The Dynamics of the Free-Rider Problem in Takeovers

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We explore the dynamics of a takeover bid. In contrast to preceding models, if the initial takeover bid is unsuccessful a raider is allowed to make a new tender offer in order to try and secure the remaining shares. Numerical analysis shows that the raider's tender offer rises over time as be accumulates more shares. The anticipation of a higher tender offer in the future makes shareholders more inclined to bold their shares and forces the raider to offer a higher premium than is predicted by static theories. As the time between tender offers goes to zero, we show analytically that the expected profit from engaging in a takeover goes to zero.

Since the work of Grossman and Hart (1980), it has generally been believed that exclusionary devices are required for takeovers to be profitable. This insight is predicated upon shareholders being atomistic. Bagnoli and Lipman (1988) show that when there is instead a finite number of shareholders, equilibria exist in which a raider engages in a takeover. For the

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case when each shareholder has one share, the symmetric equilibrium has the raider make a tender offer strictly less than the posttakeover value and each shareholder randomize in deciding whether to sell. While the takeover does not take place for certain, it does occur with positive probability, and, in the event it does occur, the raider earns strictly positive profit since his tender offer is below the posttakeover share value. As a result, the raider realizes positive expected profit from engaging in a takeover. The profitability of takeovers when the number of shareholders is finite is also established by Holmström and Nalebuff (1992).[1]

In showing that takeovers can be profitable, these theories leave a crucial issue unresolved. When randomization results in too few shareholders tendering, the takeover bid fails because the raider is not allowed to make a new tender offer. But is this a natural assumption? Might there not be an incentive for a raider to go back into the market with a new tender offer so as to try and secure the remaining shares? In fact, if it was initially profitable to make a tender offer, it would seem even more so now that the raider has a position in the firm as it will take fewer shares to complete the takeover. Of course, this depends on what happens to a shareholder's reservation price as the raider's holdings increases.

In practice, the successful takeover of a firm often does require a raider to make more than one bid. In a data set composed of U.K. acquisitions, Franks and Harris (1989) found that over 9 percent of uncontested takeovers entailed multiple bids. Multiple bids had a positive and significant impact on the abnormal returns received by the shareholders of the target firm. It would then appear that a static model creates a premature ending to the takeover bid.

In related papers, Shleifer and Vishny (1986), Hirshleifer and Titman (1990), Holmstrijm and Nalebuff (1992), and Ferguson (1992) explore a takeover bid when the bidder has a stake in the firm prior to publicly announcing his one-time tender offer. Even though shareholders are atomistic in Shleifer and Vishny (1986) and Hirshleifer and Titman (1990), attempting a takeover is profitable because the bidder can earn an above-normal return on his current holdings. Of course, he realizes no gains on shares acquired via the tender offer. The limitation to this form of analysis, however, is that it does not explain how the raider came to acquire his initial holdings. This approach constitutes only a snapshot of a dynamic setting in which a raider gradually acquires shares over time.

[1] Also see Noe (1992), who uses nonstandard analysis to prove that the raider's expected return is positive for some equilibria even when each shareholder has only an infinitestimal fraction of the total number of shares. For a review of some of the work mentioned, see Spatt (1989).

It is the objective of this article to formulate a dynamic model of a raider's quest for control of a firm. The raider is assumed to make unconditional (and uncontested) bids directly to shareholders. The takeover bid is derived as part of a symmetric Markov perfect equilibrium of an infinite horizon game. Our main objective is to explore the dynamical properties of a takeover bid and to provide new insight into the free-riding problem.

Even though a raider has more opportunities to acquire a firm in the dynamic setting, our analysis shows that his payoff is strictly less than that predicted by equilibria for the static game. When the takeover turns out to be unsuccessful in the static game, the loss to a shareholder from not tendering is considerable since he has foregone any opportunity of selling his shares at a premium. In contrast, when the raider can make future tender offers, a shareholder knows that an initially failed takeover attempt will only bring the raider back into the market with, as we will show, a more attractive tender offer. Since the value of not tendering one's shares is greater, a shareholder must be offered a higher price in order to induce him to sell today. This drives up the tender offer and drives down the raider's expected profit.

As the time between tender offers becomes arbitrarily small, it is shown that a successful takeover requires the raider to make a tender offer that gives almost all of the takeover-induced surplus to shareholders. Though the takeover is consummated very quickly, the raider receives very little of the surplus he has created. When a raider is capable of making frequent tender offers, a takeover attempt is then unlikely to generate sufficient profit so as to compensate him for any associated fixed costs. For example, with 20 identical shareholders and a takeover rule of 50 percent, the symmetric equilibrium of the static game predicts the raider's expected profit is almost 9 percent of the rise in the value of the firm from a takeover. However, with an annual real interest rate of 4 percent and a tender offer every three months, the symmetric equilibrium of the dynamic game predicts that the raider's expected profit is less than $\frac{1}{10}$ of 1 percent of the takeover-induced rise in firm value.

This finding-that the equilibrium tender offer converges to the posttakeover share value as the length of the period goes to zero-is closely related to an important theorem in the durable goods and bargaining literatures. Referred to as the Coase Conjecture (though it has been proven), it states that the monopoly price of a durable good is close to marginal cost when the length of the period is small. Thus, when agents can change their actions quickly, inefficiencies associated with price being above marginal cost or with delay in completing a takeover (due to the tender offer being less than the posttakeover share value) are small. The relationship between our model and the model of a durable goods monopolist will be discussed.

Though a dynamic theory supports the qualitative findings of Bagnoli and Lipman (1988)-in that a raider earns positive expected profit from engaging in a takeover-the free-riding problem is shown to be considerably intensified when a raider can make multiple tender offers. Quantitatively, it would appear that the expected profit from a takeover bid can be quite low and that exclusionary devices may indeed be necessary even when ownership is concentrated.

1. A Dynamic Model of a Takeover

A firm is owned by N shareholders, each of whom holds exactly one share. There is a single raider who has proprietary knowledge as to how to increase the value of the firm by $\$\theta$ per share. Normalizing, we set $\theta = 1$ and, in the absence of a takeover, make the fundamental value 0. In order for the raider to institute the necessary changes to realize this increased value, he must own at least K shares where $K \in \{1, \ldots, N-1\}$. Then (K/N) is the takeover rule.

The game starts in period 1 and proceeds as follows. In each period, the raider makes an unconditional bid for shares directly to shareholders. That is, the raider is willing to buy any number of tendered shares at the offered price, regardless of whether the number of shares suffices to take control of the firm. Given the tender offer, the shareholders simultaneously decide whether to sell their shares. In order to ensure the existence of a symmetric equilibrium (that is, where all shareholders use the same strategy), we allow shareholders to randomize. We assume that once a shareholder sells, he is out of the game. As soon as the raider owns at least K shares, the game ends and all remaining shareholders (including the raider) receive 1 for each of their shares. There is no upper bound on the number of periods.

A history of the game in period t is composed of past tender offers and number of shares sold over the preceding t - 1 periods. Let H^t denote the space of possible histories at t. A raider's strategy is a sequence of functions, $\{\psi^t(\cdot)\}_{t=1}^{\infty}$, where $\psi^t: H^t \to \mathbf{R}_+$. ψ^t tells the raider what tender offer to make in period t conditional on the history at t. A shareholder's strategy is a sequence of functions, $\{\phi^t(\cdot)\}_{t=1}^{\infty}$, where $\phi^t: H^t \times \mathbf{R}_+ \to [0, 1]$. Given he has not already sold his share, ϕ^t is the probability that the shareholder sells given the history at t and the current tender offer. Shareholders and the raider are risk neutral and have a common discount factor of $\delta \in [0, 1]$.

In light of shareholders being identical, our solution concept is symmetric Markov perfect equilibrium. The payoff-relevant state variable is the number of shares sold or, equivalently, the number of shares currently held by the raider. Let m denote the state variable. At a Markov perfect equilibrium, the raider's strategy is a function that maps from (0, 1, . . , K - 1} into $\mathbf{R}_+$, and a shareholder's strategy maps from (0, 1, . . . , K - 1} × $\mathbf{R}_+$ into [0, 1].

Under this formulation, the dynamical properties of the takeover bid are due to shareholders tendering their shares over time. A raider may then need to make several tender offers in order to secure enough shares to gain control of the firm. A distinct source of dynamics is present in models with multiple bidders; see Giammarino and Heinkel (1986), Fishman (1988), and Hirshleifer and Png (1990). In their models, bids are assumed to be conditional so that a raider does not acquire shares over time.

2. Symmetric Markov Perfect Equilibrium

In this section, the method for constructing a symmetric Markov perfect equilibrium is described. An equilibrium is defined by a recursive system of Kequations and can be solved by backward induction. Since the game ends as soon as the raider owns K or more shares, payoffs are uniquely defined when the state variable, m, lies in $\{K, K + 1, \ldots, N\}$. The first step is to solve for a symmetric Markov perfect equilibrium when $m = K - 1$. Having solved for that equilibrium, payoffs are defined for $m \geq K - 1$. Next solve for a symmetric Markov perfect equilibrium when $m = K - 2$. Continuing backwards in this manner, a symmetric Markov perfect equilibrium for the entire game can be derived.

To begin, consider the subgame when the raider has K - 1 shares and makes a tender offer of r. Let $\hat{r}_{K-1}$ denote shareholders' expectation on next period's value of a share when the state variable is unchanged; that is, no shares were sold this period. Without loss of generality, we can assume $\hat{r}_{K-1} \leq 1$ since this must be true in equilibrium. Ultimately, $\hat{r}_{K-1}$ will be derived, but for the time being it is taken as exogenous.

