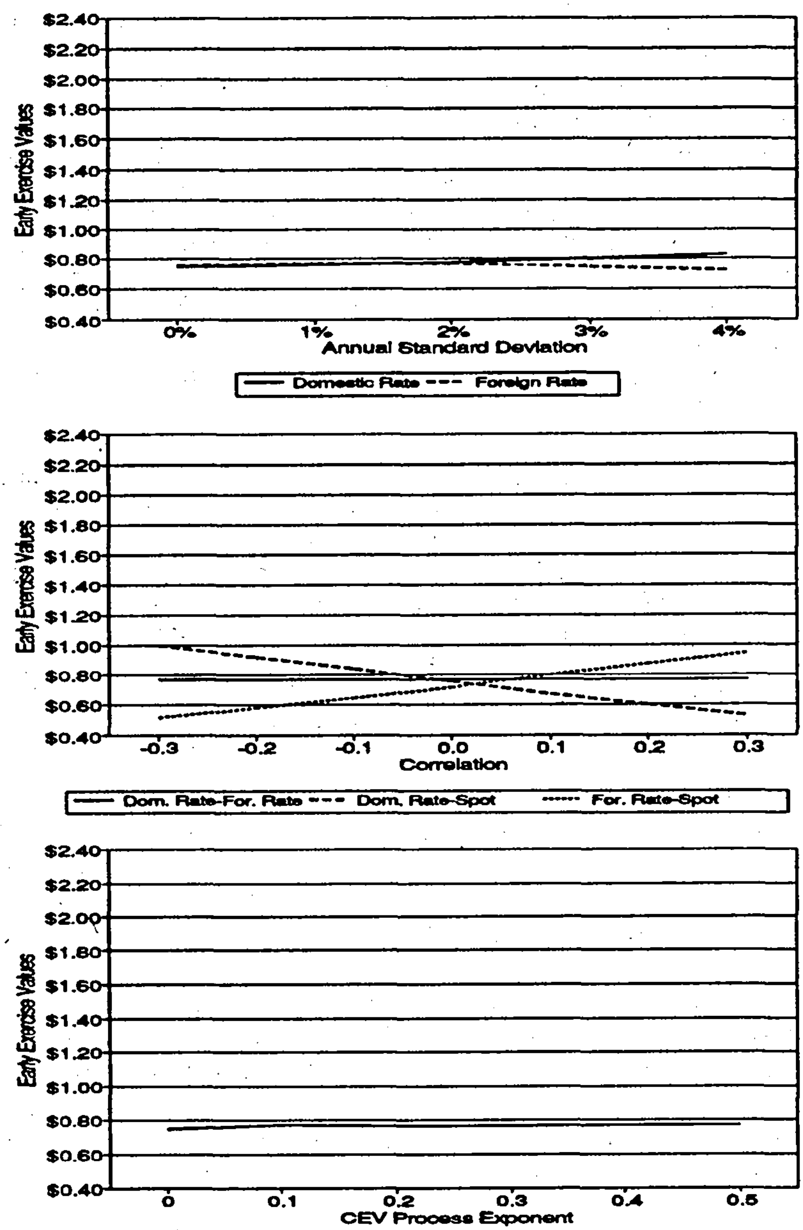

Figure 5
AMEX five-year maturity yen warrant — early exercise values
Comparative statics analysis of a Citicorp put currency warrant's April 13, 1988, offering value. We vary one parameter, while holding all others at the following benchmark values: $S(0) = 0.0079101$, $\sigma_d = 1.48$ percent, $\sigma_f = 1.53$ percent, $\sigma_s = 12.36$ percent, $\rho_{df} = 0.082$, $\rho_{ds} = -0.013$, and $\rho_{fs} = 0.063$. Strike price — $K_{tm} = 0.0075245$. The CEV process exponent, β, equals zero for the first two graphs. The warrant exercise proceeds \$50 - (50 × \$ spot price / \$ strike price).

tive correlation between the spot price and the foreign rate ($\rho_{fs} > 0$), negative correlation between the spot exchange rate and the domestic rate, ($\rho_{ds} < 0$), and positive correlation between the domestic and foreign interest rates ($\rho_{df} < 0$). For the interest rate-currency correlations, the correlations are those that lower European option values.

