Optimal Design of Securities under Asymmetric Information

David C. Nachman
Thomas H. Noe
Georgia State University

A firm must decide what security to sell to raise external capital to finance a profitable investment opportunity. There is ex ante asymmetry of information regarding the probability distribution of cash flow generated by the investment. In this setting we derive necessary and sufficient conditions for a security to be optimal (uniquely optimal), that is, for pooling at this security to be an (the unique) equilibrium outcome. Using these conditions we show that the debt contract is (uniquely) optimal if and only if cash flows are ordered by (strict) conditional stochastic dominance. Finally, we derive an equivalence relationship between optimal security designs and designs that minimize mispricing.

Consider a firm with a profitable investment opportunity that must raise external capital to finance the

---

Some of the subject matter of this paper is taken from the earlier paper, Nachman and Noe (1990). This paper was presented at workshops at Georgia Institute of Technology, University of British Columbia, McGill University, New York University, University of Pittsburgh, Columbia University, Washington University. Northwestern University, Indian Statistical Institute, New Delhi, and at the 1990 WFA Meeting in Santa Barbara. The authors are grateful to participants in those workshops for helpful discussions and especially to Larry Wall, Ron Giammarino, Art Moreau, Kose John, Larry Glosten, Phil Dybvig. Ronen Israel, and Artur Raviv. We would like to extend special thanks to the editor, Richard Green, and to an anonymous referee for their insightful comments that have greatly improved the exposition. Needless to say, we own all remaining errors and confusion. Thomas H. Noe acknowledges support from the Georgia State Business Research Council. Please address all correspondence and reprint requests to David C. Nachman, College of Business Administration, Department of Finance, Georgia State University, Atlanta, GA 30303-3083.

The Review of Financial Studies Spring 1994 Vol. 7. No. 1, pp. 144

investment. What securities should the firm sell to raise the required capital? This is the fundamental capital structure question reformulated (albeit loosely) as a security design problem. In the presence of asymmetric information between firms and suppliers of capital, firms raising external capital face an adverse selection problem in that the security designs of firms with highly productive investment opportunities may be imitated by those firms with investment opportunities of lower productivity, resulting in undervaluation of higher productivity firms and overvaluation of lower productivity firms. This adverse selection problem is the focus of this paper.[1]

The extent of misvaluations due to adverse selection may well depend on the characteristics of the securities issued by the firm. The literature on the raising of external capital in this context points to something fundamental about the debt contract design. Indeed, Myers and Majluf (1984) and Myers (1984) have argued that under asymmetric information, firms that are forced to raise external capital will prefer to issue claims with explicit promised payments (i.e., debt claims), as opposed to selling fractional claims to residual cash flows (i.e., equity financing). They argue, by appealing to option pricing theory, that the value of a debt claim is less sensitive to the characteristics of the firm's cash flows than is equity, and thus debt financing will minimize the underpricing losses from security issuance. Thus, according to this line of thought, issuing debt is "optimal" either in the weak sense that financing using debt is an equilibrium outcome or in the stronger sense that issuing debt is the only equilibrium outcome. As a consequence, the optimality of debt has been elevated in discussions of the adverse selection problem to something of a finance folklore proposition, namely that debt, in some way, minimizes the adverse selection problem of raising external capital.

There are some difficulties with this proposition. The first problem is that only a limited menu of security designs have been considered, essentially debt and equity. With such narrow choice sets, claims of optimality are weak. This is especially true given the recent proliferation of innovative security designs that have greatly expanded the menu of financing choices available to firms.

The second and perhaps the most consequential difficulty with the

[1] Alternative focuses to the optimal design of securities include the work of Allen and Gale (1988) and Harris and Raviv (1989). Allen and Gale (1988) analyze the problem of how to divide the cash flow stream from a productive investment between claimholders in the presence of incomplete markets and transactions costs for issuing new securities. In this setting, standard debt and equity contracts are never optimal securities when investors have smooth preferences. However, Madan and Soubra (1991) show that when the firm faces some uncertainty regarding the preferences of the investor set, nonextremal contracts may be optimal. Harris and Raviv (1989) look at the problem of the optimal assignment of voting rights and sharing of cash flows in the context of a conflict of interest between incumbent management and outside investors arising from private benefits from corporate control.

proposition is that Myers and Majluf's (1984, Section 3.3) argument regarding the minimization of underpricing ignores the inferences market participants may draw regarding the productivity of the issuers of various securities. As is typical in situations of asymmetric information, however, the market's beliefs regarding the productivity of the issuing firm are part of the equilibrium outcome and cannot be fixed exogenously. In particular, the equilibrium outcome depends critically on the market's beliefs regarding productivity types for securities not issued in equilibrium, so-called off-equilibrium beliefs. The somewhat arbitrary nature of off-equilibrium beliefs gives rise to a serious multiplicity of equilibria. Indeed, by utilizing the "right" off-equilibrium beliefs, virtually any pattern of security issuance can be the outcome of a Nash equilibrium.

One consequence of overlooking the endogenous nature of the market's beliefs is noted in Noe (1988, Example 2), where there exist equilibria in the Myers and Majluf setting in which both debt and equity are issued in equilibrium. In this example, there are three possible firm types ordered by productivity in which the high and the low productivity types pool and issue debt but the middle productivity type separates out and issues equity, denying the optimality of debt financing (or the universality of the pecking order between debt and equity). Here market participants correctly infer the productivity of a firm issuing equity. The high (low) productivity type encounters underpricing (overpricing) because of pooling, but the middle type is correctly priced by issuing equity and therefore suffers no mispricing. This possibility is overlooked by Myers and Majluf who presume that the absolute mispricing is always higher for equity than it is for debt regardless of the firm's productivity.

A third but related difficulty with the folklore proposition is that there is no clear delineation in the literature as to how to order the range of firm productivity so that the proposition will hold. In the original argument for the folklore proposition made by Myers and Majluf (1984), it seems that it is sufficient for the validity of this proposition that productivity types be ordered by first-order stochastic dominance of cash flows. This ordering has a natural appeal in this context and even seems necessary, since it implies and is implied by the relation that high productivity types are truly better than low productivity types in that the true value of any security (nondecreasing in firm cash flow) issued by a high productivity type is at least as great as the true value of that security issued by a lower productivity type. The Noe (1988) example shows that this ordering is not enough since there the types are ordered by (a strict version of) first-order stochastic dominance, yet the folklore proposition fails to hold. If the proposition does not hold in this generality, when does it hold? What is needed, in the case of both the second and the third difficulties with the folklore proposition, is an understanding of what the market must infer about the productivity of firms that would deviate from a debt pooling outcome in order to sustain debt pooling as an equilibrium outcome.

The aim of this paper is to address the issue of security design under asymmetric information in general, and the validity of the folklore proposition in particular. We derive necessary and sufficient conditions for the issuance of a security to be an equilibrium outcome of raising external capital and use these conditions to consider the issue of the optimality of debt. In the process, we address each of the difficulties in formalizing the folklore proposition discussed above. In Section 1.1, we formulate the basic model allowing for a rich menu of security designs, essentially any functions of firm cash flows that are nondecreasing and satisfy limited liability. The firm knows its probability distribution of investment generated cash flow but the suppliers of capital do not. The resulting game of incomplete information is called the capital raising game. The notion of equilibrium for this game is presented in Section 1.2.

In Section 1.3, we impose an ordering of productivity types in terms of first-order stochastic dominance of cash flow distributions. In the absence of costs of security design and issue, there are therefore incentives for low productivity types to mimic the design of higher productivity types and consequently, some pooling is inevitable.

Nevertheless, with multiple productivity types there may be offsetting incentives for some types to separate out in equilibrium. In Section 1.4, we introduce some language that facilitates dealing with multiple productivity types. This language concerns a *division* of the set of productivity types into an upper pool of types and a lower pool of types. To describe when the relation “is more valuable than” between securities for an upper pool of types determines the same relation for a lower pool of types, we introduce the concept of an *oriented set of securities.* This concept turns out to be much more than simply facilitating language for it characterizes the set of pooling equilibrium outcomes of the capital raising game.

This characterization is developed in Section 2. Roughly speaking, it is demonstrated in Section 2.1 that a set of securities is oriented if and only if its elements are outcomes of pooling equilibria of the game. A similar but stronger characterization is given for a strictly oriented set of securities in Section 2.2. We show that a set of securities is strictly oriented if and only if each element is the only equilibrium outcome of the appropriate game. Orientation is thus necessary and sufficient both for precluding partial pooling equilibria and for justifying off-equilibrium beliefs which identify defectors from the pooling outcome as low productivity types.

In Section 3 we turn to the specific issue of the optimality of debt financing. This we do by looking at the conditions under which the collection of debt security designs is in fact an oriented or strictly oriented set of securities. In Section 3.1, we introduce and discuss a stronger ordering of types than first-order stochastic dominance. The stronger ordering is in terms of what we call conditional stochastic dominance. In Section 3.2, we establish the characterization that the set of debt securities is (strictly) oriented if and only if the set of productivity types can be ordered by (strict) conditional stochastic dominance. Given our earlier results, this implies that debt financing is a pooling equilibrium outcome (the unique equilibrium outcome) if and only if firm types are (strictly) ordered by conditional stochastic dominance. This delineates clearly the boundaries of the validity of the folklore proposition.

In Section 4, we trace out the correspondence between equilibrium security design and the heuristic argument that firms should minimize mispricing. Our analysis of this question again demonstrates the salience of the notion of an oriented set of securities. In fact, we show in Section 4.1 that a set of securities is oriented if and only if each security in the set minimizes the true value of the security to the upper pool of types for any nontrivial division of types when the security is priced at pooled terms for the entire set of types. As a consequence, these securities minimize the ex ante absolute mispricing in any pooling equilibrium. This result, combined with the results of the previous sections establishes a one-to-one link between (i) securities which support pooling equilibria, (ii) securities which minimize mispricing, and (iii) securities which are members of an oriented set. In Section 4.2, we provide a mathematical programming characterization of the conditions under which a security solves the minimization problem of Section 4.1. This programming characterization is useful in providing insight into the reason conditional stochastic dominance is required for the optimality of debt and in identifying alternative optimal designs when conditional stochastic dominance does not hold. In Section 4.3, we use the optimality conditions given in Section 4.2 to construct a simple example of cash flow distributions which are ordered by first-order stochastic dominance but not conditional stochastic dominance where a form of layered debt is optimal but ordinary debt is not.

In Section 5, we summarize our results and discuss some directions for future research. Several technical results needed and the proofs of results in the text are collected in the Appendix.

1. The Model

1.1 A simple capital raising game

Consider a two-date model in which a firm must raise at least K dollars at the initial date 0 for the purpose of undertaking an investment opportunity that pays off at the subsequent date 1. Let x denote the total cash flow available at date 1 to pay claims sold at date 0. This cash flow may arise from assets in place at date 0, from the new investment made at date 0, or from additional financing done at date 1. The set of admissible securities is a set of functions satisfying the following limited liability and monotonicity conditions:

$$0 \leq s(x) \leq x, \qquad \text{all } x \geq 0, \tag{LL}$$

$$s \text{ is nondecreasing in } x, \qquad x \geq 0, \tag{M}$$

$$\mathbf{x} - s \text{ is nondecreasing in } x, \qquad x \geq 0, \tag{MR}$$

where $\mathbf{x}(x) = x$, for all x. The set $S = \{s: R_+ \to R: s$ satisfies (LL) and (M) and (MR)$\}$ is the set of admissible security designs.

In (LL), the restriction that $0 \leq s(x)$ for all x expresses limited liability for the new claimants, those who purchase security s. The restriction that $s(x) \leq x$ for all x expresses limited liability for the firm's shareholders of record as of date 0. Such limited liability restrictions are common features of securities sold in primary corporate capital markets. The monotonicity condition (M) requires that the payment for the new claimants, those who purchase security s, is nondecreasing in the available firm cash flows, This assumption is quite common in the finance literature on security design [for example, Brennan and Kraus (1987) and Harris and Raviv (1989)].[2,3] The other monotonicity condition (MR) requires that the residual claim, $\mathbf{x} - s$, paid to shareholders of record at date 0, be nondecreasing in the available cash flow. This assumption is also made in the extant

[2] Little attempt has been made in the finance literature to justify this assumption. Nachman and Noe (1990) formalize an argument based on the observation that if the firm had sold a claim that does not satisfy (M), then in otherwise perfect capital markets, the firm could engage in refinancing (indeed, riskless borrowing) after observing the cash flow at date 1, to raise the available cash flow to the level needed to avoid a higher payout at a low cash flow. The effect of having this refinancing opportunity is to render any security issued at date 0 a claim satisfying (M). Innis (1990). in his analysis of optimal contract design in a moral hazard context, offers a similar argument for (M).

[3] The monotonicity assumption (M) is important in establishing the optimality of debt. In the absence of (M), debt is not optimal for an important subset of the cases satisfying the conditional stochastic dominance ordering of Section 3.1. This subset consists of types satisfying a monotone likelihood ratio ordering, a stronger ordering than conditional stochastic dominance. When this monotone likelihood ratio ordering condition is satisfied and (M) is not imposed, an optimal security design is always given by a convex combination of a "do or die" contract, under which the firm pays everything to outsiders when cash flows fall below a threshold level and nothing otherwise, and a simple equity contract. See also Innes (1990) for results on the optimality of "do or die" contracts in a moral hazard context.

literature [e.g., Harris and Raviv (1989)].[4] Designs that satisfy these restrictions include all common security designs, debt, equity, warrants, convertible debt, and some rather uncommon securities such as putable stock.

Suppliers of capital are viewed as a monolith of identical investors in a competitive capital market referred to simply as “the market.” Each supplier’s strategy is a price offer P contingent on the security s offered by the firm. Beliefs are homogeneous across suppliers of capital and hence we will refer to them as the market’s beliefs.

To avoid security design motives arising from issues involved with the allocation of risk bearing or involved with the allocation of consumption across time, we assume that all investors are risk neutral with no impatience, including the shareholders and managers of the firm. We assume there are at least two suppliers of capital that can purchase any security issued and that competition among suppliers of capital takes place in the style of Bertrand [see Cho and Kreps (1987)]. The pricing strategies of suppliers of capital are therefore identical and equal to the expected value of the security given the market’s beliefs regarding the type of the firm that issued the security. Competition in the capital market therefore is assumed to result in the price response that yields zero net value to purchasers, the outcome if identical purchasers bid in the style of Bertrand for securities issued by firms.