Given $(r; \hat{r}_{K-1})$, let us derive optimal shareholder behavior. If $r \leq \delta\hat{r}_{K-1}$ then the weakly dominant strategy is not to tender. Tendering yields r, whereas not tendering yields $\delta\hat{r}_{K-1}$ when no one else tenders and 1 when at least one other shareholder tenders, so the raider owns at least K shares and can implement the takeover. The unique equilibrium is for each shareholder not to tender when $r \leq \delta\hat{r}_{K-1}$. Now suppose $r \in (\delta\hat{r}_{K-1}, 1)$. Tendering is not a symmetric equilibrium because it yields r, whereas not tendering, given everyone else is tendering, yields 1 due to the firm being taken over. Nor is not tendering a symmetric equilibrium, because it yields $\delta\hat{r}_{K-1}$ while tendering yields r. Hence, a symmetric pure-strategy equilibrium does not exist when $r \in (\delta \hat{r}_{K-1}, 1)$. A symmetric mixed-strategy equilibrium does exist, however, and it entails each shareholder tendering with probability p_{K-1} defined by

$$r = [1 - (1 - p_{K-1})^{N-K}](1) + (1 - p_{K-1})^{N-K}\delta \hat{r}_{K-1}. \qquad (1)$$

The term r is the payoff from tendering, and the r.h.s. of (1) is the payoff from not tendering. It is derived as follows. Given each of the other remaining $N - K$ shareholders tenders with probability p_{K-1}, the probability that at least one of them tenders is $1 - (1 - p_{K-1})^{N-K}$. In that event, the takeover is completed and a nontendering shareholder earns 1. If instead no one tenders, which occurs with probability $(1 - p_{K-1})^{N-K}$, he receives a payoff of $\hat{r}_{K-1}$ in the next period. The final case is when $r \geq 1$. It is easy to show that the unique symmetric equilibrium is for all shareholders to tender. It follows from the preceding analysis that a shareholder's strategy at the unique symmetric equilibrium is

$$\hat{p}_{K-1}(r; \hat{r}_{K-1}) = \begin{cases} 0 & \text{if } r \in [0, \delta \hat{r}_{K-1}], \\ 1 - \left[\dfrac{1 - r}{1 - \delta \hat{r}_{K-1}}\right]^{1/(N-K)} & \text{if } r \in (\delta \hat{r}_{K-1}, 1), \\ 1 & \text{if } r \geq 1. \end{cases} \qquad (2)$$

When he makes a tender offer of r, a raider expects a shareholder to tender his share with probability $\hat{p}_{K-1}(r; \hat{r}_{K-1})$. Let $V_{K-1}(\hat{r}_{K-1})$ denote the value function for the raider when $m = K - 1$ given $\hat{r}_{K-1}$:

$$\begin{aligned} V_{K-1}(\hat{r}_{K-1}) &= \max_{r} \sum_{j=1}^{N-K+1} \binom{N-K+1}{j} (\hat{p}_{K-1})^{j}(1 - \hat{p}_{K-1})^{N-K+1-j}(K - 1 + j - jr) \\ &\quad + (1 - \hat{p}_{K-1})^{N-K+1}\delta V_{K-1}(\hat{r}_{K-1}). \end{aligned} \qquad (3)$$

Recall that $\hat{p}_{K-1}$ is a function of r as well as $\hat{r}_{K-1}$. With probability

$$\binom{N-K+1}{j} (\hat{p}_{K-1})^{j}(1 - \hat{p}_{K-1})^{N-K+1-j}$$

exactly j of the $N - K + 1$ remaining shareholders tender their shares. Since the raider already has $K - 1$ shares, the takeover occurs when $j \geq 1$ in which case the raider earns a return of 1 on each of his $K - 1 + j$ shares. Note that these j shares cost him jr. When $j = 0$, the state variable is unchanged so that his payoff is $\delta V_{K-1}(\hat{r}_{K-1})$. Solving

(3) for $V_{K-1}(\hat{r}_{K-1})$, the optimal tender offer is that value of r that maximizes

$$\frac{\sum_{j=1}^{N-K+1}\binom{N-K+1}{j}(\hat{p}_{K-1})^j(1-\hat{p}_{K-1})^{N-K+1-j}(K-1+j-jr)}{1-\delta(1-\hat{p}_{K-1})^{N-K+1}}. \tag{4}$$

Since $\hat{p}_{K-1}$ is continuous in r, then a maximum to (4) exists. Assuming it is unique, denote it $r^0_{K-1}(\hat{r}_{K-1})$. It is straightforward to show that $r^0_{K-1} < 1$.

Thus far, equilibrium strategies for the raider and the remaining shareholders have been derived when $m = K - 1$ and $\hat{r}_{K-1}$ is shareholders' expectation on the value of a share next period given that the state variable is unchanged. The final step in deriving a symmetric Markov perfect equilibrium for this subgame is to ensure that shareholders' expectation of tomorrow's share price is correct. At a Markov perfect equilibrium the value of a share tomorrow, given that the state variable is the same as today, must equal the value of a share today. The latter is $r^0_{K-1}(\hat{r}_{K-1})$, and the former is $\hat{r}_{K-1}$. Thus, the equilibrium tender offer when m = K - 1, denoted r^*_{K-1}, is defined by the following fixed point:

$$r^*_{K-1} = r^0_{K-1}(r^*_{K-1}). \tag{5}$$

Since $r^*_{K-1} \in (\delta r^*_{K-1}, 1)$ then, according to (2), shareholders randomize when the equilibrium tender offer is made. Letting p^*_{K-1} denote the equilibrium probability of tendering a share given that the equilibrium tender offer of r^*_{K-} is made, it is defined by

$$p^*_{K-1} = 1 - \left[\frac{1 - r^*_{K-1}}{1 - \delta r^*_{K-1}}\right]^{1/(N-K)}. \tag{6}$$

The equilibrium outcome for the K - 1 subgame is then (r^*_{K-1}, p^*_{K-1}) Finally, let V^*_{K-1} denote the raider's equilibrium payoff: $V^*_{K-1} \equiv V_{K-1}(r^*_{K-1})$.

Having derived the equilibrium outcome for the K - 1 subgame, we have payoffs for all m $\geq K$ - 1. One can then solve for an equilibrium when the raider has K - 2 shares. More generally, let us define an equilibrium outcome for the generic m subgame given $r^*_{m+1}, \ldots, r^*_{K-1}$ and $V^*_{m+1}, \ldots, V^*_{K-1}$. As above, the first step is to derive the equilibrium probability of tendering one's share, $\hat{p}_m$, as a function of the tender offer, r, and shareholders' expectation on the value of a share next period when no shares are tendered today, $\hat{r}_m$. Assuming $\hat{r}_m \leq 1$ and $r \in [\delta\hat{r}_m, 1]$ (which will be true in equilibrium), $\hat{p}_m(r; \hat{r}_m)$ is implicitly defined by

$$r = \sum_{j=K-m}^{N-m-1} \binom{N-m-1}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-1-j}(1) + \sum_{j=1}^{K-m-1} \binom{N-m-1}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-1-j} \delta r^*_{m+j} + (1-\hat{p}_m)^{N-m-1} \delta \hat{r}_m. \tag{7}$$

r is the payoff from tendering one's share, and the r.h.s. of (7) is the payoff to not tendering. When j shares are tendered and $j \geq K - m$, the takeover occurs. In that case, by not tendering his share, a shareholder earns 1. If $j \in \{1, \ldots, K - m - 1\}$, the takeover is unsuccessful but a share will be worth r^*_{m+j} tomorrow. Since, in equilibrium, a shareholder is indifferent as to selling his share, his payoff is r^*_{m+j} when the raider owns $m + j$ shares. Finally, if no shares are tendered, then his share will be valued at $\hat{r}_m$.

Given that each of the remaining $N - m$ shareholders tenders with probability $\hat{p}_m$ (recall that it is a function of r and $\hat{r}_m$), the value function for the raider is

$$V_m(\hat{r}_m) = \max_r \sum_{j=K-m}^{N-m} \binom{N-m}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-j}(m+j-jr) + \sum_{j=1}^{K-m-1} \binom{N-m}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-j}(\delta V^*_{m+j} - jr) + (1-\hat{p}_m)^{N-m} \delta V_m(\hat{r}_m). \tag{8}$$

Defining $r^0_m(\hat{r}_m)$ as the optimal tender offer, it is the value of r that maximizes

$$\left[\sum_{j=K-m}^{N-m} \binom{N-m}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-j}(m+j-jr) + \sum_{j=1}^{K-m-1} \binom{N-m}{j} (\hat{p}_m)^j (1-\hat{p}_m)^{N-m-j}(\delta V^*_{m+j} - jr) \right] \div [1 - \delta(1-\hat{p}_m)^{N-m}]. \tag{9}$$

To ensure that shareholders' expectations are correct, the equilibrium tender offer, r^*_m, is defined by

$$r^*_m = r^0_m(r^*_m). \tag{10}$$

Finally, let $p^*_m = \hat{p}_m(r^*_m; r^*_m)$ and $V^*_m = V_m(r^*_m)$. Solving this recursive

Table 1
symmetric Markov perfect equilibrium

	$(N, K, \sigma) = (10, 5, .9)$			$(N, K, \sigma) = (20, 10, .9)$		
m	p_m^*	r_m^*	V_m^*	p_m^*	r_m^*	V_m^*
0	.447	.929	.160	.491	.946	.186
1	.470	.953	1.090	.494	.957	1.132
2	.481	.973	2.043	.496	.967	2.088
3	.488	.987	3.016	.497	.976	3.055
4	.492	.997	4.003	.498	.985	4.032
5				.499	.991	5.016
6				.4993	.995	6.007
7				.4995	.998	7.003
8				.4997	.9994	8.001
9				.4998	.9999	9.0001

N = number of shares/shareholders.
K = minimum number of shares required to take control of the firm.
σ = discount factor of the raider and shareholders.
m = number of shares held by the raider.
p_m^* = probability of a shareholder tendering his share when the raider has m shares.
r_m^* = tender offer when the raider has m shares.
V_m^* = raider's payoff when he has m shares.

system of K equations allows one to construct a symmetric Markov perfect equilibrium.

3. Equilibrium Behavior of the Raider

Assuming the raider can make multiple offers brings in a potentially relevant feature of the environment. The downside to this extension is that it increases the complexity of the conditions that define equilibrium. Due to the intractability of analytical methods, numerical analysis is used to explore the properties of equilibria. We would like to emphasize, however, that this equilibrium is particularly amenable to numerical analysis in that there are only three unspecified parameters: the number of shares/shareholders, N, the minimum number of shares required for a takeover, K, and the discount factor, δ. A description of our numerical method is described in Appendix A.

