Asset Prices in Dynamic Production Economies with Time-Varying Risk

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We examine the effect of changes in output uncertainty on the price of aggregate capital and on the prices of levered claims on capital. The relation between the volatility of the marginal product of capital and the price of capital depends on the level of capital adjustment costs and the elasticity of intertemporal substitution. For available estimates of this elasticity, the value of capital and risk are directly related while the value of levered equity claims on capital may be decreasing in risk. We use these results to analyze the argument that increased risk was responsible for the U.S. stock market decline of the 1970s.

The relation between risk and security prices is of fundamental importance in financial economics. In this article, we examine the effect of variation in economic risk on the equilibrium prices of aggregate capital assets. Our work is motivated by the growing evidence that overall economic risk does not remain constant over time. For example, Merton's (1980) estimates of the standard deviations of returns on the

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NYSE index vary from 11 percent p.a. to 24 percent p.a., excluding the extremely volatile period of the Great Depression. Similar observations can be made about the macroeconomy in general.[1] For example, estimates of the standard deviation of the growth rate of the index of industrial production, for nonoverlapping two-year periods between 1947 and 1987, vary from 1.6 percent p.a. to 6.8 percent p.a. As uncertainty varies over time, the price of the market portfolio and the prices of claims contingent on that asset must respond to these variations. The purpose of our work is to characterize precisely the relation between the prices of these aggregate assets and changes in uncertainty.

The question of how changes in risk affect aggregate stock prices is central to the debate about the observed behavior of stock market indices in the U.S. economy in the 1970s and 1980s. Pindyck (1984) argues that the decline, in real terms, in share prices on the New York Stock Exchange observed in the late 1970s was due to the increase in economic risk. Malkiel (1979) also attributes the decline in *q-ratios* and the sluggishness in capital accumulation in the U.S. economy in the 1970s to an increase in uncertainty. Based on their estimates of the persistence of the volatility process for market returns, Poterba and Summers (1986) argue that shifts in risk cannot account for the dramatic decline in market indices. However, these analyses are based on a partial equilibrium model. In particular, they do not explicitly account for the endogenous adjustment of interest rates and aggregate investments when considering the effect of changes in risk on asset prices. In this article, we subject the arguments of Malkiel and Pindyck to an equilibrium analysis and provide conditions under which their results hold in equilibrium.

The relation between uncertainty and the prices of aggregate assets is also of general interest in financial economics, especially in the empirical modeling of security returns. Frequently, economic risk and the prices of aggregate assets are assumed to be inversely related. This assumed inverse relation underlies the use of price-to-dividend ratios, price-earnings, ratios and market-to-book ratios (for the market portfolio) as proxies of aggregate risk. The argument for an inverse relation between asset prices and uncertainty is summed up by Black (1976), who states “if there is more risk to be borne, assuming that expected payoffs from business investments do not change, then stock prices must fall so that investors will continue to hold the existing stocks.” Of course, in an equilibrium, changes in economic uncertainty would bring about changes in aggregate investments and hence

[1] Weiss (1984) finds that autoregressive moving average (ARMA) models with autoregressive conditionally heteroskedastic (ARCH) errors are useful in modeling 13 different macroeconomic and financial time series.

change the level of expected payoffs from these investments. Our analysis shows that, although Black's conclusion holds in equilibrium in important special cases, it is not always true. Thus, our results are of interest because they identify circumstances when the supposed proxies for risk are likely to be good and conditions under which they are not.

We now describe the main contributions of our work. Our model is the first to examine the effect of changes in risk on the prices of aggregate assets in a production economy. The first result of our analysis is that the sensitivity of the price of the aggregate capital stock to shifts in risk is crucially dependent on the level of capital adjustment costs. The effect of shifts in uncertainty is significant only when capital adjustment costs are substantial. This result is important in interpreting the analyses of existing models that investigate the price-uncertainty relation. In the framework of an exchange economy, these models have been constructed by Abel (1988), Barsky (1989), and Gennotte and Marsh (1993).[2] Given the focus on modeling an exchange economy, these studies do not consider the capital accumulation process explicitly and hence cannot examine the role of the production technology in the determination of the price-uncertainty relation. The models of Abel, Barsky, and Gennotte and Marsh are special cases of ours, where the capital adjustment costs are assumed to be infinite. We show that, because of the implicit assumption of extreme capital adjustment costs, these models tend to overstate the effect of changes in risk on asset prices.

Our second contribution is to identify aspects of agents' preferences that have an important influence on how changes in risk affect the prices of aggregate assets. We show that the relation between risk and the value of aggregate capital depends on whether agents desire to save less when the future becomes more uncertain. If agents save less when they face higher uncertainty, the price of the market portfolio and economic risk are inversely related. We use the generalized recursive preferences of Epstein and Zin (1989), which permit a distinction between risk aversion and intertemporal substitution. We show that intertemporal substitution, as opposed to risk aversion, governs the direction of the effect of changes in risk on the value of capital stock. Risk aversion, however, is important in the determination of the magnitude of this effect. In particular, if agents' preferences display first-order risk aversion, the sensitivity of prices to movements in uncertainty is higher.[3]

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[2] For an analysis of the effect of changes in uncertainty on firm values in a partial equilibrium model, see Abel (1983) and Chang (1994).

[3] An investor is said to display first-order risk aversion if the premium that she is willing to pay to avoid a small gamble is proportional to its standard deviation rather than to the variance. Expected utility preferences do not generally display first-order risk aversion.

The third contribution of our work is to show that the effect of changes in aggregate risk on the market portfolio may be quite different from that on a levered claim on that asset. In the presence of corporate debt, equity shares may be thought of as in-the-money call options on a firm's assets. We show that, if the degree of leverage is not very high, the value of levered equity-type claims on the aggregate capital stock can be a *decreasing* function of production risk even when the value of the underlying asset increases because of such shifts in risk. This results from the downward adjustment in the term structure of interest rates when uncertainty increases.

We now relate our results to the debate about increased uncertainty being responsible for the market decline and sluggish capital formation in the late 1970s, as argued by Malkiel (1979) and Pindyck (1984). Suppose that we interpret these arguments as being related to the value of aggregate capital stock and can regard the productivity of capital as being roughly similar in the periods being compared. Then, our model predicts that the effects suggested by Malkiel and Pindyck hold only for large values of the elasticity of intertemporal substitution. However, the empirical estimates of this elasticity provided by Epstein and Zin (1992) tend to be rather small. For these estimates, the effect of shifts in risk on prices and investments predicted by our model are the opposite of those suggested by Pindyck and Malkiel. In addition, we show that, to explain a dramatic decline in the value of the market portfolio, one would need to assume a large capital adjustment cost. This is especially true if we account for the finding by Poterba and Summers (1986) that the persistence of the volatility process for stock market returns is not large.

The above results suggest that our model does not lend immediate support to the argument that increased risk led to a substantial decline in the value of the market portfolio. However, due to the presence of corporate debt, we could interpret the observations of Malkiel (1979) and Pindyck (1984) as applying not to aggregate capital but to levered equity claims on aggregate capital. Then, their analysis is consistent with our model. We show that because of a downward shift in interest rates caused by increases in risk, the prices of levered equity claims and risk may be negatively related, even for small elasticities of intertemporal substitution.