The concern of this article is not investment policy but rather security design. Thus to simplify the analysis we would like to impose restrictions on the profitability of projects which ensure that it is always in the interest of the firm to undertake the investment project. The most obvious condition for ensuring project funding is assuming that all projects have positive net present value. However, despite the assumed homogeneity of preferences and beliefs, there are a number of conflicts of interest among claimants to corporate cash flows that have been identified in the finance literature that may preclude the funding of positive NPV investments. Asymmetric information regarding the firm’s assets in place can lead firms to reject profitable investment opportunities [see Myers and Majluf (1984)]. Further, the presence in the firm’s capital structure at date 0 of claims senior to those of the firm’s shareholders may also lead to underinvestment if a portion of the value of the new investment is captured by these senior claimants [Myers (1977)]. In order to sidestep these

[4] (MR) is not essential for the substantive results of this article. The optimality of debt was established in Nachman and Noe (1990: Theorem 11) without assuming (MR). It does, however, simplify considerably the optimality conditions presented in Section 4.2 and is imposed for this convenience. (MR) can be justified by appealing to the presence of operating discretion on the part of the users of capital taking the form of a version of stochastic free disposal, as is done in Nachman and Noe (1993).

problems, we assume that there is no informational asymmetry regarding assets in place and the firm is all equity financed. As a result, no positive NPV investments will be foregone. These assumptions allow us to focus our attention squarely on the issue of security design without having to worry about the question of whether the project will be undertaken at all. All of the analysis will go through without these assumptions, as long as one imposes a me-first priority rule and a restriction is placed on cash flow distributions that ensures that funding the investment opportunity is a dominant strategy [see Nachman and Noe (1990, (2a))]. However, the required restrictions are, in general, complicated and contribute nothing to understanding the issues considered in this article.

The firm has private information regarding the productivity of its investment opportunity. Let T denote the set of possible outcomes of the firm's private information. Here $T = \{1, \ldots, N\}$ is just a finite set of "types," where $N > 1$ is a positive integer. The productivity of the investment opportunity of firm type t is given by the distribution function F_t representing the cumulative distribution of cash flows x available at date 1.[5] The firm learns its type $t \in T$ and hence the productivity of its investments F_t at the beginning of the game. The market is not informed about the firm's type, but there is a prior probability distribution on T, denoted by $\pi = (\pi_1, \ldots, \pi_N)$, with π_t being the prior probability that the realization of the firm's private information is t. The set T of possible types, the set of distributions $F = \{F_t\colon t \in T\}$, and the common prior π are all common knowledge.

To avoid complicated integral expressions, it will be convenient to denote the full information expectation of any variable determined by x by the notation $E_t(\cdot)$. Thus, for example, $E_t(\mathbf{x}) = \int \mathbf{x}\, dF_t$, and $E_t(s) = \int s\, dF_t$. Also let Δ_T denote the set of probability distributions on the type set T. For $\mu \in \Delta_T$, let $E_\mu(\cdot)$ denote the expectation operator when the market's beliefs regarding firm type is μ. For example, the market value of security $s \in S$, given the market's probabilistic beliefs about types is $\mu = (\mu_1, \ldots, \mu_N)$ is $E_\mu(s) = \sum_{t \in T} E_t(s)\mu_t$. The ex ante market value of any security s is then $E_\pi(s)$.

In order to minimize confusion about when two security designs are actually different in an important way, we will assume that the probabilty measures determined by the distribution functions F_t, $t \in T$ agree on sets of probability zero so that a cash flow event has probability zero for one type t if and only if it has probability zero

[5] Since x may result in part from assets in place at date 0, some components of the distribution function F_t may be symmetrically known. What is private information of the firm is the distribution of new investment generated cash flow conditional on cash flow from assets in place.

for every type $t' \in T$.[6] For convenience, we also assume the cash flows of all types are nonnegative random variables with finite expectations and that the probability of zero cash flow is the same for all types. These assumptions are formalized in the following initial conditions.

(IN) For all $t, t' \in T$, the probability measures determined by F_t and $F_{t'}$ agree on sets of probability zero and $F_t(0) = F_{t'}(0) < 1$. For all $t \in T$, $E_t(\mathbf{x}) < \infty$ and $F_t(x) = 0$, if $x < 0$.

The first part of the initial conditions **(IN)** implies that for any two securities $s, s' \in S$, the set of cash flows x where $s(x) \neq s'(x)$ has zero probability for every type $t \in T$, or it has positive probability for every type $t \in T$. The notation $s \neq s'$ will be used to abbreviate the latter condition. It follows that the set $\{x\!: x \geq 0$ and F_t (p $x) = 1\}$ is the same for all $t \in T$ since if $F_t(x) = 1$, the probability for t that cash flow is greater than x is zero and hence this probability is zero for all $t' \in T$. Hence the relative complement of this set, $\{x\!: x \geq 0$ and $F_t(x) < 1\}$, is also the same for all types t. We denote this latter set by Ω° and let Ω denote the closure of this set.

Given our assumption of risk neutrality, the assumption of positive NPV can be simply stated as follows.

(NPV) $E_t(\mathbf{x}) > K$, for all $t \in T$.

With this assumption, the presumption that the firm will sell some claim to raise the requisite investment capital K is made throughout the remainder of the article. In order to avoid the trivial case in which the firm can finance through the issuance of riskless debt, we further assume that

(RD) $F_t(K) > 0$, for each $t \in T$.

The choice of security to issue is made by the firm that seeks to maximize the full information date 0 value of the firm to its shareholders of record prior to issue. Since x represents the residual cash flow from all investments at date 1, the firm's objective function is given by

$$V(t, s, P) = P - K + E_t(\mathbf{x} - s). \tag{1}$$

If the firm's type is t and it sells security s and raises P and if $P \geq K$, the value to shareholders of record at date 0 consists of the excess of the proceeds $P - K$ from the sale of securities plus the full information value of the retained equity claim $E_t(\mathbf{x} - s)$. In this formu-

[6]In the language of measure theory, this means the probability measures determined by the F_t are mutually absolutely continuous, a condition stronger than requiring that the F_t have common support.

lation, the benefit from mimicking the high productivity type is the higher price for which the securities will sell. The cost is that the intrinsic value of the claim issued $E_t(s)$ may be higher and therefore E_t **(x** - s) may be lower. The presumption in (1) is that the excess of the proceeds $P - K$ from the sale of securities is paid to shareholders at date 0 as a dividend. Note that the quantity $V(t, s, P)$ is nonnegative so long as $P \geq K$ and s satisfies the limited liability requirement (LL).

The game described above is referred to as the capital raising game. In pursuing the analysis of this game, we need to be deliberate about some parameter variations. Recall that the prior distribution π is a member of Δ_T. In order for there to be a game of some interest, we must have that $\pi_t > 0$ for at least two distinct values of $t \in T$. For the same reason, we need positivity of the capital requirement (i.e., that $K > 0$). Also recall that K is constrained by the "positive NPV" condition **(NPV)** and by the "no riskless debt" condition **(RD).** Parameter pairs (K, π) (as well as each parameter separately) that satisfy these restrictions will be said to be *admissible.*[7] The capital raising game for such an admissible pair is denoted by $\Gamma(K, p)$.

1.2 Equilibria of the capital raising game

A candidate for an equilibrium of the game is a triple of functions $e^* = (s^*, \mu^*, P^*)$, where $s^*: T \to S$, and $s^*(t)$ is the security design chosen by firm type $t, \mu^*: S \to \Delta_T$, and $\mu^*(s)$ is the market's posterior belief on firm types given that security s is offered by the firm, and $P^*: S ® R_+$, with $P^*(s)$ being the price offered by the market for security s. A perfect Bayesian equilibrium[8] of the game is a triple $e^* = (s^*, \mu^*, P^*)$ satisfying the following conditions (SR), (BC), and (CR)

(SR) (Sequential Rationality) For each $t \in T$, $s^*(t)$ maximizes $V(t, s, P^*(s))$, subject to the constraints that $s \in S$ and $P^*(s) \geq K$.

(BC) (Belief Consistency) When security s is such that $s = s^*(t)$ for some $t \in T$, s is "on the equilibrium," then $\mu^*(s)$ is determined by Bayes' rule. When s is such that $s \neq s^*(t)$ for every $t \in T$, s is "off the equilibrium," then it is only required that $\mu^*(s) \in \Delta_T$.

(CR) (Competitive Rationality) $P^*(s) = E_{\mu \cdot (s)}(s)$, for all $s \in S$.

[7] To give the greatest flexibility in what follows, implicitly we modify the quantifier of the "positive NPV" condition and of the "no riskless debt" condition to require that these conditions hold only for those types tin the support of π —for which $\pi_t > 0$.

[8] See Fudenberg and Tirole (1991) for a definition of perfect Bayesian equilibrium in a finite type and finite action game. Note that the consistency conditions imposed in Kreps and Wilson (1982) in defining sequential equilibria have no force in our setting. See Kreps (1990, Section 3.3) and Fudenberg and Tirole (1991. Proposition 3.2).

The sequential rationality condition (SR) simply says that the security offered by type tin equilibrium must maximize the value of type *t's* firm to its shareholders of record prior to issue, given the market's beliefs μ^* and the price response P^*. The belief consistency condition (BC) places rationality restrictions on market beliefs μ^* only for securities that are offered in equilibrium. The assumption that capital markets are competitive, in the style of Bertrand competition, implies that the proceeds from a new issue will exactly equal the value of the securities issued given the market's information at the date of the issue. This is the competitive rationality condition (CR). It is intended to reflect that competition among suppliers of capital rules out profit opportunities in the equilibrium pricing of securities.

The belief consistency (BC) places no restrictions on the market's beliefs for off-equilibrium securities and this generates a serious multiplicity of equilibria. What posterior beliefs are *rational* for the market to have regarding the types who issue an off-equilibrium security is not a consequence of individual rationality postulates. What is needed here is a restriction on these beliefs that is based on the relative propensity of types to deviate from the equilibrium. Perhaps the simplest such restriction in the published literature on refinements of signalling game equilibria is the D1 refinement discussed in Cho and Kreps (1987). It is a simplified and somewhat weakened version of the universal divinity refinement of Banks and Sobel (1987).

Our results are not critically dependent on using the D1 refinement, but it is easy to use. Any of the refinements which consider the relative propensity of types to deviate to an off-equilibrium strategy are likely to yield the same set of equilibrium outcomes. One such refinement proposed in Nachman and Noe (1990) is intended to extend competitive rationality to off-equilibrium securities offerings by ruling out what are called conjectural arbitrage opportunities. In the two-type case, this refinement produces the major components of the results of this paper. On the other hand, refinements such as the intuitive criterion of Cho and Kreps (1987), which strike types only when deviation is not a weak best reply even against the most optimistic assessment of the market have no force in the context of the capital raising game. An example illustrating the latter assertion is available from the authors on request.

To state the relevant restriction, let $V^*(t)$ represent the full information value of firm type t to its shareholders in the candidate equilibrium; that is, let $V^*(t) = V(t, s^*(t), P^*(s^*(t)))$. Let $D(t \mid s)$, respectively $D^0(t \mid s)$, represent the set of responses to the choice of security s which would provide the shareholders of type t a strictly higher value than, respectively the same value as, *t's* equilibrium value; that is,

$$D(t|s) = \{P \geq K: V(t, s, P) > V^*(t)\}$$

and

$$D^0(t|s) = \{P \geq K: V(t, s, P) = V^*(t)\}.$$

The relevant restriction on off-equilibrium beliefs is as follows:

(D1) If s is such that $s \neq s^*(t)$ for every $t \in T$, s is "off the equilibrium," then $\mu^*(s)$ must satisfy the condition that for any pair $t, t' \in T$, if $D^0(t \mid s) \cup D(t \mid s) \subset D(t' \mid s)$, then $\mu^*(s)(t) = 0$.

Condition (D1) requires that the market place no probability weight on a type, say t, deviating to an off-equilibrium design if there exists another type, say t', who has a definite incentive to deviate whenever the given type t has a weak incentive to deviate. The definition of equilibrium for this paper is the following.

Definition 1. *$e^* = (s^*, \mu^*, P^*)$ is an equilibrium of the capital raising game Γ (K, p) if it satisfies (SR), (BC), (CR), and (D1)*[9]

1.3 Ordering of cash flow distributions

A natural way to say that one firm type has higher productivity than another is through requiring that any security issued by the higher productivity type be at least as valuable as the same security if issued by a lower productivity type. In the face of the monotonicity restriction (M), this entails ordering of firm types according to first-order stochastic dominance of cash flows.

(SD) If $t, t' \in T$ and $t < t'$, then $F_{t'}(x) \leq F_t(x)$, for all $x \geq 0$.

It follows from stochastic dominance **(SD)** that the higher the firm's type, the higher the value of the securities that it issues. Thus if $t, t' \in T$ and $t < t'$, then for all $s \in S$, $E_t(s) \leq E_{t'}(s)$. By monotonicity of the residual claim (MR), we also have that $E_t(\mathbf{x} - s) \leq E_{t'}(\mathbf{x} - s)$. Stochastic dominance **(SD)** is assumed to hold in the remainder of the article.

Occasionally, especially when considering uniqueness arguments, we will require a strict version of stochastic dominance to hold between firm types. The strict version of **(SD),** which we term strict stochastic dominance **(SSD),** is stated as follows:

(SSD) If $t, t' \in T$ and $t < t'$, then $F_{t'}(x) < F_t(x)$, for all $x \in \Omega^o$.

The major implication of stochastic dominance is that in the costless signaling environment of this paper, separating equilibria are rare or

---

[9]For an admissible parameter pair (K, π), equilibrium conditions of the game Γ (K, π) apply only to those types $t \in T$ for which $\pi_t > 0$ (cf. note 7).

nonexistent. An equilibrium $e^* = (s^*, \mu^*, P^*)$ of the capital raising game is separating if for all $t, t' \in T$, $t \neq t'$ implies that $s^*(t) \neq s^*(t')$. An equilibrium is a pooling equilibrium if for all $t, t' \in T$, $s^*(t) = s^*(t')$. An equilibrium is *value invariant* if $t < t'$ and $s^*(t) \neq s^*(t')$ implies that $E_t(s^*(t')) = E_{t'}(s^*(t'))$. In a value-invariant equilibrium, a security issued by a high type can differ from that issued by a lower type only if the value of the high type's security is invariant to which type, the high or low type, issues it. This condition of value invariance is quite stringent, but it does not describe a (necessarily) desirable property of an equilibrium, just a property that separating equilibria must have in the presence of stochastic dominance.

Proposition 1. *All separating equilibria are value invariant. If strict stochastic dominance* **(SSD)** *holds, no separating equilibria exist.*

The intuition behind Proposition 1 is completely straightforward. In any separating equilibrium, all claims issued must be correctly priced. This implies that the first and last terms on the right in Equation (1) are equal; that is, the price P is equal to the full information value $E_t(s)$. Thus, the equilibrium value of any type is the full information net present value of the project E_t **(x)** - K. The gain to type t from mimicking type t', when $t < t'$, is the difference between the price of $s^*(t')$ and the intrinsic value of this security if it is issued by type t. But the price of $s^*(t')$ is just its intrinsic value if issued by t', $E_{t'}(s^*(t'))$. Thus the gain to t from mimicry of t' is $E_{t'}(s^*(t')) - E_t(s^*(t'))$. In equilibrium, this difference must be nonpositive. By first-order stochastic dominance, this difference must be nonnegative. Thus, it must be the case that $E_{t'}(s^*(t')) = E_t(s^*(t'))$; that is, the equilibrium is value invariant.