Numerical analysis generated values for r_m^*, p_m^*, and V_m^* that are defined in the preceding section. For future reference, let $V^* \equiv V_0^*$ denote the raider's equilibrium payoff and $r^* \equiv r_0^*$ a shareholder's equilibrium payoff for the game. Total discounted surplus, *TS*, equals $V^* + Nr^*$, and the potential surplus is N. Note that the foregone surplus, $N - V^* - Nr^*$, is due solely to the delay in consummating the takeover. As an example of the output of our numerical analysis, Table 1 shows equilibrium values for the case of 10 and 20 share-

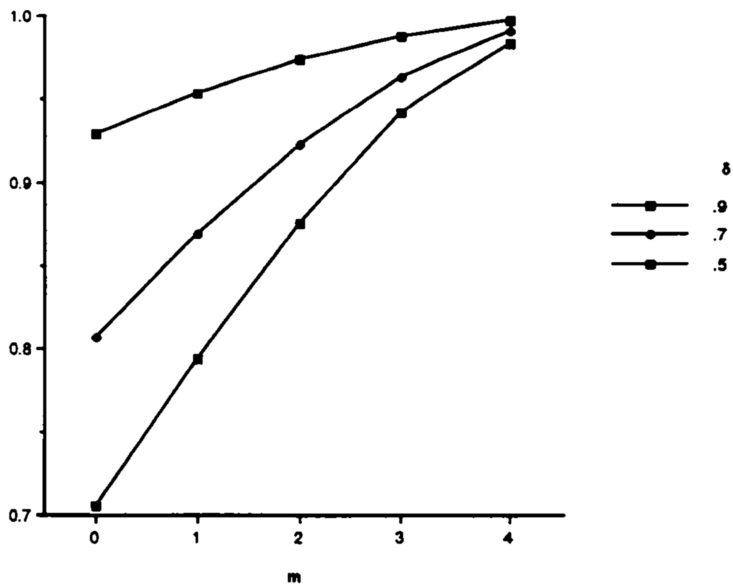

Figure 1
(N, K) = (10, 5)
Rising tender offer: $N \equiv$ number of shares/shareholders, $K \equiv$ minimum number of shares required to take control of the firm, $\delta \equiv$ discount factor of the raider and shareholders, $m \equiv$ number of shares held by the raider, $r^* \equiv$ equilibrium tender offer.

holders, given a takeover rule of 50 percent and a discount factor of .9. For $N = 10$, the raider makes an initial tender offer of .929, which is 92.9 percent of the incremental value from a takeover. In response, each of the shareholders tenders with probability .447. Out of the total potential surplus of 10, the raider's expected payoff is .160 or 1.6 percent of the value created by a takeover. The difference between 10 and .160 reflects both the considerable premium that the raider must offer to induce shareholders to tender and that there is likely to be delay in consummating the takeover. The latter is costly to the raider in that he must pay for shares upfront, but he does not reap the gains from buying those shares until he completes the takeover. The total surplus is 9.451, so the foregone value from delaying the takeover is only .549. This small loss is reflective of the quickness with which the takeover occurs.

There are several noteworthy properties of the dynamics of the takeover bid. Figures 1, 2, and 3 illustrate the equilibrium tender offer as a function of the raider's holdings when there are 10, 20, and 30 shareholders, respectively. Note that the greater the number of shares controlled by the raider, the higher is the value of the remaining shares.

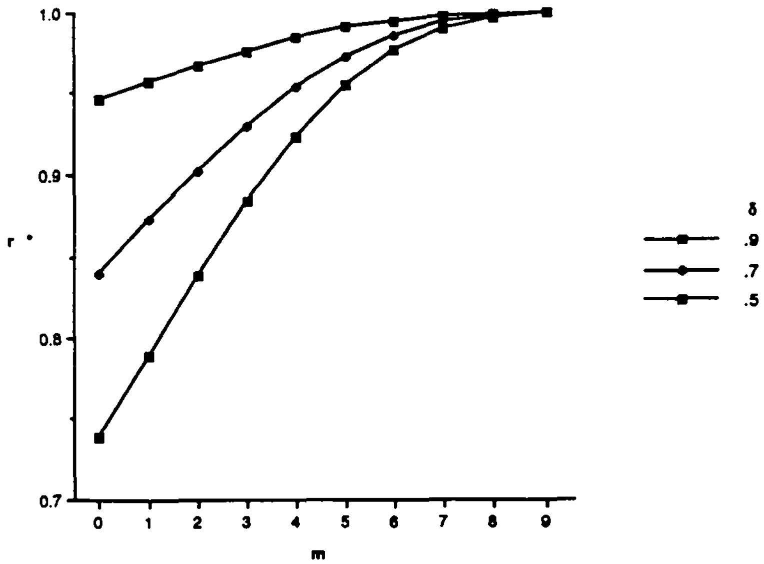

Figure 2
(N, K) = (20, 10)
Rising tender offer: $N \equiv$ number of shares/shareholders, $K \equiv$ minimum number of shares required to take control of the firm, $\delta \equiv$ discount factor of the raider and shareholders, $m \equiv$ number of shares held by the raider, $r^* \equiv$ equilibrium tender offer.

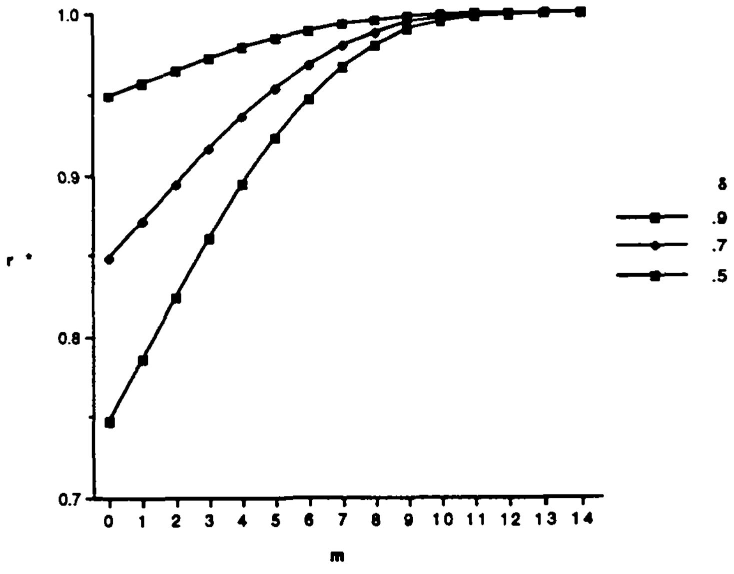

Figure 3
(N, K) = (30,15)
Rising tender offer: $N \equiv$ number of shares/shareholders, $K \equiv$ minimum number of shares required to take control of the firm, $\delta \equiv$ discount factor of the raider and shareholders, m $\equiv$ number of shares held by the raider, $r^* \equiv$ equilibrium tender offer.

Property 1. *The tender offer increases with the number of shares held by the raider.*

Property 1 was observed for all numerical runs. Since the raider's holdings do not decrease, Property 1 implies that tender offers are nondecreasing over time.

The intuition for Property 1 is as follows. When the raider owns m shares, he needs only another $K - m$ shares to complete the takeover. The benefit to him of buying those additional $K - m$ shares is K, because completing the takeover yields a return of 1 not only on his newly purchased shares but also on his current holdings. In contrast, the cost to completing the takeover is the expense of buying another $K - m$ shares since the expenditure incurred in buying the first m shares is sunk. Therefore, as the raider's holdings increase, the net benefit to completing the takeover rises. This induces the raider to make a higher tender offer in order to increase the probability of a shareholder tendering his share and thereby speed up the time until completion of the takeover. This effect is compounded by a rise in shareholders' reservation price, because a takeover has become more imminent.*

It is especially interesting to observe that the tender offer rises over time even though shareholders are homogeneous. When shareholders *are* heterogeneous and have different reservation prices for their shares, one would expect the tender offer to rise in response to an initially unsuccessful takeover bid since the raider must raise the tender offer in order to induce shareholders with higher reservation prices to sell. Shareholder heterogeneity would then result in shareholders selling at different points in time and the tender offer rising over time. Our analysis shows that such behavior occurs even in the absence of shareholder heterogeneity. Although all shareholders' reservation prices are identical at any point in time, they evolve over time in response to the raider's accumulation of shares. Thus, the endogeneity of a shareholder's reservation price to past tendering decisions causes intertermporal variation in tender offers in spite of the homogeneity of shareholders.

Not only does the tender offer rise over time, but it appears that the raider makes most of his profit on the initial purchase of shares. In Table 2, the average price is shown for the first 20 percent of the K shares purchased, the middle 60 percent, and the last 20 percent. These figures assume that no more than one share is tendered in a

[2] A similar result is derived for the static setting in Holmström and Nalebuff (1992) and Ferguson (1992). They modify the Bagnoli-Lipman model to allow the raider to have an exogenously determined stake in the firm. It is shown that as the initial number of shares held by the raider increases, the equilibrium tender offer rises. The intuition is similar to that discussed here.

Table 2
Share prices as related to the timing of purchase

(N, K, σ)	Average price of first 20% of shares	Average price of mid 60% of shares	Average price of last 20% of shares
(10, 5, .5)	.705	.870	.983
(10, 5, .9)	.929	.971	.997
(10, 5, .95)	.964	.985	.998
(10, 5, .99)	.993	.997	.9997
(20, 10, .9)	.952	.985	.9997
(30, 15, .9)	.957	.990	.99995

N = number of shares/shareholders.
K = minimum number of shares required to take control of the firm.
σ = discount factor of the raider and shareholders.

period. For example, when $(N, K, \delta) = (20, 10, .9)$, the raider gives an average premium of .952 for the first two shares purchased while giving close to full value, .9997, for the last two shares. Intuitively, once the raider is close to having enough shares to takeover the firm, the cost to delaying the takeover is sufficiently great that he finds it optimal to pay close to full value for the remaining shares.

In concluding this section, let us briefly discuss the implications of assuming there are some fixed costs in making a takeover attempt. As specified by Bhattacharya (see Spatt, 1989) and Hirshleifer and Png (1990), assume the raider incurs a fixed cost with each tender offer. It was argued above (and shown in Table 1) that the raider's payoff is increasing in the number of shares he owns as he needs to purchase fewer shares in order to complete the takeover. It follows that if it was profitable to initiate a takeover, then it must be profitable to continue it. Thus, even if there are fixed costs with each tender offer, the raider will never discontinue his takeover attempt though the presence of fixed costs might deter him from starting one.

