Equilibrium and Options on Real Assets

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In aggregate, options on real and financial assets can have very different properties. Typically, the good or service produced by a real asset has a finite elasticity of demand and developers have finite capacities. Also, the supply of options can be limited, and developers can be less than perfectly competitive. In a subgame, perfect Nash equilibrium with these properties, the optimal exercise policy, and resulting values of developed and undeveloped assets are calculated explicitly. The novel comparative statics are discussed in details

To date, options on real assets, like real estate, natural resources, and capital assets, have been analyzed only in partial equilibrium.[1] When viewed from the perspective of a single developer with a single undeveloped asset, real options have many characteristics in common with financial options. With real estate and other real assets, the option is an undeveloped property, the underlying asset is a developed property, the exercise price is the cost of development, and the maturity is generally infinite. In previous articles special characteristics of real options have been emphasized. These include the time to build, the often sto-

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[1]Options on real estate are analyzed in Titman (1985). Clarke and Reed (1988). Capozza and Sick (1991). Williams (1991), and the references cited therein. Options on natural resources are the focus of Brennan and Schwartz (1985), and Paddock, Siegel, and Smith (1988). Finally. options on capital assets are analyzed in McDonald and Siegel (1985. 1986). Majd and Pindyck (1987), Pindyck (1988). Dixit (1989, 1991), and Sick (1989).

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chastic cost of construction, the controllable quantity or density of development, and the option to abandon either developed or undeveloped property.[2]

When viewed in aggregate, options on real and financial assets can have very different properties. At least four differences are potentially significant. Exercising options to develop real assets affects the aggregate supply of developed assets and thereby the price in equilibrium of the output from each developed asset. In turn, this affects both a developer's optimal exercise policy and the value of his option to develop. Also, if contractors have increasing costs or limited capacities, then the cost of development depends on the aggregate demand for development. This not only alters each developer's optimal exercise policy **but** also may preclude simultaneous exercise of all outstanding options.[3] In addition, the supply of undeveloped assets can be constrained, possibly by the supply of developed assets. Again, this affects exercise policies and option values, Finally, with monopolistic or oligopolistic ownership of undeveloped assets, the action of any one owner can affect the values of other assets in his portfolio, as well as the actions of other owners. How each owner responds in equilibrium depends partly on the concentration of ownership. These properties of real options do not apply to most financial options, including calls and puts on stocks and bonds. With the exception of warrants and convertible bonds, financial options are exercised by trading paper that is held in zero net supply by a large number of autonomous individuals.* For the latter securities there is no downward-sloping demand by consumers, no aggregate constraint on the rate of exercise, no limited supply of options, and no monopolistic or oligopolistic exercise by holders.

How significant are these distinguishing characteristics of real options? The significance, as measured by the magnitude of the effect on optimal exercise policies and the resulting market values, depends partly on the elasticity of demand for products from the real asset, the capacities of developers, the supply of undeveloped assets, and the degree of concentration among developers. To illustrate possible magnitudes, consider one category of real assets: single-family houses within the metropolitan area of Phoenix, Arizona, between 1970 and

[2] Some financial options have related characteristics, as argued in Margrabe (1978) and Stulz (1982). Also, with real estate another special characteristic is the distance or commuting time to the central business district, as developed in Capozza and Sick (1991) and references cited therein.

[3] This problem has not occurred in previous articles on real options because new development has been limited to a negligible fraction of the current supply of developed assets. For example, in Brennan and Schwartz (1985) different mines have different costs, whereas in Capozza and Sick (1991) developers build only at the boundary of the urban area.

[4] Warrants are studied in Constantinides (1984) and Spatt and Sterbenz (1988).

1990. Despite the area's rapid growth during this period, the annual construction of new houses never exceeded 3 percent of the housing stock. Because the average time for housing permits remained roughly constant at less than two years, the supply of building lots never exceeded 6 percent of the stock.[5] In any year, half of the houses were constructed by no more than 10 developers and the remainder by no more than 200 developers.[6] Although no reliable estimates of the price elasticity of demand for housing services are available for Phoenix, most estimates for the United States have been less than 1.0.[7]

In this article options to develop real assets are analyzed in an equilibrium with the aforementioned properties. By assumption, the demand for the product from the real asset depends on its current price, as well as an exogenous factor that evolves as a geometric Wiener process. The aggregate demand function has a constant elasticity that includes as a special, limiting case the infinite elasticity from most previous models of real options.[8] Also, all developers have the same limited capacity to construct developed assets using perfectly competitive contractors. This limited capacity precludes simultaneous exercise of all real options and thereby constrains the aggregate rate of development. In addition, the aggregate supply of undeveloped assets is proportional to the supply of developed assets, with smaller constants of proportionality representing tighter restrictions on growth. Although development increases the supplies of developed and undeveloped assets at the same rate over time, and thereby adds to developers' supplies of undeveloped assets, the limited supply at any one time of options to develop affects their optimal rate of exercise. Finally, ownership of undeveloped assets is divided equally among a fixed number of identical investors, ranging from a monopolist, to oligopolists, to perfect competitors. Owners choose their rates of development in a Markov perfect equilibrium.

Under these assumptions the optimal exercise strategy and resulting values of developed and undeveloped assets are calculated explicitly. In equilibrium, development optimally occurs at all levels of current income from a developed asset that exceed or equal a lower bound. This bound is a switching point between no development and development at the maximum feasible rate. It replaces the single

[5] In fact, building lots are partially developed assets that are the product of previously incurred discretionary costs. The supply of completely raw land may be very elastic, but cannot be perfectly elastic, as argued in Section 1.

[6] Sources: *Phoenix Metropolitan Housing Study* and Arizona Republic, *Inside Metro Phoenix,* various years from 1971 to 1991.

[7] See the survey by Rosen (1985) and references cited therein.

[8] Demand curves are downward sloping in Pindyck (1988), Dixit (1989, 1991). and Bar-Han, Sulem, and Zanello (1992).

point of optimal exercise from previous articles on real options. With this rate of development, the ratio of income to value for the developed asset is everywhere increasing in income. The ratio converges to a constant discount rate as the elasticity of demand increases to infinity. In turn, the value of the undeveloped asset depends on both the optimal rate of development and the resulting value of the developed asset. In this sense, the market values of the developed and undeveloped assets are interdependent, much like stocks and warrants but unlike stocks and their traded options. Other properties of the solution are similar to previous results on options.

The novel economic insights come from the new comparative statics. For example, the optimal switching point for development, as described earlier, is increasing in three parameters of the developed asset: the elasticity of demand for its good or service, its maximum rate of development, and its rate of depreciation. It is also decreasing in both the supply of undeveloped relative to developed assets and the number of identical owners of undeveloped assets. The maximum rate of development, relativesupply of undeveloped assets, and ownership of undeveloped assets are entirely new parameters. Also, all the comparative statics have new interpretations. In standard models of options with an exogenous pricing process for the associated security, optimal exercise is deferred by changes in the values of parameters that increase the value of the option. In this model, optimal exercise is also deferred by changes in parameters that either decrease the value of the developed (optioned) asset or increase the effect of exercise on the value of the remaining undeveloped assets (options) owned by a monopolist or an oligopolist. Undeveloped assets are always held in equilibrium because the optimal rate of development is finite.

The article is organized as follows. The model is presented in Section **1,** and the equilibrium is displayed in Section 2. The numerical computations are discussed in Section 3, and the results are summarized in Section 4. All technical details appear in the Appendix.