Under strict stochastic dominance **(SSD),** this equality required for value invariance cannot obtain, and hence there can be no separating equilibrium. In the case where T contains just two elements, this implies that all equilibria are pooling equilibria. In such a case, it is reasonable to conjecture that the security issued in equilibrium minimizes the mispricing losses of the high productivity type. As we will show in what follows, this conjecture is in fact correct.

In the case of more than two productivity types, other equilibrium configurations, involving partial pooling are possible even when cash flows are ordered by strict stochastic dominance. In these instances, the criteria of minimizing the mispricing losses of the highest-quality type becomes problematic in that types other than the highest quality may suffer mispricing losses. What is needed is the language to handle pools of higher productivity types and pools of lower productivity types, while identifying sets of security designs with the property that the security design from the set that minimizes the mispricing losses of the pool of higher productivity types will simultaneously minimize the mispricing gains of the pool of lower productivity types. The relevant ideas are introduced in the following section.

1.4 Divisions and oriented sets of securities

All that is needed by way of language to handle pools of higher productivity types and pools of lower productivity types is captured in the notion of a division of the set of types. A *division* of (T, π), is determined by an element $d \in T$, the point of the division, that splits T into a pool of higher productivity types identified by the probability distribution π_{d+} concentrated on $\{d, \ldots, N)$, and the pool of lower productivity types identified by the probability distribution π_{d-} on $\{1, \ldots, d-1\}$, where π_{d+} is the conditional probability distribution under the prior π given that the type is at least d, and π_{d-} is the conditional probability distribution under the prior π given that the type is less than d. Letting $\pi^d = \Sigma_{t \geq d} \pi_t$, the unconditional probability under the prior A that the type is at least d, the division is called *nontrivial* if $0 < \pi^d < 1$. For brevity, we will refer to the pool of higher productivity types as the upper pool of types and the pool of lower productivity types as the lower pool of types.[10]

Since the set T of types is not variable in the analysis, we will refer to a division d of (T, π) simply as a division d of π. For a division d of π, the expectations with respect to the distributions π_{d+} and π_{d-} are denoted by $E_{\pi_{d+}}$ and $E_{\pi_{d-}}$, respectively. The following property describes sets of security designs where the relation "is more valuable than," when satisfied for an upper pool of types must also be satisfied for a lower pool of types. By the initial conditions (IN), for any $\mu \in \Delta_T$, and any $s \in S$, $E_\mu(s)$ is finite and nonnegative.

Definition 2. *A set of $S^o \subseteq S$ is oriented for F if for every admissible π and every nontrivial division d of π, if $s^o \in S^o$, $s \in S$, and if $E_{\pi_{d+}}(s) \leq E_{\pi_{d+}}(s^o)$, then $E_{\pi_{d-}}(s) \leq E_{\pi_{d-}}(s^o)$. A set $S^o \subseteq S$ is strictly oriented for F if it is oriented for F and if for every admissible π and every nontrivial division d of π, if $s^o \in S^0$, $s \in S$, with $s \neq s^o$, and if $E_{\pi_{d+}}(s) \leq E_{\pi_{d+}}(s^0)$, then $E_{\pi_{d-}}(s) < E_{\pi_{d-}}(s^0)$.*

Thus, an oriented set of securities is a special set for which the value orientation, expressed by the relation "is more valuable than," of the set of types is determined by the value orientation of the highest productivity types; that is, it is determined from the top of the pro-

[10] For the case when $d = 1$, the lower pool of types is of course empty and π_{d-} is not well defined. In this case, $\pi^d = 1$. $\pi_{d+} = \pi$, and the higher pool of types is the entire pool T.

ductivity spectrum down.[11] Given a division d of π, the condition that one security design, say $s^o \in S$, is more valuable than another security design, say $s \in S$, for the higher pool of types is the condition that $E_{\pi\, d+}(s) \leq E_{\pi\, d+}(s^o)$. For an arbitrary pair of designs s^o, s that satisfy this relation, it may well be and frequently would be the case that the opposite relation holds for the lower pool of types; in other words, that for the lower pool of types s^o is less valuable than s, the condition that $E_{\pi\, d-}(s) > E_{\pi\, d-}(s^o)$. However, if s^o is an element of an oriented set of securities this does not happen, the same value orientation obtains for the lower pool of types in that if $E_{\pi\, d+}(s) \leq E_{p\, d+}(s^o)$, then it must also be that $E_{\pi\, d-}(s) \leq E_{p\, d-}(s^o)$. The appeal here of an oriented set of securities is that if s^o belongs to this set and s^o minimizes the true value of the security offered by the upper pool of types, then it must be that s^o maximizes the true value of the security offered by the lower pool of types uniformly across all divisions of π (see Section 4.1).

Of course if S^o is empty, then S^o is oriented for F. There are, however, nonempty oriented sets. Trivially, the set $\{\mathbf{0}, \mathbf{x}\}$ is oriented for F, regardless of how F is ordered, where $\mathbf{0}, \mathbf{x} \in S$ are such that $\mathbf{0}(x) = 0$, for all x, the design that pays out zero for every level of cash flow, and $\mathbf{x}(x) = x$, for all x, the design that pays out all cash flows at every level of cash flow. Also, a set S^o is oriented for F if and only if $S^o \cup \{\mathbf{0}, \mathbf{x}\}$ is oriented for F. To avoid these trivial cases, in what follows, the phrase "S^o is (strictly) oriented for F" will henceforth mean that $\mathbf{x} \in S^o$ and $\mathbf{0} \in S^o$.

In order to relate orientation to equilibria of the capital raising game, we need to allow the game's parameters to vary over all admissible parameters. For such variations, for some results there will have to be security designs in the relevant set that satisfy the capital raising constraint. The set $\{\mathbf{0}, \mathbf{x}\}$ is too thin in this sense. The condition that is needed is the following.

[11] To the authors' knowledge, this concept has not been formalized in the literature on signaling games. However, there are similar results in the literature. The closest analogue to orientation is condition (A4) given by Cho and Sobel (1990) as one of a set of jointly sufficient conditions for the existence of a separating equilibrium (or an equilibrium in which all types pool at the highest possible signal level). Cho and Sobel's condition (A4) requires that higher-quality types exhibit a greater relative preference for larger signals, where large is defined by the componentwise order over a finite-dimensional vector space. [Note that because our analysis is in terms of costs instead of benefits, the inequalities in our analysis are actually in the opposite direction of those in Cho and Sobel (1990).] The natural generalization of componentwise order to our infinite-dimensional signal space is the pointwise ordering of claims. Unfortunately, the set of contracts which raise the same level of investment capital never includes two contracts which are strictly orderable under the pointwise ordering. Thus, this sort of pointwise order is of little use in a security design setting such as ours. The relation between our criteria and the classical single-crossing property is even weaker, for this criterion is only useful when the signal space is a linearly ordered space such as the real line because it requires that the marginal gain from "increased" levels of the signal be higher for higher-quality types.

Definition 3. *A set $S^o \subseteq S$ is full if for every $t \in T$ and every $K \in (0, E_t(\mathbf{x}))$, there exists $s^o \in S^o$ such that $E_t(s^o) = K$.*

2. Equilibrium Configurations

In this section we consider the equilibrium configurations of the capital raising game with a focus on the relationship between designs in an oriented set of securities and pooling equilibrium outcomes. We know from Proposition 1 that in the presence of stochastic dominance, there will be some pooling in any interesting equilibrium of the capital raising game.[12] While this does not justify looking only at pooling equilibria, our intuition tells us that if there is a rich oriented set of securities, then equilibria involving partial pooling will be as rare as separating equilibria.

There are two results in this section. The first result, Theorem 2, is a characterization of pooling equilibrium outcomes of the capital raising game in terms of oriented sets of securities in that a security is a pooling equilibrium outcome for every admissible parameter pair for which it satisfies the capital constraint if and only if this security belongs to an oriented set of securities. This result is presented in Section 2.1.

Theorem 2 does not preclude the existence of partial pooling equilibria. The second result in this section, Theorem 3, addresses this "other equilibria" problem by strengthening the ordering of types and by employing the strict version of orientation. It states that the only equilibrium outcome is a pooling equilibrium in which a security design from the oriented set is used to finance the project. This result is presented in Section 2.2.

2.1 A characterization of pooling equilibrium outcomes

Theorem 2. *$S^o \subseteq S$ is orientedfor F if and only if for any admissible parameters (K, π), if $s^o \in S^o$ and $E_\pi(s^o) = K$, then there exists an equilibrium for the game $\Gamma(K, p)$ in which all typespool by issuing s^o.*

The basic intuition behind the result in Theorem 2 is that the relative propensity of a type to deviate from a candidate equilibrium, in which a security from an oriented set is being issued, is decreasing in the type's productivity. Thus, high productivity types have the least relative propensity to deviate and low productivity types have the highest. This ordering of deviation propensities allows for the postulation of off-equilibrium beliefs, which satisfy standard refinements,

[12] This is formalized in Theorem A.8 in the Appendix.

and, at the same time, allow for the pricing of the off-equilibrium securities at unfavorable terms. This sort of market pricing makes issuing the security from the oriented set a best response for all types. Conversely, if a security is not part of an oriented set, then the ordering of deviation propensities described above fails. For some choices of π and K this failure matters in that the failure, combined with the restrictions on beliefs imposed by (D1), forces the market to assess the set of firm types selecting some off-equilibrium security in a relatively favorable fashion. This leads to favorable pricing of the off-equilibrium security and this leads to deviations from the candidate pooling equilibrium.

Note that for some specialized choices of prior probability distribution and funds requirement, it may be that other securities which do not lie in an oriented set may support pooling equilibria. Theorem 2 says that such contracts cannot do so for the entire set of parameter values for which they meet the capital raising constraint.

2.2 A stronger characterization

Under stronger assumptions on the ordering of types and orientation of S^o, it follows that for each possible specification of the game's parameters, only one security in S^o meets the capital raising constraint and the only equilibrium outcome for the corresponding game is for all types to pool by issuing this security.

***Theorem* 3.** *Assume that strict stochastic dominance* **(SSD)** *holds and that $S^o \subseteq S$ is full. Then S^o is strictly oriented for F if and only if for any admissible parameter pair (K, p), if $s^o \in S^o$ and $E_p(s^o) = K$, then in every equilibrium for the game $\Gamma(K, \pi)$, all typespool by issuing s^o.*

The intuition underlying Theorem 3 is a bit more complex than for Theorem 2 in that, to establish this result, we must also show that no partial pooling equilibria exist when the set S^o is strictly oriented. The proof of this result starts by showing that, in any equilibrium, the security choice of the highest-quality type is always mimicked by at least one other type. This implies that raising extra funds simply exposes the highest-quality type to larger losses from undervaluation without any compensating benefit. Thus, the highest-quality type never raises more funds than required to finance the project. This fact, and a number of more technical consequences of orientation, are then used to show that the highest-productivity type will choose a security design from the oriented set. Next, we show that whenever partial pooling occurs, the type pools can never be unambiguously ordered by productivity. That is, the lowest type of firm issuing one security design can never be higher than the highest type of firm issuing another security design. If this condition were violated, the lower pool of types would have an incentive to mimic the higher pool of types. Together, these results imply that any partial pooling outcome must involve a type issuing a security from the oriented set and another, higher type not issuing this security. This is then shown to contradict strict orientation. The converse result, that strict orientation is necessary for uniqueness, is established in exactly the same fashion as the necessity result in Theorem 2.

3. Debt and the Adverse Selection Problem

The results of Section 2 suggest very strongly that one way to establish that debt minimizes the adverse selection problem of raising external capital would be to show that the set of debt securities (as the promised payment varies) is an oriented set of securities. If this set is oriented, it would follow that debt is an optimal security design in that issuing debt to raise admissible capital levels is a pooling equilibrium of the capital raising game (Theorem 2). Clearly, the set of debt securities is full. If the set of debt securities is strictly oriented, then it would follow (Theorem 3) that issuing debt to raise admissible capital levels is the only equilibrium outcome of the capital raising game.

In this section, we demonstrate that when productivity types are ordered by conditional stochastic dominance, a stronger ordering than stochastic dominance, then the set of debt securities is an oriented set of security designs. Moreover, this ordering of types is necessary for the set of debt securities to be oriented. There is a corresponding characterization of strict orientation of debt in terms of a strict version of conditional stochastic dominance. The conditional stochastic dominance ordering is discussed in Section 3.1. Its characterization of the orientation of the set of debt securities is established in Section 3.2.

3.1 A stronger ordering of types

Recall that by the initial conditions (IN), the set $\Omega^{o} = \{x\colon x \geq 0$ and $F_t(x) < 1\}$ is the same for every $t \in T$. For $x \in \Omega^{o}$ and for $y \geq 0$, the conditional probability that type t's cash flows are less than or equal to the quantity $x + y$ given that the cash flow is greater than x is the quantity

$$F_t(y \mid x) = \frac{F_t(x + y) - F_t(x)}{1 - F_t(x)}. \tag{2}$$

If $F_t(x) = 1$, it is natural and convenient to define $F_t(y \mid x) = 1$ for all $y \geq 0$. Thus, for any $x \in \Omega$, $F_t(y \mid x)$ is a distribution function in y.

The assumption of first-order stochastic dominance **(SD)** is an ordering of the unconditional distribution functions determined by the inequality $F_{t'}(x) \leq Ft(_x)$ for all $x \in \Omega$ and any $t < t'$. A similar inequality for the conditional distribution functions defines *conditional stochastic dominance* **(CSD).**

(CSD) If $t, t' \in T$ and $t < t'$, then $F_{t'}(y \mid x) \leq F_t(y \mid x)$, for all $x \in \Omega$, and all $y \geq 0$.

The strong version of this ordering needed in this article is called strict conditional stochastic dominance **(SCSD).**

(SCSD) If $t, t' \in T, t < t'$, then $F_{t'}(y \mid x) < Ft(y \mid x)$, all $x \in \Omega^{o}$, and all $y > 0$ with $x + y \in \Omega^{o}$.

Notice also that by the initial conditions **(IN),** $F_t(y \mid 0) = aF_t(y) + b$, where $a = (1 - F_t(0))^{-1} > 0$ and where $b = - F_t(0)\, a$, independently of t. Thus conditional stochastic dominance **(CSD)** implies that stochastic dominance **(SD)** holds. Similarly, strict conditional stochastic dominance **(SCSD)** implies that strict stochastic dominance (SSD) holds.[13]

Fairly simple constructions of families of random variables ordered by conditional stochastic dominance are readily available. For example, suppose that the cash flow distribution for all types has a common two-point support, x_b and x_1, with $x_b > x_1 \geq 0$. In this case the distribution of cash flows for a given type is completely specified by the probability that a type has the big cash flow x_b. Let $p_t = 1 - F_t(x_1)$ be the probability that type *t's* cash flow is x_b. In this case, (strict) conditional stochastic dominance is satisfied if and only if (strict) stochastic dominance is satisfied, which holds if and only if $p1 \leq (<) \ldots \leq (<) p_N$. This example is perhaps at the heart of the third difficulty regarding the folklore proposition mentioned in the introduction and is discussed further in Section 3.2.