4. Free Riding in the Dynamic Setting

Numerical analysis of the infinite horizon setting showed that a raider earns positive expected profit from attempting a takeover when the number of shareholders is finite. This supports the central finding of Bagnoli and Lipman (1988). However, the significance of this result depends on the amount of expected profit. If it is very small, then it is unlikely to be adequate to cover any fixed costs associated with attempting a takeover. In that case, takeovers are unprofitable ex ante and thus will not be attempted.

To explore this issue, define At as the length of the period, and let $\delta = e^{-\rho\Delta t}$ where $\rho > 0$ is the common discount rate. Given Δt, $p^*_m(\Delta t)$, $r^*_m(\Delta t)$,and $V^*_m(\Delta t)$ denote the values for the probability of tendering, the tender offer, and the raider's payoff, respectively, for some equilibrium, $m \in \{0, \ldots, K-1\}$. As before, $r^*(\Delta t) \equiv r_0^*(\Delta t)$ and $V^*(\Delta t) \equiv V_0^*(\Delta t)$ are the equilibrium payoffs for a shareholder and the raider.

For any sequence of equilibria associated with the time between tender offers going to 0, Theorem 1 shows that the raider's tender offers converge to the posttakeover share value. Thus, as $\Delta t \rightarrow 0$, the shareholders get all of the surplus from the takeover.

Theorem 1. For all sequences of equilibria, $\lim_{\Delta t \rightarrow 0} r_m^*(\Delta t) =$ *for all m Î {0, ... K- 1).*

Sketch of Proof Consider an arbitrary sequence of values for Δt from $\mathbf{R}_{++}$ for which $\Delta t \rightarrow 0$. By selecting an equilibrium for each value of Δt, we generate a sequence of equilibria. Associated with this sequence of equilibria are the sequences $\{\{p_m^*\}_{m=0}^{K-1}\}$, $\{\{r_m^*\}_{m=0}^{K-1}\}$, and $\{\{V_m^*\}_{m=0}^{K-1}\}$. To prove Theorem 1, we need to show that for any arbitrary sequence of equilibria, $\lim_{\Delta t \rightarrow 0} r_m^*(\Delta t) = 1 \; \forall \; m \in \{0, \ldots, K-1\}$.

The strategy of the proof is as follows. We begin by supposing that there exists a sequence of equilibria such that $r_{K-1}^*(\Delta t)$ does not converge to 1 as $\Delta t \rightarrow 0$. It is shown to follow from this supposition that there exists a sequence of equilibria such that $\{p_{K-1}^*\} \rightarrow 0$ and $\{r_{K-1}^*\}$ is bounded below 1. A contradiction is then established by showing that if $\{p_{K-1}^*\} \rightarrow 0$, then $\{r_{K-1}^*\} \rightarrow 1$. From this contradiction, we conclude that our original supposition, that $\{r_{K-1}^*\}$ does not converge to 1, is false. The proof of Theorem 1 is completed by induction. It is shown that if, for all sequences of equilibria, $\{r_{m+j}^*\} \rightarrow 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$, then, for all sequences of equilibria, $\{r_m^*\} \rightarrow 1$. This step is proven by applying the same method used for showing $\{r_{K-1}^*\} \rightarrow 1$. The proof is provided in Appendix B.

Since the tender offer made by the raider goes to one as the time between tender offers goes to zero, it follows that the raider's payoff goes to zero since he is paying a premium that approaches the full posttakeover rise in share value.

Corollary. $\lim_{\Delta t \rightarrow 0} V^*(\Delta t) = 0.$

Even when the number of shareholders is finite, the raider's expected profit is small when tender offers can be made frequently. Intuitively, the argument is as follows. Let r_{K-1}^* denote the equilibrium tender offer when the raider holds K - 1 shares. The value of a share tomorrow will either be r_{K-1}^* if no shares are sold today, or 1, if at least one share is sold so that the takeover is completed. A shareholder then expects the value of his share to be at least as high tomorrow and, with some probability, to be strictly higher. Hence, the only cost to holding on to one's share is the delay in being paid. As the time between tender offers goes to 0, the cost of delay goes to 0 so that a shareholder strictly prefers not to tender unless $r^*_{K-1} \rightarrow 1$. Thus, in order to induce shareholders to tender, the raider must make a tender offer that is arbitrarily close to the posttakeover share value. By induction, this argument can be applied to show that regardless of the raider's current holdings, the equilibrium tender offer goes to 1 as $\Delta t \rightarrow 0$. For if the raider holds m shares and the equilibrium tender offer is bounded below 1 as $\Delta t \rightarrow 0$, shareholders anticipate the expected value of a share being higher tomorrow because, by the inductive assumption, $r^*_{m+j} \rightarrow 1$ as $\Delta t \rightarrow 0$, $j \in \{1, \ldots, K-1-m\}$. As a result, the lone cost to not tendering today is the delay from being paid, and this cost goes to 0 as $\Delta t \rightarrow 0$. To induce shareholders to tender requires that the raider give all of the surplus to the shareholders.

It is also an implication of Theorem 1 that the time until completion of a takeover goes to 0 as the time between tender offers goes to 0. To establish this result, define $[T/\Delta t]$ as the integer part of $T/\Delta t$, where $T > 0$ and $\Delta t > 0$. For the symmetric Markov equilibrium, define $\Psi(T, \Delta t)$ as the probability that at least K shares are tendered over periods $\{1, \ldots, [T/\Delta t]\}$.

Theorem 2. *$\lim_{\Delta t \rightarrow 0} \Psi(T, \Delta t) = 1$ for all $T > 0$.*

Proof Since, in equilibrium, a shareholder randomizes, it must then be true that he is indifferent between tendering his share at $r^*_m(\Delta t)$ and not tendering his share, for all $m \in \{0, 1, \ldots, K-1\}$. When $m = 0$, it follows that a shareholder is indifferent between tendering at a price of $r^*_0(\Delta t)$ and never tendering his share (that is, waiting for a takeover). Let the payoff associated with the latter strategy be denoted $s(\Delta t)$ Since, by Theorem 1, $\lim_{\Delta t \rightarrow 0} r^*_0(\Delta t) = 1$, it follows from $s(\Delta t) = r^*_0(\Delta t)$ for all $\Delta t > 0$ that $\lim_{\Delta t \rightarrow 0} s(\Delta t) = 1$.

For any $T > 0$, note that the r.h.s. of (11) is an upper bound on $s(\Delta t)$:

$$s(\Delta t) \leq \Psi(T, \Delta t)(1) + [1 - \Psi(T, \Delta t)]e^{-\rho \Delta t [T/\Delta t]}(1). \qquad (11)$$

This follows from the fact that with probability $\Psi(T, \Delta t)$ a takeover occurs over periods $\{1, \ldots, [T/\Delta t]\}$ while with probability $[1 - \Psi(T, \Delta t)]$ it occurs over $\{[T/\Delta t] + 1, \ldots\}$. The r.h.s. expression presumes that if it occurs over $\{1, \ldots, [T/\Delta t]\}$ then it occurs in period 1, whereas if it occurs over $\{[T/\Delta t] + 1, \ldots\}$ then it occurs in period $[T/\Delta t] + 1$.

Now suppose that Theorem 2 is not true. This means that there exists a value for T, which we denote T', and an $\epsilon > 0$ such $\Psi(T', \Delta t)$

$< 1 - \epsilon$ for all $\Delta t > 0$. In conjunction with (11) it follows that

Since the r.h.s. is bounded above by $(1 - \epsilon) + \epsilon e^{-\rho(T-\Delta t)}$, it follows from (12) that $\lim_{\Delta t \to 0} s(\Delta t) \leq (1 - \epsilon) + \epsilon e^{-\rho T}$, which contradicts $\lim_{\Delta t \to 0} s(\Delta t) = 1$. We conclude that Theorem 2 is true. ■

To explore the actual size of a raider's payoff, numerical analysis was performed. Table 3 shows the results for when there are 10, 20, and 30 shareholders with a 50 percent takeover rule: V^* is the raider's payoff, r^* is a shareholder's payoff, and $TS = V^* + Nr^*$ is total surplus. When there are 10 shareholders and the raider can only make one tender offer ($\delta = 0$), he gets a payoff of 1.230, which is over 12 percent of the additional value created by a takeover. Increasing δ is equivalent to shrinking the time between tender offers. Note that a raider's expected profit is observed to fall. For reasonable values of the real interest rate and the time between tender offers, δ will exceed .9. For example, even if the real interest rate is 20 percent, the discount factor will be at least .9 when the time between tender offers is six months or less. A real interest rate of 4 percent and a tender offer every three months yields a discount factor of about .99. When δ = .99 and each shareholder has 10 percent of the firm, the raider can expect to get less than two-tenths of 1 percent of the additional value created by a takeover. The payoff in the static game is over 70 times as large. When there are 20 shareholders, a raider's expected profit shrinks by almost 99 percent relative to the static game. This yields him a payoff that is less than one-tenth of 1 percent of the takeover-induced rise in firm value. For example, if the target firm is currently worth $1 billion and the raider anticipates being able to improve value by 10 percent, his expected profit from attempting a takeover is $100,000.[3]

In the static game, a shareholder stands to lose a lot by not selling his share. If the takeover bid fails, a shareholder has lost any opportunity of selling his share at a premium. This creates a strong propensity to sell and results in the raider making a tender offer considerably below the posttakeover value of a share. When $(N, K) = (10, 5)$, it is half of that value. However, when the takeover bid is not a one-time affair ($\delta > 0$), the downside to not selling today is lessened. Even if the takeover is not initially successful, it may succeed in the

[8] An important caveat is that these numbers assume that the only determinants of δ are the time between tender offers and the real interest rate. However, another determinant is the possibility that the takeover breaks down for exogenous reasons; for example. the raider loses financing, or a change in managerial conditions eliminates the potential gains from a takeover. It is unclear what are reasonable values for the probability of exogenous breakdown.

