Asymmetric Information and Options

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In an extension of the Kyle (1985) model of continuous insider trading, it is shown that asymmetric information can make it impossible to price options by arbitrage. Even when an option would appear to be redundant, its introduction into the market can cause the volatility of the underlying asset to become stochastic. This eliminates the potential for dynamically replicating the option. The change in the price process of the asset reflects a change in the information transmitted by volume andprices when the option is traded.

The usual approach to option valuation is to posit a price process for the underlying asset and then value the option by arbitrage arguments. The option can be priced by arbitrage if it is redundant-that is, if it can be created synthetically by trading the underlying asset and other assets. If the option is not redundant, then its existence may have an effect on the underlying asset price, because prices will adjust to a new equilibrium when a nonredundant asset is created. This aspect of incomplete markets is well understood [see, e.g., Detemple and Selden (1991)]. In this article I describe a similar but more subtle phenomenon, linked to asymmetric information. I describe a market in which an option appears to be redundant-it can

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be synthesized by dynamic trading. Nevertheless, the existence of the option affects the underlying asset price, because it affects the flow of information. When the option is traded, the volatility of the underlying asset becomes stochastic as a result of the change in the information flow. Thus, the option is not redundant when it is actually traded.

The fact that real and synthetic options can differ in the amount of information they transmit has previously been noted by Grossman (1988,1989). Grossman (1988) shows that a traded index option can appear to be redundant, but its removal from the market would have real consequences because its price reveals information about trading plans. I show that a nontraded option can appear to be redundant, but adding it to the market can have real consequences because its price reveals information about the fundamental value of the asset.

The framework that will be used to demonstrate the informational role of options is Kyle's (1985) model of continuous insider trading. Kyle assumes that the asset value is normally distributed and shows that in the unique linear equilibrium the price process is a Brownian motion. Back (1992a) extends Kyle's uniqueness result beyond the linear class and shows that, when the asset value is lognormally distributed, the equilibrium price process is a geometric Brownian motion. In either the normal or lognormal case, one would expect an option on the asset to be priced by arbitrage.[1] As is well known, a market for an option cannot exist in this environment unless there are agents who desire to trade it for noninformational reasons, commonly called noise or liquidity traders. We assume that the value of the underlying asset (which for brevity we call a stock) is normally distributed and that the liquidity trades in the stock and option have a constant correlation coefficient $\rho \neq \pm 1$. Under these assumptions, in the equilibrium I have found, the stock has stochastic volatility and the option is not redundant.

One motivation for imperfect correlation of liquidity trades is that some agents trade only in the stock and others use options for hedging; that is, they hold calls to hedge short positions or write covered calls. More generally, there might simply be some differences across agents in the extent to which they use the options market, perhaps due to differential transaction costs. If these various agents have independent liquidity shocks, then this structure will generate imperfect correlation of liquidity trades. The correlation will be negative, as will be assumed in Sections 3.1-3.3 and 4, if it results from agents

[1] The arbitrage trading in the model is done by market makers. Being risk neutral and competitive, they force prices to equal discounted expected values. Hence, taking the risk-free rate to be zero, a call option with exercise price K and maturity T on a stock with price P will have the price $E[(P_T - K)^+]$. When P is a geometric Brownian motion, this is the Black-Scholes formula.

putting on or undoing hedged positions.[2] Obviously, it would be desirable to model the liquidity trades and determine their correlation structure endogenously, but that is beyond the scope of this article.

It is not surprising that the introduction of the option affects the stock price process in this model, because we are essentially assuming a different structure for liquidity trades when the option is traded. In the single-asset model, Back (1992b) shows how the volatility pattern of an asset depends on the pattern of liquidity trades. What is interesting in the current model is that the option is no longer spanned by the stock and risk-free asset once it is introduced; that is, the stock and option prices are imperfectly correlated. The correlation of the prices depends on two factors: the correlation of orders and the matrix of coefficients that relate prices to order flows. The former is taken to be exogenous here, but the latter is determined endogenously. If it were singular, the option would be redundant. However, it is nonsingular in the equilibrium that will be constructed.

The matrix relating prices to orders would be singular if market makers in both markets were to base prices on the same weighted average of stock and option orders. This would happen if the payoff of the derivative were an affine function of the stock price. In that case, an order for the derivative would convey the same information to the market as an order for the stock, so orders would simply be added to define an aggregate volume measure that drives both prices. However, in the equilibrium that will be constructed, an order for an option conveys different information than an order for the stock. For example, a buy order for a call option has little effect on the relative probabilities of values below the exercise price, but it increases the probability of the option finishing in the money to a disproportionate extent, relative to a buy order for the stock. As a result, the option price is comparatively more sensitive to option orders than is the stock price. Hence, there is no aggregate volume measure that drives both prices, and consequently, the prices are not locally perfectly correlated.

There is some empirical literature concerning whether the introduction of an option affects the volatility of an asset. After controlling for systematic changes in idiosyncratic risks, Lamoureux (1991) finds that there is no change in volatilities on average. This agrees with the prediction of this model. The volatility becomes stochastic when the option is introduced, but the expected average volatility does not change.

The theoretical literature on Kyle models with multiple assets includes Bhushan (1991), Caballé and Krishnan (1993), Chowdhry

[2]Other reasons for imperfect correlation of liquidity trades could be tax effects (e.g., sales of specific assets to realize losses) and option trades made to close out positions.

and Nanda (1991), and Subrahmanyam (1991). These are primarily one-period models. In each of these articles the assets are assumed to have a joint normal distribution, which excludes options. In each article, liquidity trades across the markets are assumed to be imperfectly correlated. The articles differ with regard to whether market makers see the orders for other assets or at other market locations. It is assumed here that market makers see the orders for both assets. This makes the result that the prices do not move together more striking.

An important motivation for informed traders to use options is to gain additional leverage or to avoid short-sale constraints. This motivation is not captured in this article. It is assumed, for tractability, that there are no margin requirements or short-sale constraints. Therefore, the choice of which asset to trade is based entirely on the relative mispricing and price pressure in the two markets.

The model is described in the following section. Section 2 shows that the following three conditions are equivalent and would hold if the derivative had a linear or affine payoff: local perfect correlation of prices, existence of an aggregate volume measure that drives both prices, and singularity of the matrix relating prices to orders. Sections 3.1-3.3 present an equilibrium under certain parametric assumptions. Section 3.4 explains how these assumptions can be generalized. Section 4 describes some properties of the equilibrium presented in 3.1-3.3. Included in Section 4 are several graphs of the conditional density of the stock value. These demonstrate the different signaling effects of stock and option orders. They also illustrate the stochastic volatility of the stock. Section 5 briefly concludes.