Next we consider the case where type *t's* cash flows are distributed as X_t, with $X_t = a_t + b_t Z$, where a_t and b_t are positive scalars and Z is a nonnegative random variable with continuous density and increasing hazard rate. Again, it is easy to verify that as long as a_t and b_t are increasing in t, then conditional stochastic dominance is satisfied.

[13] Although the conditional stochastic dominance ordering defined above was developed independently in the context of proving the optimality of debt financing in Nachman and Noe (1990, Section 7), it is a special case of the uniform conditional stochastic order developed by Whitt (1980). Conditional stochastic dominance is also a generalization of the hazard rate ordering to distributions which may not have densities. As is well known, ordering by hazard rate is a weaker requirement than the well-known monotone likelihood ratio ordering. While the reliability language is suggestive of the above intuition for conditional stochastic dominance **(CSD),** such a rate does not exist even for all distributions that are continuous much less for those with jumps. Consequently, we state the ordering in terms of the more primitive conditional likelihood function given in Equation (2). For further discussions of hazard rate orderings and monotone likelihood ratio orderings, see Ross (1983, Sections 8.3 and 8.4).

Distributions which have an increasing hazard rate include many of the classical statistical distributions, including the uniform, the unimodal (finite mode) gamma and Weibull distributions, and the truncated normal distribution.

On the other hand, because the lognormal need not have a monotone hazard rate, if the cash flow distributions of the type set are translations of a common lognormal variate then it is possible to have the type set ordered by stochastic dominance but not by conditional stochastic dominance. For example, let $Z = \exp(Y)$, where Y has the standard normal distribution. Then the hazard rate of Z is unimodal, first increasing then decreasing. Let $X_t = a_t + Z$. Then the X_t are ordered by stochastic dominance as long as a, is increasing in t, but the hazard rates of the X_t are not ordered and it follows that the X_t are not ordered by conditional stochastic dominance (see note 13).

This last example illustrates the difficulty in Myers and Majluf's original appeal to option pricing theory to argue for the optimality of debt. In this example, the productivity of firm type, which conforms to the distributional basis for the original option pricing model, is unambiguously increasing in firm type. Hence the cost to a higher productivity type of increasing payout at any level of cash flow is greater than the same cost for a lower productivity type. Nevertheless, the relative cost for the high productivity type vis-à-vis the low productivity type of increasing payout at any level of cash flow is not increasing in the level of cash flow. It is this monotonicity of the relative cost of increasing payout that is important for the optimality of debt. This interpretation is developed in the remainder of this subsection, in Section 3.2, and in Section 4.2.

The conditional probability formulation of conditional stochastic dominance given above is quite natural given the usual formulation of stochastic dominance and it provides insight into the interpretation of the ordering. Another equivalent formulation useful from the point of view of deriving new results can be given in terms of functions $r_{t,t'}$, where for $t, t' \in T$ with $t < t'$,

$$r_{t,t'}(x) = \frac{1 - F_{t'}(x)}{1 - F_t(x)}, \qquad x \in \Omega^o. \tag{3}$$

In Section 4.2, we will argue that for a given type t, the quantity $1 - F_t(x)$ is a measure of the marginal cost of increasing payout at the level of cash flow x. The ratio $r_{t,t'}(x)$ then expresses the relative marginal cost of increasing payout at x for type t' vis-à-vis type t. Note that under stochastic dominance, $r_{t,t'}(x) \geq 1$ for all $x \in \Omega^o$; that is, the marginal cost of increasing payout at any cash flow is always at least as great for the higher productivity type. This is another interpretation of stochastic dominance. For debt to always be the optimal security design, however, the relative marginal cost of increasing payout will have to be nondecreasing in the level of cash flow. This property is synonymous with conditional stochastic dominance.

Proposition 4. *Conditional stochastic dominance* **(CSD)** *holds if and only if for every t, $t' \in T$ with $t < t'$, the function $r_{t,t'}$ is nondecreasing on Ω^o. Similarly, strict conditional stochastic dominance* **(SCSD)** *holds if and only if for every t, $t' \in T$ with $t < t'$, the function $r_{t,t'}$ is strictly increasing on Ω^o.*

In the next subsection, we characterize orientation of debt in terms of conditional stochastic dominance. The folklore proposition regarding the optimality of debt is given as a corollary. The role of the relative marginal cost function of Equation (3) in understanding the limits of the intuition underlying this result is also discussed.

3.2 Optimality of debt

Let $S_D = \{s \in S: s(x) = \min\{x, b\}, 0 \leq b \leq \infty\}$, The parameter b is the promised payment (or face value) of the debt contract $\min\{\mathbf{x}, b\}$. Then S_D is the set of pure discount debt security designs. For any $t \in T$, the value $E_t(\min\{\mathbf{x}, b\})$ varies continuously in b from 0 to $E_t(\mathbf{x})$ as b varies from 0 to ∞. Consequently, the set S_D is full. We include the cases $b = 0$ and $b = \infty$ in the definition of S_D so that $\mathbf{0} \in S_D$ and $\mathbf{x} \in S_D$. The following result demonstrates the role of conditional stochastic dominance in determining the optimality of debt financing.

Theorem 5. *Conditional stochastic dominance* **(CSD)** *holds if and only if S_D is oriented for F. Similarly, strict conditional stochastic dominance* **(SCSD)** *holds if and only if S_D is strictly oriented for F.*

We state the implications of S_D being oriented, using the results of Section 2, in the following corollary.

Corollary 6. *Conditional stochastic dominance* **(CSD)** *holds if and only if for any admissible parameters (K, π), there exists s^* Î S_D with $E_p(s^*) = K$ and s^* is an equilibrium outcome for the game $\Gamma(K, p)$ in which all types pool by issuing s^*. Strict conditional stochastic dominance* **(SCSD)** *holds if and only if for any admissible parameters (K, p), there is $s^* \in S_D$ with $E_p(s^*) = K$ and in every equilibrium for the game $\Gamma(K, p)$, all types pool by issuing s^*.*

The above results establish the limits of validity of the folklore proposition that debt minimizes the adverse selection problem of raising external capital. Our results also shed some light on the validity of the intuition underlying the folk wisdom. The received intuition is as follows.[14] There is a lemons problem arising from the mimicry of the high productivity type's security design by the low productivity type. This lemons problem is minimized, so the received intuition goes, by designing the security to pay out the maximum possible in the low-cash-flow states because the likelihood is lower for the high productivity type of being in these low-cash-flow states. While this reasoning is appealing, it suggests that ordering types simply by stochastic dominance **(SD)** should suffice for the debt design to be an equilibrium outcome of the capital raising game. The above result makes it clear that this intuition misses something since in general, stochastic dominance does not imply conditional stochastic dominance. One instance in which it does, of course, is the two-cash-flow example presented in Section 3.1. In this case, the received intuiton is correct.

In the case when there are more than two possible cash flows, it is no longer clear that "paying out the maximum possible in low-cash-flow states" does indeed minimize the adverse selection problem. Although stochastic dominance implies that the total probability of a cash flow below any given value is lower for the high productivity type, any design that pays out a given value must pay out that value or more in all cash flow states above it. As a consequence one needs to examine the conditions under which it is efficient to leave payout constant or to increase it. In doing so, it may well be that optimal security design entails more subtle trade-offs involving the marginal cost to the higher productivity type relative to the lower productivity type in increasing payout at various payout levels. This relative marginal cost is precisely the function defined in Equation (3) above. We take up the task of developing this interpretation in Section 4.

4. Security Design and Mispricing

In the previous section, we provided necessary and sufficient conditions for debt financing to be an (the unique) equilibrium outcome. Thus, we have determined the exact extent to which the folklore proposition holds. However, the equilibrium arguments we used to establish these conditions have a very different flavor from the heuristic argument in favor of debt financing: that debt minimizes the mispricing losses to the high productivity type. The aim of this section is to explore the link between efficiency in terms of mispricing and

[14] The intuition expressed in the text is an extended version of that expressed to us privately by Kose John. We are also grateful to Art Raviv for help with this intuition.

the equilibrium outcomes of the capital raising game. We use the insights derived from this linkage to delineate the conditions under which other security designs may be optimal.

In Section 4.1 we demonstrate the intimate link between orientation and the minimization of mispricing. It is shown in Theorem 7 that a security is a member of an oriented set of securities if and only if it minimizes the ex ante value of the security to the pool of high productivity types, for any division of the set of types, subject to the security raising the requisite capital when priced as though it is issued by the entire pool (set) of types. In addition, the set of securities that are minimizing in this sense form the largest oriented set of securities. This result is important because, combined with Theorems 2 and 3, it implies that pooling equilibrium outcomes are also efficient in that they minimize the ex ante mispricing and because they can be found as solutions to a mathematical programming problem. In Section 4.2 we present a general formulation of this problem and give necessary and sufficient conditions for it that are useful in identifying optimal security designs and in explaining why the intuition of the folk wisdom, discussed in Section 3.2, may not extend to problems with more than just two cash flow possibilites.

In Section 4.3, we utilize the optimality conditions of Section 4.2 to provide a simple counterexample to the folk wisdom where types are ordered by stochastic dominance yet debt fails to be an optimal design. It turns out that other extreme point designs look like forms of layered debt and one such design is optimal in this example. Thus while the minimization of mispricing should not necessarily be the focus of security design in raising external capital, it turns out to be helpful in understanding the limits of the intuition supporting the folk wisdom and in providing the needed corrections to that reasoning.

4.1 Efficiency and oriented sets of securities

As discussed above, the idea that optimal security designs minimize the mispricing of securities issued by high-quality types has a prominent place in the capital structure literature. Such "optimal" designs minimize the true value of the security issued subject to the constraint that it raise the requisite capital. This optimization can be formalized as follows. For a division d of π and a security $s \in S$, the true value or ex ante value of s if issued by the pool T with prior distribution π is $E_{\pi}(s)$, while the true value of s if issued by the upper pool $\{d, \ldots, N\}$, is $E_{p_{d+}}(s)$. In the following we parameterize the amount of capital to be raised by the family of securities that raise it at pooled terms. For the parameters (s, π, d), one description of efficiency with respect to the adverse selection problem is embedded in the optimization problem

$$\mathbf{MV}(s, \pi, d) \qquad \inf_{s^o \in S} E_{\pi d^+}(s^o)$$
$$\text{subject to } E_\pi(s^o) = E_\pi(s).$$

Given a division d of π, a solution to **MV** (s, π, d) minimizes the true value of the security issued by the upper pool of types of the division, subject to its value if issued by the entire pool equaling the value of s. The quantity **MV** $(s, \pi, d) = \inf\{ E_{\pi\, d+}(s^o): s^o \in S, E\pi(s^o) = E_\pi(s)\}$ is the least amount of true value the upper pool of types must give up among all such designs. Note that s is always a feasible solution for **MV** (s, π, d). A relevant question is, when is s a solution of **MV** (s, π, d)? The set of such security designs is of sufficient importance to give it a name. Let

$$S^\wedge = \{s \in S\text{: } s \text{ solves } \mathbf{MV}(s, \pi, d) \text{ for every admissible } \pi \text{ and any nontrivial division } d \text{ of } \pi\}. \tag{4}$$

Clearly, $S^\wedge \neq \emptyset$, since $\mathbf{x} \in S^\wedge$. The following result shows that not only is $S^\wedge$ oriented for F, it is the largest oriented set of securities for F.

Theorem 7. *$S^\wedge$ is oriented for F and if S^o is oriented for F, then S^o Í $S^\wedge$.*

It is useful to interpret this result through the use of Theorem 2. If S^o is an oriented set of securities, then by Theorem 2, if $s^o \in S^o$, for any admissible pair (K, π) with $K = E\pi(s^o)$, s^o is a pooling equilibrium outcome of the game $\Gamma(K, \pi)$. By Theorem 7, $S^o \subseteq S^\wedge$ and it follows that s^o also minimizes $E_{p\, d+}(s)$ subject to $E\pi(s) = K$ $(= E\pi(s^o))$, for any admissible π and any division d of π. In particular, for the division $d(s^o) = \min\{ t \in T: E_t(s^o) \geq K\}$, s^o minimizes the true value of the security issued in equilibrium to the pool of types that are underpriced in this equilibrium. It follows from the capital constraint that s^o also maximizes the true value of the security issued in equilibrium to the pool of types who are overpriced in equilibrium and hence also that s^o minimizes the absolute ex ante mispricing in the equilibrium.

Conversely, suppose that for any admissible prior π and any nontrivial division d of π, s^o minimizes $E_{p\, d+}(s)$ subject to $E p(s) = K$, with $K = E_p(s^o)$. Then by Theorem 7, s^o belongs to an oriented set of securities, namely the largest oriented set of securities $S^\wedge$. Hence, it follows from Theorem 2 that for any admissible pair (K, π) with $K = E\pi(s^o)$, s^o is a pooling equilibrium outcome of the game $\Gamma(K, \pi)$.

Thus, Theorem 7 offers us a way to identify pooling equilibrium outcomes of the capital raising game by solving a parametric family of linear programming problems. We turn to this task in the following section.

4.2 Optimality conditions for security design

The problems **MV** (s, π, d) have a very simple structure. In fact, for a given division d of A, the problem **MV** (s, π, d) depends only on the expectation operators $E_{p\ d+}$ and E_p or the distributions corresponding to them. Hence it suffices to determine the optimality conditions for a problem of the following type. Let L and H denote two cumulative distribution functions on the set Ω. We can think of H as the cash flow distribution of the higher pool of productivity types corresponding to $\{d, \ldots N\}$, and L as the cash flow distribution of the lower pool of productivity types corresponding to $\{1, \ldots, d - 1\}$. Let p be the probability of the higher pool of types, corresponding to π^d. Corresponding to nontrivial divisions d of π, we require only that $0 < \bar{p} < 1$. Let $J = pH + (1 - p)L$ denote the distribution function for the entire pool, and let E_H, E_L, and E_J denote expectations with respect to the distributions H, L, and J, respectively.