1. Model

Consider a category of real assets, such as single-family houses within a metropolitan area. For notational simplicity, all properties within the category are assumed to be identical, after adjusting for measurable differences, such as the number of square feet in a house or its distance from the central business district. Also, the category is sufficiently broad so that the supply and demand for other categories of real assets can be ignored, except as specified later. For this category, q is the current aggregate supply of developed assets; x is the current value of an exogenous variable affecting the demand for the good or service produced by each developed asset; and *y* is the current income from that asset. For example, with single-family houses the exogenous variable *x* summarizes such factors as the current ratio of buyers to sellers, as well as the current income and net worth of buyers, and the income *y* equals the current rent or implicit rent minus operating costs.

The good or service from the developed asset can be sold in a perfectly competitive, continuous spot market. At each instant of time, atomistic consumers maximize their utility and thereby generate an aggregate demand for the good or service that has a constant price elasticity. This constant elasticity holds with an instantaneously perishable commodity, such as rental services, if, for example, all consumers have identical, time-additive, isoelastic utilities. In addition, the output from each developed asset is produced with an identical, convex, Cobb-Douglas cost function. Finally, control of the developed assets is sufficiently dispersed that producers of the good or service are perfect competitors in the spot market. As in all option pricing models, these assumptions are introduced mainly for analytical convenience. When combined with subsequent assumptions, they permit an explicit solution in Section 2.

Under the foregoing assumptions the clearing condition in the spot market has an analytically convenient form. Specifically, the aggregate demand for the good or service, the optimal output from each developed asset, and the resulting income per developed asset *y* are proportional to power functions of the current price of the good or service. Thus, the aggregate demand for the good or service and its supply by each developed asset are proportional to power functions of the income *y*. As a result, the exogenous factor *x* can be resealed to capture the constants of proportionality, and the requirement that aggregate demand always equals aggregate supply in the spot market can be simplified to the clearing condition:[9]

$$q = (x/y)^{\alpha}, \tag{1}$$

with constant a. The elasticity a depends on two constant elasticities for the good or service: one in the aggregate demand by consumers and another in the cost of production. Details, including the derivation of (1), appear in the Appendix. Given (1), the income per unit of developed assets is $y = xq^{-1/\alpha}$. In most previous models of real options, income has been assumed to be exogenous: $y = x$. Exoge-

[9] Alternatively, with the additive changing condition $q = \alpha(x - y)$ the subsequent problem can be solved explicitly if the subsequent assumptions of proportionality ate replaced everywhere by additive specifications.

nous income is a special case of (1) in which demand is perfectly elastic: $\alpha = \infty$.

The exogenous factor affecting demand changes stochastically through time. By assumption, it evolves continuously in a proportional random walk. Also, the mean and variance of its growth rate are constant per unit of time. In this case, the exogenous variable x follows a geometric Wiener process:

$$dx/x = \mu\, dt + \sigma\, dz. \tag{2}$$

In (2) the variables t and z are time and a standard Wiener process, and the constants μ and σ^2 are the mean and variance of the growth rate in the exogenous demand x. The stochastic increments in (2) can be correlated with changes in other state variables affecting the demands for other categories of real assets. Thereby, the rates of development to be derived here can be correlated in equilibrium for different categories of real assets.

All owners of undeveloped assets are identical. At all times the number of such owners is constant at v, with $v \in \{1, \ldots, \infty\}$. These identical owners have equal numbers of undeveloped assets.[10] When developing an asset, an owner acts as his own developer and hires a contractor in the perfectly competitive market for contractors. Hence, owners of undeveloped assets are also called developers. To produce a developed asset, each owner or developer must use one undeveloped asset and pay his contractor the perfectly competitive, constant cost, $\gamma > 0$. This cost is constant up to each developer's capacity of $\beta q/v$ assets per unit of time. For example, a developer of single-family houses in a metropolitan area with q houses can build over a very short period of length At at most $(\beta q/v)\Delta t$ houses. With this analytically convenient assumption, each developer's capacity is identical and proportional to the current aggregate supply of developed assets q.[11] Also, as in most models of real options, development is instantaneous once it is started.[12]

[10]This assumption permits a symmetric solution in Section 2. Without the assumption owners would maximize the market value of their undeveloped assets by selling only to other owners and ownership would become concentrated over time. See the discussion in Section 4.

[11]This simple cost function-a constant cost γ per developed asset up to capacity-is consistent with the casual empirical observation that the real price of housing is roughly constant over time at the expanding boundaries of metropolitan areas without major physical or legal constraints on development. Under these conditions housing costs are determined largely by the highly elastic supply of new construction and not demand.

[12]In models of partial equilibrium with isolated developers, instantaneous development involves no significant loss of generality. Exercising the option to develop produces a newly developing asset rather than a fully developed asset. This developing asset is priced at the risk-adjusted, expected present value of a developed asset. With downward-sloping demand, as in (1), the aggregate supply of developed assets q must be modified to include construction in progress, as in Bar-Ilan, Sulem, and Zanello (1992).

The aggregate supply of the real asset also changes through time, increasing with development and decreasing with depreciation. If developer n builds $b_n q$ units per unit of time, then he adds to the aggregate supply q at the rate b_n, for $n = 1, \ldots, v$. Development then occurs at the rate $b \equiv \Sigma_n b_n$. The aggregate building rate b cannot exceed the maximum building rate β, $0 \leq b \leq \beta$. Simultaneously, each developed property depreciates at the constant rate δ per unit of time. The depreciation rate δ is also less than the maximum building rate β, $0 \leq \delta < \beta$. Net of depreciation the aggregate supply q then grows at the rate:

$$\dot{q}/q = b - \delta, \tag{3}$$

with $0 \leq b \leq \beta$.

The values of developed and undeveloped assets depend on the aggregate supplies of developed and undeveloped assets, as well as the demand for developed assets, as reflected in the clearing condition (1). The supply and demand for developed assets are determined by the state variables, q and x. To avoid adding another state variable, we must restrict the aggregate supply of undeveloped assets to depend at most on these two variables. Also, for an explicit solution under these assumptions, the aggregate supplies of developed and undeveloped assets must be proportional. Proportional supplies are motivated later. Given proportionality, the aggregate supply of the undeveloped asset is θq, with the constant, $0 < \theta < \beta/\delta$. Smaller values of the constant θ represent tighter restrictions on growth. With proportionality the supplies of developed and undeveloped assets grow at the same rate over time. At any time the supply of undeveloped assets is finite. As shown subsequently, this finite supply affects developers' optimal exercise policies. Realistically, the restrictive assumption of proportional supplies should be viewed in the same context as the previous assumptions of proportional building capacity and proportionality in (1), (2), and (3).[13] Under the foregoing assumptions, the current state of the market for both developed and undeveloped assets is summarized by the state variables, q and x. In this case, the current values of each developed and undeveloped asset can be specified as $V^d(q, x)$ and $V^u(q, x)$, respectively.

Proportional supplies of developed and undeveloped assets may be a reasonable representation of a market in steady state. An example of developed and undeveloped assets is single-family houses and prepared building lots. Over time as houses are constructed, building lots are withdrawn from the stock of undeveloped assets at the rate

[13] An explicit solution is also possible with constant supply, constant capacity, linear demand in (1). a random walk in (2), and constant change in (3). This alternative appears to be less realistic than proportionality everywhere.