We could also interpret the arguments of Malkiel (1979) as saying that the *average q-ratios* and the *average* investment-to-capital ratios are lower in regimes with higher risk. Our results are partly consistent with this conclusion. In our model, the average *q-ratios* for aggregate capital are lower in high-risk regimes for a range of parameters, where the average is being taken over different levels of productivity of capital in a given regime. However, we do not find a theoretical reason to expect the same for average investment-to-capital ratios.

The rest of the article is organized in four sections. Section 1 outlines our model of a production economy with time-varying uncertainty. Section 2 examines the relation between output risk and the value of the aggregate capital stock. In Section 3, we consider levered claims on the capital stock, and Section 4 presents conclusions.

1. The Model

Below, we describe our model of a production economy. The crucial features of the model are (i) time-varying uncertainty of the marginal product of capital, (ii) the presence of capital adjustment costs, (iii) the separation of intertemporal substitution and risk aversion, and (iv) the allowance for first-order risk aversion in investors' preferences. In the following sections, we show that these features are the key determinants of the effect of changes in risk on asset prices and aggregate investment.

1.1 A firm facing capital adjustment costs and time-varying risk

We consider a finite horizon economy where time takes values in the set $\mathcal{T} := \{0, 1, \ldots, T\}$. There is a representative firm in the economy. The firm produces output $q(t)$ on date t as a function of its capital stock $k(t)$ and a random production shock $x(t)$. It pays out a quantity $d(t)$ as dividends and reinvests the rest of the output.[4] We assume that the firm's production and capital accumulation technologies are $q(t) = k(t)x(t)$ and $[k(t+1)]/k(t) = g[i(t)]$ where $k(t)$ is the capital stock at time t and $i(t) \equiv [q(t) - d(t)]/k(t)$ is the ratio of investment to capital at time t. The initial capital $k(0) > 0$ and the initial output shock $x(0)$ are exogenously given. The capital accumulation function $g(\cdot)$ is

$$g(i) = \begin{cases} i - \theta_u(i-1) & \text{if } i \geq 1 \\ i - \theta_l(1-i) & \text{if } i < 1 \end{cases} \tag{1}$$

where $0 \leq \theta_u \leq 1$ and $0 \leq \theta_l \leq \infty$. Thus, the rate of growth in the capital stock is a concave function of the ratio of investments to the existing capital stock. There is no cost of making replacement investments to maintain the capital at its existing level. However, there are costs associated with building new capital or dismantling existing

[4] All stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a given filtration $\{\mathcal{F}_t, t \in \mathcal{T}\}$ with $\mathcal{F} = \mathcal{F}_T$.

capital. Thus, the supply of capital is not perfectly elastic. The parameters θ_u and θ_l govern the cost of adjusting capital upward and downward, respectively. An important special case occurs when $\theta_u = 0$ but $\theta_l = \infty$.[4] This implies that capital is irreversible. When $\theta_u = \theta_l = 0$, the capital stock is perfectly flexible.

To model time-varying volatility in aggregate output, we assume that the dynamics of the production shock process $\{x(t), t \in \mathcal{T}\}$ are given by $x(t+1) = \mu + \sigma(t)\epsilon(t+1)$. Thus, an increase in $\sigma(t)$ implies a mean preserving spread in the output at time $t+1$. The random variable $\epsilon(t)$ is independently and identically distributed over time with mean 0 and, for $t \in \mathcal{T}$, $0 \leq \sigma_{\min} \leq \sigma(t) \leq \sigma_{\max}$ and $|\epsilon(t)| \leq \epsilon_{\max}$. To ensure that the process $\{x(t), t \in \mathcal{T}\}$ and the capital stock process $\{k(t)\ t \in \mathcal{T}\}$ are positive, we assume that $\mu - \sigma_{\max}\epsilon_{\max} > \theta_l/(1 + \theta_l) \geq 0$.

To model an economy that goes through cycles of high-uncertainty epochs and relatively stable times, we assume that $\{\sigma(t), t \in \mathcal{T}\}$ is a positive and mean reverting Markov process.[5] The transition distribution of $\{\sigma(t), t \in \mathcal{T}\}$ is denoted $F(\sigma' | \sigma)$, which equals $\text{Prob}(\sigma(t+1) \leq \sigma' \mid \sigma(t) = \sigma)$. Also, $x(t+1)$ and $\sigma(t+1)$ are assumed to be conditionally independent given $\sigma(t)$. We denote the state space of $\{\sigma(t), t \in \mathcal{T}\}$ as Σ and the state space of $\{x(t), t \in \mathcal{T}\}$ as X.[6]

Given the production function described above, one could interpret the process $\{x(t), t \in \mathcal{T}\}$ as the marginal productivity of capital that changes randomly. It could also be interpreted as an exogenously given random supply of noncapital production inputs. Thus, $\sigma(t)$ measures the volatility at time t of the future productivity of aggregate capital or of the future supply of production inputs. We keep the mean return on real capital constant, and changes in $\sigma(t)$ bring about mean-preserving spreads in future output. In this way, we ensure that the equilibrium effects of changes in risk remain isolated from the effects of changes in the mean.

The firm is financed by one equity share. This share is available for trading in a competitive stock market. The decision problem of the firm is to choose a technologically feasible dividend stream to maximize the market value of its equity share. We denote the ex-dividend price of the share of the firm at time t as $S_0(t)$. Besides the share of the firm, investors can trade in N other zero net supply claims. The cash flow at any date t on each of these claims is exogenously given.

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[5] For $t \in \mathcal{T}$, we take $\mathcal{F}_t$ to be $\sigma(\epsilon(s), \sigma(s), 0 \leq s \leq t) = \sigma(x(s), \sigma(s), 0 \leq s \leq t)$.

[6] We restrict our model to the following two cases. The first case is when Σ is a finite set and $\epsilon(t)$ can take a finite number of values. The second case is when $\Sigma = [\sigma_{\min}, \sigma_{\max}]$, for $t \in \mathcal{T}$, $\epsilon(t) \in [-\epsilon_{\max}, \epsilon_{\max}]$, the probability density of $\epsilon(t)$ exists, and the conditional density of $\sigma(t+1)$ given $\sigma(t) = \sigma$ also exists for every $\sigma \in \Sigma$. In both of these cases, $F(\sigma' | \sigma)$ and the conditional distribution of $\{\epsilon(t+1), \sigma(t+1)\}$ given $\sigma(t) = \sigma$ [also denoted $F(\epsilon', \sigma' | \sigma)$] exist for every $\sigma \in \Sigma$. We denote the conditional distribution functions $F(\cdot \mid \sigma)$ also by $F(\cdot \mid \sigma(t) = s)$.

The consumption good is the numeraire. All security prices are determined in a competitive equilibrium.

1.2 A representative investor with recursive preferences

There exists a representative investor who chooses an optimal consumption stream subject to feasibility constraints.[7] The agent seeks to maximize the utility of her lifetime consumption. We assume that the representative agent has generalized recursive preferences of the type assumed by Epstein and Zin (1989). This specification of preferences permits a partial separation of the elasticity of intertemporal substitution and risk aversion—two distinct aspects of preferences which are confounded in the time-additive expected utility model.[8] Also, our specification permits the modeling of first-order risk aversion. In the following sections, these aspects of preferences are shown to be important in the determination of the price-uncertainty relation.