1. The Model

The basic model is that of Kyle (1985), generalized to include a call option. There are risk-neutral market makers, a risk-neutral informed trader, and liquidity traders. Trading occurs continuously over an interval [0, 1]. The stock is assumed to pay no dividends during this period. There is to be a public information release at time 1. The informed trader knows at time 0 the price v^s at which the asset will trade after the information release. The rest of the market views v^s as being normally distributed with mean K equal to the exercise price of the call option.[3] Denote the variance of v^s by σ_v^2. More general

[3] In equilibrium, the price of the stock at time 0 will be the mean of v^s. We can think of the option as being created at time 0. Therefore, what we are assuming here is that the exercise price is set equal to the stock price when the option is created. This corresponds roughly to practice.

distributions of v^s will be considered in Section 3.4. We assume the option expires just after the information release, so its price at expiration will be $v^o \equiv (v^s - K)^+$. The market makers are assumed to be competitive and to observe the order flows in both assets. The risk-free rate is normalized to zero.

Prices will be denoted by P, the informed trader's orders by X, liquidity orders by Z, and total orders $(X + Z)$ by Y. We use the superscript s to denote the stock and the superscript o to denote the option. Unscripted symbols will denote vectors: thus $P = (P^s, P^o)^\top$, $v = (v^s, v^o)^\top$, and so on.

The cumulative liquidity trades are assumed to be Brownian motions with a constant covariance matrix

$$\Sigma \equiv \begin{pmatrix} \sigma^{ss} & \sigma^{so} \\ \sigma^{so} & \sigma^{oo} \end{pmatrix}.$$

We assume Z^s and Z^0 are imperfectly correlated, so Σ is nonsingular. The process Z is assumed to be independent of v.

Denote by $C[0, 1], \Re^2)$ the space of continuous $\Re^2$--valued functions on [0, 1]. A pricing rule is defined to be a nonanticipative function

$$(t, Y) \mapsto P(t, Y): [0, 1] \times C([0, 1], \Re^2) \rightarrow \Re^2.$$

By "nonanticipative" it is meant that, for each t, $P(t, \cdot)$ depends only on $(Y(s))_{s \leq t}$ (i.e., is measurable with respect to the u-field on C([O, 1], $\Re^2$) generated by the family of projection maps $\{x \mapsto x(s) \mid s \leq$ t}). This captures the idea that the market makers at each time t have only observed the history of the aggregate order process Y

The price vector just before the announcement is $P(1, Y)$. It should be kept in mind that this is not the price after the announcement. The price vector after the announcement is v, and we are allowing for a jump at the announcement, so v can, in principle, differ from $P(1, Y)$. It will be proved that there is no jump in equilibrium.

The informed trader's order for asset i ($i = s$ or $i = o$) at any time t is assumed to be of the form $dX^i(t) = \alpha^i(t)\, dt$ for some stochastic process α^i.[4] The informed trader's information at any time t consists of v and the history of prices up to that time. Hence the order rates $\alpha^i(t)$ must be measurable with respect to that information. We use the device of an "artificial market" employed in the rational expectations literature [e.g., Grossman (1981)]. In the artificial market, the informed trader is assumed to observe the order process Y Hence, he can infer the liquidity trades Z. We construct an equilibrium of

[4] More general strategies were considered in Back (1992a) for the single-asset case and shown not to occur in equilibrium.

the artificial market and show that this is an equilibrium of the actual market, because the price process in this equilibrium reveals the order process to the informed agent.

A technical requirement is also needed to exclude doubling strategies; namely, we must constrain a to be such that the stochastic integrals $\int P^i(t, Y)\, dZ^i(t)$ are martingales. As in Back (1992a), this can be interpreted as stating that liquidity traders would not lose money on average if they could trade at the midpoint of the spread.

An equilibrium consists of an order strategy $\alpha = (\alpha^s, \alpha^o)^{\mathsf{T}}$ and a pricing rule P such that, given P, a maximizes the expected value of the informed agent's profits

$$\int_0^1 [v - P(t, Y)]^{\mathsf{T}} \alpha(t)\, dt. \tag{1}$$

subject to the measurability and "no-doubling-strategies" conditions and such that, given a, P satisfies the zero-profit condition

$$(\forall\ t)\ P(t, Y) = E[v \mid (Y_u)_{u \le t}] \quad \text{a.s.} \tag{2}$$

In this conditioning, market makers are assumed to know a, though they cannot observe the actual orders of the informed trader.

2. Signaling, Spanning, and Aggregation

It is useful to briefly review the Kyle model here. Assume that the option is not traded. We may want to allow the variance of liquidity trades in the stock to be different in this case, so denote the variance by σ_*^2. Given the distributional assumptions on v^S, the equilibrium pricing rule is

$$P^s(t, Y) = K + \lambda Y^s(t),$$

where $\lambda = \sigma_v/\sigma_*$. In equilibrium, the process Y^S is a Brownian motion on the filtration it generates (i.e., given the market makers' information) with zero drift and instantaneous variance σ_*^2. This is a consequence of the particular strategy chosen by the informed trader. Therefore P^s is a Brownian motion with zero drift and instantaneous variance σ_v^2, given the market makers' information. The conditional distribution of v^S is always normal, and the variance decreases linearly in time. Orders for the stock cause the conditional distribution to shift in a parallel fashion, buy orders shifting it right and sell orders shifting it left.

2.1 Affine payoffs and signaling

The distribution of the stock price process would be unaffected by introducing a traded derivative with a linear or affine payoff. Suppose, for example, that the payoff were $v^s - K$ rather than $(v^s - K)^+$. Denote values related to this derivative by the superscript d. The equilibrium would have the following form. Market makers would consider the combined order process $Y^* \equiv Y^s + Y^d$. The equilibrium pricing rule would be

$$P^s(t, Y) = K + \lambda Y^*(t),$$
$$P^d(t, Y) = \lambda Y^*(t),$$

where $\lambda = \sigma_v/\sigma_*$ and σ_* now denotes the instantaneous standard deviation of $Z^s + Z^d$. The informed agent's trading strategy would cause the process Y^* to be a Brownian motion with zero drift and instantaneous variance σ_*^2, relative to the market makers' information. Therefore, P^s would be a Brownian motion with zero drift and instantaneous variance σ_v^2, just as if the derivative were not traded.