(3). This rate, $b - \delta$, is measured net of the depreciation rate 6 at which houses are returned to the undeveloped state of building lots. If the city is to stay in steady state, then building lots must be created at the urban boundary at a rate that is equal to the current net use of building lots, $b - \delta$. In practice, this could reflect a regulatory rule of planning officials or the capture of regulators by developers. Given this growth rate of building lots, the aggregate supply of undeveloped assets at time t is θq_t. These new building lots could be created at virtually constant cost from nonurban land at the ever-expanding boundary of the city. However, the supply of raw land cannot be perfectly elastic because land at the boundary is closer to urban centers of employment and therefore more valuable than land away from the boundary.[14]

With some additional assumptions, developed and undeveloped assets can be valued by using standard techniques from the literature on option pricing. The first assumption is common to all models of real options. Either all investors are risk neutral, or the stochastic evolution in (2) of the exogenous variable x is spanned by the concurrent, instantaneous returns from a portfolio of securities that can be traded continuously without transaction costs in a perfectly competitive capital market. With spanning, trading in liquid securities substitutes for trading in the illiquid real asset.[15] With single-family houses the substitute portfolio could include listed securities such as real estate investment trusts, master limited partnerships, and common stocks of home builders and their suppliers. For the substitute portfolio the excess mean rate of return per unit of standard deviation is assumed to be a constant λ. This constant is zero if all investors are risk neutral and positive if investors are risk averse. Also, the riskless rate of interest is constant at ι per unit of time. In this familiar case, the risk-adjusted expected rate of growth per unit of time for income y is another constant: $\rho = \mu - \lambda\sigma$. Alternatively, with risk neutrality and no spanning, the risk-adjusted expected rate of growth ρ equals μ. To ensure that the values of developed and undeveloped assets are positive and finite, the risk-adjusted mean ρ must be bounded above: $\rho < \iota - \delta/\alpha$.

The description of the valuation problem begins with a developed

[14] If the supply of raw land were perfectly elastic, then no parcel would have an option value. Premia for the option to develop are commonly observed.

[15] It may be possible to generalize the assumption of spanning as follows. The second term in (2) can be decomposed into a systematic component and a residual. The systematic component must be spanned by the stochastic returns on traded securities, whereas the residual is completely diversifiable by construction. In this case, the Brownian residual can be written as the limit of a sequence of Poisson variates, and the argument in Naik (1992) can be applied to the Poisson variates.

asset. By a now-familiar argument from the literature on option pricing, the valuation function for a developed asset V^d must satisfy the partial differential equation:

$$0 = \frac{1}{2}\sigma^2 x^2 V^d_{xx} + (b - \delta) q V^d_q + \rho x V^d_x - (\delta + \iota) V^d + x q^{-1/\alpha}.$$

In (4) the subscripts denote partial derivatives. Although this valuation equation is well specified for all feasible building rates 6, it is solved subsequently only for the optimal building rates in equilibrium. Also, since development is irreversible, (4) must be satisfied for all feasible values of the variables q and x: $q, x \geq 0$.[16]

A valuation equation for the developed asset, such as (4), appears in other models of real options but not financial options. With financial options the analogue of the developed asset is the underlying security-for example, the stock on which a call option is exercised-and the value of this security is specified exogenously as one of the state variables. Here, as in other models of real options, the state variables q and x are not traded assets. This causes no complication with respect to the aggregate supply q because it evolves continuously and deterministically in (3). However, the exogenous demand x is a geometric Wiener variate in (2); hence, substitute securities or special preferences must be invoked to identify the coefficient ρ in (4). As a result, the risk-adjusted expected rate of growth ρ that appears in (4) as the coefficient of xV^d_x replaces the riskless rate of interest ι that appears at this position in valuation equations for financial options. In addition, developed assets depreciate at the constant rate 6, so that the sum $\delta + \iota$ replaces the riskless rate of interest ι in valuation equations for financial options. Finally, the last term in (4) is the asset's current income: $y = xq^{-1/\alpha}$.

The value of a developed asset must also satisfy two boundary conditions. If the exogenous variable affecting demand is zero ($x = 0$), then the asset's income y must be zero from that time forward, under (1) and (2). In this case, a developed asset has no value:

$$V^d(q, 0) = 0. \tag{5}$$

Also, if the developed asset is to have a finite price-earnings ratio, then its value per unit of income must be bounded above by some positive constant v:

$$V^d(q, x) \leq v x q^{-1/\alpha}, \tag{6}$$

[16] If developed assets could be abandoned, then the valuation equation (4) would be satisfied only at those values of the state variables q and x at which abandonment is not optimal. If developed assets could be redeveloped, then (4) must be modified to reflect the issues discussed below (7).

as $x \to \infty$. The limiting condition (6) precludes bubbles in prices. Together, the valuation equation (4) and the boundary conditions (5) and (6) determine the valuation function for the developed property, V^d.[17]

Undeveloped assets are valued similarly. In a Markov perfect equilibrium, each player has a strategy that depends on the history of the game only through state variables that follow a Markov process. Moreover, these Markov strategies must yield a Nash equilibrium in every proper subgame. In this equilibrium each owner or developer n conjectures correctly that each other owner-developer currently builds at his optimal rate b^*_m, for $m = 1, \ldots, n-1, n+1, \ldots, \nu$. The optimal building rate b^*_m can depend only on the current state variables, q and x Thereby, each developer n correctly anticipates in equilibrium that all other developers currently build at the aggregate rate: $b^*_{-n} \equiv \Sigma_{m \neq n} b^*_m$. Given this conjecture, each developer then maximizes his wealth by maximizing the current market value of his inventory of undeveloped assets. In this case, developer n solves the problem:

$$0 = \max_{b_n} \left\{ \frac{1}{2}\sigma^2 x^2 V^u_{xx} + \left(\frac{\nu - 1}{\nu} b^* + b_n - \delta \right) q V^u_q + \rho x V^u_x - \iota V^u + \frac{\nu}{\theta} b_n (V^d - V^u - \gamma) \right\}, \qquad (7)$$

subject to $0 \leq b_n \leq \beta/\nu$, for all $n = 1, \ldots, \nu$. In equilibrium each developer always builds at his optimal rate, equal to the conjectured rate b^*_n, for n = 1, . . . , ν. Development then occurs at the aggregate rate: $b^* \equiv \Sigma_n b^*_n$. Without loss of generality, each developer is assumed to build in (7) whenever he is indifferent between building and not. Because the rates of development b_n are bounded above by the maximum building rate β/ν, he valuation equation (7) must be satisfied for all feasible states: $q, x \geq 0$.[18]

This valuation equation for an undeveloped asset differs somewhat from the previous equation for a developed asset. At the solution to (7), all developers optimally build at the rate $b_n = b^*$, in which case the terms qV^d_q in (4) and qV^u_q in (7) have the same coefficient, b^* - 6. Also, only developed assets depreciate, so that the constant δ appears as a coefficient of V^d in (4) but not of V^u in (7). To understand the final terms in (7), note that developer n builds $b_n q$ units per unit of

[17] The remaining boundary condition, (6) as $q \to 0$, is superfluous under the subsequent transformation of variables.

[18] In standard models with optimal exercise at a point or along a boundary, the partial differential equation for the option must hold only on a connected region where the values of the state variables do not induce optimal exercise.

time, using his inventory of undeveloped assets. Because all undeveloped assets θq are owned by developers and all ν developers are identical, all building by each developer can be restricted to his portfolio of $\theta q/\nu$ undeveloped assets. Measured per undeveloped asset, developer n then builds per unit of time the number of developed assets: $b_n q/(\theta q/\nu) = (\nu/\theta) b_n$. Finally, each developed asset yields the capital gain, $V^d - V^u - \gamma$. This gain reflects the dependence of the value of undeveloped assets V^u on the concurrent value of developed assets V^d. With monopolistic or oligopolistic developers, $\nu < \infty$, the gain may be positive, whereas with perfect competition, $\nu = \infty$, it should be zero.[19] In fact, as shown in Section 3, the gain is positive with a monopoly or oligopoly and zero otherwise.