Specifically, we assume that the utility at date 0 of a feasible consumption sequence $\{c(t), t \in \mathcal{T}\}$ is $J_c(0) \equiv J(\{c(t)\}_{t=0}^{T}, 0)$, which is recursively given by

$$J_c(t) \equiv [c(t)^v + \phi\{m[J_c(t+1) \mid \mathcal{F}_t]\}^v]^{1/v} \tag{2}$$

for $t = 0, 1, 2, \ldots, T-1$ and with $J_c(T) \equiv c(T)$. The random variable $m[J(t+1) \mid \mathcal{F}_t]$ is the certainty equivalent, conditional on the information available at t, of the utility of consumption from $t+1$ onward, $J(t+1)$. The objective of the investor is to maximize $J_c(0)$ over all feasible consumption streams. The certainty equivalent function $m(\cdot)$ provides the information about the agent's risk preferences. The parameters ϕ and v affect the time aggregation of consumption sequences and thus contain information about the agent's time preferences. Specifically, $\kappa \equiv 1/(1-v)$ measures the agent's elasticity of intertemporal substitution. That is, κ indicates whether the agent treats current and future consumption more as substitutes or as complements.

We follow Epstein and Zin (1989, 1990) in the modeling of the certainty equivalent $m(\cdot)$ used in the definition of $J_c(0)$. Formally, the certainty equivalent function is defined for distribution functions of strictly positive random variables. For a given strictly positive random variable $\tilde{x}$, the certainty equivalent $m(\tilde{x})$ is defined implicitly

[7] A consumption plan, $\{c(t), t \in \mathcal{T}\}$, is feasible if it is generated by a predictable and self-financing trading strategy that is budget feasible given the investor's endowment at $t = 0$. In addition, we require that $0 < \epsilon < c(t) \leq K < \infty$, $t \in \mathcal{T}$. The agent's endowments are assumed to equal one share of the firm and none of the zero net supply assets.

[8] However, as noted by Detemple and Selden (1989), a clear separation between these two aspects In the preference specification of Epstein and Zin (1989) depends on the nonstandard notion that risk preferences are defined over random induced value functions rather than over random consumption.

as the solution to the equation $\int \beta[x/m(\tilde{x})]\, dG(x) = 0$, where $\beta(\cdot)$ is a strictly increasing and strictly concave function on $\Re_{++}$, $\beta(1) = 0$, and $G(\cdot)$ denotes the distribution function of $\tilde{x}$.[9] The certainty equivalent function is similarly defined for the distribution of a random variable conditional on a given information set. Thus, $m[J(t+1) \mid \mathcal{F}_t]$ is the certainty equivalent of the conditional distribution of $J(t+1)$ given $\mathcal{F}_t$. It can be formally defined as an $\mathcal{F}_t$-measurable random variable $m(t)$ that satisfies $E[\beta(J(t+1)/m(t)) \mid \mathcal{F}_t] = 0$ with probability 1.[10]

1.3 Equilibrium security prices

Brock (1982) shows that the value-maximizing dividend stream for the representative firm is the one that maximizes the utility of a central planner whose preferences are the same as the representative investor and who faces the same technological restrictions as the firm. Thus, the equilibrium aggregate consumption and investment processes can be obtained by solving the central planner's dynamic optimization problem. Given our assumptions about the economy, the value function for the central planner's problem is a function of the capital stock k, output shock x, production uncertainty σ, and time t. We show in the Appendix that the value function equals $V(k, x, \sigma, t) = kH(x, \sigma, t)$ with $H(\cdot, t)\colon X \times \Sigma \to R$ given recursively by

$$H(x, \sigma, t) = \max_{\theta_t/(1+\theta_t) \le i \le x}[(x - i)^v + \phi g^v(i) h(\sigma, t)]^{1/v} \qquad (3)$$

where $h(\sigma, t) = m^v[H(x(t+1), \sigma(t+1), (t+1) \mid \sigma(t) = \sigma]$ and $m[H(\cdot, t+1) \mid \sigma]$ is the certainty equivalent of the distribution function of $H(\cdot, t+1)$ given $\sigma(t) = \sigma$.[11] The above recursion is subject to the boundary condition, $H(x, \sigma, T) \equiv x$.

Denote the dividend sequence that maximizes the central planner's utility function as $\{\delta_*(t), t \in \mathcal{T}\}$. Then, the equilibrium security prices

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[9] In a static setting, if the preferences of an agent over uncertain outcomes satisfy constant relative-risk aversion, the betweenness axiom, and certain other standard axioms, then these preferences can be represented by the certainty equivalent function $m(\cdot)$, as defined above. Such preferences were originally proposed by Dekel (1986) and Chew (1989). In our numerical analysis, we use the following parameterization of the function $\beta(\cdot)$:

$$\beta(y) = \begin{cases} \dfrac{1}{\gamma}[y^\gamma - 1] & \text{if } y \le 1 \\ \dfrac{A}{\gamma}[y^\gamma - 1] & \text{if } y \ge 1 \end{cases}$$

for $0 < A \le 1$, $\gamma < 0$. When $A = 1$ and $\gamma = v$, we obtain time-additive expected utility with constant relative-risk aversion. When $\gamma \ne v$ (but $A = 1$), we get a special case of Kreps and Porteus' (1978) utility functions, which is used by Epstein and Zin (1992) in an asset-pricing context. When $A < 1$, $m(\cdot)$ displays first-order risk aversion.

[10] Given the boundedness assumptions on feasible consumption plans. such a random variable m(r) exists in the two cases mentioned in note 6.

[11] That is, for $\sigma \in \Sigma$, $m[H(\cdot, t+1) \mid \sigma]$ satisfies $\int \beta(H(\mu + \sigma\epsilon', \sigma', t+1)/m[H(\cdot, t+1) \mid \sigma])\, dF(\epsilon', \sigma' \mid \sigma) = 0$.

in our production economy are those prevailing in a representative agent exchange economy in which the aggregate dividend process is exogenously given to be equal to $\{\delta_\cdot(t), t \in \mathcal{T}\}$. Equations (5-7) of Epstein and Zin (1990) imply that the (cum-dividend) price of the share of the firm is given by

$$S_0(t) + \delta_\cdot(t) = \frac{V^\rho(t)}{\delta_\cdot^{\rho-1}(t)} \tag{4}$$

where, for all $t \in \mathcal{T}$, $V(t)$ is the value function at t of the representative agent (and the central planner) computed using Equation (3). Since there is only one firm that owns the capital stock of the economy, we can interpret the share of the firm as the portfolio of all capital assets in the economy, or the market portfolio.

2. Output Uncertainty and the Value of Aggregate Capital

In this section, we first derive the optimal investment policy of the firm and the equilibrium price of its capital stock. Then we use these results to investigate the effect of changes in risk on the value of the capital stock.

2.1 Value of the capital stock and the optimal investment policy

The aggregate investment and the capital stock processes are endogenous in a production economy. In such an environment, changes in uncertainty affect not only the value of the capital stock but aggregate investment as well. This is seen in the equations for the value of capital stock and the optimal investment policy, which are documented in the following proposition.