Let **MV** (L, H) be the problem **MV** (s, π, d), for some s such that $E_p(s) = K$. This is an infinite-dimensional linear programming problem that can be analyzed as follows. The limited liability and monotonicity restrictions (LL), (M), and (MR) defining S imply that S is a subset of the space of Lipschitz continuous and hence absolutely continuous functions on Ω. It follows that for every $s \in S$, the derivative s' exists and is Lebesgue measurable with $0 \leq s' \leq 1$, and with $s(x) = \int_0^x s'(r)\,dr$, for each $x \geq 0$. More importantly, for any distribution function F on the nonnegative reals, by integration by parts we have that $E_F(s) = \int s'(1 - F)\,d\mathbf{x}$, where $d\,x$ denotes integration with respect to Lebesgue measure. We can then restate the problem **MV** (L, H) as follows:

$$\inf \int_\Omega w(1 - H)\, d\mathbf{x} \tag{5}$$

$$\text{subject to:}\quad 0 \leq w \leq 1, \tag{6}$$

$$K \leq \int_\Omega w(1 - J)\, d\mathbf{x}. \tag{7}$$

In this formulation, the security design problem is stated in terms of the derivative w of the security. This derivative is implicitly restricted to be a Lebesgue measurable function on Ω. The explicit bounds in the constraint (6) are a restatement of the limited liability constraint (LL) and the monotonicity conditions (M) and (MR). This implicit restriction and explicit bounds constrain the derivative to a convex set of measurable functions that can be ignored in forming the Lagrangian function for **MV** *(L, H).*

The constraint (7) is a restatement that the security valued at pooled terms be worth at least the required capital *K,* the capital constraint. Note that we have expressed this constraint here as an inequality for convenience in referencing optimality conditions. It has no significance for the set of solutions. The capital constraint is a single scalar-valued constraint that must be included in the Lagrangian function for **MV** *(L, H).* After some simplification, this Lagrangian is the function

$$\Lambda(w, \gamma) = \int_{\Omega} w(x) Q(\gamma, x)\, dx + \gamma K, \tag{8}$$

where γ is a nonnegative real number that is the multiplier for the capital constraint (7), and where

$$Q(\gamma, x) = 1 - H(x) - \gamma(1 - J(x)), \qquad x \in \Omega. \tag{9}$$

In order to develop as much intuition as possible, it is helpful at this point to interpret the components of the function Q. From the objective function (5), the quantity $1 - H(x)$ is the cost to the high productivity pool of an infinitesimal increment $w(x)\,dx$ in payout at x, since by monotonicity *(M)* such an increment will have to be paid out at all $x' \geq x$. Thus $1 - H(x)$ is the marginal cost of increasing payout at x. From the capital constraint (7), the term $1 - J(x)$ is the benefit of an infinitesimal increment $w(x)\,dx$ in payout at x, where benefit here is in terms of raising the required capital at pooled terms. Thus $1 - J(x)$ is the marginal benefit of increasing payout at x. Interpreting the Lagrange multiplier γ as measuring the change in intrinsic value foregone by the high productivity pool per unit of change in required capital, the function $Q(\gamma, x)$ in (9) is the net marginal cost of increasing payout at x. Determining the infimum of the Lagrangian $\Lambda(w, \gamma)$ over w gives the following characterization of solutions to **MV** *(L, H).*

Theorem 8. *s* solves* ***MV*** *(L, H) if and only if there exists* g^{*} ^{3} *0 such that for almost all* $x \in W$ *(with respect to Lebesgue measure), the derivative w* of s* satisfies the conditions that*

$$w^*(x) \in \begin{cases} \{1\} & \text{if } Q(\gamma^*, x) < 0, \\ [0, 1] & \text{if } Q(\gamma^*, x) = 0, \\ \{0\} & \text{if } Q(\gamma^*, x) > 0, \end{cases} \tag{10}$$

and $E_J(s^*) = K$. *In this case the minimum value is* $MV(L, H) = \int_{\{Q(\gamma^*)\leq 0\}} Q(\gamma^*, x)\, dx + \gamma^* K$, *where* $\{Q(\gamma^*) \leq 0\} = \{x \in \Omega: Q(\gamma^*, x) \leq 0\}$.

The characterization in Theorem 8 shows that optimal security designs have a simple form. In determining when to increase the payout in a security, there are two considerations involved. One is that an increase in payout increases the instrinsic value of the security sold to outside suppliers of capital. The cost of doing this is that the high productivity pool looses this intrinsic value. The other consideration is that an increase in payout increases the pooled value of the security, contributing toward the attainment of the capital constraint. Optimal security designs are determined by comparing the marginal cost and the marginal benefit of increasing payout at each level of cash Bow at the shadow cost of capital $\gamma*$. By Expression (10), these designs have the character that they are either not increasing ($w* = 0 \leftrightarrow Q(g*) > 0 \leftrightarrow$ marginal cost of increasing payout is greater than the marginal benefit), or they are increasing at the maximum rate ($w* = 1 \leftrightarrow Q(\gamma*) < 0 \leftrightarrow$ marginal cost of increasing payout is less than the marginal benefit).[15]

This economic interpretation of the optimality conditions for security design can help in explaining why the intuition of the folk wisdom, discussed in Section 3.2, may not extend to problems with more than just two cash flow possibilities. The character of optimal designs depends critically on the behavior of the net marginal cost function $Q(\gamma*)$ or equivalently on the relative marginal cost to the higher productivity pool over the lower productivity pool in increasing payout, given by the ratio $(1 - H)/(1 - L)$. For example, debt will be optimal if and only if the net marginal cost of increasing payout is nonpositive below the promised payment and positive above the promised payment. The net marginal cost of increasing payout as a function of the level of cash flow has this monotonic behavior if and only if the relative marginal cost ratio $(1 - H)/(1 - L)$ (or equivalently, $(1 - H)/(1 - J)$) is nondecreasing.[16] The relative marginal cost ratio $(1 - H)/(1 - L)$ (respectively, $(1 - H)/(1 - J)$) is the ratio $r_{t,t'}$, given in (3) for the case when the distribution for t' is H and the distribution for t is L (J). Proposition 4 says that conditional

[15] There are no restrictions on $w*$ imposed by the optimality conditions of Theorem 8 for those x such that $Q(\gamma*, x) = 0$, save that $0 \leq w*(x) \leq 1$ and $E_j(s*) = K$, where $s*$ is the integral of $w*$, because the value of the lagrangian function $\Lambda(w*, \gamma*)$ is unaffected. While there may be solutions to **MV** (L, H) that have $0 < w* < 1$ on $(Q(\gamma*) = 0\}$, if there is. there is also a solution $s*$ with **derivative** $w* = I_{\{Q(\gamma*)<0\} \cup A}$, for some measurable subset $A \subseteq \{Q(\gamma*) = 0\}$—in other words, a solution whose derivative $w* \in \{0, 1\}$.

[16] Whether or not the high productivity pool is compared to the low productivity pool or to the entire pool is of no significance here since H conditionally stochastically dominates L if and only if if conditionally stochastically dominates J. The ratio $(1 - H)/(1 - L)$ may also be interpreted as the relative advantage of the high productivity pool over the low productivity pool in supporting promised payments at various levels of cash flows. For debt to be optimal, this relative advantage must be increasing in the level of cash flow. We are grateful to the referee for suggesting this interpretation.

stochastic dominance is equivalent to this ratio being nondecreasing with the level of cash flow x. In this case, the debt contract design solves **MV** *(L, H)*.

When there are more than two possible cash flows, the monotone behavior of $r_{t,t'}$ required for optimality of debt may fail when the distributions are only ordered by stochastic dominance. Indeed, it may be that at higher levels of cash flows, following payout not increasing for some levels of cash flow, the relative marginal cost of increasing payout may fall below the shadow cost of capital γ^* making increasing payout again optimal. In the next subsection, we present an example where this is the case.

4.3 Optimality of other security designs

Our initial intuition in studying the optimality of debt was that it is natural to look at extreme points of S for possible solutions. It is easy to see that each debt security is an extreme point of S. Note also that equity, which is of the form b **x,** is never an extreme point of S and hence is not likely to be an optimal security design. But if debt is not optimal, are there any other extreme points of S that could be optimal? The answer is yes. From the *optimality* conditions of Theorem 8, an optimal security design could increase at the maximum rate of one or not increase at all.

Consider a security that acts this way in successive intervals. For example, suppose that $x_i \in \Omega$, $i = 1, 2, 3$, with $x_1 < x_2 < x_3$, and consider the design $s(x) = x$, for $0 \leq x \leq x_1$; $s(x) = x_1$, for $x_1 \leq x < x_2$; $s(x) = x - (x_2 - x_1)$, for $x_2 \leq x < x_3$; and $s(x) = x_3 - (x_2 - x_1)$, for $x_3 \leq x$. Clearly $s \in S$, and is an extreme point of S since $s'(x) \in \{0, 1\}$ (whenever $s'(x)$ exists). This security design looks like a combination of senior debt with promised payment x_1, with a second layer of junior debt with promised payment $x_3 - x_2$, but for which equity or the residual claim is not subordinate. Equity is subordinate to the senior claim, but equity and the junior claim share in the cash flows in excess of x,, with the junior claim getting nothing until $x > x_2$, then the junior claim gets the residual $x - x_2$ and equity gets the fixed payment of $x_2 - x_1$, for $x_2 \leq x < x_3$. At x_3 and above, the junior claim gets the promised payment of $x_3 - x_2$ and equity gets the residual $x - (x_3 - x_2) - x_1$.

We will construct an example here where such a design is optimal for **MV** *(L, H)*. Take $\Omega = [0, 4]$. For simplicity, we will describe the cash flows for the high productivity pool and for the low productivity pool by giving the distributions H and L. There are four possible cash flows, $x = i$, $i = 1, \ldots, 4$. Given ε, $0 < \varepsilon < ¼$, the distribution L will have probability ½ - ε each on 1 and 3 and ε each on 2 and 4. The distribution H will have probability ½ - ε each on 2 and 4 and ε each on 1 and 3. Clearly H dominates L in the sense of first-degree stochastic dominance, but the ratio $(1 - H)/(1 - L)$ is not monotonic. On the interval [0, 1), $(1 - H)/(1 - L) = 1$. On the interval [1, 2), $(1 - H)/(1 - L) = 2(1 - \varepsilon)/(1 + 2\varepsilon) > 1$. On the interval [2, 3), $(1 - H)/(1 - L) = 1$. On the interval [3, 4), $(1 - H)/(1 - L) = (1 - 2\varepsilon)/2\varepsilon > 1$.

We take $p = ½$ and $K = 1.5$, to get one specific version of **MV** (L, H). Then $J = .5H + .5L$ has probability ¼ assigned to each possible cash flow. It is easy to verify from (10) that the net marginal cost of increasing payout Q behaves as follows: $Q(\gamma, x) = 1 - \gamma$, for $0 \leq x < 1$; $Q(g, x) = 1 - 3\gamma/4 - \varepsilon$, for $1 \leq x < 2$; $Q(g, x) = (1 - \gamma)/2$, for $2 \leq x < 3$; and $Q(\gamma, x) = ½ - \gamma/4 - \varepsilon$, for $3 \leq x < 4$. For γ such that $1 < \gamma < 4(1 - \varepsilon)/3$, it follows that $Q(\gamma)$ is first negative, then positive, then negative, then positive again. For any such value γ^* of γ, the optimality conditions of Theorem 8 suggest that the optimal security design s^* will have derivative equal to one on the intervals [0, 1) and [2, 3) and equal to zero on the intervals [1, 2) and [3, 4). It is easily verified in this case that $E_H(s^*) = E_J(s^*) = E_L(s^*) = 1.5$; that is, s^* is value invariant. For a promised payment of $\frac{5}{3}$, we have that $E_J(\min\{\mathbf{x}, \frac{5}{3}\}) = 1.5$. However, $E_H(\min\{\mathbf{x}, \frac{5}{3}\}) = (5 - 2\varepsilon)/3 > 1.5$, for the assumed range of ε. Thus an optimal security design in this case is an extreme point of S that has some of the features of layered debt, but simple debt is not an optimal security design despite the fact that H stochastically dominates L.

If we take $T = \{1, 2\}$ and $F_1 = L$ and $F_2 = H$, where L and H are given above, then admissible capital levels are K for which $1 < K < E_L(\mathbf{x}) = 2 + 2\varepsilon$ and admissible priors are parameter values of p, $0 < p < 1$. The set S^ of (4) and Theorem 7 then includes all of the layered debt such as securities described above.

5. Summary and Directions for Future Research

We have developed a model of raising external capital under asymmetric information and used this model to analyze the optimal design of securities in general in the presence of adverse selection and the folklore proposition that debt is an optimal design in particular. Our framework accommodates a very general class of cash flow distributions, a multiplicity of firm types, and a rich menu of possible security designs. In this framework, we demonstrated that for a class of securities to be optimal in the sense of supporting pooling equilibrium outcomes of the capital raising game, it is necessary and sufficient for the class to be an oriented set of securities. Further, we showed that the family of pure discount debt designs is oriented if and only if productivity types are ordered by conditional stochastic dominance.

Next, we characterized the largest oriented set of securities in terms of minimization of mispricing, which permitted us to characterize optimal security designs in an efficiency sense using mathematical programming. This allowed us to construct a simple but otherwise unremarkable example in which a form of layered debt is optimal but pure discount debt is not, despite the fact that the cash flow distribution of the high productivity type(s) stochastically dominates the distribution of the low productivity type(s).

While we have delineated in a precise sense the limits of the folklore proposition that debt minimizes the adverse selection problem of raising external capital, there are extended versions of this proposition in the folklore that need further clarification. For example, in addition to arguing that debt is a dominant security design, Myers and Majluf (1984; p. 207) go further to assert that "the general rule seems to be: better to issue safe securities than risky ones." A natural question to ask is, for what senses of "safe" versus "risky" is this extended version valid? Indeed, this "general rule" supports the pecking order hypothesis of capital structure that has considerable intuitive appeal and descriptive power. We feel that the framework developed in this article is an important starting point for the clarification of this hypothesis and the associated extended versions of the folklore proposition.

Another direction for extending this work that would be useful in the above clarifications would be to determine distributional restrictions that yield other interesting oriented sets of security designs. By Theorems 2 and 3 these sets of securities are optimal security designs. In our analysis we focused our attention on the debt contract and determined the distributional restrictions required for the optimality of this design. An alternative approach would be to start with a set of reasonable distributional restrictions and then determine the set of oriented designs $S^\wedge$ defined in Equation (4), as we did in the example of Section 4.3. This is especially important given that the distributional assumptions required to ensure the optimality of the debt contract, while not terribly restrictive, by no means exhaust the set of reasonable parameterizations of uncertainty. Our results relating mispricing minimization to orientation, should also prove useful in extending the analysis in this direction.

A major emphasis in correctly delineating the scope of the folklore proposition that debt minimizes the adverse selection problem of raising external capital has been the role of off-equilibrium inferences that support debt pooling outcomes as equilibria of the capital-raising game. Another question for future research is how to characterize firm issuance behavior in the absence of interesting oriented sets of securities [when $S^\wedge$ of (4) is not interesting]. In this case, partial pooling equilibria will obtain. The structure of these partial pooling equilibria will in general be quite complex as the productivity of types in one pool will never uniformly dominate that of any other. In fact, even if types are ordered by stochastic dominance, the cash flow distributions of the pools will not be ordered by first-order stochastic dominance. This fact will greatly complicate any attempt to provide simple characterizations of equilibrium behavior. Nevertheless, a characterization of issuance behavior in the absence of interesting pooling behavior will depend even more critically on off-equilibrium inferences and consequently would be a natural extension of this work and a significant contribution to the security design literature.

Appendix. Proofs and Technical Results

In this appendix, we give outlines of proofs of the results in the text and of technical results required by them. Detailed proofs are available from the authors on request. An outline of the proof for Proposition 1 is given in the discussion following its statement.

Proofs of results in Section 2

To establish the remaining results in the text, we need some basic results about divisions and oriented sets. The first is an alternative characterization of oriented sets of securities in terms of divisions involving just two types. The other two results give useful implications of orientation.