Table 3
Effect of the time between tender offers

	$(N, K) = (10, 5)$				$(N, K) = (20, 10)$				$(N, K) = (30, 15)$			
σ	$V^*(\sigma)/N$	$V^*(\sigma)/V^*(0)$	r^*	TS/N	$V^*(\sigma)/N$	$V^*(\sigma)/V^*(0)$	r^*	TS/N	$V^*(\sigma)/N$	$V^*(\sigma)/V^*(0)$	r^*	TS/N
0	.123	1.000	.500	.623	.088	1.000	.500	.588	.072	1.000	.500	.572
.1	.114	.923	.536	.650	.080	.905	.546	.626	.065	.902	.549	.614
.2	.103	.841	.575	.678	.071	.809	.593	.664	.058	.802	.598	.656
.3	.093	.755	.615	.708	.063	.712	.641	.704	.051	.703	.648	.698
.4	.082	.665	.659	.740	.054	.614	.689	.744	.044	.604	.697	.741
.5	.070	.570	.705	.775	.045	.515	.739	.784	.036	.504	.747	.783
.6	.058	.470	.754	.812	.036	.414	.789	.826	.029	.403	.797	.826
.7	.045	.363	.807	.852	.028	.313	.840	.868	.022	.303	.848	.869
.8	.031	.250	.866	.896	.018	.209	.893	.911	.015	.202	.898	.913
.9	.016	.130	.929	.945	.009	.106	.946	.955	.007	.101	.949	.956
.95	.008	.066	.964	.972	.005	.053	.973	.977	.004	.051	.974	.978
.99	.0017	.014	.993	.994	.001	.011	.995	.996	.0007	.010	.995	.996

$N \equiv$ number of shares/shareholders.
$K \equiv$ minimum number of shares required to take control of the firm
$\sigma \equiv$ discount factor of the raider and shareholders.
$V^*(\sigma) \equiv$ raider's equilibrium payoff.
$V^*(0) \equiv$ raiders equilibrium payoff in the static game
$r^* \equiv$ shareholder's equilibrium payoff.
$TS \equiv$ equilibrium surplus $(TS = V^*(\sigma) + Nr^*)$.

Table 4
Effect of the takeover rule $(N, \sigma) = (10, .9)$

K	V^*	r^*	TS
5	.160	.929	9.451
6	.184	.932	9.506
7	.222	.932	9.546
8	.288	.930	9.583
9	.449	.918	9.628

$N \equiv$ number of shares/shareholders.
$K \equiv$ minimum number of shares required to take control of the firm
$\sigma \equiv$ discount factor of the raider and shareholders.
$V^* \equiv$ raider's equilibrium payoff.
$r^* \equiv$ shareholder's equilibrium payoff.
$TS \equiv$ equilibrium surplus $(TS = V^*(\sigma) + Nr^*)$.

future, and, in any event, the future tender offer is likely to be higher (Tables 1 and 2). Thus, as the frequency of tender offers rises, the shareholder's reservation price rises. This forces the raider to set a higher tender offer, which cuts into the raider's profit.

Property 2. *As the time between tender offers shrinks, the raider's payoff declines and a shareholder's payoff and total surplus rise.*

It follows that if a raider could commit, he would choose to make a one-time tender offer.

Assuming there are some fixed costs associated with either attempting or consummating a takeover, a dynamic theory predicts there are fewer targets that are ex ante profitable in comparison to that predicted by a static theory.

Property 3. *As the time between tender offers shrinks, there are fewer attempted takeovers.*

Some additional findings are presented in Tables 4 and 5 that, besides being of intrinsic interest, provide further sensitivity analysis. Table 4 describes the influence of the takeover rule on the raider's payoff. As expected, a more stringent takeover rule increases the expected profit from attempting a takeover. Since there are more shares required, there is less of an incentive to free-ride, since each shareholder is more likely to be crucial to the success of the takeover. Table 5 shows that as ownership concentration falls, the percentage of value taken by the raider falls. With more shareholders, the free-

Table 5
Effect of the number of shareholders $(K, \sigma) = (N/2, .9)$

N	V^*/N	r^*	TS/N
10	.0160	.929	.9450
18	.0100	.944	.9540
20	.0093	.946	.9553
26	.0079	.948	.9559
30	.0073	.949	.9563
34	.0068	.949	.9558
40	.0063	.950	.9563

$N \equiv$ number of shares/shareholders.
$K \equiv$ minimum number of shares required to take control of the firm.
$\sigma \equiv$ discount factor of the raider and shareholders.
$V^* \equiv$ raider's equilibrium payoff as a percentage of the value created by the takeover.
$r^* \equiv$ shareholder's equilibrium payoff.
$TS/N \equiv$ equilibrium surplus per share.

riding problem is more severe, since an individual shareholder's decision is less likely to determine the success of a takeover.

Special case of (N, K) = (2, 1). Though analytical methods are intractable for the general case, one can derive closed-form solutions for the special case of two shareholders and a takeover rule of 50 percent. Setting $(N, K) = (2, 1)$ in Equation (4), we derive the equilibrium tender offer, r^*, by maximizing (4) and then solving for the fixed point in (5). We find r^* and the equilibrium probability of tendering, p^*, to be

$$r^* = \frac{1}{1 + \sqrt{1-\delta}}, \qquad p^* = \frac{\sqrt{1-\delta}}{1+\sqrt{1-\delta}}. \tag{13}$$

Consistent with our numerical analysis, the equilibrium tender offer is increasing as the time between tender offers shrinks (δ rises). Since $p^* = 1 - r^*$, then p^* is decreasing in δ. Note that as the time between tender offers shrinks, the probability of tendering falls even though the tender offer is higher.

Substituting r^* into (4), one derives the raider's equilibrium payoff:

$$V^* = \frac{\sqrt{1-\delta}}{1+\sqrt{1-\delta}}. \tag{14}$$

The raider's payoff is decreasing in the discount factor:

$$\frac{\partial V^*}{\partial \delta} = \frac{-1}{2\sqrt{1-\delta}[1+\sqrt{1-\delta}]^2} < 0. \tag{15}$$

Finally, total surplus is an increasing function of δ:

$$TS = V^* + 2r^* = \frac{2 + \sqrt{1 - \delta}}{1 + \sqrt{1 - \delta}}; \tag{16}$$

$$\frac{\partial TS}{\partial \delta} = \frac{1}{2\sqrt{1 - \delta}[1 + \sqrt{1 - \delta}]^2} > 0. \tag{17}$$

Consistent with the results from our numerical analysis (summarized as Property 2), we find that as the time between tender offers shrinks the tender offer (and thus a shareholder's payoff) and total surplus rise and the raider's payoff falls.

5. Relationship with the Coase Conjecture

Our main findings are that the equilibrium tender offer rises over time and that, as the time between tender offers goes to zero, the equilibrium tender offer converges to the posttakeover share value. In the limit, shareholders get all of the surplus and the takeover occurs in finite time with probability 1. Those familiar with the durable goods pricing literature will note a striking similarity between these results and the Coase Conjecture. It has been shown by Stokey (1981) and Gul, Sonnenschein, and Wilson (1986) that a durable goods monopolist's price falls over time and, when the time between price changes is small, price is close to marginal cost. This result also has implications for bargaining under asymmetric information, where it has been shown that as the time between offers goes to zero, the delay in consummating an agreement goes to zero; see Gul and Sonnenschein (1988) and Ausubel and Deneckere (1992).

In spite of the similarity of these results, there are distinct forces at play in the takeover game and the durable goods pricing game. This is evidenced by the fact that the results for the takeover game are derived with homogeneous shareholders, whereas consumer heterogeneity is a necessary condition for the analogous results to hold in the durable goods setting. When consumers have different valuations of the good, the monopolist lowers its price over time so as to price discriminate. It initially sells to those consumers who value the good the most and then lowers price so as to sell to consumers with lower valuations. Since consumers anticipate the ensuing price decline, as the time between price changes shrinks, consumers are more willing to postpone their purchases. This forces the monopolist to shift its price path down. When δ is close to 1 so that consumers are very patient, this logic forces price to be close to marginal cost. If, instead, consumers are homogeneous, then the monopolist's optimal strategy is to price at the (common) valuation (assuming it exceeds unit cost). In that case, price does not fall over time and the monopolist receives a positive payoff that is bounded away from 0 as $\delta \to 1$.

In comparison, shareholder heterogeneity is not required in order to have a rising tender offer and for shareholders to get all of the surplus as the time between tender offers goes to zero. In its place is the presence of an externality between shareholders. Since per share income is 1 when at least K shareholders tender (so that a takeover occurs) and 0 otherwise, a shareholder's payoff depends not only on his decision to tender but also on how many other shareholders decide to sell their shares. (In comparison, a consumer's payoff in the durable goods setting depends only on his decision to buy.) This externality makes a shareholder's valuation of a share dependent on the raider's current holdings. In equilibrium, as the raider's holdings approach K, the market value of a share rises because the expected time until completion of the takeover is shorter. This causes the reservation price of a (remaining) shareholder to rise over time and accounts for the rising tender offer in equilibrium. The externality then endogenously generates a rising supply schedule for shares (in an intertermporal sense), which is analogous to the downward-sloping demand schedule in the durable goods setting. As the time between tender offers shrinks, shareholders increasingly prefer to wait for the takeover to occur rather than to tender their shares. This forces the raider to set a higher tender offer. As $\delta \to 1$, this tender offer converges to the full posttakeover value of a share.

6. Concluding Remarks

In actuality, when a tender offer fails to induce enough shareholders to sell, a raider can return to the market with a new bid in order to try and secure additional shares. This article explored the implications of this possibility for the dynamical properties of a takeover bid and the intensity of the free-riding problem.

As found by preceding analyses in the static setting, a raider can expect to earn positive expected profit when the number of shareholders is finite. However, the return to engaging in a takeover is very small when the time between tender offers is very short. For plausible assumptions on the interest rate and the time between tender offers, numerical analysis suggests that the payoff to a raider is quite low. With at least 10 (symmetric) shareholders, a 50 percent takeover rule, an annual real interest rate of 4 percent or less, and a time between tender offers of no more than three months, the expected profit to a raider is less than two-tenths of 1 percent of the additional value created by a takeover. It would appear that the free-riding problem can be quite severe even when ownership is concentrated.