Proposition 1. *The cum-dividend price of the firm's share, $S_0(t) + \delta_\cdot(t)$, equals $Q(t)k_\cdot(t)$ where $\{k_\cdot(t), t \in \mathcal{T}\}$ is the capital stock process induced by $\{\delta_\cdot(t), t \in \mathcal{T}\}$ and*

$$Q(t) = \begin{cases} x(t) - \theta_l/(1+\theta_l) & \text{if } x(t) \le L[\sigma(t), \\ x(t) + \theta_u/(1-\theta_u) & \text{if } x(t) \ge U[\sigma(t), \\ [x(t) - 1] + \phi[x(t) - 1]^{1-\rho} b[\sigma(t), t] & \text{otherwise} \end{cases}$$

where $b[\sigma(t), t]$ is recursively defined in Equation (3) and

$$L[\sigma(t), t] = 1 + [\phi(1+\theta_l) b(\sigma(t), t)]^{1/(\rho-1)},$$

$$U[\sigma(t), t] = 1 + [\phi(1-\theta_u) b(\sigma(t), t)]^{1/(\rho-1)}.$$

The optimal investment-to-capital ratio at time t, $i_\cdot(t)$ is given by

$$\dot{\iota}(t) = \begin{cases} A_l[\sigma(t), t]x(t) + B_l[\sigma(t), t] & \text{if } x(t) \leq L[\sigma(t), t] \\ A_u[\sigma(t), t]x(t) - B_u[\sigma(t), t] & \text{if } x(t) \geq U[\sigma(t), t] \\ 1 & \text{otherwise} \end{cases}$$

where

$$A_l[\sigma(t), t] = [(1 + \theta_l)L[\sigma(t), t] - \theta_l]^{-1},$$

$$B_l[\sigma(t), t] = \frac{\theta_l L[\sigma(t), t] - \theta_l}{(1 + \theta_l)L[\sigma(t), t] - \theta_l},$$

$$A_u[\sigma(t), t] = [(1 - \theta_u)U[\sigma(t), t] + \theta_u]^{-1},$$

$$B_u[\sigma(t), t] = \frac{\theta_u U[\sigma(t), t] - \theta_u}{(1 - \theta_u)U[\sigma(t), t] + \theta_u}.$$

Proof. See the Appendix.

The above proposition shows that, in the presence of adjustment costs, the optimal investment policy can be described in terms of an interval whose upper and lower limits depend on the current level of risk. If the current production shock, $x(t)$, lies in the interval, it is optimal not to make any new investment, or to divest. Divestments are optimal when the production shock is below the lower limit $L[\cdot, t]$. New investment takes place when $x(t)$ is above the upper limit $U[\cdot\ , t]$. The region of no new investment or divestment is also the region where the price of the market portfolio in the economy is explicitly a function of output uncertainty. As a result, there is a complementarity between aggregate investments and the value of capital with respect to changes in risk. Only when no investment or divestment takes place can the value of capital respond to changes in uncertainty.

2.2 The effect of capital adjustment costs

We now compare the effect of changes in risk in our model with that documented in the exchange economy models of Abel (1988), Barsky (1989), and Gennotte and Marsh (1993). These models take the aggregate dividend process as given and constrain this process not to respond to changes in risk. Our model reduces to such a situation when $\theta_l = \infty$ and $\theta_u = 1$. Thus, the exchange models can be thought of as models where capital adjustment costs are infinite and the supply of capital is fixed. This implies that these models would tend to overstate the effect of shifts in risk on the price of the firm's capital.

An example of the effect of adjustment costs on the price-uncertainty relation is provided in panel A of Table 1. In this table, we report computations of the solution to our model under a particular

Table 1
The value of the capital stock and changes in output risk

A: The effect of capital adjustment costs

% change in $S_o(0) + \delta_\cdot(0)$ for an increase of .01 in $\sigma(0)$

	Adjustment cost parameters	Level of the output shock $x(0)$					
		1.005	1.015	1.025	1.045	1.055	1.065
	$\theta_u = 1, \theta_l = \infty$	−.60	−.59	−.58	−.58	−.58	−.58
	$\theta_u = 0, \theta_l = \infty$	−.22	−.22	0	0	0	0
	$\theta_u = 0.5, \theta_l = 0.5$	0	−.53	−.53	−.53	−.52	−.52
$\nu = 0.5$	$\theta_u = 0.25, \theta_l = 0.25$	0	0	−.34	−.34	0	0
	$\theta_u = 1, \theta_l = \infty$	11.3	12.5	12.6	12.7	12.7	12.7
	$\theta_u = 0, \theta_l = \infty$	14.9	17.4	0	0	0	0
	$\theta_u = 0.5, \theta_l = 0.5$	0	0	2.4	0	0	0
$\nu = -1.5$	$\theta_u = 0.25, \theta_l = 0.25$	0	0	0	0	0	0

B: The effect of risk aversion

% change in $S_o(0) + \delta_\cdot(0)$ for an increase of .01 in $\sigma(0)$

	Preference parameters	Level of the output shock $x(0)$					
		1.005	1.015	1.025	1.045	1.055	1.065
	$\gamma = 0.5, A = 1$	−.43	−.43	−.43	0	0	0
	$\gamma = -1.5, A = 1$	−.51	−.51	−.51	0	0	0
	$\gamma = 0.5, A = 0.3$	−1.8	−1.7	−1.7	−1.7	0	0
$\nu = 0.5$	$\gamma = -1.5, A = 0.3$	−1.9	−1.9	−1.8	−1.8	−1.2	0
	$\gamma = 0.5, A = 1$	2.3	2.3	2.4	0	0	0
	$\gamma = -1.5, A = 1$	2.5	2.5	2.5	0	0	0
	$\gamma = 0.5, A = 0.3$	3.0	3.1	0	0	0	0
$\nu = -0.5$	$\gamma = -1.5, A = 0.3$	3.0	3.1	0	0	0	0

Panel A of this table provides the percentage change in the value of capital stock at $t = 0$, $S_o(0) + \delta_\cdot(0)$, as $\sigma(0)$ is increased from its long run mean by .01. The computations are reported under various assumptions for the level of capital adjustment costs. The representative agent is assumed to maximize time-additive expected utility of her lifetime consumption. Thus, $1 - \nu$ is the coefficient of the agent's relative risk aversion. The time discount factor ϕ is .975. The mean of the output shocks is 1.035, for all t, $\epsilon(t)$ can take values +l or -1 with equal probability and $\Sigma = \{.01, .02, .03\}$.

The transition matrix of $\{\sigma(t), t \in T\}$ is

$$\begin{bmatrix} \alpha & 1-\alpha & 0 \\ \frac{(1-\alpha)}{2} & \alpha & \frac{1-\alpha)}{2} \\ 0 & 1-\alpha & \alpha \end{bmatrix}$$

with $\alpha = .7$. The long run mean of $\sigma(t) = .02$.

Panel B of the table provides the percentage change in the value of capital stock at $t = 0$, $S_o(0) + \delta_\cdot(0)$, as $\sigma(0)$ is increased from its long run mean by .01. The computations are reported under various assumptions for the certainty equivalent function m(·). When $\nu = \gamma$, the utility function is given by time-additive expected utility with constant relative risk aversion. When $\nu \neq \gamma$, the utility function corresponds to that modeled in Epstein and Zin (1992). When $A < 1$, the certainty equivalent function displays first-order risk aversion. The adjustment cost parameters are $\theta_u = .2$ and $\theta_l = \infty$. Other parameters arc as in panel A.

parameterization. The entries in the table denote the percentage change in $S_o(0) + \delta_\cdot(0)$ for an increase in $\sigma(0)$ of 0.01, for fixed $x(0)$.