Increasing the CEV parameter (β) yields only a very small increase in the early exercise premium, \$0.02. As argued in the previous subsection, the effect of an increase in β is analogous to that of an increase in the forward rate volatility. Therefore, the valuation impact of changes in β is similar to that of Changes in σ_d or σ_f. Overall, we conclude that the significant change in the warrant value due to changes in β is due to the large changes in the value of the European warrant component.

7.2 Cross-currency contingent claims

To our knowledge, no uniform approach currently exists for valuing cross-currency contingent claims with values that explicitly depend on the stochastic nature of interest rates. Yet, many claims of this type are traded, and the market is growing rapidly.

A new, fundamental, and actively dealt cross-currency claim is the cross-rate swap. In this contract, foreign currency payments are convetted into the domestic currency at a fixed exchange rate, typically 1:1, independent of the exchange rate at the time of the payment. Another rapidly growing segment of international financial markets is quanto derivatives. These claims have nominal payments in the foreign currency, but these payments are converted into the domestic currency at a fixed value independent of the exchange rate. Cross-rate swap valuation is a basic component in valuing many quantos.

7.2.1 Cross-rate swap characteristics. A cross-rate swap is an agreement to exchange cash flows denominated in two different currencies on a periodic basis, usually every six months. The cash flows in each currency can be either fixed or tied to the prevailing interest rates (floating). Unlike a cross-currency swap, the cash flows in the foreign currency are converted to the domestic currency at a fixed, exchange rate independent of the prevailing exchange rate.

We consider a specific example. On July 22, 1991, Sallie Mae entered into a two-year maturity cross-rate swap agreement requiring semiannual payments with CSFP The swap was tied to a Deutschemark floating-rate note (FRN) issue; Sallie Mae’s objective was to use the note-swap package to lower their all-in floating dollar cost of funds. Specifically, Sallie Mae was required to pay the six-month dollar LIBOR less 50 basis points and receive the six-month Deutschemark LIBOR less 212.5 basis points on a notional principal of $100 million.[18] Four payments were to be made at six-month intervals through July 22, 1993. The Deutschemark rate-related payment was to be converted to dollars at an exchange rate of 1:1, despite the Deutschmark value of 1.80 to the dollar on July 22, 1991, and notwithstanding its future values.

7.2.2 Cross-rate swap values. The benchmark valuation parameters are the July 22, 1991, Eurodollar and Euro-Deutschemark term structures (plotted in the second panel of Figure 2), the 1.80 Deutschemark to dollar exchange rate, and the standard deviation and correlation estimates listed in Table 2. Based on these parameter values, we value the swap to Sallie Mae at -0.626 percent per annum.[19] Adding this spread to Sallie Mae's payments equates the swap value to zero. This large negative spread is difficult to reconcile with market efficiency. One possible explanation is that the terms of the contract were finalized on a date earlier than the reported announcement date when the yields were different. However, *Investment Dealer's Digest* [Liebowitz (1991)] notes that the rate achieved by Sallie Mae on its FRN issue was lower than all other usual alternatives and was reported to be at least 50 basis points below LIBOR.

To further analyze the cross-rate swap value; we conduct a comparative statics analysis with respect to forward rate process standard deviations, the correlations and changes in β. This analysis is similar to our analysis of the long-term warrant. The results of this analysis are presented in Figure 6.

The first panel in Figure 6 shows that foreign rate risk is more important than domestic rate risk. Raising foreign rate risk from 1 to 4 percent effectively lowers CSFP's risk-adjusted foreign rate-linked payment costs by 4 basis points. Increasing domestic rate risk also lowers the value of their dollar rate-linked receipts (but by only 1.7 basis points).

The foreign rate standard deviation has a larger effect than the domestic rate standard deviation for the following reason. The value of

[18] The payments on both sides were also capped. However, these caps are sufficiently high so that their influence on swap values is negligible.

[19] For our current application, if the volatility of the exchange rate is assumed constant, the swap an be valued by creating a tree only for the domestic and foreign term structures under $\bar{Q}$. However, this simplification is not possible for many other applications or for other exchange rate volatility functional forms.