The equation for an undeveloped asset also differs from familiar valuation equations for financial options. In (7) the option is exercised on a developed asset, the value of which depends on the elasticity a. Most financial options have zero net supply, so their exercise has no effect on cash flows from real assets.[20] Second, options to develop a real asset can be exercised in (7) no more rapidly than the maximum building rate of each developer: $b_n \le \beta/\nu$. In contrast, options on securities are exercised by exchanging paper and hence have no economically significant restrictions on rates of exercise. In addition, the supply of undeveloped assets is restricted in (7) relative to the supply of developed assets by the ratio θ. In familiar models of financial options, the supply of options is unlimited because writers of options are not required to provide collateral. Finally, the number of developers ν measures the concentration of ownership of undeveloped assets. Owners can be monopolistic ($\nu = 1$), oligopolistic ($1 < \nu < \infty$), or competitive ($\nu = \infty$). In all previous models of real options, the representative developer is endowed with one undeveloped asset. He then chooses his exercise policy to maximize his gain on that asset: $V^d - V^u - \gamma$. This matches the maximand in (7) with perfectly competitive developers ($\nu = \infty$).[21]

Finally, the boundary conditions for developed and undeveloped assets are identical. For an undeveloped asset the initial condition corresponds to (5):

[19]The latter condition, no immediate gain at the optimal stopping point, is imposed in models of real options in partial equilibrium. There. prices or values are functions only of aggregate states with exogenous growth *rates,* so that the representative developer is effectively a perfect competitor.

[20]Some securities. such as warrants and convertible bonds. have positive net supply. Previous analyses of such securities have focused on the financial effects of positive net supply and ignored the effects on real assets. For example, in Constantinides (1984) a firm's proceeds from the exercise of its warrants are either distributed as extraordinary dividends or repurchases of stock or reinvested in projects with zero net present value.

[21]Because only the competitive case is realistic with most financial options, it has been analyzed almost exclusively in the literature on option pricing. Exceptions are the analyses of warrants and sinking fund bonds in Constantinides, Dunn and Spatt (1984), and the references cited herein.

$$V^u(q, 0) = 0. \tag{8}$$

Also, the limiting condition matches (6):

$$V^u(q, x) \leq vxq^{-1/\alpha}, \tag{9}$$

as $x \to \infty$. This limiting condition replaces the optimal stopping conditions in previous articles on options with unbounded rates of exercise. Together, (7) through (9) determine the value of an undeveloped property.[22]

2. Equilibrium

The previous problem is solved in three steps. First, the region in which development is optimal is assumed to satisfy a simple switching condition. Conditional on this assumption, the value of a developed asset V^d is then calculated from (4) through (6). Next, this solution is used with (7) through (9) to derive both the interval of optimal development and the resulting value of an undeveloped asset V^u. Finally, the solution is shown to satisfy the initial switching condition under two constraints on the parameters. All results are displayed in this section and derived in the Appendix.

The analysis begins with two preliminaries. First, the variables in (4) through (9) are transformed as follows: $F(y) \equiv V^d(q, x)$ and $G(y) \equiv V^u(q, x)$. With this transformation F(y) and G(y) are the current market values of a developed and an undeveloped asset, conditional only on the income per unit of the developed asset y. Next, the optimal building rate b_n^* from problem (7) is hypothesized to satisfy the switching condition:

$$b_n^* = \begin{cases} 0, & 0 \leq y < y^*, \\ \beta/\nu, & y^* \leq y < \infty, \end{cases} \tag{10}$$

for some nonnegative switching point y^* and all developers $n = 1, \ldots, \nu$. If (10) is correct, then the Markov perfect equilibrium is symmetric with the aggregate building rate: $b^* = \nu b_n^*$ In other words, all developers optimally build at the same rate: $b_n^* = b^*/\nu$. This equilibrium is characterized by no development $(b^* = 0)$ at all incomes y below the switching point y^*, and building at the maximum rate $(b^* = \beta)$ otherwise. As shown in the Appendix, problem (4) through (9) has a unique solution under the previous transformation that satisfies hypothesis (10) if the parameters satisfy two inequalities. The inequalities are derived in the Appendix, displayed in this section, and evaluated numerically in the next section.

[22] Again, the remaining boundary condition, (9) as $q \to \infty$, is superfluous under the subsequent transformation of variables.

The value of a developed asset follows from (4) through (6). Conditional on the previous transformation, problem (4) through (6) has the unique solution:

$$F(y) = \begin{cases} \phi_1 y - \chi_1 y^*(y/y^*)^{\zeta_1}, & 0 \le y < y^*, \\ \phi_2 y + \chi_2 y^*(y/y^*)^{\zeta_2}, & y^* \le y < \infty. \end{cases} \tag{11}$$

In (11) the new parameters are

$$\zeta_1 \equiv 1 - \left(\frac{1}{2} + \frac{\rho + \delta/\alpha}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{\rho + \delta/\alpha}{\sigma^2}\right)^2 + \frac{2}{\phi_1 \sigma^2}} > 1,$$

$$\zeta_2 \equiv \frac{1}{2} - \frac{\rho + (\delta - \beta)/\alpha}{\sigma^2} - \sqrt{\left[\frac{1}{2} - \frac{\rho + (\delta - \beta)/\alpha}{\sigma^2}\right]^2 + 2\frac{\delta + \iota}{\sigma^2}} < 0,$$

$$\phi_1 \equiv \frac{1}{\delta + \iota - \rho - \delta/\alpha} > 0, \qquad \phi_2 \equiv \frac{1}{\delta + \iota - \rho + (\beta - \delta)/\alpha} > 0,$$

and

$$\chi_1 \equiv \frac{\beta}{\alpha}\frac{1 - \zeta_2}{\zeta_1 - \zeta_2}\phi_1\phi_2 > 0, \qquad \chi_2 \equiv \frac{\beta}{\alpha}\frac{\zeta_1 - 1}{\zeta_1 - \zeta_2}\phi_1\phi_2 > 0.$$

The value of a developed asset is displayed in Figure 1. The valuation function F is nonnegative and increasing, strictly concave below the switching point y^*, and strictly convex above y^*. It is bounded above by $\phi_1 y$ and below by its asymptote $\phi_2 y$. As the income y increases, the ratio of income to value, $y/F(y)$, increases. At y = 0, this ratio, commonly called the capitalization rate, attains its lower bound: $\delta + \iota - \rho - \delta/\alpha$. The lower bound is the risk-adjusted, discount rate for future income when no building is currently optimal: $b^* = 0$. With $\dot{q}/q = -\delta$, this bound equals the riskless rate of interest ι minus the expected growth rate of income y from (1), adjusted for both risk and depreciation, $\rho + \delta/\alpha - \delta$. Also, as $y \rightarrow \infty$, the ratio $y/F(y)$ approaches its upper bound: $\delta + \iota - \rho + (\beta - \delta)/\alpha$. This asymptote is the same risk-adjusted, discount rate, but with building at the maximum rate: $b^* = \beta$. In addition, as the elasticity of demand a increases the growth rate of income y becomes less sensitive to the growth rate of aggregate supply q, and the difference between the above two discount rates decreases. In the limit the capitalization rate converges to a constant: $y/F(y) \rightarrow \delta + \iota - \rho$ as $\alpha \rightarrow \infty$. This constant discount rate appears in previous solutions for real options with exogenous income.[23]

[23] An example with real estate is Williams (1991).