The reason that trading this derivative would have no effect on the stock is that orders for this derivative would convey the same information as orders for the stock. Because it may have come from the informed trader, a buy order for the derivative would indicate that the derivative may be undervalued (i.e., that $P^d < v^s - K$). Given pricing as above, this is the same as the stock being undervalued. The result is that both types of orders would induce parallel shifts in the conditional distribution of v^s; for example, a unit buy order for either asset would shift the distribution right by the amount λ.

In contrast, orders for a call option can convey different information than orders for the stock. Suppose that it tends to be the case that the informed agent buys the call option only when it is undervalued in the sense that $P^o < v^o$. Then a buy order for the option is a signal that it will finish in the money. The market will rationally respond to this order by increasing the probability assigned to the event $v^s > K$ to a disproportionate extent, relative to its response to a stock buy order. We will see in Section 4 that this occurs in equilibrium. Thus, when there is a traded call option there is a richer class of signals the market can receive. Unless stock and option orders are locally perfectly correlated, this enlargement of the local state space will preclude spanning of the option by the stock and risk-free asset. An equivalent perspective is that the change in the stock price alone will not determine the change in the option price; one must also know the source of the stock-price change (whether it was stimulated by stock or option orders) in order to determine the option price change. Hence, a risk-free hedge cannot be formed from the stock and option.

2.2 Order aggregation and spanning

In the previous subsection, where the derivative has an affine payoff, market makers base prices on the sum of orders in the two markets. If the units of the stock and derivative differ, then the sum is a weighted sum.[5] This raises the question of whether there could be a weighted sum of orders, possibly with random time-varying weights, which drives the price processes in the option model. In this subsection we show that this type of order aggregation is equivalent to the option being spanned by the stock and risk-free asset or, equivalently (given certain regularity conditions), to local perfect correlation of the stock and option.

We assume that in equilibrium

$$dP(t) = \mu(t)\ dt + \Lambda(t)\ dY(t) \tag{3}$$

for some vector-valued process μ and matrix-valued process Λ adapted to Y This assumption is unrestrictive.

Theorem 1. Assume that (3) holds. Then the following conditions are equivalent:

1. *P^s and P^o are locally perfectly correlated almost everywhere.*
2. *$\Lambda(t)$ is singular* almost everywhere.
3. *There exists a scalar process Y^* satisfying*

$$dY^*(t) = \gamma(t)\ dt + \beta(t)^\top dY(t), \tag{4}$$

for some processes γ and $\beta = (\beta^s, \beta^o)^\top$ adapted to Y, such that

$$dP(t) = \kappa(t)\ dt + \eta(t)\ dY^*(t) \tag{5}$$

for some processes $\kappa = (\kappa^s, \kappa^o)^\top$ and $\eta = (\eta^s, \eta^o)^\top$ adapted to Y.

Proof Given (3), the local covariance matrix of dP is $\Lambda\Sigma\Lambda^\top$. This is singular iff Λ is singular, because Σ is positive definite. Therefore, (1) is equivalent to (2). If (4) and (5) hold, then

$$dP = \begin{pmatrix} \kappa^s + \eta^s\gamma \\ \kappa^o + \eta^o\gamma \end{pmatrix} dt + \begin{pmatrix} \eta^s\beta^s & \eta^s\beta^o \\ \eta^o\beta^s & \eta^o\beta^o \end{pmatrix} dY,$$

so Λ is singular. For the converse, suppose that Λ is singular. Let $dY^*(t) = \lambda(t)^\top\ dY(t)$, where $\lambda(t)^\top$ denotes a nonzero row of $\Lambda(t)$ if there is one and is zero otherwise. The other row of $\Lambda(t)$ is of the form $\delta(t)\lambda(t)^\top$, so (5) holds. ∎

The weighted sum (4) is a generalization of the type of weighted sum considered by Jarrow (1992). Jarrow assumes that the stock and

[5] For example, if $v^d = 2v^s$ + constant, then $Y^* = Y^s + 2Y^d$ would drive the price processes.

option prices satisfy $dP^o(t) = \delta(t)\ dP^s(t)$ for some process δ adapted to Y and uses the weighted sum

$$Y^*(t) = Y^s(t) + \delta(t)\, Y^o(t)$$

to drive the stock and option prices. However, Jarrow does not assume that there is asymmetric information.

2.3 An example

In Sections 3 and 4, we construct an equilibrium and show that the three equivalent conditions of Theorem 1 do not hold in that equilibrium. I do not know whether the equilibrium is unique, so it is possible that there are other equilibria in which the conditions do hold. The following example illustrates some of the issues involved in determining whether the conditions are consistent with any equilibrium.

Let $\lambda^\mathsf{T} = (\lambda^s, \lambda^o)^\mathsf{T}$ denote the first row of the matrix Λ in (3). Suppose that the price-response coefficients λ^s and λ^c are both positive, as one would expect them to be. Suppose that $P^s(t) < v^s < K$. Even though the stock is underpriced, relative to its true value, the option must be overpriced here, because it will finish out of the money; that is, $P^o(t) > 0 = v^o$. Therefore, the informed trader can buy the stock and sell the option and make money on both trades. If these trades are made in the proportion

$$\alpha^s = -(\lambda^o/\lambda^s)\alpha^o, \tag{6}$$

then they will not move the stock price. If the matrix Λ is singular, then the second row is proportional to λ^T; hence, trades made in the proportion (6) will not move the option price either. This suggests that the trades could be be repeated indefinitely, thereby generating unbounded profits for the informed trader.

It is not compatible with equilibrium for profits to be unbounded, so in equilibrium either or both of the following must hold: (i) Λ is nonsingular, so any vector of trades must move one of the prices; (ii) the ratio λ^o/λ^s changes even when the prices do not. Either of these could prevent the profitable vector of trades described in the last paragraph from being repeated indefinitely. However, if only (ii) holds, then there will be another vector of profitable trades satisfying (6) after λ^o/λ^s changes. It is possible therefore that (i) must always hold. If so, then there is no equilibrium in this model in which the option is spanned by the stock and risk-free asset.

3. Equilibrium

Back (1992a) constructs an equilibrium of the Kyle model for a general distribution of the asset value and proves that it is unique within a certain class. The same technique will be used here to define an equilibrium with a traded option. Whether the equilibrium is unique in this model is an open question.