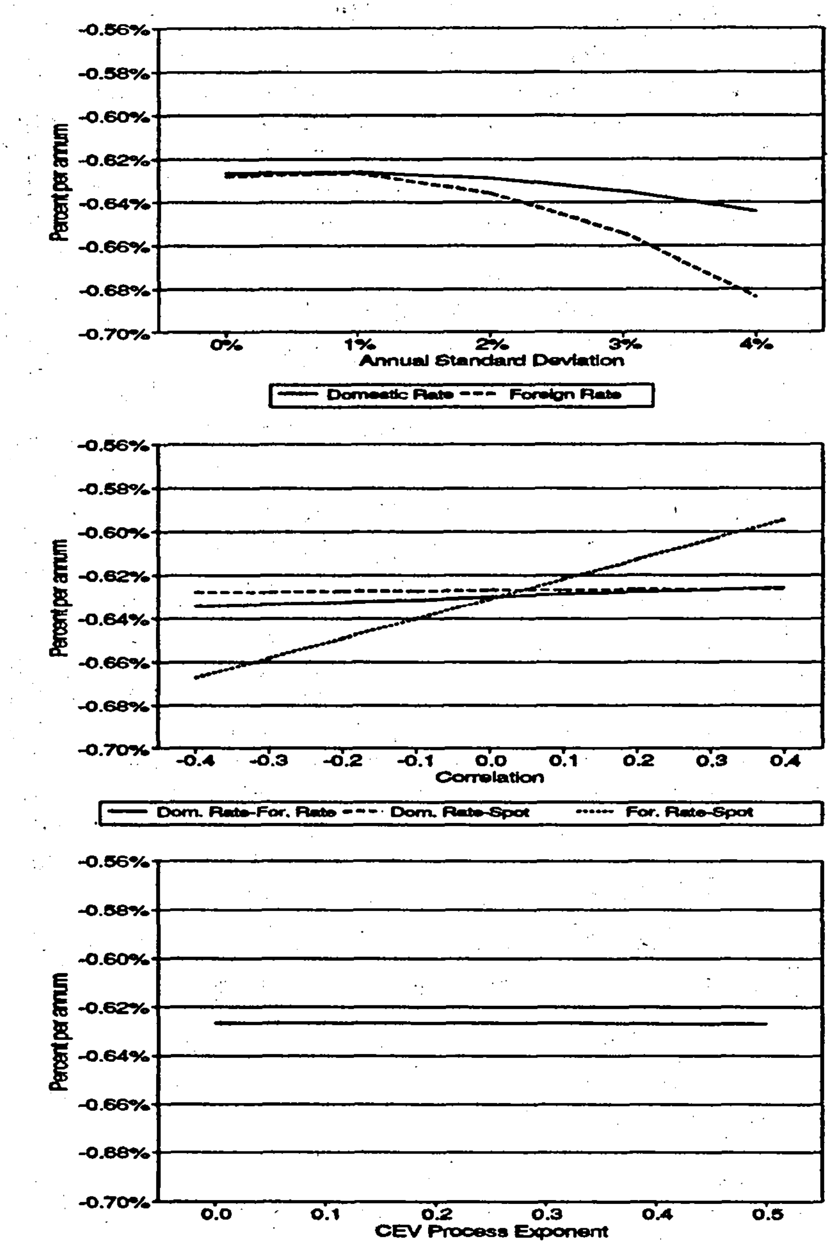

Figure 6
Sallie Mae – CSFP two-year maturity cross-rate swap
Comparative statics analysis of a Sallie Mae-CSFP cross-rate swap's July 22, 1991, offering value. We vary one parameter, while holding all others at the following benchmark values: $S(0)$ = 0.5555, $\sigma_d = 1.48$ percent, $\sigma_f = 1.13$ percent, $\sigma_s = 11.80$ percent, $\rho_{df} = 0.295$, $\rho_{ds} = -0.023$, and $\rho_{fs} = 0.045$. The CEV process exponent, β, equals zero for the first two graphs. Semiannual dollar payments are [(six-month \$ LIB0R - 0.5 percent) - (six-month Dm LIBOR - 2.125 percent)]/2.