The value of an undeveloped asset is determined by the solution to problem (7) through (9). With (10) and the transformation above (10), the unique solution is

$$G(y)=\begin{cases}\beta\left[\dfrac{\gamma}{\beta+\iota\theta}\dfrac{\eta_2}{\eta_1-\eta_2}\right.\\ \qquad\left.+\left(\dfrac{1-\eta_2}{\eta_1-\eta_2}\dfrac{\phi_2\psi_2}{\theta}+\dfrac{\zeta_2-\eta_2}{\eta_1-\eta_2}\dfrac{\chi_2}{\beta-\delta\theta}\right)y^*\right]\left(\dfrac{y}{y^*}\right)^{\eta_1}, & 0\le y<y^*,\\ \beta\left[\dfrac{\gamma}{\beta+\iota\theta}\dfrac{\eta_1}{\eta_1-\eta_2}\right.\\ \qquad\left.-\left(\dfrac{\eta_1-1}{\eta_1-\eta_2}\dfrac{\phi_2\psi_2}{\theta}+\dfrac{\eta_1-\zeta_2}{\eta_1-\eta_2}\dfrac{\chi_2}{\beta-\delta\theta}\right)y^*\right]\left(\dfrac{y}{y^*}\right)^{\eta_2}\\ \qquad+\beta\left[\dfrac{\phi_2\psi_2}{\theta}y+\dfrac{\chi_2}{\beta-\delta\theta}y^*\left(\dfrac{y}{y^*}\right)^{\zeta_2}-\dfrac{\gamma}{\beta+\iota\theta}\right], & y^*\le y<\infty.\end{cases}\tag{12}$$

In (12) the new parameters are

$$\eta_1\equiv 1-\left(\frac{1}{2}+\frac{\rho+\delta/\alpha}{\sigma^2}\right)+\sqrt{\left(\frac{1}{2}+\frac{\rho+\delta/\alpha}{\sigma^2}\right)^2+\frac{2}{\psi_1\sigma^2}}>1,$$

$$\eta_2\equiv\frac{1}{2}-\frac{\rho+(\delta-\beta)/\alpha}{\sigma^2}-\sqrt{\left[\frac{1}{2}-\frac{\rho+(\delta-\beta)/\alpha}{\sigma^2}\right]^2+2\frac{\iota+\beta/\theta}{\sigma^2}}<0,$$

and

$$\psi_1\equiv\frac{1}{\iota-\rho-\delta/\alpha}>0,\qquad \psi_2\equiv\frac{1}{\iota+\beta/\theta-\rho+(\beta-\delta)/\alpha}>0.$$

The value of an undeveloped asset is also displayed in Figure 1. Under the subsequently specified constraints on the parameters, the valuation function G is nonnegative, increasing, and bounded below by its asymptote. Also, it is strictly convex on the initial interval, $0 < y < y^*$, strictly concave on some intermediate interval, $y^* < y < y^{**}$, and strictly convex on the subsequent interval, $y^{**} < y < \infty$.

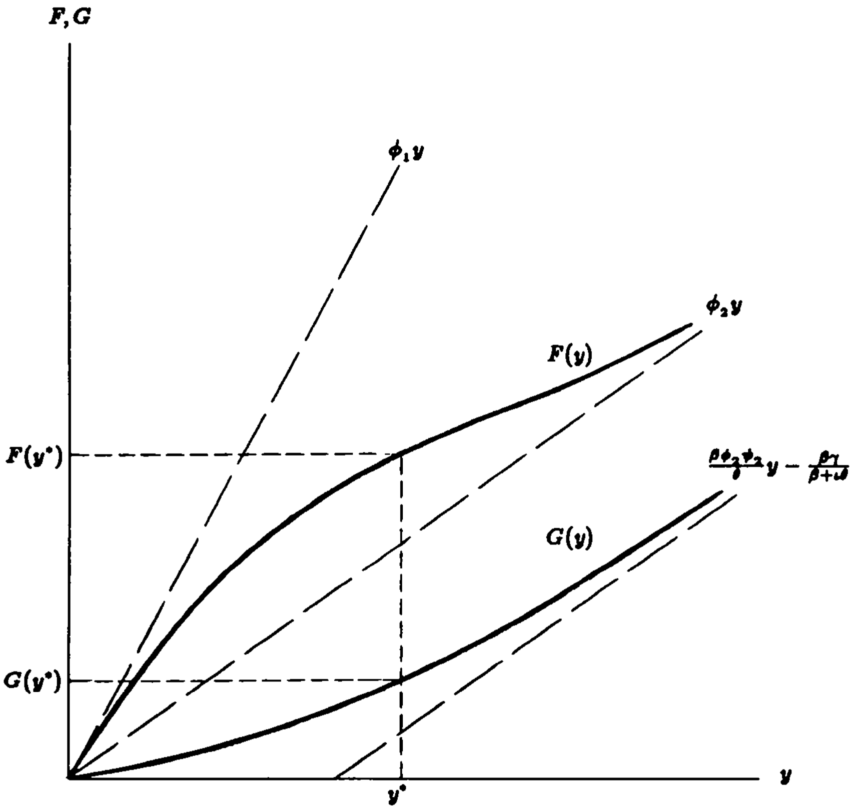

Figure 1
The values of a developed and undeveloped asset, *F(y)* and *G(y)*. depend on the current income *y* from a developed asset. In equilibrium, developers do not build, $b^* = 0$, when 0 £ y £ y*, and build at the minimum rate. $b^* = \beta$. when $y^* \leq y < \infty$.

Depending on the values of the parameters, the intermediate interval may not exist: $y^* \leq y^{**} < \infty$. The interpretation of the asymptote is familiar, even though its parameters are new. In the asymptote the constant $\beta\gamma/(\beta + \iota\theta)$ is an exercise price. As $y \rightarrow \infty$, development occurs asymptotically at the constant rate β per unit of developed assets or, equivalently, at the rate β/θ per unit of undeveloped assets, with θ again representing the ratio of undeveloped to developed assets. Also, the development cost is γ per unit. To compute the present value, the riskless cost per unit of time, $\beta\gamma/\theta$, must be discounted at the riskless rate $\iota + \beta/\theta$, reflecting the rate β/θ at which undeveloped assets are eliminated by development. In the asymptote the remaining term, $(\beta\phi_2\psi_2/\theta)y$, is the asymptotic value of development. As previously discussed following (11), developed properties have the asymptotic present value $\phi_2 y$ per unit. With development at the asymptotic rate β/θ per unit of time, development then yields the risky income $(\beta\phi_2/\theta)y$ per unit of time. This risky income has the present value $(\beta\phi_2\psi_2/\theta)y$, calculated with the present-value factor ψ_2 or, equivalently, the discount rate $\iota + \beta/\theta - \rho + (\beta - \delta)/\alpha$. This risky discount rate equals the riskless discount rate from above, $\iota + \beta/\theta$, minus the risk-adjusted, expected growth rate of income, $\rho - (\beta - \delta)/\alpha$. With undeveloped assets there is no adjustment for depreciation.

The solution to problem (4) through (9) also includes the optimal switching point in (10). This switching point is calculated from (7) under the transformation above (10) as follows:

$$y^* = \gamma \frac{1 + \left(1 + \dfrac{\eta_1\theta}{\alpha v}\right)\dfrac{\beta}{\beta + \iota\theta}\dfrac{\eta_2}{\eta_1 - \eta_2}}{\phi_1 - \chi_1 - \beta\left(1 + \dfrac{\eta_1\theta}{\alpha v}\right)\left(\dfrac{1 - \eta_2}{\eta_1 - \eta_2}\dfrac{\phi_2\psi_2}{\theta} + \dfrac{\zeta_2 - \eta_2}{\eta_1 - \eta_2}\dfrac{\chi_2}{\beta - \delta\theta}\right)}. \tag{13}$$

Under the constraints on the parameters to be identified, the switching point (13) is positive and finite. Below the switching point y^*, no developer builds in the symmetric Nash equilibrium: $b^* = 0$ for all $0 \le y < y^*$. Above the switching point all developers build at the maximum rate: $b^* = \beta$ for all $y^* \le y < \infty$. The switching point (13) replaces the single point of optimal exercise from previous models of real options in partial equilibrium because the capacity constraints of developers preclude simultaneous exercise of all options at one level of income y. This switching point has economically plausible comparative statics, as illustrated numerically in the next section.