In Sections 3.1-3.3 we assume that $\sigma^{so} = -\sigma^{oo}/2$. This parametric assumption simplifies the solution of the model, in a way that will be explained in Section 3.4.

3.1 Pricing rule

We begin with the definition of a function $\pi^s: \Re^2 \rightarrow \Re$. This function will appear in the definition of the equilibrium pricing rule. In particular, we will have $P^s(1) = \pi^s(Y(1))$.

Figure 1 shows the level curves of π^s. When $y \in \mathcal{A} \equiv \{(y^s, y^o) \mid 2y^s + y^o = 0\}$, then $\pi^s(y) = K$. Thus, when $Y(1) \in \mathcal{A}$, then $P^s(1) = K$; is, the option is at the money just before the announcement. Let $\mathcal{O}$ denote the region below $\mathcal{A}$ and $\mathcal{I}$ the region above $\mathcal{A}$. The definition of π^s is such that $\pi^s(y) > K$ in $\mathcal{I}$ and $\pi^s(y) < K$ in $\mathcal{O}$, so the option is in (respectively, out of) the money just before the announcement when $Y(1)$ is in $\mathcal{I}$ (respectively, $\mathcal{O}$). We show that in equilibrium the informed agent will trade in such a way that $P^s(1) = v^s$ a.s.,[6] market believes that the announcement will not cause a jump in the price. Hence, for example, if the option is out of the money just before the announcement, then it will finish out of the money.

The level curves of π^s in $\mathcal{I}$ have slope - 1, and the level curves in $\mathcal{O}$ are vertical lines. This means that π^s is a function of $y^s + y^o$ and a function of y^s only in $\mathcal{O}$, so $P^s(1)$ depends on $Y^s(1) + Y^o(1)$ when the option finishes in the money, and depends on $Y^s(1)$ when the option finishes out of the money. The reason for this construction is to ensure that a certain type of arbitraging cannot be done by the informed trader. If it were known with certainty that the option would finish in the money, then the prices of the two assets would move exactly together. In this situation, there would be an arbitrage opportunity if there were differential price pressure across the two markets. For example, if both assets happened to be priced below their true values, then the informed trader could arbitrage by buying the one with the lowest price pressure and selling the one with the highest. Due to the definition of π^s, the pricing rule to be defined here will ensure that the price pressure becomes approximately equal when it becomes highly probable that the option will finish in the money. Also, for a similar reason, it will ensure that the price pressure from option orders becomes nearly zero when it becomes highly probable that the option will finish out of the money.[7]

[6] It is intuitively obvious that this is required for optimality; for example, if $P^s(1) < v^s$, then the informed trader should have bought more just before time 1.

[7] The exact formulas for the price pressure coefficients are given in equation (A13) in the proof of Theorem 3.

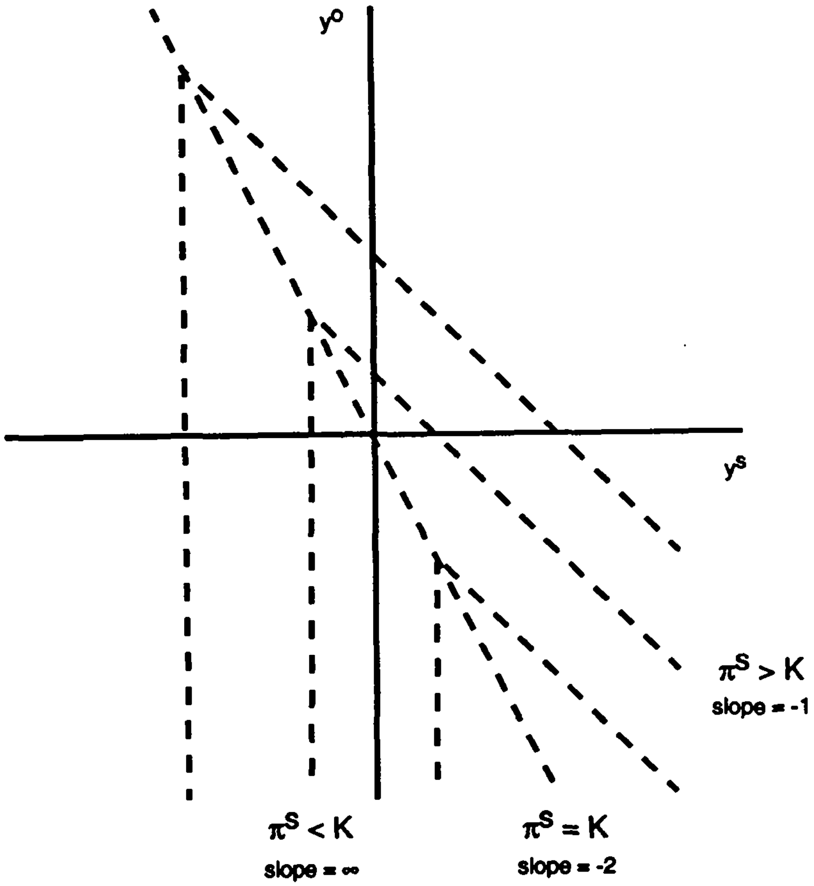

Figure 1
This figure shows typical level curves of π^s. For $p < K$, $(\pi^s)^{-1}(p) = \{y \mid 2y^s + y^o < 0 \text{ and } y^s = a\}$ for some a, $(\pi^s)^{-1}(K) = \{y \mid 2y^s + y^o = 0\}$, and for $p > K$, $(\pi^s)^{-1}(p) = \{y \mid 2y^s + y^o > 0 \text{ and } y^s + y^o = b\}$ for some b.

Figure 2 illustrates in more detail how π is defined. A change of variables has been made in going from Figure 1 to Figure 2. The vertical axis in Figure 2 shows the variable $y^c \equiv y^s + y^o$. With this change of variables, the line A is the -45° line, and the level curves above A are horizontal. Also, we can write $\mathcal{I} = \{y \mid y^s + y^c > 0\}$ and $\mathcal{O} = \{y \mid y^s + y^c < 0\}$. To reduce the notational burden in the following, let $\zeta = (Z^s(1), Z^c(1))$, where $Z^c(1) \equiv Z^s(1) + Z^o(1)$.