This finding offers new insight into the empirical regularity that the acquiring firm's shareholders earn minimal returns from a takeover, whereas the target firm's shareholders receive a substantial premium for their shares.[4] One explanation for this phenomenon is that competition among raiders dissipates the rents from a takeover. Our dynamic theory of takeovers suggests an alternative explanation, which is that rent dissipation occurs because a raider is effectively competing with himself. A shareholder knows that even if the takeover does not succeed today, the raider will make as good or a better offer tomorrow. This makes a shareholder more inclined not to tender today which forces the raider to offer a greater premium. When a raider can make multiple tender offers, he is competing with his future self and this results in the transfer of surplus to the shareholders of the target firm.

Clearly, further analysis of the dynamic setting is required. There are at least two substantive limitations to our analysis. First, we assume shareholders are homogeneous and either each shareholder has one share or, if he has multiple shares, he must sell his shares as a block. In light of the findings of Holmström and Nalebuff (1992), generalizing our model to allow shareholders to hold multiple shares and to sell any number of their shares would be an important extension. In a static model, they show that the free-riding problem is less severe when shareholders can sell a portion of their holdings. Unfortunately, it appears quite difficult to characterize even symmetric equilibria for the infinite horizon game when shareholders are allowed to sell a portion of their holdings. In that setting, a state includes not only the number of outstanding shares but also how those shares are distributed among the remaining shareholders.

The other limitation is that we have only considered symmetric Markov perfect equilibria. There are also asymmetric Markov perfect equilibria with quite different predictions.[5] In light of shareholder homogeneity, these equilibria seem less compelling. Perhaps more interesting is what non-Markov perfect equilibria might predict. As shown by Ausubel and Deneckere (1989) in the context of durable goods pricing, non-Markov perfect equilibria can have very different behavior. Characterizing the general class of perfect equilibria in the dynamic setting would be an interesting extension of our analysis.

Finally, several related issues are worthy of investigation. Since the

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[4] For reviews of the empirical evidence, see Jensen and Ruback (1983) and Jarrell, Brickley, and Netter (1988).

[5] There exist asymmetric Markov perfect equilibria in which the raider's payoff is bounded away from 0 even as $\delta \rightarrow 1$. Such equilibria entail the raider always offering a price $\bar{P}$, where $\bar{P} \in (0, 1)$, until at least K shares are tendered. Equilibrium has exactly K shareholders tendering in the first period. There are many such equilibria, and they differ according to $\bar{P}$ and to the identity of the shareholders that tender. In light of the multiplicity of asymmetric equilibria and that shareholders are identical, there would appear to be a serious coordination problem.

profitability of a takeover is reduced when one can make multiple tender offers, this raises the issue of how a raider can commit to making a limited number of bids. In fact, the raider's payoff is maximized when he makes a one-time tender offer. One device is for a raider to build a reputation for making take-it-or-leave-it offers to target firms. What other devices might allow a raider to credibly commit to making a limited number of bids is a challenging issue for future research.

Another issue is how results might change if a raider could make a conditional offer. With a conditional offer, a raider would have to decide each period whether to keep the previous offer open or make a new offer. One potential advantage for a raider from making conditional bids is that they may avoid the opportunistic holdup by shareholders when all a raider requires is a few more shares to complete the takeover, as occurs with unconditional offers. A virtue to solving our model for the case of conditional offers is that it would then allow one to explore the relative merits of conditional and unconditional offers in the dynamic setting.

Appendix A

Our procedure for calculating numerical examples followed the construction in Section 3 quite closely, with one exception. Instead of substituting $\hat{p}_m(r; \hat{r}_m)$ into the raider's objective function and maximizing with respect to r, we substituted the expression for r given in (7) into the raider's objective function and maximized with respect to p_m. This is legitimate since $\hat{p}_m(\cdot)$ is a monotonic function of r.

The steps pursued in our numerical analysis were as follows. With (7) substituting for r, denote the expression in (9) as $W(p_m, \hat{r}_m)$ and calculate $\partial W(p_m, \hat{r}_m)/\partial p_m$. Set $r = \hat{r}_m$ and solve (7) for r. Denoting the solution $\hat{r}_m(p_m)$, substitute $\hat{r}_m(p_m)$ for $\hat{r}_m$ in $\partial W(p_m, \hat{r}_m)/\partial p_m$ and solve $\partial W(p^*_m, \hat{r}_m(p^*_m))/\partial p_m = 0$ for p^*_m. Check the second-order condition, $\partial^2 W(p^*_m, \hat{r}_m(p^*_m))/\partial p^2_m < 0$. Derive $r^*_m = \hat{r}_m(p^*_m)$ and $V^*_m = W(p^*_m, r^*_m)$. Perform this procedure sequentially for $m \in \{0, 1, \ldots, K - 1\}$, starting with $m = K - 1$.

This procedure was repeated for different values for (N, K, δ) using Mathematics. It was run for a variety of combinations of $N \in \{10, 18, 20, 26, 30, 34, 40\}$, $K \in \{N/2, \ldots, N - 1\}$, and $\delta \in (0, .1, .2, .3, .4, .5, .6, .7, .8, .9, .95, .99\}$.

Appendix B

Let us continue where we left off in sketching the proof of Theorem 1. To begin, in the ensuing analysis we use the following result.

Lemma 1. *Let* $\{q_i\}$ *be a sequence where* q_i *Î (0, I]. If* $\{q_i\}$ *does not converge to 1, then there exists a subsequence of* $\{q_i\}$ *that is bounded below 1.*

Proof By the definition of a limit, $\{q_i\}$ converges to one if any neighborhood of 1 contains all but a finite number of elements of (q_i). Hence, if $\{q_i\}$ does not converge to 1, there exists $\epsilon > 0$ such that $\{q \in \{q_i\} \mid q < 1 - \epsilon\}$ contains an infinite number of elements. This set of elements forms a subsequence of $\{q_i\}$ that is bounded below 1 by ϵ. ∎

Contrary to Theorem 1, suppose $\{r^*_{K-1}\}$ does not converge to 1. By Lemma 1, there exists a subsequence of $\{r^*_{K-1}\}$, which we denote $\{\bar{r}^*_{K-1}\}$, that is bounded below 1. Consider the equilibria associated with this subsequence as a sequence of equilibria. Let $\{\bar{p}^*_{K-1}\}$ be the associated sequence of probabilities. By (6), $\bar{p}^*_{K-1}$ is defined by

$$\bar{p}^*_{K-1} = 1 - \left[\frac{1 - \bar{r}^*_{K-1}}{1 - e^{-\rho\Delta t}\bar{r}^*_{K-1}}\right]^{1/(N-K)} \tag{B1}$$

It follows from (B1) and $\{\bar{r}^*_{K-1}\}$ being bounded below 1 that $\{\bar{p}^*_{K-1}\} \rightarrow 0$. Lemma 2 allows us to generate a contradiction.

Lemma 2. *If* $\{p^*_{K-1}\} \rightarrow 0$*, then* $\{r^*_{K-1}\} \rightarrow 1$*.*

Proof. Contrary to Lemma 2, let us suppose that $\{p^*_{K-1}\} \rightarrow 0$ and $\{r^*_{K-1}\}$ does not converge to 1. It follows from Lemma 1 that there exists a subsequence of $\{r^*_{K-1}\}$, (denoted $\{\bar{r}^*_{K-1}\}$,) that is bounded below 1. Let us consider the equilibria associated with $\{\bar{r}^*_{K-1}\}$ as a sequence of equilibria. Note that the sequence of associated probabilities of tendering, $\{\bar{p}^*_{K-1}\}$, converges to 0. This follows from the assumption that $\{p^*_{K-1}\} \rightarrow 0$ and that every subsequence of a convergent sequence is convergent and has the same limit.

Letting $\Pr_{K-1}(j \mid r)$ denote the probability that exactly j shareholders tender (given a tender offer of r and the raider has K - 1 shares), we define the raider's value function as

$$V_{K-1}(\hat{r}_{K-1}) = \max_r \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid r)(K - 1 + j - jr) + \left[1 - \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid r)\right] e^{-\rho\Delta t} V_{K-1}(\hat{r}_{K-1}), \tag{B2}$$

where

$$\Pr_{K-1}(j \mid r) = \binom{N-K+1}{j} (\bar{p}^*_{K-1})^j (1 - \bar{p}^*_{K-1})^{N-K+1-j}. \quad \text{(B3)}$$

By the definition of $\bar{r}^*_{K-1}$, it satisfies the first-order condition

$$\sum_{j=1}^{N-K+1} \left[\frac{\partial \Pr_{K-1}(j \mid \bar{r}^*_{K-1})}{\partial r}\right][K - 1 + j - j\bar{r}^*_{K-1} - e^{-\rho\Delta t}\bar{V}^*_{K-1}],$$

$$- \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid \bar{r}^*_{K-1}) j = 0, \quad \text{(B4)}$$

where $\bar{V}^*_{K-1} \equiv V_{K-1}(\bar{r}^*_{K-1})$ and we have set $\hat{r}_{K-1} = \bar{r}^*_{K-1}$.

We want to evaluate $\bar{r}^*_{K-1}$ as $\Delta t \to 0$. To do so requires evaluating (B4) as $\Delta t \to 0$. This shall be done by analyzing each part of (B4) separately. From (B3), we have

$$\frac{\partial \Pr_{K-1}(j \mid r)}{\partial r} = \binom{N-K+1}{j} [j(\bar{p}^*_{K-1})^{j-1}(1 - \bar{p}^*_{K-1})^{N-K+1-j} - (N - K + 1 - j)(\bar{p}^*_{K-1})^j \times (1 - \bar{p}^*_{K-1})^{N-K-j}]\left(\frac{\partial p_{K-1}}{\partial r}\right), \quad \text{(B5)}$$

where $\partial p_{K-1}/\partial r$ is evaluated at $r_{K-1} = \bar{r}^*_{K-1}$ and $\hat{r}_{K-1} = \bar{r}^*_{K-1}$. From (6), one derives that

$$\frac{\partial p_{K-1}}{\partial r} = \left(\frac{1}{N-K}\right)(1 - \bar{r}^*_{K-1})^{-(N-K-1)/(N-K)}(1 - e^{-\rho\Delta t}\bar{r}^*_{K-1})^{-1/(N-K)}. \quad \text{(B6)}$$

Since $\bar{r}^*_{K-1}$ is bounded below 1, it follows from (B6) that $\partial p_{K-1}/\partial r$ is bounded above 0. Using this fact in (B5) along with $\{\bar{p}^*_{K-1}\} \to 0$, we see that $\lim_{\Delta t \to 0} \partial \Pr_{K-1}(j \mid \bar{r}^*_{K-1})/\partial r = 0 \ \forall\, j \geq 2$ and $\partial \Pr_{K-1}(1 \mid \bar{r}^*_{K-1})/\partial r$ is bounded above 0.