The symmetric Nash equilibrium is completely characterized by (11) through (13) if assumption (10) is correct. This assumption is correct if (10) satisfies the first-order condition from (7) under the transformation above (10). As shown in the Appendix, (10) and (13) uniquely satisfy the first-order condition if the switching point (13) satisfies the upper bound in the following inequalities:

$$\frac{\eta_2\gamma/(\beta + \iota\theta)}{(\eta_2 - 1)\phi_2\psi_2/\theta + (\eta_2 - \zeta_2)\chi_2/(\beta - \delta\theta)} \le y^* < \frac{\gamma\eta_1}{(\eta_1 - 1)\phi_1 + (\zeta_1 - \eta_1)\chi_1}. \tag{14}$$

With the lower bound in (14), the valuation function (12) is everywhere nonnegative. Because the switching point (13) depends only on the parameters, the combination of (13) and (14) constrains the values of the parameters. These constraints are too complicated to evaluate analytically. Fortunately, (14) is not binding for plausible values of the parameters, as shown numerically in the next section.

3. Discussion

Comparative statics are discussed in this section. Because the solution, (11) through (13), is a complicated function of the parameters, most results must be calculated numerically. However, numerical computations have the advantage of revealing magnitudes that can help to answer interesting questions. For example, do realistic elasticities of demand generate significantly different exercise policies and market values than the infinite elasticity from most previous models of real assets in partial equilibrium? Also, do constraints on the rate of development and the supply of undeveloped assets significantly affect rates of growth in equilibrium? Finally, how many developers are necessary for the optimal rate of development to approximate the competitive solution?

Numerical values of the optimal switching point and resulting values of the developed and undeveloped assets are displayed in Table 1 for various values of all parameters. In each vertical triplet the top number is the optimal switching point y^*, the middle number is the resulting value of the developed asset $F(y^*)$, and the bottom number is the value of the undeveloped asset $G(y^*)$. The column heading is the value of the parameter in question, and all other parameters are the values from the base case. The constraints (14) are binding nowhere in the table.

Of all the results reported in the table, the most interesting comparative statics relate to the optimal switching point and the parameters that do not appear in models of financial options. These parameters are the elasticity of demand α, the maximum building rate β, the depreciation rate δ, the relative supply of undeveloped assets θ, and the number of developers v. As indicated in the table, the optimal switching point y^* is increasing in α, β, and δ, but decreasing in θ and v. Moreover, these comparative statics hold for all possible combinations of the three different values of each parameter in the table.

The comparative statics are economically plausible. With larger elasticities α, income y is less sensitive in (1) to the aggregate supply of developed assets q, so the option to develop is more valuable in (12). Thus, the exercise of that option is optimally deferred in (13). Similarly, with larger maximum building rates β developers are less constrained from building only when building is most profitable. As a result, development is again deferred in (13). Also, with larger depreciation rates δ the developed asset is less valuable in (11) and development is retarded in (13). In contrast, with larger relative

Table 1
Optimal switching point y^* and the resulting values of developed and undeveloped assets $F(y^*)$ and $G(y^*)$

Parameter values for the base case			α elasticity of demand		
			1.0	**2.0**	**∞**
$\alpha = 1.0$	$\beta = .03$	$\gamma = 5.0$	.5448	.5921	.6464
$\delta = .02$	$\theta = 1.0$	$\iota = .10$	5.116	5.216	5.386
$\nu = 10$	$\rho = 0$	$\sigma^2 = .05$	.0939	.1908	.3013
β maximum building rate			**γ construction cost per unit**		
.02	**.03**	**.04**	**4.0**	**5.0**	**6.0**
.5415	.5448	.5454	.4358	.5448	.6538
5.186	5.116	5.028	4.093	5.116	6.139
.1510	.0939	.0227	.0751	.0939	.1126
δ depreciation rate			**θ relative supply of undeveloped asset**		
.01	**.02**	**.03**	**0.5**	**1.0**	**1.5**
.5437	.5448	.5507	.5500	.5448	.5438
5.077	5.116	5.198	5.165	5.116	5.107
.0609	.0939	.1629	.1306	.0939	.0873
ι riskless rate of interest			**ν number of developers**		
.08	**.10**	**.12**	**1**	**10**	**∞**
.4448	.5448	.6443	.5792	.5448	.5421
5.117	5.116	5.109	5.440	5.116	5.091
.0957	.0939	.0872	.1301	.0939	.0911
ρ risk-adjusted, expected rate of growth in exogenous demand			**σ^2 variance of growth rate in exogenous demand**		
−.02	**.00**	**.02**	**.04**	**.05**	**.06**
.6449	.5448	.4446	.5373	.5448	.5517
5.071	5.116	5.186	5.082	5.116	5.147
.0551	.0939	.1563	.0651	.0939	.1197

In each vertical triplet, the top number is the optimal switching point y^* above which development is optimal, the middle number is the value of a developed asset $F(y^*)$. and the bottom number is the value of an undeveloped asset $G(y^*)$. The column heading is the value of the indicated parameter. and values of all other parameters are indicated in the base case.

supplies θ the undeveloped asset is relatively less abundant and thereby less valuable in (12), so development is accelerated in (13). Finally, with more developers ν each has a smaller inventory of undeveloped assets. Hence, each developer is less concerned in (7) about the effect of his development on the values of the remaining undeveloped assets, and development is accelerated in (13).

For a deeper understanding of these results, some preliminaries are necessary. For each developer n the maximand in (7) is linear in the control b_n. Under the transformation above (10), the coefficient in (7) of the control b_n is proportional to the value H(y) from the function:

$$H \equiv F - G - (\theta/\alpha\nu)yG' - \gamma \tag{15}$$

for all $y \geq 0$. Let π represent a generic parameter, and consider parameter values that satisfy the inequalities in (14). In this case, the switching point, y^* in (13), is the unique root of the function (15):

$H(y^* \mid \pi) = 0$. Moreover, this root satisfies the inequality $H'(y^* \mid \pi) > 0$. Under these conditions the switching pointy* increases with small increments in the parameter π if and only if $H(y^* \mid \pi)$ decreases similarly in π.

Conditions under which the function H is decreasing in a parameter can be seen by examining (15). In (15) the first term F is the value of the developed asset, and the second term G is the value of the undeveloped asset or, equivalently, the option to develop. The third term, $(\theta/\alpha\nu)yG'$, is the effect of current construction by each developer on the income y and thereby the values of his remaining undeveloped assets. In models with an infinite elasticity of demand, $\alpha = \infty$, or perfectly competitive developers, $\nu = \infty$, the third term is zero. However, with finite elasticity and imperfect competition, this term is positive. In standard models of financial options other than warrants and sinking fund bonds, the first term is independent of all parameters, and the third term is zero. Hence, for financial options a larger value of the parameter π increases the switching pointy* and thereby defers optimal exercise of the option if and only if it also increases the value of the option $G(y^* \mid \pi)$ This is a familiar result. With real options and either infinite elasticity of demand or perfect competition among developers, a larger value of the parameter p increases the switching pointy* if and only if it also increases the difference between the values of the undeveloped and developed assets, $G(y^* \mid \pi) - F(y^* \mid \pi)$. Finally, with real options, finite elasticity, and imperfect competition, a larger value π defers optimal exercise if and only if it increases the weighted difference, $(\eta_1\theta/\alpha\nu)G(y^* \mid \pi) - F(y^* \mid \pi)$, from (13) and (14).