For each real a, the set $\{y \mid y^s = a\} \cap$ is a level curve of π^s. Figure 2 shows one such level curve, labeled $\pi^s = p^L$. The number p^L is defined by

$$\text{prob}[\zeta \in L] = \text{prob}[v^s \leq p^L], \tag{7}$$

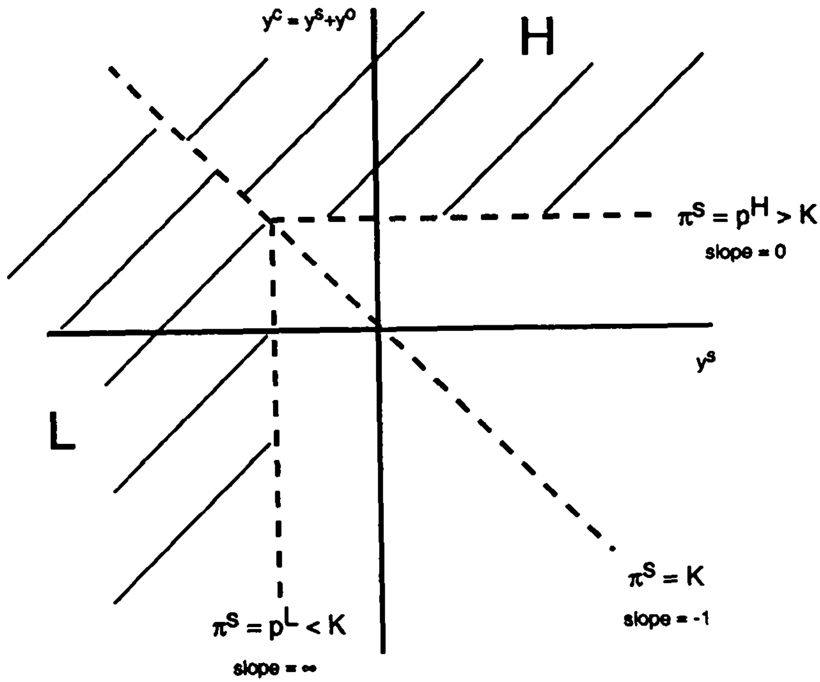

Figure 2
For $p^L < K$, $(\pi^s)^{-1}(p^L) = \{y \mid y^s + y^c < 0 \text{ and } y^s = a\}$, where a and p^L are related by prob$[(Z^s(1), Z^c(1)) \in L] = $ prob$[v^s \leq p^L]$. Here, prob denotes the unconditional probability, and $L = \{y \mid y^s + y^c < 0 \text{ and } y^s \leq a\}$. For $p^H > K$, $(\pi^s)^{-1}(p^H) = \{y \mid y^s + y^c > 0 \text{ and } y^c = b\}$, where b and p^H are related by prob$[(Z^s(1), Z^c(1)) \in H] = $ prob$[v^s > p^H]$. Here, $H = \{y \mid y^s + y^c > 0 \text{ and } y^c \geq b\}$.

where prob denotes the unconditional probability, and $L = \{y \mid y^s \leq a\} \cap \mathrm{O}$. We have

$$\begin{aligned} \text{prob}[\zeta \in L] &= \text{prob}[\zeta \in L \mid \zeta \in \mathrm{O}] \cdot \text{prob}[\zeta \in \mathrm{O}] \\ &= \tfrac{1}{2}\,\text{prob}[Z^s(1) \leq a \mid \zeta \in \mathrm{O}] \\ &= G_O(a)/2, \end{aligned}$$

where G_O denotes the cdf of $Z^s(1)$ conditional on $\zeta \in \mathrm{O}$. Also

$$\begin{aligned} \text{prob}[v^s \leq p^L] &= \text{prob}[v^s \leq p^L \mid v^s \leq K] \cdot \text{prob}[v^s \leq K] \\ &= F_O(p^L)/2, \end{aligned}$$

where F_O denotes the cdf of v^s conditional on $v^s \leq K$. Therefore (7) is equivalent to $F_O(p^L) = G_O(y^s)$, or, equivalently, $p^L = F_O^{-1}(G_O(a))$. In general, we define

$$(\forall\, y \in \mathrm{O})\pi^s(y) = F_O^{-1}(G_O(y^s)).$$

Similarly, for each real b, the set $\{y \mid y^c = b\} \cap \mathcal{I}$ is a level curve of π^s. Figure 2 shows such a level curve labeled as $\pi^s = p^H$. The number p^H is defined by

$$\text{prob}[\zeta \in H] = \text{prob}[v^s > p^H],$$

where $H = \{y \mid y^c \geq b\} \cap \mathcal{I}$. This is equivalent to $p^H = F_I^{-1}(G_I(b))$, where F_I denotes the cdf of v^s conditional on $v^s > K$, and G_I denotes the cdf of $Z^c(1)$ conditional on $\zeta \in \mathcal{I}$. In general, we define

$$(\forall\, y \in \mathcal{I})\pi^s(y) = F_I^{-1}(G_I(y^s + y^o)).$$

For future reference, note that our assumption that $\sigma^{so} = -\sigma^{oo}/2$ implies that $Z^c(1)$ has the same variance as $Z^s(1)$, so the vector ζ is symmetrically distributed about the line $\mathcal{A}$. This implies that

$$\text{prob}[\zeta \in L] = \text{prob}[\zeta \in H],$$

when $a + b = 0$ as in Figure 2. Hence, the definitions of p^L and p^H imply

$$\text{prob}[v^s \leq p^L] = \text{prob}[v^s > p^H].$$

It follows from this that $(p^L + p^H)/2 = K$.

The stock price at time t is defined by averaging over the values of π^s. The average (expectation) is taken with respect to the distribution of the random vector $y + Z(1) - Z(t)$, where $y = Y(t)$ is known at time t. Formally, we define

$$P^s(t) = E[\pi^s(Y(t) + Z(1) - Z(t)) \mid (Y(s))_{s \leq t}]. \tag{8}$$

The option price is defined by

$$P^o(t) = E[\pi^o(Y(t) + Z(1) - Z(t)) \mid (Y(s))_{s \leq t}], \tag{9}$$

where $\pi^o \equiv (\pi^s - K)$ The motivation for averaging over the distribution of $y + Z(1) - Z(t)$ will be explained following Lemma 2. Note that even though π^s is discontinuous at the line $\mathcal{A}$, the prices are smooth functions of the state variable $Y(t)$ at each time $t < 1$, due to the averaging in (8) and (9).