Sallie Mae's dollar-based interest rate payments is essentially the value of a dollar FRN and does not depend on interest rate risk. However, since CSFP's payments depend on Deutschemark interest rates, they are most sensitive to foreign interest rate volatility. These stochastic cash flows must be discounted back to the current date using domestic interest rates, so that their current value depends on the relationship between domestic rates and both foreign rates and the currency. Because the swap has only a two-year maturity, the discounting related effect is not strongly dependent on domestic interest rate uncertainty. Therefore, the swap value is only moderately sensitive to domestic interest rate risk. However, for swaps with maturity greater than two years, domestic interest rate risk will also be a significant valuation factor.

Increases in all correlations increase the cross-rate swap value. The foreign rate-currency correlation has the largest of the valuation effects, seven basis points. As the correlation between the foreign and domestic rate is changed from -0.3 to 0.3, the cross-rate swap value increases by about one basis point. The domestic rate-spot currency correlation has the smallest valuation impact, roughly one-tenth of a basis point.

The final panel of Figure 6 plots the pricing impact of changing the assumed distribution of forward rates by varying β. Since the average maturity of the swap payments is one year, we change σ_d and σ_f to maintain the standard deviation of the one-year discount yield as β is varied. As β is raised from zero to one-half, the cross-rate swap value falls imperceptibly. As argued for the warrant, the effect of increasing β is similar to an increase in the forward interest rate volatility. Here, the effect of varying β is much smaller because the maturity of the swap is only two years.[20]

A notable feature of our analysis is that interest rate and exchange rate uncertainty cause small valuation effects in comparison to the estimated spread of -0.626 percent for the swap. However, as the market matures, the spread is likely to narrow significantly and these valuation effects will be relatively more important.

8. Conclusions

We have developed an approach to valuing a broad range of instruments with exchange rate (or asset price) risk and interest rate risk along two different term-structures. Our discrete-time model can be

[20] The final payment is actually determined afterone and a half years since the cash flow at any date depends on interest rates six months before the payment date.

implemented in practice for up to five-year maturity options. The general model is path dependent and permits arbitrary volatility and covariance functions for both domestic and foreign interest rates and for the exchange rate. With specific volatility functions, we also provide a path-independent implementation of the model. Though our analysis has focused on international money market instruments, the model may also be used to value regular equity and commodity derivatives or even Asian and other "exotic" path-dependent options with some modifications.

The model is illustrated with applications to a five year maturity. currency warrant and a cross-rate swap. We find that listed long-term put warrant values are significantly affected by interest rate risk parameters. For example, the pricing biases in long-term currency warrant premia demonstrated by Rogalski and Seward (1991) can be substantially reduced by incorporating interest rate risk. We also provide a first analysis of currency cross-rate or "fanciful" swaps.[21] These claims are of interest both in their own right and as basic building blocks of a new class of claims known as *quantos*. We evaluate a particular Sallie Mae-CSFP cross-rate swap in which all cash flows are denominated in dollars. The value of the swap is affected predominantly by foreign interest rate standard deviation and the correlation between the exchange rate and foreign interest rates.

Appendix: Proof of Lemma 1

By definition,

$$\frac{B_d(t+h)}{B_d(t)} = \exp(r_d(t)h) = \exp(f_d(t,t)h), \tag{16}$$

$$\frac{P_d(t+h,T)}{P_d(t,T)} = \frac{\exp\left(-\sum_{j=\frac{t}{h}+1}^{\frac{T}{h}-1} f_d(t+h,jh)h\right)}{\exp\left(-\sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} f_d(t,jh)h\right)},$$

$$= \exp\left[f_d(t,t)h - \sum_{j=\frac{t}{h}+1}^{\frac{T}{h}-1}(f_d(t+h,jh)-f_d(t,jh))h\right]. \tag{17}$$

[21] This term was coined by Litzenberger (1992) in his presidential address to the American Finance Association in 1991.