Although these results can be used to interpret all the comparative statics in the table, for brevity the discussion is limited to the new parameters α, β, δ, θ, and ν. From a numerical evaluation of the valuation functions for developed and undeveloped assets, F and G, the following conclusions can be drawn. Increments in the elasticity a do not affect the developed value $F(y^*)$ but do increase the undeveloped value $G(y^*)$ sufficiently to offset the decreased weight $\eta_1\theta/\alpha\nu$ in (15) and, thereby, increase the switching point y^*. Increments in either the maximum building rate β or the depreciation rate δ also increase the switching point y^*, but do so by decreasing the developed value $F(y^*)$ without affecting the undeveloped value $G(y^*)$. In contrast, increases in the relative supply of undeveloped assets θ do not change the developed value $F(y^*)$ but greatly decrease the undeveloped value $G(y^*)$ and, thereby, decrease the switching point y^*. Finally, an increased number of developers v affects neither the developed nor the undeveloped values, $F(y^*)$ or $G(y^*)$, but decreases the weight $\eta_1\theta/\alpha\nu$ and thereby decreases the switching pointy*.

Conclusions concerning numerical magnitudes can also be drawn from the table. If the true elasticity of demand is unity ($\alpha = 1$) but income is falsely hypothesized to be exogenous ($\alpha = \infty$), then the value of the option to develop at the point of first exercise $G(y^*)$ is falsely overstated by the multiple: 0.3013/0.0939 = 3.2. With smaller elasticities α, the error is even greater. What are realistic values for the elasticity α? Consider two constants: the elasticity of aggregate demand with respect to the price of the good or service, $\xi > 0$, and the exponent from the convex, Cobb-Douglas cost function for output from each developed asset, $\kappa > 1$. As shown in the Appendix, the elasticity in the clearing condition (1) is $\alpha = \xi + (1 - \xi)/\kappa$. Since $\kappa > 1$, the elasticities α and ξ have the relationship $\alpha \leq 1$ if and if $\xi \leq 1$, with $\alpha = 1$ whenever $\xi = 1$. For housing services most estimates of the elasticity ξ are concentrated in a small interval bounded above by 1.[24] For houses a realistic elasticity of demand a is then slightly less than or equal to 1. Also, as illustrated in Table 1 increments in the maximum building rate β have a substantial effect on the value of the option $G(y^*)$ when $\alpha = 1$. With exogenous income ($\alpha = \infty$), this effect disappears completely. Finally, imperfect competition among developers can greatly decrease the value of the option to develop at the point of first exercise $G(y^*)$. However, effects of major magnitude require a very small number of developers v. For example, with duopolists ($v = 2$) the switching point, $y^* = 0.5574$, and resulting option value, $G(y^*) = 0.1071$, are much closer to the corresponding values in Table 1 for perfect competition ($v = \infty$) than the values for monopoly ($v = 1$).

4. Conclusion

In this article real options are valued under conditions that are characteristic of real assets but not most financial assets. Here, real assets have four distinguishing characteristics. Each real asset produces goods or services that consumers demand with a finite elasticity. The rate at which assets can be developed is limited by developers' capacity, so all options to develop cannot be exercised simultaneously. Also, the supply of undeveloped assets is limited relative to the current supply of developed assets. Finally, the ownership of undeveloped assets is monopolistic, oligopolistic, or competitive. For analytical convenience, the elasticity of demand is constant, the cost of development is constant up to a capacity constraint that is proportional to the supply of developed assets, the supplies of developed and undeveloped assets are proportional, and the number of owners or devel-

[24] Again, see the survey of Rosen (1985).

opers is fixed. Under these assumptions owners develop their assets in a Markov perfect equilibrium.

For this equilibrium the optimal exercise policy of developers and the resulting values of developed and undeveloped assets are calculated explicitly. In equilibrium, development is optimal at all values of income per developed asset above a critical value or switching point. Below this optimal value no developer builds; above the switching point all developers build at the maximum feasible rate. This region of optimal exercise replaces the single point of optimal exercise in all previous models of real options. Also, unlike previous models the ratio of earnings to value for developed assets depends on the income per asset. Here, the ratio decreases as income increases. Finally, the solution has both new comparative statics and new interpretations of familiar results. The comparative statics are computed numerically and discussed in detail.

Some straightforward extensions with explicit solutions are possible. Assumptions can be introduced from models of single developers in partial equilibrium. For example, construction can take time, construction costs can evolve in a geometric Wiener process, and development can occur at an optimal density or scale with Cobb-Douglas cost functions. Also, rents on real estate can depend on such factors as commuting time to the central business district. In these cases explicit solutions are possible with either Wiener or geometric Wiener processes if the option to abandon property is ignored. Other explicit solutions are possible if the aggregate demand in (1) is replaced by a linear demand, the geometric Wiener process (2) is replaced by a Wiener process, and the assumptions of proportional capacity and proportional supplies are replaced with additive capacity and additive supplies. With numerical solutions still other extensions are possible.

Explicit solutions are also possible with different ownership structures of either developed or undeveloped assets. In this article, developers own equal numbers of all undeveloped assets but no developed assets. Instead, imagine that developers retain some of their developed assets. This has two effects on the previous analysis if the number of developers v is finite. First, the spot market for the good or service from a developed asset is no longer perfectly competitive. With imperfect competition in the product market, the clearing condition (1) must be changed. Also, the representative developer no longer chose his building rate to maximize the market value of his portfolio of undeveloped assets. Instead, the residual term in his maximand (7) must be altered to include the impact of his building rate b_n on his portfolio of developed assets. These changes affect the optimal switching point between no development and development y^* and thereby the market values of developed and undeveloped assets, but do not alter the structure of the previous solutions.

Endogenizing the ownership of either developed or undeveloped assets by allowing the number of developers to be determined in equilibrium is much more difficult. For example, assume again that developers own equal numbers of all undeveloped assets but no developed assets. In the numerical solutions from Section 3, the profit from development increases as the ownership of undeveloped assets becomes more concentrated. As shown in Table 1, the direct gain from development when development begins, $F(y^*) - G(y^*) - \gamma$, decreases in the number of developers or owners v. It is positive with imperfect competition ($v < \infty$) and zero with perfect competition ($v = \infty$). Also, numerical evaluation of the valuation functions F and G verifies that the gain, $F - G - \gamma$, decreases in the number of developers v for all fixed incomes $y \geq y^*$ and increases in the income y for all incomes $y \geq y^*$. As a result, sellers of undeveloped assets receive the highest price from other owners of undeveloped assets, and ownership becomes more concentrated over time. In this model, as in other applications of option pricing, portfolio diversification is not an issue because, by assumption, all investors are either risk neutral or the capital market is complete. For example, with risk neutrality the concentration of ownership is determined in equilibrium by the distribution ofwealth. For this reason realistic predictions about the ownership of real options in equilibrium require a very different model with risk aversion and an incomplete capital market.