Let us now use this information to evaluate (B4) as $\Delta t \to 0$. First, since $\{\bar{p}^*_{K-1}\} \to 0$ implies that $\lim_{\Delta t \to 0} \Pr_{K-1}(j \mid \bar{r}^*_{K-1})\} = 0 \ \forall\, j \geq 1$, the last term in (B4) goes to zero as $\Delta t \to 0$. Second, it follows from $[K - 1 + j - j\bar{r}^*_{K-1} - e^{-\rho\Delta t}\bar{V}^*_{K-1}]$ being finite and $\lim_{\Delta t \to 0} \partial \Pr_{K-1}(j \mid \bar{r}^*_{K-1})/\partial r = 0 \ \forall\, j \geq 2$ that

$$\lim_{\Delta t \to 0} \left[\frac{\partial \Pr_{K-1}(j \mid \bar{r}^*_{K-1})}{\partial r}\right](K - 1 + j - j\bar{r}^*_{K-1} - e^{-\rho\Delta t}\bar{V}^*_{K-1}) = 0 \quad \forall\, j \geq 2. \quad \text{(B7)}$$

By the preceding two steps, (B4) holds as $\Delta t \to 0$ if:

$$\lim_{\Delta t \to 0} \left[\frac{\partial \Pr_{K-1}(1 \mid \bar{r}^*_{K-1})}{\partial r} \right] (K - \bar{r}^*_{K-1} - e^{-\rho \Delta t} \bar{V}^*_{K-1}) = 0 \qquad \text{(B8)}$$

Since we have proven that $\partial \Pr_{K-1}(1 \mid \bar{r}^*_{K-1})/\partial r$ is bounded above 0, (B8) holds iff $\lim_{\Delta t \to 0} (K - \bar{r}^*_{K-1} - e^{-\rho \Delta t} \bar{V}^*_{K-1}) = 0$. Since $\lim_{\Delta t \to 0} e^{-\rho \Delta t} = 1$, this can be stated as

$$\lim_{\Delta t \to 0} (K - \bar{r}^*_{K-1} - \bar{V}^*_{K-1}) = 0. \qquad \text{(B9)}$$

To summarize, we have shown that if $\{\bar{p}^*_{K-1}\} \to 0$ and $\{\bar{r}^*_{K-1}\}$ is bounded below 1, then $\lim_{\Delta t \to 0} (K - \bar{r}^*_{K-1} - \bar{V}^*_{K-1}) = 0$. Next define $q(T)$ as the equilibrium probability that at least one share is sold over $(0, T]$ where $T > 0$. If T is a multiple of At, then $q(T) = 1 - (1 - \bar{p}^*_{K-1})^{(N-K)(T/\Delta t)}$. Note that $q(T)$ depends directly on At and also indirectly through $\bar{p}^*_{K-1}$. Consider the following upper bound on $\bar{V}^*_{K-1}$:

$$\bar{V}^*_{K-1} \le \{q(T) + [1 - q(T)]e^{-\rho T}\}$$
$$\times \left\{ K - 1 + \sum_{j=1}^{N-K+1} \left[\Pr_{K-1}(j \mid \bar{r}^*_{K-1}) \Big/ \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid \bar{r}^*_{K-1}) \right] \right.$$
$$\left. \times j(1 - \bar{r}^*_{K-1}) \right\}. \qquad \text{(B10)}$$

The second { } on the r.h.s. of (B10) is the raider's undiscounted payoff conditional on selling at least one share. The r.h.s. expression is then the raider's payoff when, with probability $q(T)$, he immediately sells at least one share and, with probability 1 - $q(T)$, he does not sell any shares over $[0, T)$ and sells at least one share at T. To evaluate the r.h.s. expression in (B10) as $\Delta t \to 0$, first note that $\{\bar{p}^*_{K-1}\} \to 0$ implies

$$\lim_{\Delta t \to 0} [\Pr_{K-1}(j \mid \bar{r}^*_{K-1}) \Big/ \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid \bar{r}^*_{K-1})] = 0 \qquad \forall\ j \ge 2$$

(that is, as the probability of an individual shareholder tendering his share goes to 0, the probability that exactly one share is tendered in a period becomes arbitrarily large relative to the probability that two or more shares are tendered). It follows that

$$\lim_{\Delta t \to 0} \left\{ [q(T) + (1 - q(T))e^{-\rho T}] \right.$$

$$\times \left[K - 1 + \sum_{j=1}^{N-K+1} \left[\Pr_{K-1}(j \mid \bar{r}^*_{K-1}) \Big/ \sum_{j=1}^{N-K+1} \Pr_{K-1}(j \mid \bar{r}^*_{K-1}) \right] \right.$$
$$\left. \times\, j(1 - \bar{r}^*_{K-1}) \right]$$
$$- [q(T) + (1 - q(T))e^{-\rho T}](K - \bar{r}^*_{K-1}) \Bigg\} = 0. \qquad \text{(B11)}$$

It follows from (B10) and (B11) that $\forall\, \epsilon > 0\ \exists\, \tau > 0$ such that

$$V^*_{K-1} - \{q(T) + [1 - q(T)]e^{-\rho T}\}(K - \bar{r}^*_{K-1}) < \epsilon\, \forall\, \Delta t \in (0, \tau). \qquad \text{(B12)}$$

By (B11) and (B12), we conclude that $\lim_{\Delta t \to 0} q(T) = 1$. Since T is arbitrary, this is true for all $T > 0$.

The preceding step showed that if $\lim_{\Delta t \to 0} (K - \bar{r}^*_{K-1} - \bar{V}^*_{K-1}) = 0$ then the probability that the takeover occurs over $(0, T]$ goes to 1 as $\Delta t \to 0$, for all $T > 0$. Consider the situation faced by a shareholder as $\Delta t \to 0$. He can tender his share today and receive a payoff of $\bar{r}^*_{K-1}$, or he can hold on to his share and wait for a takeover. If the takeover occurs at time T, his realized payoff is $e^{-\rho T}$. Since $\lim_{\Delta t \to 0} q(T) = 1$ for all $T > 0$, the probability that a takeover occurs in finite time goes to 1 as $\Delta t \to 0$. Therefore, as $\Delta t \to 0$, a shareholder's payoff from holding on to his share is at least $e^{-\rho T}$ for all $T > 0$. In order to induce a shareholder to tender his share as $\Delta t \to 0$, it must then be true that $\lim_{\Delta t \to 0} (\bar{r}^*_{K-1} - e^{-\rho T}) \geq 0$ for all $T > 0$. However, this is only true if $\{\bar{r}^*_{K-1}\} \to 1$, which contradicts our earlier result that $\{\bar{r}^*_{K-1}\}$ is bounded below 1. Going back to our original supposition, we conclude that if $\{p^*_{K-1}\} \to 0$ then $\{r^*_{K-1}\} \to 1$. n

To summarize, we supposed there existed a sequence of equilibria such that $\{r^*_{K-1}\}$ does not converge to 1. Using Lemma 1, it followed that there exists a sequence of equilibria such that $\{r^*_{K-1}\}$ is bounded below 1. It was then proven that if $\{r^*_{K-1}\}$ is bounded below 1, then $\{p^*_{K-1}\} \to 0$. Finally, in Lemma 2, it was shown that if $\{p^*_{K-1}\} \to 0$ then $\{r^*_{K-1}\} \to 1$, which contradicts the earlier result that $\{r^*_{K-1}\}$ is bounded below 1. Therefore, our original supposition that $\{r^*_{K-1}\}$ does not converge to 1 must be false. We conclude that, for all sequences of equilibria, $\{r^*_{K-1}\} \to 1$.

The proof of Theorem 1 is completed by induction. Using the same method as above, we will show that if, for all sequences of equilibria, $\{r^*_{m+j}\} \to 1\ \forall\, j \in \{1, \ldots, K - 1 - m\}$, then, for all sequences of equilibria, $\{r^*_m\} \to 1$. Let us suppose it is false: for all sequences of equilibria $\{r^*_{m+j}\} \to 1\ \forall\, j \in \{1, \ldots, K - 1 - m\}$ and there exists a sequence of equilibria such that $\{r^*_m\}$ does not converge to 1. By

Lemma 1, if $\{r_m^*\}$ does not converge to 1, then there exists a subsequence $\{\bar{r}_m^*\}$ bounded below 1. Let us consider as a sequence of equilibria, the equilibria associated with $\{\bar{r}_m^*\}$, and let $\{\bar{p}_m^*\}$ be the associated sequence of probabilities. Setting $\hat{r}_m = \bar{r}_m^*$ in (7), $\bar{p}_m^*$ is defined by

$$\begin{aligned}\bar{r}_m^* = &\sum_{j=K-m}^{N-m-1} \binom{N-m-1}{j} (\bar{p}_m^*)^j (1-\bar{p}_m^*)^{N-m-1-j} \\ &+ \sum_{j=1}^{K-m-1} \binom{N-m-1}{j} (\bar{p}_m^*)^j (1-\bar{p}_m^*)^{N-m-1-j} e^{-\rho\Delta t} \bar{r}_{m+j}^* \\ &+ \left[1 - \sum_{j=1}^{N-m-1} \binom{N-m-1}{j} (\bar{p}_m^*)^j (1-\bar{p}_m^*)^{N-m-1-j}\right] e^{-\rho\Delta t} \bar{r}_m^*. \end{aligned} \tag{B13}$$

Since $\{\bar{r}_m^*\}$ is bounded below 1 and, by the inductive assumption, $\{r_{m+j}^*\} \to 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$ it follows from (B13) that $\{\bar{p}_m^*\} \to 0$. Lemma 3 establishes the necessary contradiction.