Using the symmetry of the distribution of $\zeta - (Z^s(t), Z^c(t))$ about the line $\mathcal{A}$ for each t, we can easily show that (8) and (9) imply that the option is in the money at time t if $Y(t) \in \mathcal{I}$, at the money if $Y(t) \in \mathcal{A}$, and out of the money if $Y(t) \in \mathcal{O}$.

3.2 Trading strategy

We now turn to analyzing the informed agent's optimal strategies when prices are defined by (8) and (9). Lemma 1 shows that the informed agent will have many optima in equilibrium. The situation

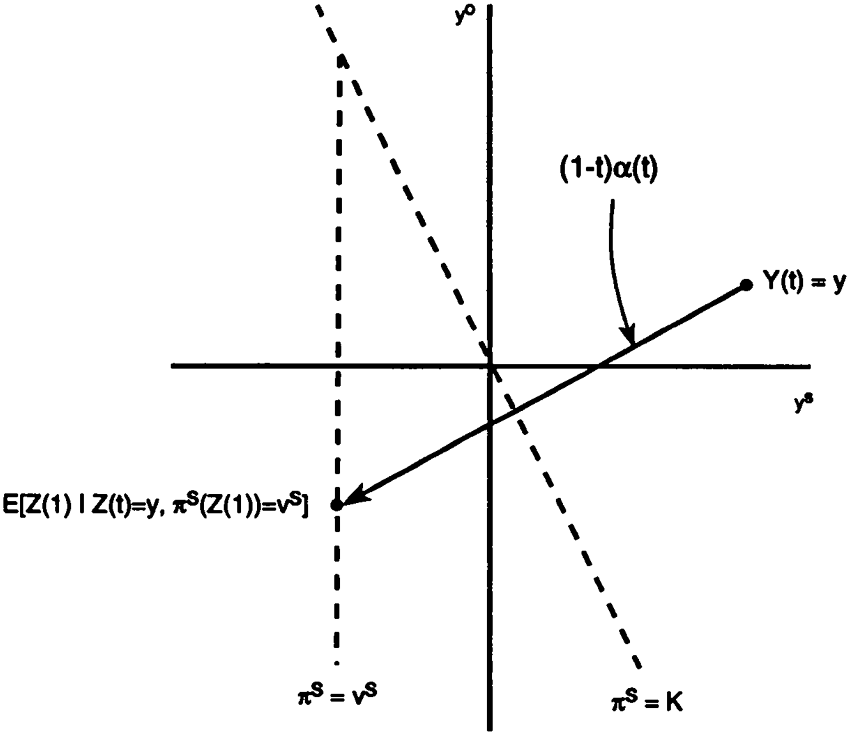

Figure 3
The equilibrium trade vector for the informed agent at any time t, when $Y(t) = y$, is $\alpha(t) = (1 - t)^{-1}(E[Z(1) \mid Z(t) = y, \pi^s(Z(1)) = v^s] - y)$. This figure illustrates $\alpha(t)$ when $v^s < K$.

here is analogous to that of a risk-neutral agent in a competitive model-he has an optimum only if all expected returns are equal, but in that case anything is optimal. Here anything is optimal provided it causes the prices to be right just before the announcement. All the results in Sections 3 and 4 are proved in Appendix A.

Lemma 1. Given the pricing rule defined by (8) and (9), any strategy for the informed agent that implies P(1) = v a.s. is an optimal strategy in the artificial market.

The next lemma asserts the existence of a particular optimal strategy that will subsequently be shown to be an equilibrium strategy. This is a generalization of the Brownian bridge strategy followed by the informed agent in the single-asset model (Back 1992a). The strategy is illustrated in Figure 3. The order rate is the vector shown divided by 1 - t. The effect of dividing by 1 - t is to make the vector very large near time 1 whenever Y is away from the level curve $\pi^s = v^s$.

This forces the stock price to v^s at time 1, so there is no jump in price following the announcement.

Lemma 2. Given the pricing rule defined in the theorem, there exists a trading strategy for the informed agent in the artificial market that guarantees P(1) = v a.s. and which has the property that the distribution of Y; given the market makers' information, is that of a Brownian motion with zero drift and instantaneous covariance matrix Σ.

It may seem surprising that Y and Z can have the same distribution, as this lemma states. This is simpler to see in the single-asset model. Consider the terminal distributions. From Back (1992a), we have $X(1) = -Z(1) + \pi^{-1}(v)$, where A and v are scalars. The ability to continuously monitor the liquidity trades (which can be done simply by monitoring the price) and to move continuously up or down the residual supply curve makes it optimal for the informed agent to completely offset the liquidity trades, as shown by the formula given for $X(1)$. The definition of π, which the construction in this article generalizes, is such that $\pi^{-1}(v) \sim Z(1)$, so $Y(1) \sim Z(1)$.

One can prove that Y and Z have the same distribution simply by showing that Y is a martingale on the filtration it generates. This is an intuitively appealing condition, because it means that market makers cannot predict whether the next orders arriving will be buy or sell orders. This is plausible because in equilibrium the assets are neither underpriced nor overpriced on average. Back (1992a) shows that under some assumptions it is a necessary condition for equilibrium in the single-asset model. The distribution of a continuous martingale is completely determined by its instantaneous variance, and the instantaneous variance of Y is the same as that of Z because dX is of smaller order than dZ (dt rather than $\sqrt{dt}$). Hence, Y must be distributed like Z if it is a martingale (and it is a martingale on its own filtration when the informed trader follows the strategy in Lemma 2).

3.3 Eqilibrium

The properties of the trading strategy in Lemma 2 justify defining the prices according to (8) and (9). These prices are the conditional expectations of v^s and $v°$, respectively, because $v = \pi(Y(1))$ a.s., and $Y(1) \sim Y(t) + Z(1) - Z(t)$ conditional on $Y(t)$. This leads immediately to the following.

Lemma* 3. *The pricing rule (8) and (9) and strategy in Lemma 2 constitute an equilibrium in the artificial market.

The final step is to show that in this equilibrium the price process reveals Y (hence Z) to the informed agent, so the equilibrium is also an equilibrium of the actual market.

Theorem 2. *The pricing rule (8) and (9) and strategy in Lemma 2 constitute an equilibrium.*

3.4 Generalizations

The purpose of this subsection is to explain the role of the assumptions that v^s is normally distributed with mean K and $\sigma^{so} = -\sigma^{oo}/2$ and to show how to relax them. Hopefully, this will also provide additional insight into the construction of the equilibrium. Suppose now that the support of v^s is an interval (possibly a half-line or the whole real line) and its distribution function F is continuous. This implies that F^{-1} is well-defined on (0, 1). Let Σ be an arbitrary symmetric positive-definite matrix.