From panel A of Table 1, we see that the response of the value of the capital stock to shifts in risk depends crucially on θ_u and θ_l. When capital supply is fixed because of extreme adjustment costs (rows 1, 2, 5, and 6 of panel A in Table 1), the effect of changes in $\sigma(0)$ is quite large. When adjustment costs are lower, the interval of no new investments is smaller. As a result the price is not sensitive to changes in σ (0) for many values of $x(0)$. As adjustment costs fall, even for values of $x(0)$ that continue to remain in the region of no new investments, the sensitivity of $S_o(0) + \delta_*(0)$ to shifts in uncertainty decreases.

Notice that $Q(t)$, as defined in Proposition 1, can be interpreted as the equilibrium *q-ratio* at time t of the representative firm. This is because $Q(t)$ is the ratio of the market value of the firm to the number of units of the numeraire needed to replace the physical units of the installed capital. Consequently, the above discussion on the effect of capital adjustment costs is important in the interpretation of the stock market decline in the U.S. economy in the 1970s. Malkiel (1979) argues that the *q-ratio* of corporate assets fell from around 1.25 in the mid 1960s to about 0.75 in 1977 as a result of an increase in uncertainty.[12] Our results suggest that we need to assume that capital adjustment costs were quite significant during this episode of price decline before attributing it to an increase in risk.

2.3 The role of the elasticity of intertemporal substitution

It is also seen in panel A of Table 1 that the effect of changes in σ on $S_o(t) + \delta_*(t)$ are positive when $\kappa \equiv 1/(1 - v) < 1$ and vice versa for $\kappa > 1$. This suggests that the elasticity of substitution determines the direction of the effect of changes in risk on the value of capital stock. The following proposition shows that this conjecture is indeed correct.

***Proposition* 2.** *Suppose that (i)* q_u *and* q_l *are not both equal to zero, (ii) for* $\sigma_1 > \sigma_0$, $F(\sigma' \mid \sigma_1)$ *stochastically dominates* $F(\sigma' \mid \sigma_0)$ *in the first order, and (iii)* $x(t+1)$ *and* $s(t+1)$ *are conditionally independent given* $s(t)$. *If* $K = 1/(1 - v) > (<) 1$, *then the q-ratio of the aggregate capital stock,* $Q(t)$, *is decreasing (increasing) in* $s(t)$ *for fixed* $x(t)$. *The same conclusions hold for the equilibrium investment-to-capital ratio,* $i_*(t)$, *and for the price-to-dividend ratio of aggregate capital.*

Proof. See the Appendix.

The intuition for the response of $Q(t)$ and $i_*(t)$ to changes in risk in the above proposition is simple. It is related to the results of Selden (1979). In a static setting, Selden shows that the elasticity of intertemporal substitution, as opposed to risk aversion, determines whether an agent desires to save more as uncertainty increases. Thus, when $\kappa < 1$ in our model, there is an increase in investment with an increase

[12] The behavior of the NYSE index, the Dow Jones index, and the price-earning ratio of the S&P 500 portfolio has also been cited in support of this market decline.

in risk. However, with capital adjustment costs, it is not optimal to increase investment as much as in the zero adjustment cost case. Thus, increases in risk lead to a rise in the demand for capital goods, but adjustment costs impede an offsetting increase in their supply. Thus, the price of the capital good or of the firm that owns it also increases in equilibrium. A similar argument holds when $\kappa > 1$.[13]

Proposition 2 can also be related to the analysis of Malkiel (1979). As stated before, Malkiel argues that an increase in the risk in the U.S. economy caused a sluggishness in capital formation and a decline in q-ratios in the U.S. economy in the 1970s. Holding the productivity of capital constant, *q-ratios* and investment-to-capital ratios across regimes with different levels of risk would exhibit this pattern in our model when $\kappa > 1$. For $\kappa < 1$, we should expect exactly the opposite pattern. Empirical estimates of κ have been provided in Epstein and Zin (1992). These estimates invariably tend to be less than one. Given these estimates, Malkiel's conclusions about the effect of changes in risk on the value of aggregate capital and aggregate investments are the opposite of the predictions of our model.

2.4 Risk aversion and the effect of changes in risk

Proposition 2 shows that risk aversion (or, more generally, the nature of the function $m(\cdot)$) does not influence the direction of the effect of changes in risk on the value of capital. The certainty equivalent function, however, has an important influence on the magnitude of this effect. This can be seen from panel B of Table 1. It is evident here that, for higher levels of risk aversion $(1 - \gamma)$, the change in $S_o(0) + \delta_*(0)$ caused by shifts in $\sigma(0)$ is greater, especially with first-order risk aversion $(A < 1)$.

2.5 Average *q-ratios* and average price-to-dividend ratios

It is important to remember that in the above analysis, which considers the effect of changes in risk on asset prices and investments, the productivity of capital is held fixed. However, changes over time in the investment-to-capital ratio, the *q-ratio*, and the price-to-dividend ratio for aggregate capital come from changes in risk as well as from changing productivity of capital. Consequently, to detect the effect of shifts in risk in an econometric analysis, it is necessary to know how such shifts alter the time series behavior of prices and investment. Our analysis reveals that the stochastic behavior of the q-ratio is significantly different in regimes with different levels of risk.

In Table 2 (columns 2-4), we report the expected value of *Q(t +*

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[13] The stochastic dominance condition on $F(\cdot \mid \cdot)$ ensures that increases in risk are sufficiently persistent. Our numerical analysis reveals that this condition is sufficient but not necessary.

Table 2
Output risk, average q-ratios, and average price dividend ratios

(θ_u, θ_l)	Average q-ratio $\sigma = .01$	$\sigma = .02$	$\sigma = .03$	Average investment-to-capital ratio $\sigma = .01$	$\sigma = .02$	$\sigma = .03$	Average price-to-dividend ratio $\sigma = .01$	$\sigma = .02$	$\sigma = .03$
A: $\nu = .5$ $(\kappa = 2)$									
$\theta_u = 0$ $\theta_l = \infty$	1.04	1.01	0.81	1.02	1.02	1.02	60.2	62.0	84.7
$\theta_u = .1$ $\theta_l = .1$	1.15	1.05	1.05	1.01	1.02	1.02	50.1	55.5	55.3
$\theta_u = .25$ $\theta_l = .25$	1.26	1.13	1.10	1.00	1.01	1.01	41.8	48.2	49.9
$\theta_u = .5$ $\theta_l = .5$	1.32	1.27	1.24	1.00	1.00	1.00	39.2	43.8	49.7
B: $\nu = -.5$ $(\kappa = .67)$									
$\theta_u = 0$ $\theta_l = \infty$	0.99	0.75	0.58	1.01	1.01	1.02	38.2	34.4	28.8
$\theta_u = .1$ $\theta_l = .1$	1.05	1.05	1.05	1.01	1.01	1.01	36.7	36.8	36.8
$\theta_u = .25$ $\theta_l = .25$	1.14	1.10	1.10	1.01	1.01	1.01	38.7	38.3	38.4
$\theta_u = .5$ $\theta_l = .5$	1.50	1.37	1.37	1.00	1.00	1.00	43.5	41.8	42.0

This table provides the average q-ratios and the average price-to-dividend ratios for the aggregate capital stock, in regimes with different levels of output risk. The average is taken over realizations of output shocks $\{x(t)\}$ within a given regime. The representative agent is assumed to maximize time-additive expected unitily of her lifetime consumption. Thus, $1 - \nu$ is the coefficient of the agent's relative risk aversion. The parameter $\kappa = 1/(1 - \nu)$. The time discount factor ϕ is .975. The mean of the output shocks is 1.035; for all t, $\epsilon(t)$ can take values + 1 or -1 with equal probability and $\Sigma = \{.01, .02, .03\}$. The transition matrix of $\{\sigma(t), t \in T\}$ is $\begin{bmatrix} \alpha & 1-\alpha & 0 \\ \frac{(1-\alpha)}{2} & \alpha & \frac{(1-\alpha)}{2} \\ 0 & 1-\alpha & \alpha \end{bmatrix}$ with $\alpha = .7$. The long-run mean of $\sigma(t) = .02$.