Lemma 3. *If $\{p_m^*\} \to 0$, then if $\{r_{m+j}^*\} \to 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$ then $\{r_m^*\} \to 1$.*

Proof. Suppose, contrary to Lemma 3, that $\{p_m^*\} \to 0$, $\{r_{m+j}^*\} \to 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$, and $\{r_m^*\}$ does not converge to 1. By Lemma 1, there exists a subsequence $\{\bar{r}_m^*\}$ bounded below 1. Consider as a sequence of equilibria, the equilibria associated with $\{\bar{r}_m^*\}$. Note that $\{\bar{p}_m^*\} \to 0$ and $\{\bar{r}_{m+j}^*\} \to 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$. These two properties follow from the assumptions that $\{p_m^*\} \to 0$ and $\{r_{m+j}^*\} \to 1$ and that $\{\bar{p}_m^*\}$ is a subsequence of $\{p_m^*\}$ and $\{\bar{r}_{m+j}^*\}$ is a subsequence of $\{r_{m+j}^*\}$, $j \in \{1, \ldots, K-1-m\}$. To prove Lemma 3, we will show that $\{\bar{r}_m^*\} \to 1$, which establishes the necessary contradiction to prove the falsity of the claim that $\{r_{m+j}^*\} \to 1 \; \forall \; j \in \{1, \ldots, K-1-m\}$ and $\{r_m^*\}$ does not converge to 1.

Given the raider has m shares, his value function can be defined as

$$\begin{aligned} V_m(\hat{r}_m) = \max_r &\sum_{j=K-m}^{N-m} \Pr_m(j \mid r)(m+j-jr) \\ &+ \sum_{j=1}^{K-m-1} \Pr_m(j \mid r)(e^{-\rho\Delta t}\bar{V}_{m+j}^* - jr) \\ &+ \left[1 - \sum_{j=1}^{N-m} \Pr_m(j \mid r)\right] e^{-\rho\Delta t} V_m, \end{aligned} \tag{B14}$$

where $P_m(j \mid r)$ is the probability that j shareholders tender. The optimal value for r_m, $\bar{r}_m^*$, satisfies the first-order condition

$$\begin{aligned}\sum_{j=K-m}^{N-m} &\left[\frac{\partial \Pr_m(j \mid \bar{r}_m^*)}{\partial r}\right][m + j - j\bar{r}_m^* - e^{-\rho\Delta t}\bar{V}_m^*] \\ &+ \sum_{j=1}^{K-m-1}\left[\frac{\partial \Pr_m(j \mid \bar{r}_m^*)}{\partial r}\right][e^{-\rho\Delta t}\bar{V}_{m+j}^* - j\bar{r}_m^* - e^{-\rho\Delta t}\bar{V}_m^*] \\ &- \sum_{j=1}^{N-m} \Pr_m(j \mid \bar{r}_m^*)j \\ &= 0. \end{aligned} \tag{B15}$$

It follows from $\{\bar{p}_m^*\} \rightarrow 0$ that $\lim_{\Delta t \rightarrow 0} \Pr_m(j \mid \bar{r}_m^*) = 0 \ \forall \ j \in \{1, \ldots, N - m\}$. Therefore, the last term in (B15) goes to 0 as $\Delta t \rightarrow 0$. Next note that

$$\begin{aligned}\frac{\partial \Pr_m(j \mid \bar{r}_m^*)}{\partial r} &= \binom{N-m}{j}[j(\bar{p}_m^*)^{j-1}(1 - \bar{p}_m^*)^{N-m-j} \\ &\quad - (N - m - j)(\bar{p}_m^*)^j(1 - \bar{p}_m^*)^{N-m-j-1}]\frac{\partial p_m}{\partial r},\end{aligned} \tag{B16}$$

where $\partial p_m/\partial r$ is evaluated at $r_m = \bar{r}_m^*$ and $\hat{r}_m = \bar{r}_m^*$. Since, by assumption, $\{\bar{p}_m^*\} \rightarrow 0$, if $\partial p_m/\partial r$ is bounded then $\lim_{\Delta t \rightarrow 0} \partial \Pr_m(j \mid \bar{r}_m^*)/\partial r = 0 \ \forall \ j \in \{2, \ldots, N - m\}$. From (B16),

$$\begin{aligned}\frac{\partial \Pr_m(1 \mid \bar{r}_m^*)}{\partial r} &= (N - m)[(1 - \bar{p}_m^*)^{N-m-1} \\ &\quad - (N - m - 1)(\bar{p}_m^*)(1 - \bar{p}_m^*)^{N-m-2}]\frac{\partial p_m}{\partial r},\end{aligned} \tag{B17}$$

so that $\lim_{\Delta t \rightarrow 0} \partial \Pr_m(1 \mid \bar{r}_m^*)/\partial r = \lim_{\Delta t \rightarrow 0} (N - m)\partial p_m/\partial r$. Hence, if $\partial p_m/\partial r$ is bounded above 0, then $\partial \Pr_m(1 \mid \bar{r}_m^*)/\partial r$ is bounded above 0.

We now want to show that $\partial p_m/\partial r$ is finite and bounded above 0. For this purpose, take the total derivative of (7) with respect to r and set $r_m = \bar{r}_m^*$ and $\hat{r}_m = \bar{r}_m^*$:

$$\begin{aligned}1 = \sum_{j=K-m}^{N-m-1}\binom{N-m-1}{j}&[j(\bar{p}_m^*)^{j-1}(1 - \bar{p}_m^*)^{N-m-1-j} - (N - m - 1 - j) \\ &\times (\bar{p}_m^*)^j(1 - \bar{p}_m^*)^{N-m-2-j}]\frac{\partial p_m}{\partial r}\end{aligned}$$

$$+\sum_{j=1}^{K-m-1}\binom{N-m-1}{j}[j(\bar{p}_m^*)^j(1-\bar{p}_m^*)^{N-m-1-j}-(N-m-1-j)$$
$$\times(\bar{p}_m^*)^j(1-\bar{p}_m^*)^{N-m-2-j}]e^{-\rho\Delta t}\bar{r}_{m+j}^*\frac{\partial p_m}{\partial r}$$
$$-(N-m-1)(1-\bar{p}_m^*)^{N-m-2}e^{-\rho\Delta t}\bar{r}_m^*\left(\frac{\partial p_m}{\partial r}\right). \tag{B18}$$

In (B18), there are three expressions multiplied by $\partial p_m/\partial r$. The first expression goes to zero as $\Delta t \rightarrow 0$. Using the inductive assumption that $\{\bar{r}_{m+1}^*\} \rightarrow 1$, the second expression goes to $(N-m-1)$ as $\Delta t \rightarrow 0$. The third expression goes to $-(N-m-1)\bar{r}_m^*$ as $\Delta t \rightarrow 0$. Combining these three expressions, we find that they go to $(N-m-1)(1-\bar{r}_m^*)$ as $\Delta t \rightarrow 0$. It then follows from (B18) that

$$\lim_{\Delta t \rightarrow 0}(N-m-1)(1-\bar{r}_m^*)\frac{\partial p_m}{\partial r}=1. \tag{B19}$$

Since $\bar{r}_m^*$ is bounded below 1, then (B19) implies that $\partial p_m/\partial r$ is finite and bounded above 0 as $\Delta t \rightarrow 0$.

Using the above facts, we conclude that for (B15) to hold as $\Delta t \rightarrow 0$, it must be true that

$$\lim_{\Delta t \rightarrow 0}(e^{-\rho\Delta t}\bar{V}_{m+1}^* - \bar{r}_m^* - e^{-\rho\Delta t}\bar{V}_m^*)=0. \tag{B20}$$

Since $\lim_{\Delta t \rightarrow 0} e^{-\rho\Delta t}=1$ then (B21) follows:

$$\lim_{\Delta t \rightarrow 0}(\bar{V}_{m+1}^* - \bar{r}_m^* - \bar{V}_m^*)=0. \tag{B21}$$

Using the same type of argument as in Lemma 2, one can show that it follows from (B21) that the probability of a takeover occurring in finite time goes to 1 as $\Delta t \rightarrow 0$. Intuitively, if $\bar{V}_m^*$ converges to $\bar{V}_{m+1}^* - \bar{r}_m^*$), then, as $\Delta t \rightarrow 0$, the raider must be buying the $(m+1)$st share infinitely fast since his payoff when he has m shares converges to his payoff when he has $m+1$ shares less the cost of the $(m+1)$st share (that is, there is no delay in buying this share). Since this share is being purchased in finite time with probability 1, then $\bar{r}_m^*$ must be converging to $\bar{r}_{m+1}^*$ as $\Delta t \rightarrow 0$, since a shareholder can always receive $\bar{r}_{m+1}^*$ by holding his share. However, by assumption, $\{\bar{r}_{m+1}^*\} \rightarrow 1$ from which it follows that $\{\bar{r}_m^*\} \rightarrow 1$. Since this contradicts the earlier result that $\{\bar{r}_m^*\}$ is bounded below 1, we conclude that our original supposition that $\{p_m^*\} \rightarrow 0$, $\{r_{m+j}^*\} \rightarrow 1\ \forall\ j \in \{1, \ldots, K-1-m\}$, and $\{r_m^*\}$ does not converge to 1 is false. We have thus shown that if $\{p_m^*\} \rightarrow 0$, then, if $\{r_{m+j}^*\} \rightarrow 1\ \forall\ j \in \{1, \ldots, K-1-m\}$, $\{r_m^*\} \rightarrow 1$. ■

Recall that we began with the supposition that, for all sequences of equilibria, $\{r^*_{m+j}\} \to 1 \ \forall \ j \in \{1, \ldots, K-1-m\}$ and there exists a sequence of equilibria such that $\{r^*_m\}$ does not converge to 1. We then showed that this implied that there exists a sequence of equilibria for which $\{p^*_m\} \to 0$, $\{r^*_{m+j}\} \to 1 \ \forall \ j \in \{1, \ldots, K-1-m\}$, and $\{r^*_m\}$ is bounded below 1. By Lemma 3, we have a contradiction that allows us to conclude that our original supposition is false.

To summarize, we have proven (1) for all sequences of equilibria, $\{r^*_{K-1}\} \to 1$; and (2) if, for all sequences of equilibria, $\{r^*_{m+j}\} \to 1 \ \forall \ j \in \{1, \ldots, K-1-m\}$, then, for all sequences of equilibria, $\{r^*_m\} \to 1$. By induction, for all sequences of equilibria, $\{r^*_m\} \to 1 \ \forall \ m \in \{0, \ldots, K-1\}$.

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