The candidate for an equilibrium is the pricing rule defined by (8) and *(9)* and the trading strategy in Lemma 2. The only change that needs to be made to accommodate this more general setup is in the definition of the curve $\mathcal{A}$. Continue to denote the region below $\mathcal{A}$ by $\mathcal{O}$ and the region above $\mathcal{A}$ by $\mathcal{I}$, and take the level curves of $\pi^s(y^s, y^c)$ to be vertical lines in $\mathcal{O}$ and horizontal lines in $\mathcal{I}$.

Lemma 1 goes through if

$$(p^H - K)/(p^L - K) = \text{slope } \mathcal{A}, \tag{10}$$

in the notation of Figure 2, where p^H and p^L are defined as in Section 2.1. In general the slope of $\mathcal{A}$ need not be constant. Lemmas 2 and 3 remain valid here, so we have at least an equilibrium of the artificial market when $\mathcal{A}$ satisfies (10). I conjecture that Theorem 2 and Theorems *3* and 4 also remain valid, but I have not verified the details.

Because p^L and p^H are defined in terms of $\mathcal{A}$, Equation (10) is an equation in the curve $\mathcal{A}$. There are various ways to write this as an integral or differential equation (one of which appeared in a previous version of this article). These suggest numerical methods for analyzing the equilibrium. However, I have been able to solve the equation analytically only under the assumptions that v^s is symmetrically distributed about K and that $Z^s(1)$ and $Z^c(1)$ have the same variance. These imply that (10) is satisfied by the -45° line.

4. Properties of the equilibrium

The local dynamics of the equilibrium price processes are described by the following theorem.

Theorem 3. In the equilibrium of Theorem 2,

$$dP(t) = \Lambda(t, Y(t))\, dY(t), \tag{11}$$

where $\Lambda(t, y)$ is a symmetric positive-definite matrix for each (t, y) and depends continuously on (t, y). The instantaneous covariance matrix of P is the symmetric positive-definite matrix $\Lambda\Sigma\Lambda$. Writing

$$\Lambda = \begin{pmatrix} \lambda^{ss} & \lambda^{so} \\ \lambda^{os} & \lambda^{oo} \end{pmatrix},$$

we have $\lambda^{ss} > \lambda^{so} = \lambda^{os} > \lambda^{oo}$ for all (t, y).

The positive definiteness of the covariance matrix of ***P*** means, of course, that the stock and option prices are not locally perfectly correlated. Hence, the option is not spanned; equivalently, one cannot form a risk-free hedge by using the stock and option.

The positive definiteness of the price-response matrix Λ can be interpreted as follows. Suppose that we were dealing with discrete orders and discrete price changes and had $\Delta P = \Lambda \Delta Y$. Consider a stock buy order ΔY^s and an option buy order ΔY^o normalized so that either one alone would cause a $1 increase in the stock price. Then the option-price change would be greater in response to the option order than in response to the stock order. The reverse is also true: If the orders were normalized so that either would induce a $1 change in the option price, the stock-price change would be greatest in response to the stock order. Therefore, each market is comparatively more sensitive to its own orders. In a multiasset extension of the static Kyle model, assuming joint normality of the asset values, Caballé and Krishnan (1993) also show that the price-response matrix is symmetric and positive definite.

The fact that the top row of A is larger than the bottom means that, for any vector of orders, the stock-price change will always be greater than the option-price change, as one would expect. Of course, this does not mean that the stock ***return*** is greater than the option's. The fact that the left column of A is bigger than the right means that an order for one share of the stock will have a bigger effect on each price than will an order for one option.

Theorem 2 shows that the stock and option are not locally perfectly correlated, so the option price is not given by a Black-Scholes-type formula $P^o(t) = f(t, P^s(t))$. However, prices are martingales, due to the existence of risk-neutral competitive market makers. This implies there are no arbitrage opportunities for price-taking traders with public information. Hence, the failure of arbitrage-based pricing must be due to the presence of stochastic volatility. This is established by the following result.

Theorem 4. In the equilibrium of Theorem 2, the stock-price process P^s is not a Markov process relative to the market makers' information.

Because P reveals Y in this equilibrium, the information received by market makers is public information. Hence, the stock price is not a Markov process relative to public information either. The vector process (P, Y) is Markov in this equilibrium, however.

Even though the volatility process is much different when the option is traded, the expected average volatility is the same. Because P^s is a martingale, relative to the market makers' information, and $P^s(1) = v^s$ a.s., we have[8]

$$E\left[\int_0^1 (dP^s)^2\right] = \text{var } P^s(1) = \text{var } v^s.$$

The same formula applies when the option is not traded.

Theorems 3 and 4, as well as other properties of the equilibrium, are illustrated in Figures 4-9. These show the conditional density of v^s at time $t = \frac{1}{2}$, given the market makers' information (or, equivalently, public information) for various values of $Y(\frac{1}{2})$. These figures assume that $\sigma^{ss} = \sigma^{oo} = 1$, $K = 100$, and the unconditional standard deviation of v^s is 10. The assumption that $\sigma^{ss} = 1$ is implicitly an assumption about the unit in which the stock is measured. The calculation of the densities is explained in Appendix B.

A characteristic of each of Figures 4-9 is that the density has two peaks. This is difficult to discern in Figure 4, nevertheless it is true. This is a consequence of the trading strategy of the informed agent. Whenever v^s is near K, the informed agent sells options while simultaneously buying the stock. The scale of this trading plan increases with the proximity of v^s to K. Even if such a plan has apparently been carried out on some scale and the conditional probability mass is concentrated around K, as in Figure 4, there will be values of v^s very near K at which the conditional density is relatively low, because the scale of the plan has not been large enough to indicate that v^s is that close to K. This is true for every $t > 0$, though not at $t = 0$.[9]

[8] As usual, the real meaning of $(dP^s)^2$ is $d[P^s, P^s]$, where the brackets denote the quadratic variation process. See, for example, Dothan (1990). This formula is a consequence of the fact that $M^2 - [M, M]$ is a martingale, whenever M is a centered continuous square-integrable martingale [Karataas and Shreve (1988, Definition 1.5.3) and Dothan (1990, Definition 10.33)].