1) given that $\sigma(t) = \sigma(t + 1) = \sigma$. These values can be interpreted as the mean q-ratios in regimes with different levels of risk. It is seen that changes in risk can change the mean of the *q-ratios* considerably. Also, there is an inverse relation between mean *q-ratios* and uncertainty. In the following proposition, we prove this result for an important special case, the case where capital formation is irreversible.

Proposition 3. *Let the hypotheses (i), (ii), and (iii) of Proposition 2 bold. Also, assume that capital formation is irreversible* $(\theta_l = \infty)$ *and* $\kappa \equiv 1/(1 - \nu) > 1$. *Let* $F(\epsilon')$ *be the distribution function of* $\epsilon(t)$, $t \in \mathcal{T}$. *If* $\sigma < \sigma'$, *then*

$$\int Q(\mu + \sigma\epsilon', \sigma, t)\, dF(\epsilon') \geq \int Q(\mu + \sigma'\epsilon', \sigma', t)\, dF(\epsilon')$$

Proof. The proposition follows from the facts that $Q(\cdot)$ is concave in x and decreasing in σ under the stated restrictions.

Our numerical calculations in Table 2 indicate that the conclusions of Proposition **3** may hold even when $\kappa \equiv 1/(1 - \nu) < 1$ and for adjustment cost parameters not covered by the above proposition. Table 2 also shows that the mean of the price-to-dividend ratio may also be affected significantly by the level of risk in the economy, especially for large values of κ. When κ is small, the relation between risk and the average price-to-dividend ratio seems to be weak. Moreover, this relation may not even be monotonic. The average investment-to-capital ratio is also not very sensitive to σ. It may also be seen that, even for parameter values where $i.(t)$ is decreasing in σ, its expected value conditional on the level of risk may be increasing in risk.

Propositions 2 and 3 above provide guidelines for the selection of proxies for aggregate economic uncertainty. Our results suggest that the *q-ratio* for the aggregate capital stock, its price-to-dividend ratio, and the investment-to-capital ratio could all proxy for aggregate risk in the economy. Whether or not these ratios are good proxies for risk depends on whether or not the marginal productivity of capital is held fixed when comparing these ratios across regimes. If the productivity of capital is controlled for, then all of these ratios are monotonic (and given the available estimates of κ, increasing) functions of σ. If the productivity of capital is not controlled for, then the average *q-ratio* for aggregate capital seems to be a robust proxy for economic risk. However, the average price-to-dividend ratio and the average investment-to-capital ratio do not seem to have this property.

3. Levered Claims on Capital Stock and Time-Varying Risk

It is often argued that increases in risk generate "flights to safety," leading to a fall in equity prices and an increase in the prices of high-grade bonds. However, the analysis of the previous section has shown that, for reasonable savings behavior by the agents, the firm value may actually increase as uncertainty increases. Here, we examine if this response of the value of aggregate capital can be reconciled with the "flight to safety" argument, which suggests exactly the opposite effect. We show below that a reconciliation can be obtained by taking into account the fact that, in the presence of corporate debt, equity shares are levered claims on the firm's capital stock.

Equity claims may be viewed as in-the-money call options on the firm's capital stock. Our analysis below shows that the prices of such in-the-money call options are likely to fall with an increase in the uncertainty in the marginal product of capital. This might seem counter-intuitive, given the well-known direct relation between option prices and the variance of returns on the underlying asset. However, this direct relation is contingent on the interest rate remaining constant. In an equilibrium model like the present one, an increase in the variance of the marginal product of capital reduces the riskless rates of interest. This endogenous adjustment of the term structure is what lends validity to the "flight to safety" argument.

To cast the above discussion in the present framework, suppose that one of the zero net supply claims in the economy has cash flows that resemble the cash flows on equity shares. Specifically, the claim yields a cash flow of $\max[\delta.(t) - \eta k(t), 0]$ for some constant η, for every $t \in \mathcal{T}$. The constant η measures the extent to which the firm is levered. In our model, for values of η low enough to be consistent with equity claims being in-the-money claims, increases in risk lead to a fall in equity prices. The example in Table 3 makes this idea clear. This example is of a static economy with an investor with logarithmic utility. Table 3 shows that, as σ increases, the payoff on levered equity increases in the good state but decreases in the bad state. However, the value of an additional unit of the consumption good in the bad state is much higher than the value of an additional unit in the good state. Hence, the increase in the value of the levered equity, because of the higher payoff in the good state, is less than the decrease in its value resulting from a lower payoff in the bad state. This leads to a lower price for the levered equity share when σ is high.

Our numerical analysis (not reported) shows that, when the leverage parameter η is not too high, the conclusions of Example 1 remain true in a dynamic set-up with adjustment costs and recursive preferences. Output risk and the value of the levered claim on capital may be inversely related even if the underlying asset's price is an increasing function of risk. For high values of η, the usual direct relation between risk and values of levered equity is observed. Our calculations also show that higher levels of risk aversion lead to a higher absolute value of the elasticity of the price of levered equity with respect to changes in risk.

4. Conclusions

We develop a general equilibrium model to analyze the relation between the prices of aggregate assets and the uncertainty in the production process. We identify the elasticity of the supply of capital and the elasticity of intertemporal substitution as the key determinants of the price-uncertainty relation. We use our model to analyze the argument of Malkiel (1979) and Pindyck (1984) that increases in risk led to the decline in aggregate equity values in the 1970s. Given

Table 3
An example of the effect of changes in output risk on the value of the levered equity

	1 Time-state	2 $k(t)$	3 $\delta.(t)$	4 Equity payoff $\max[\delta.(t) - \eta k(t), 0]$	5 State price	6 4×5
			Degree of leverage: $\eta = .5$			
$\sigma(0) = .2$	0	1	.5	0	1	0
	(1, u)	0.5	.6	.35	0.5/1.2	.15
	(1, d)	0.5	.4	.15	0.5/0.8	.09
Price at $t = 0$						.24
$\sigma(0) = .4$	0	1	.5	0	1	0
	(1, u)	0.5	.7	.45	0.5/1.4	.16
	(1, d)	0.5	.3	.05	0.5/0.6	.04
Price at $t = 0$						.20

This table contains a numerical example to show that the value of a levered equity type claim an be a decreasing function of the output risk in a general equilibrium model. Suppose $\gamma = \nu = 0$ (expected utility framework with logarithmic preferences). Let T *equal* 1 (a static economy). Let $k(0)$, $x(0)$, ϕ, the time discount factor, and μ all equal 1. Let $x(1) = [\mu + \sigma(0)\epsilon(1)]$ where $\epsilon(1)$ can take values 1 or - 1 with equal probabilities. Let $\theta_l = \theta_u = 0$ (no adjustment costs). These parameter values imply that $\delta.(0) = I(0) = k(1) = .5$ and $\delta.(1) = k(1)x(1)$. Time-state (1, u) refers to the case when $\epsilon(1) = 1$ and time-state (1, d) refers to the case when $\epsilon(1) = -1$.

the available estimates of the elasticity of intertemporal substitution, our analysis yields the opposite conclusion, if we interpret their argument as being applicable to the value of the market portfolio. However, if we view a broad-based equity portfolio as a levered claim on the aggregate capital stock, then our analysis shows that the effects predicted by Malkiel and Pindyck are indeed consistent with a general equilibrium model.