Another interesting issue is the relationship in equilibrium between earnings and the ratio of price to earnings for developed assets. In previous articles on real assets, this ratio is a constant, independent of earnings per developed asset; here, it is increasing in income or earnings. In particular, higher earnings per developed asset increase the probability of future earnings above the critical value at which construction occurs. This decreases the value or price of developed property and thereby the ratio of price to earnings. In fact, casual observation suggests the reverse relationship: with higher average operating income for developed assets, which is characteristic of cyclical peaks versus troughs, the ratio of price to earnings is typically higher, not lower. Clearly, this procyclical relationship between P/E ratios and earnings cannot be explained by the behavior of rational, value-maximizing investors in an equilibrium with cyclical returns on real assets. If returns are recognized by all investors to be cyclical, then P/E ratios should be countercyclical, not procyclical. Instead, explaining procyclical P/E ratios seems to require a model with agency and/or asymmetric information. Indeed, this perplexing procyclicality appears from casual observation to be associated with the persistent entry into markets late in business cycles of relatively uninformed, institutional investors and lenders.

Appendix

Derivation of (1)

In the perfectly competitive spot market, the consumption good c has current price p. With respect to this price, the aggregate demand by consumers is $(x/p)^{\xi}$, with constant elasticity $\xi > 0$. Also, the cost of producing c is proportional to c^{κ} with constant exponent $\kappa > 1$. For each unit of the developed asset, the profit-maximizing production of c is proportional to $p^{1/(\kappa-1)}$, and the resulting income y is proportional to $p^{\kappa/(\kappa-1)}$. Thus, the aggregate supply of the product from the developed asset is proportional to $qp^{1/(\kappa-1)}$ or, equivalently, $qy^{1/\kappa}$. Similarly, the aggregate demand is proportional to $(x/y^{1-1/\kappa})^{\xi}$. Equating aggregate supply with aggregate demand and rescaling the geometric Wiener variable x yields (1) with the constant, $\alpha \equiv \xi + (1-\xi)/\kappa$.

Hypothesis

With the transformation above (10) and the function H from (15), the unique optimal control in (7) is

$$b^* = \begin{cases} 0, & H(y) < 0, \\ \beta/\nu, & H(y) \geq 0. \end{cases} \tag{A1}$$

Thus, the unique subgame perfect Nash equilibrium is symmetric with the aggregate building rate: $b^* = \nu b_n^*$. Now, suppose that the function H satisfies the inequalities: $H(y) < 0$ for all $0 \leq y < y^*$ and $H(y) > 0$ for all $y^* < y < \infty$. In this case, (10) and (Al) are equivalent. That is, there is a unique switching point y^* in (10) satisfying $H(y^*) = 0$. For this case, the indicator function I is defined as follows: $I(y) = 0$ on $0 \leq y < y^*$ and $I(y) = 1$ on $y^* \leq y < \infty$.

Derivation of (11)

With (10) and the transformation above (10), the valuation function (4) simplifies to

$$0 = \frac{1}{2}\sigma^2 y^2 F'' + \left(\rho + \frac{\delta - \beta I}{\alpha}\right) yF' - (\delta + \iota)F + y. \tag{A2}$$

Also, the initial condition (5) becomes

$$F(0) = 0. \tag{A3}$$

Finally, the limiting condition (6) simplifies to

$$F(y) \leq vy$$

as $y \to \infty$.

The differential equation (A2) has a unique solution of the form

$$F(y) = \begin{cases} a_1 y^{\zeta_1} + a_1' y^{\zeta_1'} + \phi_1 y, & 0 \leq y < y^*, \\ a_2 y^{\zeta_2} + a_2' y^{\zeta_2'} + \phi_2 y, & y^* \leq y < \infty. \end{cases} \tag{A5}$$

The parameters ζ_1 and ζ_2 are defined below (11). The remaining constants are

$$\zeta_1' \equiv \frac{1}{2} - \frac{\rho + \delta/\alpha}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\rho + \delta/\alpha}{\sigma^2}\right)^2 + 2\frac{\delta + \iota}{\sigma^2}} < 0$$

and

$$\zeta_2' \equiv 1 - \left[\frac{1}{2} - \frac{\rho + (\delta - \beta)/\alpha}{\sigma^2}\right] + \sqrt{\left[\frac{1}{2} - \frac{\rho + (\delta - \beta)/\alpha}{\sigma^2}\right]^2 + \frac{2}{\phi_2 \sigma^2}} > 1.$$

The initial condition (A3) requires that $a_1' = 0$, and the limiting condition (A4) requires that $a_2' = 0$.Also, F is continuously differentiable on $y > 0$, so the constants, a_1 and a_2, in (A5) are determined by the conditions:

$$F(y^* + \epsilon) - F(y^* - \epsilon) \to 0, \quad F'(y^* + \epsilon) - F'(y^* - \epsilon) \to 0, \tag{A6}$$

$\epsilon \to 0$. Thereby, (A5) simplifies to (11).

Derivation of (12) and (13)

With assumption (10) and the transformation above (10), the valuation equation (7) simplifies to

$$0 = \frac{1}{2}\sigma^2 y^2 G'' + \left(\rho + \frac{\delta - \beta I}{\alpha}\right) yG' - \left(\iota + \frac{\beta}{\theta} I\right) G + \frac{\beta}{\theta}(F - \gamma) I, \tag{A7}$$

since $b_n = b^*/v = \beta/v$. Again, the initial condition (8) becomes

$$G(0) = 0. \tag{A8}$$

Also, the limiting condition (9) simplifies to

$$G(y) \leq vy \tag{A9}$$

as $y \to \infty$.

Under assumption (10), problem (A7) through (A9) has the unique solution (12) and (13). The valuation equation (12) is derived much like (11). Again, one of the boundary conditions, (A8) and (A9), eliminates one root of the characteristic polynomial on each of the two intervals $[0, y^*)$ and $[y^*, \infty)$. This result, combined with the particular solution to the inhomogeneous equation on the upper interval, $[y^*, \infty)$, and the continuous differentiability of G at y^*, as in (A6), yields the solution (12). Finally, the switching point (13) is calculated using (11) and (12). Under assumption (10), the switching point y^* is the root of the function H from (15): $H(y^*) = 0$. With the inequalities (14) this root is positive and finite.

Verification of (10)

To complete the derivation, (10) through (13) must be shown to satisfy the switching condition (Al). The function H in (15) satisfies the corner conditions $H(0) < 0 < H(\infty)$. The latter inequality holds under the assumption above (4): $\rho < \iota - \delta/\alpha$. Also, at y^* it satisfies

$$y^* H'(y^*) = \gamma\eta_1 - [(\eta_1 - 1)\phi_1 + (\zeta_1 - \eta_1)\chi_1]y^*,$$

from (11) and (12). Because the term in brackets is positive, H is increasing at y^* if the upper bound in (14) is satisfied by (13). In this case, it is sufficient to show that (13) can be the only root of the function H on the interval $[0, \infty)$. On the lower interval, $[0, y^*)$, the function H is strictly concave, given (11) and (12). Because $H'(y^*) > 0$, there is no root below y^*. On the upper interval, (y^*, ∞), the function H is bounded below:

$$H(y) > \phi_2 y - G(y) - (\theta/\alpha\nu)yG'(y) - \gamma \equiv B(y).$$

The function B is convex on an initial interval, (y^*, y^0), and concave on the subsequent interval, (y^0, ∞), with an inflection point y^0 satisfying $y^* \leq y^0 \leq \infty$. Also, B satisfies $B'' < H''$ for all $y^* < y < \infty$. Given this lower bound Band the initial condition $H'(y^*) > 0$, there can be no root on the upper interval (y^*, ∞) if $B(y)/y \geq 0$ as $y \rightarrow \infty$. With (12) and (15), the desired result follows if $\rho < \iota - \delta/\alpha$, as assumed above (4).

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