Now, using the definition of $Z_d(t, T)$ and combining with Equations (16) and (17) yields

$$\begin{aligned}\frac{Z_d(t+h, T)}{Z_d(t, T)}\\ &= \exp\left[-\sum_{j=\frac{t}{h}+1}^{\frac{T}{h}-1}(f_d(t+h, jh) - f_d(t, jh))h\right]\\ &= \exp\left[-\sum_{j=\frac{t}{h}+1}^{\frac{T}{h}-1}(\alpha_d(t, jh)h + \sigma_d(t, jh, \cdot)X_d((j+1)h)\sqrt{h})h\right].\end{aligned}$$

Taking expectations on both sides and equating to 1 yields the first equation in Lemma 1. To obtain the second expression, apply the $(\tilde{Q})$ martingale condition on $Z_f(t)$, that is,

$$\tilde{E}_t\left[\frac{Z_f(t+h)}{Z_f(t)}\right] = \tilde{E}_t\left[\frac{B_f^*(t+h) \times S(t+h) \times B_d(t)}{B_f^*(t) \times S(t) \times B_d(t+h)}\right] = 1.$$

Substitute the stochastic difference equation for $S(t)$ and the fact that $\frac{B_d(t+h)}{B_d(t)} = \exp(r_d(t)h)$ and $\frac{B_f^*(t+h)}{B_f^*(t)} = \exp(r_f(t)h)$ into the above equation. This yields

$$\tilde{E}_t\left[\exp\left(\alpha_s(t)h + \sigma_s(t)X_s(t+h)\sqrt{h} - (r_d(t) - r_f(t))h\right)\right] = 1. \quad (18)$$

Rearranging this equation yields the second expression in Lemma 1. To prove the final expression, we use the fact that $Z_f(t, T)$ must be $\tilde{Q}$-martingale. Hence,

$$\tilde{E}_t\left[\frac{Z_f(t+h, T)}{Z_f(t, T)}\right] = 1. \quad (19)$$

By multiplying and dividing the definition for $Z_f(t, T)$ by $B_f^*(t)$, we get

$$\begin{aligned}Z_f(t, T) &= \frac{S(t)P_f^*(t, T)}{B_d(t)} = \frac{S(t)B_f^*(t)}{B_d(t)} \times \frac{P_f^*(t, T)}{B_f^*(t)}\\ &= Z_f(t)\frac{P_f^*(t, T)}{B_f^*(t)}.\end{aligned} \quad (20)$$

By definition

$$\frac{B_f^*(t)}{B_f^*(t+h)} = P_f^*(t, t+h) = \exp[-r_f(t)]. \tag{21}$$

Combining Equations (20) and (21),

$$\frac{Z_f(t+h, T)}{Z_f(t, T)} = \frac{Z_f(t+h)}{Z_f(t)} \times \frac{P_f^*(t, t+h)P_f^*(t+h, T)}{P_f^*(t, T)} \tag{22}$$

Given the definition of $Z_f(t)$, the foreign money market accumulation account, and the stochastic difference equation for $S(t)$, we obtain

$$\frac{Z_f(t+h)}{Z_f(t)} = \exp\left[\alpha_s(t)h + \sigma_s(t, \cdot)X_s(t+h)\sqrt{h}\right] \tag{23}$$

Now

$$P_f^*(t+h, T) = \exp\left[-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} f_f(t+h, ih)h\right] \tag{24}$$

and

$$\frac{P_f^*(t, T)}{P_f^*(t, t+h)} = \exp\left[-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f_f(t, ih)h\right]\exp[f_f(t, t)]$$

$$= \exp\left[-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} f_f(t, ih)h\right]. \tag{25}$$

Using Equations (24) and (25),

$$\frac{P_f^*(t, t+h)P_f^*(t+h, T)}{P_f^*(t, T)}$$

$$= \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} [f_f(t+h, ih) - f_f(t, ih)]h\right\} \tag{26}$$

$$= \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[\alpha_f(t, ih)h + \sigma_f(t, \cdot)\sqrt{h}X_f(t+h)\right]h\right\} \tag{27}$$

Now substituting Equations (23), (24), and (27) into Equation (22) yields

$$\frac{Z_f(t+h, T)}{Z_f(t, T)} = \exp\left(\alpha_s h + \sigma_s(t, \cdot)X_s(t+h)\sqrt{h}\right.$$
$$\left. - \left\{ \sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[\alpha_f(t, ih)h + \sigma_f(\cdot)X_f(t+h)\sqrt{h}\right] \right\} \right).$$

Taking expectations with respect to $\tilde{Q}$ and rearranging yields the desired result. ■

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