Appendix

1.1 Dynamic programming solution to the central planner's problem

Because of the recursive structure of preferences, we can use dynamic programming to find the utility-maximizing dividend sequence $\{\delta.(t), t \in \mathcal{T}\}$. The dynamic programming solution is described here. Let $V(k, x, \sigma, T) \equiv kx$. For $t = 0,\ldots,T-1$, define a family of functions $V(\cdot, t)\colon (0, \infty) \times X \times \Sigma \rightarrow \Re$ recursively as follows:

$$V(k, x, \sigma, t) = \max_{\theta_l/(1+\theta_l)k \le I \le kx}[[Akx - I]^\rho + \phi M^\rho(k, I, \sigma, t)]^{1/\rho} \quad \text{(A1)}$$

where $M(\cdot, t)$ is equal to $a \in \Re$ such that $\int \beta[V(k', \mu + \sigma\epsilon', \sigma', t+1)/a]\, dF(\epsilon', \sigma' \mid \sigma) = 0$ with $k' = kg[I/k]$. It can be checked by backward substitution that, for all t, $V(k, x, \sigma, t)$ is of the form $H(x, s, t) \times k$ where $H(\cdot, t)$ is recursively defined in Equation (3).[14]

Given the above, it can be verified that $V(k(0), 0) \geq J(0, \{\gamma(t), t \in \mathcal{T}\})$ for an arbitrary feasible dividend sequence $\{\gamma(t), t \in \mathcal{T}\}$ and that $V(k(0),0) = J(0, \{\delta_*(t), t \in \mathcal{T}\})$. This implies that $\{\delta_*(t), t \in \mathcal{T}\}$ is the consumption policy that yields the maximum time 0 utility value of $J_C(0)$. It is also true that the utility from following the policy $\{\delta_*(t), t \in \mathcal{T}\}$ from time t onward, $J_C(t)$, is equal to $V[k_*(t), t]$, where $\{k_*(t), t \in \mathcal{T}\}$ denotes the capital stock process induced by $\{\delta_*(t), t \in \mathcal{T}\}$ and $V(\cdot,\cdot,\cdot)$ is defined above in Equation (A1). Also, if the state space of $\{x(t), t \in \mathcal{T}\}$ and $\{\sigma(t), t \in \mathcal{T}\}$ is finite, then $\{\delta_*(t), t \in \mathcal{T}\}$ is bounded above and bounded below by some $\epsilon > 0$. Under the hypotheses of Proposition 2, this conclusion is also true when the state space of the underlying state variables is infinite.

1.2 Proof of Proposition 1

For a given t, consider the maximization problem defining $H(\cdot, t)$ in Equation (3). The solution to this problem provides the optimal investment-to-capital ratio at time t, as a function of the current values of the state variables. Recall that the capital accumulation function $g(\cdot)$ is not differentiable at 1. Thus, the optimality conditions for the problem defining $H(\cdot, t)$ are as follows. If $\phi b(\sigma, t)(1 - \theta_u) \leq (x - 1)^{\gamma-1} \leq \phi b(\sigma, t)(1 + \theta_l)$, then $i_*(x, \sigma, t) = 1$.[15] The functions $U(\cdot, t)$ and $L(\cdot, t)$ are obtained by rewriting these inequalities. If $\phi b(\sigma, t)(1 - \theta_u) \geq (x - 1)^{\gamma-1}$, then the first-order condition holds as an equality. Then, $i_*(t)$ is determined as the solution to the following equation:

$$(x - i)^{\gamma-1} = \phi b(\sigma, t)(1 - \theta_u)[i - \theta_u(i - 1)]^{\gamma-1} \tag{A2}$$

Similarly, when $\phi b(\sigma, t)(1 + \theta_l) \leq (x - 1)^{\gamma-1}$, the optimum investment-to-capital ratio is given by the solution to the following equation:

$$(x - i)^{\gamma-1} = \phi b(\sigma, t)(1 + \theta_l)[i - \theta_l(1 - i)]^{\gamma-1} \tag{A3}$$

The expression for $i_*(t)$ is obtained using Equations (A2) and (A3). The expression for $Q(t)$ is obtained by using Equation (4) and the first-order conditions stated above.

1.3 A preliminary result

We first prove a lemma that is needed subsequently.

[14] The constraint that $I \geq [\theta_l/(1 + \theta_l)]k$ i in Equation (A1) is a technological constraint. It ensures that the capital stock process remains positive at all times.

[15] This is the case where the right-hand side of Equation (3) is maximized at 1, since the above-mentioned inequalities ensure that the left derivative of the maximand at $i = 1$ is positive and its right derivative at $i = 1$ is negative.

Lemma 1. *Under the hypotheses of Proposition 2, $H(x, s, t)$ defined by Equation (3) is increasing and concave in its first argument and decreasing in its second argument for all $t \in \mathcal{T}$.*

Proof: The proof is by induction. Assume, as the induction hypothesis, that $H(x, \sigma, t + 1)$ is decreasing in σ and increasing and concave in x for some $t \leq T - 1$. Standard arguments imply that $H(x, \sigma, t)$ is increasing and concave in x. The function $H(x, \sigma, t)$ is decreasing in σ if $a(s, t) \equiv m[H(x(t + 1), \sigma(t + 1), t + 1) \mid \sigma(t) = \sigma]$ is decreasing in σ. The following arguments show that $\alpha(\cdot, t)$ has this property. Let $\sigma_l < \sigma_b$. We show that $a(\sigma_b, t) \leq a(\sigma_l, t)$ if the induction hypothesis is true. For any given $(\sigma, \sigma') \in \Sigma \times \Sigma$, define $Z(\sigma, \sigma', t + 1) = \int \beta[H(\mu + \sigma\epsilon', \sigma', t + 1)/a(\sigma_l, t)]\, dF(\epsilon')$.[16] The function $b(\cdot)$ is strictly increasing and concave and, by the induction hypothesis, $H(x, \sigma, t + 1)$ is concave in x. Thus, $\beta[H(x, \sigma', t + 1)/a(\sigma_l, t)]$ is a concave function of x. This implies that $Z(\sigma_l, \sigma', t + 1) \geq Z(\sigma_b, \sigma', t + 1)$ because an increase in σ implies a mean preserving spread in $x(t + 1) = \mu + \sigma\epsilon(t + 1)$. This implies that $\int Z(\sigma_l, \sigma', t + 1)\, dF(\sigma' \mid \sigma_l) \geq \int Z(\sigma_b, \sigma', t + 1)\, dF(\sigma' \mid \sigma_l)$ where the required integrals exist and are finite because $|Z(\cdot, \sigma', t + 1)| < \infty$. Moreover, as $b(\cdot)$ is increasing and $H(x, \sigma, t + 1)$ is decreasing in its second argument by the induction hypothesis, we get $\int Z(\sigma_b, \sigma', t + 1)\, dF(\sigma' \mid \sigma_l) \geq \int Z(\sigma_b, \sigma', t + 1)\, dF(\sigma' \mid \sigma_b)$ by the stochastic dominance condition. Since $\sigma(t + 1)$ and $\epsilon(t + 1)$ are conditionally independent given $\sigma(t)$, we have that