Proposition A.1 *A set S^o Í S is orientedfor F if and only if for every t, t' Î T, $t < t'$, if s^o Î S^o, s Î S, and if $E_{t'}(s)$ £ $E_{t'}(s^o)$ then $E_t(s)$ £ $E_t(s^o)$. A set S^o Í S is strictly orientedfor F if and only if it is oriented for F and for every t, t' Î T, $t < t'$, if s^o Î S^o and s Î S, with $s \neq s^o$, and if $E_{t'}(s)$ £ $E_{t'}(s^o)$ then $E_t(s) < E_t(s^o)$.*

Proof Follows directly from Definition 2. □

Lemma A.2. *If d is a nontrivial division of p and S^o is oriented for F, then for s^o Î S^o such that s^o ¹ **x**, if s Î S and $E_{p\ d-}(s - s^o)$ ³ 0, then $E_{p\ d+}(s - s^o)$ ³ 0. If S^o is (strictly) oriented for F, then for every s^o Î S^o, if $E_{p\ d-}(s - s^o)$ ³ 0, then $E_{p\ d+}(s - s^o)$ ³ $(>)\ 0$, for all s Î S, $s \neq S^o$.*

Proof Let d be a nontrivial division of π and suppose that S^o is oriented for F. By Definition 2, if $s^o \in S^o$ and $s \in S$, we may assume that $E_{p\ d-}(s - s^o) = 0$. Suppose that $s^o \neq$ **x.** For each positive integer n, let $s_n = (1 - 1/n)s + (1/n>$ **x.** It follows that $E_{p\ d-}(s^o) <$

$(1 - 1/n)E_{\pi d^-}(s) + (1/n)E_{\pi d^-}(\mathbf{x}) = E_{\pi d^-}(s_n)$, and hence that $E_{\pi d^+}(s_n - s^o) > 0$, $n \geq 1$. By the Lebesgue dominated convergence theorem, we have $\lim_n E_{\pi d^+}(s_n - s^o) = E_{\pi d^+}(s - s^o) \geq 0$.

Under the condition that $s \in S$, $s \neq s^o$, it follows that either $s^o \neq \mathbf{x}$ or $s \neq \mathbf{x}$. In either case, we can proceed as above to get the conclusion that $E_{\pi d^+}(s - s^o) \geq 0$. The strict inequality follows by Definition 2. □

Lemma A.3. *If d is a nontrivial division of p and S^o is oriented for F, then for $s^o \in S^o$ such that $s^o \neq \mathbf{x}$, if $s \in S$, $E_{\pi d^-}(s - s^o) \geq 0$, then $E_{\pi d^-}(s - s^o) \leq E_{\pi d^+}(s - s^o)$. If S^o is (strictly) oriented for F, then for $s^o \in S^o$, $E_{\pi d^-}(s - s^o) \geq 0$, then $E_{\pi d^-}(s - s^o) \leq (<) E_{\pi d^+}(s - s^o)$, for all $s \in S$ with $bs \neq s^o$, for every b such that $0 < b \leq 1$.*

Proof: For d, S^o, s^o, s as hypothesized, Lemma A.2 implies that $E_{\pi d^+}(s - s^o) \geq 0$. Thus there exists b, $0 \leq b \leq 1$, such that $E_{\pi d^+}(bs - s^o) = 0$. Since S^o is oriented, we have $E_{\pi d^-}(bs - s^o) \leq 0$. Thus,

$$\begin{aligned}
&E_{\pi d^+}(s - s^o) - E_{\pi d^-}(s - s^o)\\
&\quad = \{E_{\pi d^+}(s - s^o) - E_{\pi d^+}(bs - s^o) - [E_{\pi d^-}(s - s^o) - E_{\pi d^-}(bs - s^o)]\}\\
&\qquad + [E_{\pi d^+}(bs - s^o) - E_{\pi d^-}(bs - s^o)]\\
&\quad = (1 - b)\{E_{\pi d^+}(s) - E_{\pi d^-}(s)\} + [E_{\pi d^+}(bs - s^o)\\
&\qquad\qquad - E_{\pi d^-}(bs - s^o)].
\end{aligned}$$

By stochastic dominance, the first term on the right is nonnegative. By the choice of b, the second term on the right is also nonnegative. The first claim is then proved. Under the condition that $bs \neq s^o$, all $b \in (0, 1]$, by Lemma A.2, we get that $E_{\pi d^+}(s - s^o) \geq 0$. Choosing b as above and using Lemma A.2 again we get that $E_{\pi d^-}(bs - s^o) \leq (<)$ 0, giving (strict) positivity as above. □

Proof of Theorem 2. (Sufficiency). Suppose that S^o is oriented for F. Let (K, π) be an admissible parameter pair and suppose that there is $s^o \in S^o$ with $E_p(s^o) = K$. Let (μ^*, s^*, P^*) be such that

$$\mu^*(s^o) = \pi,\ \mu^*(s) = q,\ s \neq s^o,$$

$$\text{where } q = (q_1, \ldots, q_N)$$

$$\text{and } q_1 = 1,\ q_t = 0,\ t > 1; \tag{A1.a}$$

$$s^*(t) = s^o, \qquad \text{all } t \in T; \tag{A1.b}$$

$$P^*(s) = E_{\mu^*(s)}(s). \tag{A1.c}$$

We claim that (s^*, μ^*, P^*) is an equilibrium for $\Gamma(K, \pi)$. Clearly (s^*, μ^*, P^*) satisfies (BC) and (CR). That s^o is a best response for every $t \in T$ given that every other type issues s^o follows from the fact that by conditions (A1.a), any other response by t is viewed as though t is the lowest type. Thus (s^*, μ^*, P^*) satisfies (SR) as well. It remains to show that (D1) is satisfied.

To do this, we will show that for every $s \neq s^o$, it is never the case that $D(1 \mid s) \cup D^o(1 \mid s)$ is strictly contained in $D(t \mid s)$ for any $t > 1$. If $E_1(s - s^o) < 0$, then $D(1 \mid s) = [K, \infty)$, and hence for every t, $D(t \mid s) \subseteq D(1 \mid s) \cup D^o(1 \mid s)$. If $E1(s - s^o) \geq 0$, then by Lemma A.3, $E_t(s - s^o) \geq E_1(s - s^o) \geq 0$, all $t \in T$. But for every $s \in S$ with $P^*(s) \geq K$,

$$(V(t,s,P^*(s)) - V(t,s^o,P^*(s^o))) - (V(1, s,P^*(s)) - V(1, s^o,P^*(s^o))) \\ = E_1(s - s^o) - E_t(s - s^o) \leq 0,$$

implying that

$$V(t,s,P^*(s)) - V(t,s^o,P^*(s^o)) \leq V(1, s,P^*(s)) - V(1, s^o,P^*(s^o)).$$

Thus $D(t \mid s) \subseteq D(1 \mid s)$, for all $t > 1$, and the proof of sufficiency is complete.

(Necessity). Suppose that for any admissible parameters (K, p), if $s^o \in S^o$ and $E_p(s^o) = K$, there is an equilibrium for the game $\Gamma(K, p)$ in which all types pool by issuing s^o. If S^o is not oriented for F then by Proposition A.1, there is a pair of types $t, t' \in T$, with $t < t'$, and some $s^o \in S^o$ and $s \in S$ such that $E_{t'}(s - s^o) \leq 0$, but $E_t(s - s^o) > 0$. By scaling s down if necessary, we can assume that both inequalities are strict; that is, $E_{t'}(s - s^o) < 0 < E_t(s - s^o)$. By stochastic dominance and the initial conditions **(IN),** $0 < E_t(s^o) < E_t(s) \leq E_{t'}(s) < E_{t'}(s^o)$. The second and fourth inequalities imply that $E_t(s^o) < E_t$ **(x)** and that $E_{t'}(s) < E_{t'}$ **(x),** respectively. Thus there is a number p_t, $0 < p_t$, < 1, such that $0 < \pi_t E_t(s^o) + (1 - \pi_t) E_{t'}(s^o) \leq E_{t'}(s)$ and $\pi_t E_t(s^o) + (1 - \pi_t) E_{t'}(s^o) < E_t$ **(x).** Let π be the prior that assigns probability π_t, to type t and $1 - \pi_t$ to type t', and take $K = E_p(s^o)$. Clearly (K, p) is admissible. We claim that the outcome in which all types issue s^o and raise K is not an equilibrium outcome of the game $\Gamma(K, p)$.

It follows that for the given security s,

$$D(t \mid s) \cup D^0(1 \mid s) = \{P \geq K: P \geq K + E_t(s - s^o)\} \subsetneq (K, \infty),$$

because by assumption $E_t(s - s^o) > 0$. But since $E_{t'}(s - s^o) < 0$, we have that $D(t' \mid s) = [K, \infty)$. Thus, we have that $D(t \mid s) \cup D^o(t \mid s) \subset D(t' \mid s)$, the containment being strict. By (D1), $\mu^*(s)(t) = 0$. It follows that $P^*(s) = E_{t'}(s) \geq K$ and hence $V(t', s, P^*(s)) = E_{t'}$ **(x)** $- K$. But $K = E_p(s^o) \leq E_{t'}(s) < E_{t'}(s^o)$. Thus

$$V^*(t') = E_{t'}(\mathbf{x}) - E_{t'}(s^o) < E_{t'}(\mathbf{x}) - K = V(t', s, P^*(s)).$$

This contradicts the sequential rationality of s^o and hence pooling at s^o and raising K could not be an equilibrium outcome. □

To prove the strict version of this result, Theorem 3, we need a series of technical results that help establish that there are only pooling equilibria and that there is only one pooling equilibrium outcome when there is a strictly oriented set of securities. The first few results are geared at discerning when strict stochastic dominance yields strict inequalities. In particular, the first result gives sufficient conditions for a strict inequality in the value of a security across types. Recall that the monotonicity and limited liability restrictions (M), (MR), and (LL) entail that each security design $s \in S$ is absolute continuous. For $s \in S$, let $s^+ = \{x \in \Omega^o : s'(x) > 0\}$, the set (in Ω^o) where the derivative s' of s is positive.

Lemma A.4. *Assume strict stochastic dominance* **(SSD).** *If $s \in S$ and s^+ has positive Lebesgue measure, then for any t, t' Î T, with $t < t'$, we have that $E_t(s) < E_{t'}(s)$.*

Proof By integration by parts $E_{t'}(s) - E_t(s) = \int_{s^+} s'(F_t - F_{t'})\,dx$. □

The next result shows that the sufficiency condition in Lemma A.4 is automatically satisfied for securities issued in equilibrium. For the sake of brevity, all parameter pairs (K, π) appearing in the results below are assumed to be admissible.

Lemma A.5 *If (s^*,μ^*,P^*) is an equilibrium for $\Gamma(K, p)$, then for each $t \in T$, $s^*(t)^+$ has positive Lebesgue measure.*

Proof Follows from competitive rationality (CR), integration by parts, and the initial conditions **(IN).** □

Lemma A.6. *Suppose (s^*,μ^*,P^*) is an equilibrium for $\Gamma(K, \pi)$. Let $s', s'' \in s^*(T)$, with $s' \neq s''$. If $t' \in s^{*-1}(s')$ and $t'' \in s^{*-1}(s'')$ are such that $E_{t''}(s'') \leq P^*(s'')$ and $P^*(s') \leq E_{t'}(s')$ and at least one of these inequalities is strict, then $t'' < t'$. If strict stochastic dominance* **(SSD)** *holds and if $t' \in s^{*-1}(s')$ and $t'' \in s^{*-1}(s'')$ are such that $E_{t''}(s'') \leq P^*(s'')$ and $P^*(s') \leq E_{t'}(s')$, then $t'' < t'$.*

Proof: If $t' < t''$, by stochastic dominance, $E_{t'}(s'') \leq E_{t''}(s'')$. Then $P^*(s'') - E_{t'}(s'') \geq P^*(s'') - E_{t''}(s'') \geq 0 \geq P^*(s') - E_{t'}(s')$, and one of the last inequalities is strict. Hence t' can do better by mimicking t'', contradicting the fact that s' is a best response for t'. We conclude that $t'' < t'$. The same conclusion holds under **(SSD)** because in that case, if $t' < t''$, then by Lemmas A.4 and A.5, $E_{t'}(s'') < E_{t''}(s'')$. □

Lemma A.6 says that if there are two distinct securities issued in equilibrium, one of which is overvalued and one of which is under-valued, then the overvalued one must be issued by the lower productivity type because of the lower type's incentive to mimic. The existence of types $t' \in s^{*-1}(s')$ and $t'' \in s^{*-1}(s'')$ such that $E_{t''}(s'') \leq P^*(s'')$ and $P^*(s') \leq E_{t'}(s')$ is not a restrictive condition since for the capital raising game, for any security s, $P^*(s)$ is the average value of s over those types in the support of $\mu^*(s)$. Hence if $s \in s^*(T)$, there is at least one type $t \in s^{*-1}(s)$ such that $P^*(s) \leq E_t(s)$ and at least one type $t \in s^{*-1}(s)$ such that $P^*(s) \geq E_t(s)$. The statement that a security is *mispriced* in an equilibrium will mean that (it is issued in the equilibrium and) at least one of these inequalities is strict. This is the hypothesis of the first claim of Lemma A.6. Restated, this gives the following result.

Lemma A.7. *Suppose (s^*,μ^*,P^*) is an equilibrium for $\Gamma(K, \pi)$. If s', $s'' \in s^*(T)$, with $s' \neq s''$, and if either s' or s'' is mispriced, then $\max s^{*-1}(s') > \min s^{*-1}(s'')$. If strict stochastic dominance* ***(SSD)*** *holds and s', $s'' \in s^*(T)$, with $s' \neq s''$, then $\max s^{*-1}(s) > \min s^{*-1}(s'')$.*

Proof Follows directly from Lemma A.6. □

We can use Lemma A.7 to show that there will be some pooling in any equilibrium of the capital raising game if there is any mispricing. There is mispricing, in particular, if strict stochastic dominance **(SSD)** holds.