$$\int Z(\sigma, \sigma', t + 1)\, dF(\sigma' \mid \sigma) = \int \beta\left[\frac{H(\mu + \sigma(t)\epsilon', \sigma', t + 1)}{a(\sigma_l, t)}\right] dF(\epsilon', \sigma' \mid \sigma(t) = \sigma)$$

Thus, the above calculations imply that

$$\int \beta\left[\frac{H(x', \sigma', t + 1)}{a(\sigma_l, t)}\right] dF(\epsilon', \sigma' \mid \sigma(t) = \sigma_l) \geq \int \beta\left[\frac{H(x', \sigma', t + 1)}{a(\sigma_l, t)}\right] dF(\epsilon', \sigma' \mid \sigma(t) = \sigma_b)$$

with $x' \equiv \mu + \sigma(t)\epsilon'$. However, by the definition of $m(\cdot)$, the left-hand side of the above inequality equals $0 = \int \beta[H(x', \sigma', t +$

[16] Note that, since (i) $|\epsilon(t)| \leq \epsilon_{max}$, (ii) $\mu - \sigma_{max}\epsilon_{max} > 0$, and (iii) $H(\cdot, t + 1)$ is increasing in its first argument, $|\beta[H(\mu + \sigma\epsilon', \sigma', t + 1)/a(\sigma_l, t)]|$ is less than

$$\max\left\{\beta\left[\frac{H(\mu + \sigma\epsilon_{max}, \sigma', t + 1)}{a(\sigma_l, t)}\right], -\beta\left[\frac{H(\mu - \sigma\epsilon_{max}, \sigma', t + 1)}{a(\sigma_l, t)}\right]\right\} < \infty.$$

Thus, $|Z(\sigma, \sigma', t + 1)| < \infty$.

$1)/a(\sigma_b, t)]\, dF(\epsilon', \sigma' \mid \sigma(t) = \sigma_b)$. Thus, we get that 0 equals

$$\int \beta\left[\frac{H(x', \sigma', t+1)}{a(\sigma_b, t)}\right] dF(\epsilon', \sigma' \mid \sigma(t) = \sigma_b)$$

$$\geq \int \beta\left[\frac{H(x', \sigma', t+1)}{a(\sigma_l, t)}\right] dF(\epsilon', \sigma' \mid \sigma(t) = \sigma_b)$$

with $x' \equiv \mu + \sigma(t)\epsilon'$. This implies that $a(\sigma_l, t) > a(\sigma_b, t)$. Therefore, if the induction hypothesis is true for $t + 1$, then it is true for t. To start the induction, by the boundary condition on $H(\cdot, t)$, the induction hypothesis is clearly true for $t = T$.

1.4 Proof of Proposition 2

Let $v \geq 0$ and let $\sigma_b > \sigma_l$. Fix $x(t) = x$. We show that $Q(x, \sigma_l, t) \geq Q(x, \sigma_h, t)$. Recall that the function $\alpha(\sigma, t)$ defined in the proof of Lemma 1 is decreasing in σ. This implies that when $v \geq 0$, $L(\sigma, t)$ and $U(\sigma, t)$ (defined in Proposition 1) are increasing in σ. This is because the function $h(\sigma, t)$ used in the definitions of $U(\cdot)$ and $L(\cdot)$ equals $\alpha(\sigma, t)$. This implies that one of following two sets of inequalities must be true: (i) Case 1: $L(\sigma_l, t) \leq L(\sigma_b, t) \leq U(\sigma_l, t) \leq U(\sigma_b, t)$ or (ii) Case 2: $L(\sigma_l, t) \leq U(\sigma_l, t) \leq L(\sigma_b, t) \leq U(\sigma_b, t)$. Suppose first that Case 1 holds. Then, there are five possibilities, which we consider now: (i) Suppose $x \leq L(\sigma_l, t)$. Then $Q(x, \sigma_l, t) = Q(x, \sigma_b, t) = x - \theta_l/(1 + \theta_l)$. (ii) Suppose $x \in (a, b)$ where $a = L(\sigma_l, t)$ and $b = L(\sigma_b, t)$. Note that $Q(a, \sigma_b, t) = Q(a, \sigma_l, t) = a - \theta_l/(1 + \theta_l)$. Moreover, $Q_1(x, \sigma_b, t) = 1$ and $Q_1(x, \sigma_l, t) \geq 1$ on (a, b) where $Q_1(\cdot)$ denotes the derivatives of Q with respect to its first argument. This implies that $Q(x, \sigma_b, t) \leq Q(x, \sigma_l, t)$ on (a, b). (iii) Suppose $x \in [a, b]$ where $a = L(\sigma_h, t)$ and $b = U(\sigma_l, t)$. Then, $Q(x, \sigma_h, t) = (x - 1) + \phi(x - 1)^{1-v} h(\sigma_b, t)$ and $Q(x, \sigma_l, t) = (x - 1) + \phi(x - 1)^{1-v} h(\sigma_l, t)$. Thus, $Q(x, \sigma_b, t) \leq Q(x, \sigma_l, t)$ because $h(\cdot, t)$ is decreasing in σ from Lemma 1 and the fact that $v \geq 0$. (iv) Suppose that $x \in (a, b)$ where $a = U(\sigma_l, t)$ and $b = U(\sigma_h, t)$. In this range, $Q(x, \sigma_h, t) = (x - 1) + \phi(x - 1)^{1-v} h(\sigma_b, t)$ but $Q(x, \sigma_l, t) = x + \theta_u/(1 - \theta_u)$. Recall that if $\sigma(t) = \sigma_h$ then this range is in the interval of no new investment. The optimality conditions stated in Section 1.2 of this Appendix imply that, for x in this range, $\phi(1 - \theta_u) h(\sigma_b, t) \leq (x - 1)^{v-1}$. This means that, for $x \in (a, b)$, $Q(x, \sigma_b, t) \leq Q(x, \sigma_l, t) = x + \theta_u/(1 - \theta_u)$. (v) If $x \geq U(\sigma_h, t)$ then $x, \sigma_b, t) = Q(x, \sigma_l, t) = x + \theta_u/(1 - \theta_u)$. Thus, we obtain that if Case 1 holds, $v \geq 0$, and $\sigma_l < \sigma_b$, then $Q(x, \sigma_b, t) \leq Q(x, \sigma_l, t)$. The arguments are similar if Case 2 holds and when $v \leq 0$. The conclusions about $i.(t)$ and the price-to-dividend ratio can be obtained similarly.

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