Theorem A.8. *Let (s^*,μ^*,P^*) be an equilibrium for $\Gamma(K, p)$. If there is some mispricing in this equilibrium, then there exists $t \in s^{*-1}(s^*(N))$ such that $t \neq N$. If there is no mispricing in this equilibrium, then $E_t(s^*(N)) = E_{t'}(s^*(N))$, all $t, t' \in T$. If strict stochastic dominance* ***(SSD)*** *holds, there is $t \in s^{*-1}(s^*(N))$ such that t ¹ N and $E_t(s^*(N)) < P^*(s^*(N)) < E_N(s^*(N))$.*

Proof If $s^{*-1}(s^*(N)) = \{N\}$, then $P^*(s^*(N)) = E_N(s^*(N))$; that is, $s^*(N)$ is not mispriced. Then there exists $s' \in s^*(T)$, such that $s' \neq s^*(N)$ and s' is mispriced. By Lemma A.7, we must have that $\max s^{*-1}(s') > \min s^{*-1}(s^*(N)) = N$, a contradiction. This proves the first statement of the theorem. To prove the second statement, suppose there is no mispricing, but there is a $t \neq N$ such that $E_t(s^*(N)) \neq E_N(s^*(N))$. By **(SD)** $E_t(s^*(N)) < E_N(s^*(N))$. For this type t, however,

$$\begin{aligned} &V(t, s^*(N), P^*(s^*(N))) - V(t, s^*(t), P^*(s^*(t))) \\ &\quad = P^*(s^*(N)) - E_t(s^*(N)) - (P^*(s^*(t)) - E_t(s^*(t))) \\ &\quad = P^*(s^*(N)) - E_t(s^*(N)) > P^*(s^*(N)) - E_N(s^*(N)) = 0, \end{aligned}$$

where the second and last equality follow from the assumption that there is no mispricing. This denies that $s^*(t)$ is a best response for t. Hence $E_t(s^*(N))$ must be constant in t. The final statement follows from Lemmas A.4 and A.5 and the first two statements of the theorem. □

The first statement of Theorem A.8 says that in any equilibrium where there is some mispricing, some type mimics the security design of the highest type. The second statement says that to have an equilibrium where there is no mispricing (and hence at most only trivial pooling), it must be that the high type must issue a value invariant security. This implies that there cannot be a separating equilibrium in which the high type issues a security that is not value invariant. Of course, when the types are ordered by strict stochastic dominance **(SSD),** there can be no value invariant securities issued in equilibrium, and hence there will be underpricing and pooling with the highest type. This underpricing of the highest type in equilibrium means that the security issued by the highest type will raise only the requisite capital. This is the next result.

Lemma A.9. *Assume strict stochastic dominance* ***(SSD)*** *holds. If (s^*, μ^*, P^*) is an equilibrium for $\Gamma(K, p)$, then $P^*(s^*(N)) = K$.*

Proof. Given an admissible parameter pair (K, π), let (s^*,μ^*,P^*) be an equilibrium for $\Gamma(K, \pi)$. Clearly, $P^*(s^*(N)) \geq K$. Consider the scaled contracts $\gamma\, s^*(N)$, for $\gamma \in (0, 1)$. It is possible (but tedious) to show that if $P^*(S^*(N)) > K$, then there is at least one value of $\gamma \in (0, 1)$ such that $\mu^*(\gamma\, s^*(N))(N) = 1$, and N would be better off deviating to the scaled contract $\gamma\, s^*(N)$ than issuing $s^*(N)$. This contradiction establishes the result. □

The final three results in preparation for the proof of Theorem 3 deal with consequences of strictly oriented sets of securities. The first one states that in this context, security designs that satisfy the capital constraint are unique.

Lemma A.10. *If S^o is strictly oriented for F, then for any admissible parameterpair (K, p), the solutions s^o Î S^o with $E_p(s^o) = K$ are unique.*

Proof Let (K, π) be admissible. Suppose that $s', s'' \in S^o$, $s' \neq s''$, and $E_p(s') = E_p(s'') = K$. If $E_N(s') \leq E_N(s'')$, because S^o is strictly oriented for F, it follows from Proposition A.1 that $E_t(s') < Et(s'')$, for all $t < N$. This implies that $E_\pi(s') < E_p(s'')$, a contradiction. Assuming that $E_N(s'') \leq E_N(s')$, yields the same conclusion. □

The next result states that security designs from a strictly oriented set of securities that are issued in equilibrium cannot be priced below any other security design issued in equilibrium when one of the designs in question is mispriced.

Lemma A.11. *Suppose S^o is strictly oriented for F. Let (s^*, μ^*, P^*) be an equilibrium for Γ (K, p) for which there is $s' \in s^*(T) \cap S^o$. If $s'' \in s^*(T)$ and $P^*(s') \leq P^*(s'')$ and if either s' or s'' is mispriced, then $s' = s''$.*

Proof Suppose that $s' \in s^*(T) \cap S^o$, but that there is $s'' \in s^*(T)$ such that $s' \neq s''$ and $P^*(s'') \geq P^*(s')$. If either s' or s'' are mispriced, if $t' \in s^{*-1}(s')$ and $P^*(s') \geq E_{t'}(s')$, by Lemma A.7, we can find $t'' \in s^{*-1}(s'')$ so that $t' < t''$. Because s' is a best response for t', $E_{t'}(s'' - s') \geq P^*(s'') - P^*(s')$. Similarly, because s'' is a best response for t'', $E_t''(s'' - s') \leq P^*(s'') - P^*(s')$. We then have that $E_{t''}(s'' - s') \leq E_{t'}(s'' - s')$ and $0 \leq P^*(s'') - P^*(s') \leq E_{t'}(s'' - s')$. Since S^o is strictly oriented for T, it follows from the second statement in Lemma A.3 that $s'' = bs'$, some b with $0 < b \leq 1$. But if $b < 1$, then we have that $E_{t'}(s'' - s') = (b - 1) E_{t'}(s') < 0$, a contradiction. Thus if $s'' \in s^*(T)$ and if $P^*(s'') \geq 0$, it follows that $s'' = s'$. □

The requirement that $s \in s^*(T) \cap S^o$ in the hypothesis of Lemma A.11 is in fact satisfied by $s = s^*(N)$ in any equilibrium when strict stochastic dominance **(SSD)** holds.

Theorem A.12. *Assume that S^o is full and strictly oriented for F. Then in any equilibrium (s^*, μ^*, P^*) for Γ (K, π) for which $s^*(N)$ is not value invariant, $s^*(N) \in S^o$.*

Proof. Assume that (s^*, μ^*, P^*) is an equilibrium for Γ (K, π). If $s^*(N)$ is not value invariant, then by Theorem A.8, there exists $t' \in s^{*-1}(s^*(N))$ such that $t' \neq N$ and such that $E_{t'}(s^*(N)) < P^*(s^*(N)) < E_N(s^*(N))$, and hence $s^*(N)$ is mispriced. If $s^*(N) = \mathbf{x}$, we are done since $\mathbf{x} \in S^o$. Suppose that $s^*(N) \neq \mathbf{x}$, but $s^*(N) \in S^o$. Then $E_N(s^*(N)) < E_N(\mathbf{x})$. Let $s^o \in S^o$ be such that $E_N(s^o) = E_N(s^*(N))$. Such a design exists because S^o is full. By assumption, $s^o \neq s^*(N)$. If $s^o \in s^*(T)$, then it must be that $P^*(s^o) > P^*(s^*(N))$, by Lemma A.11. But then

$$
\begin{aligned}
&V(N, s^o, P^*(s^o)) \\
&\quad = P^*(s^o) - K + E_N(\mathbf{x}) - E_N(s^o) \\
&\quad > P^*(s^*(N)) - K + E_N(\mathbf{x}) - E_N(s^*(N)) \\
&\quad = V(N, s^*(N), P^*(s^*(N))),
\end{aligned}
$$

denying that $s^*(N)$ is a best response for N.

If $s^o \notin s^*(T)$, considering off-equilibrium pairs (P, s^o), $P \geq K$, shows that

$$\begin{aligned} &D(t \mid s) \cup D^0(t \mid s) \\ &\quad = [\inf\{D(t \mid s) \cup D^0(t \mid s)\}, \infty) \\ &\quad \subseteq [\max\{K, P^*(s^*(N)) + E_t(s^o) - E_t(s^*(N))\}, \infty). \end{aligned}$$

Since $E_N(s^o) - E_N(s^*(N)) = 0$, it follows from Proposition A.1 that $E_t(s^o) - E_t(s^*(N)) > 0$. Since $P^*(s^*(N)) \geq K$, for $t \neq N$,

$$\begin{aligned} \inf\{D(t \mid s) \cup D^0(t \mid s)\} &\geq P^*(s^*(N)) + E_t(s^o) \\ &\quad - E_t(s^*(N)) > P^*(s^*(N)). \end{aligned}$$

Thus

$$\begin{aligned} &D(t \mid s) \cup D^0(t \mid s) \subset (P^*(s^*(N)), \infty) = D(N \mid s), \\ &\qquad \text{for all } t \in T,\ t \neq N, \end{aligned}$$

and the inclusion is strict. Since (s^*, μ^*, P^*) is an equilibrium, it follows from (D1) that $\mu(s^o)(N) = 1$, and hence $P^*(s^o) = E_N(s^o) > P^*(s^*(N))$. But then again,

$$\begin{aligned} &V(N, s^o, P^*(s^o)) \\ &\quad = P^*(s^o) - K + E_N(\mathbf{x}) - E_N(s^o) \\ &\quad > P^*(s^*(N)) - K + E_N(\mathbf{x}) - E_N(s^*(N)) \\ &\quad = V(N, s^*(N), P^*(s^*(N))), \end{aligned}$$

denying that $s^*(N)$ is a best response for N. Hence if $s^*(N) \neq \mathbf{x}$, we must have that $s^*(N) \in S^o$. □

Proof of Theorem 3. Let S^o be a full set of securities and assume that strict stochastic dominance **(SSD)** holds. (Necessity). Suppose that it is true that for any admissible parameter pair (K, π), if $s^o \in S^o$ and $E_\pi(s^o) = K$, then in every equilibrium for the game $\Gamma(K, \pi)$, all types pool by issuing s^o. Then by Theorem 2, it follows that S^o is oriented for F. Suppose S^o is not strictly oriented. Then by Proposition A.1, there exist $t, t' \in T$, with $t < t'$, and $s^o \in S^o$, $s \in S$, such that $s \neq s^o$ and $E_{t'}(s - s) \leq 0$, but $E_t(s - s^o) = 0$. Since $s \neq s^o$, it must be that either $s \neq \mathbf{x}$ or $s^o \neq \mathbf{x}$. It follows that both $s \neq \mathbf{x}$ and $s^o \neq \mathbf{x}$ and $E_t(s^o) < E_t(\mathbf{x})$ and $E_{t'}(s) \leq E_{t'}(s^o) < E_{t'}(\mathbf{x})$.

Suppose $E_{t'}(s - s^o) = 0$. Let $\pi_{t}, = a$, $\pi_{t'} = 1 - a$, and $\pi_{t'} = 0$, $t'' \neq t, t'$. Also let $K = E_p(s^o)$. We can choose a, $0 < a < 1$, but sufficiently close to 1 so that (K, π) is admissible. By Theorem 2, there is an equilibrium (s^*, μ^*, P^*) for the game $\Gamma(K, \pi)$ in which all types pool by issuing s^o; that is, $s^*(t'') = s^o$ for $t'' = t, t'$, and $P^*(s^o) = K$. By assumption, in every equilibrium for the game Γ (K, π), all types pool by issuing s^o and $P^*(s^o) = K$ Also by assumption $E_t(s) = E_t(s^o)$. We claim that s is also the outcome of a pooling equilibrium for Γ (K, π). Consider the candidate equilibrium $(s^\wedge, P^\wedge, \mu^\wedge)$, where $s^\wedge(t'') = s$ for $t'' = t, t'$, $\mu^\wedge(s'') = \mu^*(s'')$, and $P^\wedge(s'') = P^*(s'')$ for all $s'' \in S$, $s'' \neq s$. Let $\mu^\wedge(s) = \pi$ and $P^\wedge(s) = K$. By construction and the fact that (s^*, μ^*,P^*) is an equilibrium, it follows that $(s^\wedge,P^\wedge,\mu^\wedge)$ satisfies (SR), (BC), and (CR). It is also the case that the sets $D(t'' \mid s'')$ and $D^o(t'' \mid s'')$ are the same for the conjectured equilibrium $(s^\wedge,\mu^\wedge,P^\wedge)$ as for the equilibrium (s^*,μ^*,P^*), for all $s'' \in S$ and all $t'' \in T$. Thus, the fact that (s^*,μ^*,P^*) satisfies (D1) implies that $(s^\wedge,\mu^\wedge,P^\wedge)$ also satisfies (D1). It follows that $(s^\wedge,\mu^\wedge,P^\wedge)$ is an equilibrium for Γ (K, π).

Suppose therefore that $E_{t'}(s - s^o) < 0$. It then follows that $E_t(s^o) = E_t(s) \leq E_{t'}(s) < E_{t'}(s^o)$. Let $s'' = (1 - h)s + h$ **x**, where $h \in (0, 1)$ is chosen so that $E_{t'}(s'') = (1 - h)E_{t'}(s) + hE_{t'}$ **(x)** $< E_{t'}(s^o)$. Then we have as in the proof of necessity in Theorem 2 that $E_{t'}(s'' - s^o) < 0 < E_t(s'' - s^o)$. The construction there gives an admissible pair (K, π) such that $E_\pi(s^o) = K$, but the outcome in which all types issue s^o and raise K is not an equilibrium outcome of the game Γ (K, π). This contradiction completes the proof of necessity here.

(Sufficiency). Assume that S^o is strictly oriented for F. By Theorem 2, for any admissible parameter pair (K, π), if $s^o \in S^o$ and $E_p(s^o) = K$, then there is an equilibrium for the game Γ (K, π) in which all types pool by issuing s^o. By strict stochastic dominance **(SSD),** Theorem A.8, Lemma A.10, and Theorem A.12, in every pooling equilibrium for Γ (K, π), all types issue s^o. It remains to show that all equilibria are pooling equilibrium.

To show this, let (s^*,μ^*,P^*) be an equilibrium for the game Γ (K, π) and let $s^o = s^*(N)$. By Theorem A.8 and Lemma A.9, there is $t' \in s^{*-1}(s^o)$ such that $t' \neq N$ and $E_{t'}(s^o) < P^*(s^o) = K < E_N(s^o)$. By Theorem A.12, $s^o \in S^o$. If there exists $s^\wedge \in s^*(T)$ with $s^\wedge \neq s^o$, then by Lemma A.7, we can choose $t' \in s^{*-1}(s^o)$ and $t'' \in s^{*-1}(s^\wedge)$ such that $t' < t'' <$ N. Because t' chooses s^o in the equilibrium, and since $P^*(s^o) = K$, it follows that $P^*(s^\wedge) - E_{t'}(s^\wedge) \leq K - E_{t'}(s^o)$. Rearranging gives that $0 \leq P^*(s^\wedge) - K \leq E_{t'}(s^\wedge - s^o)$. Since S^o is strictly oriented for F, Lemma A.3 gives that $E_{t'}(s^\wedge - s^o) < E_{t'}(s^\wedge - s^o)$. By an identical argument, the fact that t'' chooses $s^\wedge$ in the equilibrium implies that $P^*(s^\wedge) - E_{t''}(s^\wedge) \geq K - E_{t''}(s^o)$, and hence that $P^*(s^\wedge) - K \geq E_{t''}(s^\wedge - s^o)$. Putting these inequalities together, however, gives that $E_{t''}(s^\wedge - s^o) \leq E_{t'}(s^\wedge - s^o)$, a contradiction. Thus, we have shown that if $s^\wedge \in S^*(T)$, then $s^\wedge = s^o = s^*(N)$; that is, all equilibria for Γ (K, π) are pooling equilibria. □

Proofs of results in Section 3

Proof of Proposition 4. Let $\in \Omega^{o}$ and let $y > 0$ with $x + y \in \Omega^{o}$. Then by cross-multiplying in the expression $r_{t,t'}(x + y) - r_{t,t'}(x)$, it is easily seen that this expression has the same sign as the expression

$$(1 - F_{t'}(x + y))(1 - F_t(x)) - (1 - F_{t'}(x))(1 - F_t(x + y)).$$

Adding and subtracting $(1 - F_{t'}(x))\ (1 - F_t(x))$, this expression has the same sign as

$$(1 - F_t(x))[F_{t'}(x) - F_{t'}(x + y)] - (1 - F_{t'}(x))[F_t(x) - F_t(x + y)].$$

Divide this last expression by $(1 - F_t(x))\ (1 - F_{t'}(x))$ to see that it has the same sign as the expression $F_t(y \mid x) - F_{t'}(y \mid x)$. It follows that $r_{t,t'}(x + y) \geq (>)\ r_{t,t'}(x)$ if and only if $F_t(y \mid x) \geq (>)\ F_{t'}(y \mid x)$. □

Proof of Theorem 5. For $b \geq 0$ given, let $s^{o} = \min\{\mathbf{x}, b\}$. For any distribution function F and any $s \in S$, by integration by parts, $\int s\, dF = \int w(1 - F)\, d\mathbf{x}$ where w is the derivative of s ((LL), (M), and (MR) imply s is absolutely continuous). Note that for w^{o} the derivative of s^{o}, $w^{o}(x) = 1$, $0 \leq x < b$, and $w^{o}(x) = 0$, $b < x$. Thus

$$\begin{aligned}\int_{\Omega} (s - s^{\circ})\, dF &= \int_{\Omega} (w - w^{\circ})(1 - F)\, d\mathbf{x} \\ &= \int_{[0,b]\cap\Omega} (w - 1)(1 - F)\, d\mathbf{x} \\ &\quad + \int_{[b,\infty)\cap\Omega} w(1 - F)\, d\mathbf{x}. \end{aligned} \tag{A.2}$$

Henceforth, to simplify notation, the range of integration in all integrals, indicated by limits of integration, will be the part of the range indicated that is inside the set Ω, but specific reference to Ω will be suppressed. No limits of integration will mean integration over all of Ω.

*[Sufficiency of **(CSD)**].* Suppose conditional stochastic dominance **(CSD)** holds. By Proposition 4, for $t, t' \in T$, $t < t'$, the function $r_{t,t'}$ is increasing on Ω^{o}. Hence, for $0 \leq x \leq b \leq x'$, $r_{t,t'}(x) \leq r_{t,t'}(b) \leq r_{t,t'}(x')$. Consequently, since $0 \leq w \leq 1$ (almost everywhere with respect to Lebesgue measure), $w(x')\ r_{t,t'}(x') \geq w(x')\ r_{t,t'}(b)$, $b \leq x'$, and $(w(x) - 1)\ r_{t,t'}(x) \geq (w(x) - 1)\ r_{t,t'}(b)$, $0 \leq x \leq b$. Using (A.2) then we get that

$$\begin{aligned}\int (s - s^{o})\, dF_{t'} &= \int_{0}^{b} (w - 1)(1 - F_{t'})\, d\mathbf{x} \\ &\quad + \int_{b}^{\infty} w(1 - F_{t'})\, d\mathbf{x} \\ &= \int_{0}^{b} (w - 1)r_{t,t'}(1 - F_{t})\, d\mathbf{x} \\ &\quad + \int_{b}^{\infty} wr_{t,t'}(1 - F_{t})\, d\mathbf{x} \\ &\geq r_{t,t'}(b)\left\{\int_{0}^{b} (w - 1)(1 - F_{t})\, d\mathbf{x}\right. \\ &\quad \left. + \int_{b}^{\infty} w(1 - F_{t})\, d\mathbf{x}\right\} \\ &= r_{t,t'}(b) \int (s - s^{o})\, dF_{t}. \end{aligned} \tag{A.3}$$

It follows from (A.3) that if $E_{t'}(s) \leq E_{t'}(s^o)$, then $E_t(s) \leq E_t(s^o)$, and consequently, by Proposition A.1, S_D is oriented for F. Note that if strict conditional stochastic dominance **(SCSD)** holds and s $\neq s^o$, then $r_{t,t'}$ is strictly increasing on Ω^0 yields a strict inequality in (A.3). Again by Proposition A.1, it follows that S_D is strictly oriented for F.

[Necessity of ***(CSD)].*** Conversely, suppose that S_D is oriented for F. We will show that for $t < t'$, $r_{t,t'}$ is increasing on Ω^0. First consider the case of $a, b \in \Omega^0$, with $a < b$, and such that both a and b are continuity points of F,. Consider a family of alternative security designs with derivative of the following form: $w(x; a, b, n) = 1 - [1 - F_{t'}(b)]/[1 - F_{t'}(a)]$, if $x \in [a - 1/n, a]$, $w(x; a, b, n) = 1$, if $x \in$ [0, b + 1/ n]/[a - 1/ n, a], and $w(x; a, b, n) = 0$, otherwise. Let $w^o(\cdot; b)$ represent the derivative of the debt security with face value b. Then for n sufficiently large, $w(x; a, b, n) - w^o(x; b) = [1 - F_{t'}(b)]/[1 - F_{t'}(a)]$, if $x \in [a - 1/n, a]$, $w(x; a, b, n) - w^o(x; b) = 1$, if $x \in (b, b + 1/n]$, and $w(x; a, b, n) - w^o(x; b) = 0$, otherwise. Thus,

$$\begin{aligned}&\int (w - w^{o})(1 - F_{t'})\, d\mathbf{x} \\ &\quad = (1 - F_{t'}(b))\left\{\int_{b}^{b+1/n} \frac{1 - F_{t'}(x)}{1 - F_{t'}(b)}\, dx\right. \\ &\quad \left. - \int_{a-1/n}^{a} \frac{1 - F_{t'}(x)}{1 - F_{t'}(a)}\, dx\right\} \leq 0, \end{aligned} \tag{A.4}$$

since $(1 - F_{t'}(x))/(1 - F_{t'}(b)) \leq 1$ whenever $x \in [b, b + 1/n)$ and $(1 - F_{t'}(x))/(1 - F_{t'}(a)) \geq 1$ whenever $x \in [a - 1/n, a]$. Since S_D is oriented, Equation (A.4) implies that

$$\int (w - w^{\circ})(1 - F_t)\, d\mathbf{x} = (1 - F_t(b))\left\{\int_b^{b+1/n} \frac{1 - F_t(x)}{1 - F_{t'}(b)}\, dx - \int_{a-1/n}^{a} \frac{1 - F_t(x)}{1 - F_{t'}(a)}\, dx\right\} \leq 0. \tag{A.5}$$

Multiplying both sides of Expression (A.5) by $1/n$, dividing by $1 - F_{t'}(b)$ and taking limits as $n \to \infty$, gives

$$\frac{1 - F_t(b)}{1 - F_{t'}(b)} - \frac{1 - F_t(a)}{1 - F_{t'}(a)} \leq 0, \tag{A.6}$$

which implies that $r_{t,t'}(a) \leq r_{t,t'}(b)$. Since F_t and $F_{t'}$ are right continuous, so is $r_{t,t'}$. Since F_t has only discontinuities of the first kind and at most a countable number of these, if a and b are any two points in Ω^{o}, let a_n and b_n be sequences such that $a_n \downarrow a$, $b_n \downarrow b$, $a_n < b_n$, and $\{a_n\}$ and $\{b_n\}$ are continuity points of F_t. Then Expression (A.6) holds for each n and hence it holds in the limit; that is,. $r_{t,t'}(a) \leq r_{t,t'}(b)$ for all $a, b \in \Omega^{o}$, $a \leq b$.

Suppose now that S_D is strictly oriented. By the above proof, $r_{t,t'}$ is increasing on Ω^{o}. We must show that $r_{t,t'}$ is strictly increasing everywhere on Ω^{o}. If $r_{t,t'}$ is not strictly increasing on Ω^{o} then there exists a, $b \in \Omega^{o}$, $a < b$, such that $r_{t,t'}$ is constant on the interval (a, b). Let c be the midpoint of this interval and consider a security s with the following derivative: $w(x) = 1$, if $x \leq a$; $w(x) = d$, if $x \in [a, c)$; $w(x) = e$, if $x \in [c, b)$; $w(x) = 0$, if $x \geq b$, where $0 \leq d, e \leq 1$. Let w^{o} be the derivative of a debt contract with face value of c. Then for any distribution function G,

$$\int (w - w^{\circ})(1 - G)\, d\mathbf{x} = (d - 1)\int_a^c (1 - G)\, d\mathbf{x} + e\int_c^b (1 - G)\, d\mathbf{x}. \tag{A.7}$$

Choose $d \in [0, 1)$ and $e \in (0, 1]$ such that (A.7) is zero for $G = F_{t'}$. Because $r_{t,t'}$ is constant on (a, b) and $r_{t,t'}(x) = r_{t,t'}(c)$, for all $x \in (a, b)$, by (A.7),

$$0 = \int (w - w^\circ)(1 - F_{t'})\, d\mathbf{x} = \int (w - w^\circ) r_{t,t'}(1 - F_t)\, d\mathbf{x}$$

$$= r_{t,t'}(c) \int (w - w^\circ)(1 - F_t)\, d\mathbf{x},$$

a contradiction, by Proposition A.1. □

The proof of Corollary 6 follows directly from Theorems 2, 3, and 5.

Proofs of the results in Section 4

Proof of Theorem 7. Let S^o be oriented for F and let d be a nontrivial division of π. If $s^o \in S^o$, but s^o does not solve **MV** (s^o, π, d), there is an $s \in S$ such that s is feasible for **MV** (s^o, π, d) but is better than s^o; that is, $E_\pi(s - s^o) = 0 > E_{\pi^{d+}}(s - s^o)$. Since S^o is oriented for F, it follows that $E_{\pi^{d-}}(s - s^o) \le 0$. But then $E_\pi(s - s^o) = 0 = \pi^d E_{\pi^{d+}}(s - s^o) + (1 - \pi^d) E_{\pi^{d-}}(s - s^o) < 0$, since $0 < \pi^d < 1$, a contradiction. Thus s^o must solve **MV** (s^o, π, d) for any nontrivial division d of π; that is, $S^o \subseteq \hat{S}$.

To show that $\hat{S}$ is oriented for F, let d be a nontrivial division of π and suppose that for some $s^o \in \hat{S}$, and some $s \in S$, $E_{\pi^{d+}}(s - s^o) \le 0 < E_{\pi^{d-}}(s - s^o)$. As in the proof of necessity in Theorem 2, we may conclude that $E_{\pi^{d-}}(s^o) < E_{\pi^{d-}}(s) \le E_{\pi^{d+}}(s) < E_{\pi^{d+}}(s^o)$. Let $a = (E_{\pi^{d-}}(s) - E_{\pi^{d-}}(s^o))/((E_{\pi^{d-}}(s) - E_{\pi^{d-}}(s^o) + E_{\pi^{d+}}(s^o) - E_{\pi^{d+}}(s))$, and let $\pi' = a\pi^{d+} + (1 - a)\pi^{d-}$. It is easily verified that $\pi'^{d+} = \pi^{d+}$ and $\pi'^{d-} = \pi^{d-}$, and therefore that d is a nontrivial division of π', and that $E_{\pi'}(s) = E_{\pi'}(s^o)$. Notice, however, that s^o is not a solution to **MV** (s^o, π', d), because $E_{\pi'^{d+}}(s) < E_{\pi'^{d+}}(s^o)$. Thus, it must be that if $E_{\pi^{d+}}(s - s^o) \le 0$, then $E_{\pi^{d-}}(s - s^o) \le 0$; that is, $\hat{S}$ is oriented for F. □

Proof of Theorem 8. This result follows from the usual necessary and sufficient optimality conditions such as Luenberger (1969: Theorem 1, p. 217 and Theorem 1, p. 220) upon noting that for every $\gamma \ge 0$, $Q(\gamma, \cdot)$ is a measurable function and consequently Expression (8) is minimized over w by setting $w(x) = 0$ whenever $Q(\gamma, x) > 0$, and by setting $w(x) = 1$ whenever $Q(\gamma, x) < 0$. This infimum is the expression $MV(L, H) = \int_{\{Q(\gamma^*, \cdot) \le 0\}} Q(\gamma^*, x)\, dx + \gamma^* K$. □

References

Allen, F., and D. Gale, 1988, "Optimal Security Design," *Review of Financial Studies*, 1, 229-263.

Banks, J., and J. Sobel, 1987, "Equilibrium Selection in Signaling Games," *Econometrica*, 55, 647-661.

Brennan, M., and A. Kraus, 1987, "Efficient Financing under Asymmetric Information," *Journal of Finance*, 42, 1225-1244.

Cho, I., and D. Kreps, 1987, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics,* 102, 179-221.

Cho. I., and J. Sobel, 1990, "Strategic Stability and Uniqueness in Signaling Games," *Journal of Economic Theory,* 50, 381-413.

Fudenberg, D., and J. Tirole, 1991, "Perfect Bayesian Equilibrium and Sequential Equilibrium," *Journal of Economic Theory,* 53, 236-260.

Harris, M., and A. Raviv, 1989. "The Design of Securities," *Journal of Financial Economics,* 24, 255-287.

Innes, R.D., 1990, "Limited Liability and Incentive Contracting with Ex Ante Action Choices," *Journal of Economic Theory,* 52, 45-67.

Kreps, D. M., 1990, "Out-of-Equilibrium Beliefs and Out-of-Equilibrium Behaviour." in F.H. Hahn (ed.), *The Economics of Missing Markets, Information, and Games,* Clarendon Press, Oxford,

Kreps, D.M., and R. Wilson, 1982, "Sequential Equilibria," *Econometrica,* 50, 863-894

Luenberger, D. G., 1969, *Optimization by Vector Space Methods,* Wiley, New York

Madan. D., and B. Soubra, 1991, "The Design and Marketing of Corporate Securities," *Review of Financial Studies,* 4, 361-384.

Myers, S., 1977, "Determinants of Corporate Borrowing," *Journal of Financial Economics,* 5, 147-175.

Myers, S., 1984, "Presidential Address: The Capital Structure Puzzle," *Journal of Finance,* 39, 575-592.

Myers, S., and N. Majluf. 1984, "Corporate Financing and Investment Decisions When Firms Have Information Investors Do Not Have," *Journal of Financial Economics,* 13, 187-221.

Nachman, D.C., and T. H. Noe, 1990, "Design of Securities under Asymmetric Information," working paper, School of Management, Georgia Institute of Technology, Atlanta.

Nachman. D.C., and T.H. Noe, 1993, "Operating Efficiency and Output Insensitive Employment Contracts for Capital Management," forthcoming in *Economic Theory.*

Noe, T.H., 1988, "Capital Structure and Signaling Game Equilibria," *Review of Financial Studies,* 1, 331-355.

Ross, S., 1983, *Stochastic Processes,* Wiley, New York

Whitt, W., 1980, "Uniform Conditional Stochastic Order," *Journal of Applied Probability,* 17, 112-123.