[9] The difference between $t > 0$ and $t = 0$ is mathematically a result of the fact that the normal $(\mu, (1 - t)\Sigma)$ density is asymptotically smaller than the normal $(0, \Sigma)$ density-that is, the ratio of the two densities is arbitrarily close to zero whenever the norm of the argument is sufficiently large-no matter what μ is.

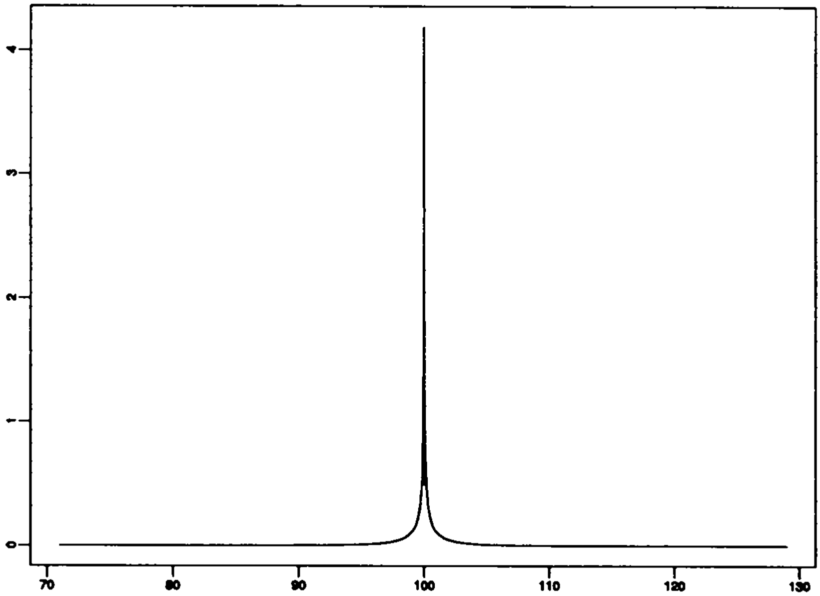

Figure 4
This is the conditional density of v^s given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) = 1$ and $Y^o(\frac{1}{2}) = -2$, assuming that $\sigma^{ss} = \sigma^{oo} = 1$, $\sigma^{so} = -\frac{1}{2}$, $K = 100$, and the unconditional variance of v^s is 100. The stock price in this circumstance is 100, and the option price is 0.53. The conditional variance of v^s is approximately 4.

In Figures 4-6, the option is at the money, because $Y(\frac{1}{2})$ is on the line A. These cases differ with regard to the location of Y(f) on A. This location affects the conditional variance and the option price. The fact that the conditional variance and option price can vary without the stock price varying shows that the stock has stochastic volatility and that the option is not spanned by the stock. In Figure 4, option orders through time ½ have on net been sells, and stock orders have been buys. In Figure 5, net orders for both assets are zero, and in Figure 6 option orders have on net been buys and stock orders have been sells.

Both the variance and the option price increase with option buy orders. The conditional variance of v^s is approximately 4 in Figure 4, 52 in Figure 5, and 243 in Figure 6 (these numbers are biased downward because the tails of the distributions were neglected in computing them). The option price is 0.53 in Figure 4, 2.98 in Figure 5, and 7.41 in Figure 6. Clearly, option buy orders can increase the conditional variance substantially. Recall that the unconditional variance in this example is only 100, so the conditional variance in Figure 6 is higher than the unconditional variance. This feature of the model

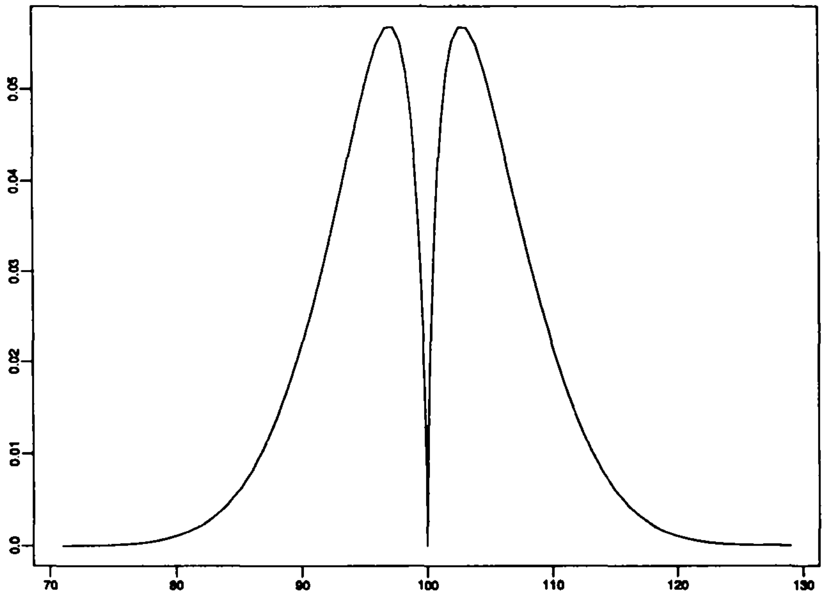

Figure 5
This is the conditional density of v^s given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) = 0$ and $Y^o(\frac{1}{2}) = 0$, assuming that $\sigma^{ss} = \sigma^{oo} = 1$, $\sigma^{so} = -\frac{1}{2}$, $K = 100$, and the unconditional variance of v^s is 100. The stock price in this circumstance is 100, and the option price is 2.98. The conditional variance of v^s is approximately 52.

differs radically from the single-asset Kyle model, in which the conditional variance decreases linearly with time.

In Figures 7-9, $Y(\frac{1}{2}) \in O$, so the option is out of the money. In Figure 7, the stock price is 90. Figures 8 and **9** illustrate the different signaling effects of stock and option buy orders. Taking Figure 7 as a benchmark, Figure 8 (respectively, Figure 9) shows how the conditional density would differ if there had been enough stock (respectively, option) buy orders to raise the stock price to 95. Consistent with Theorem 3, it takes more option buy orders than stock buy orders to raise the stock price to 95. Also consistent with Theorem 3, the resulting effect on the option price is greater with the option buy orders. In fact, relative to stock buy orders, option buy orders significantly increase the conditional variance of v^s, the conditional probability of the option finishing in the money, and the option price.

5. Conclusion

I have described an equilibrium in a market in which an agent with private information can trade in an option and the underlying asset. Trades in the option and underlying asset convey different informa-

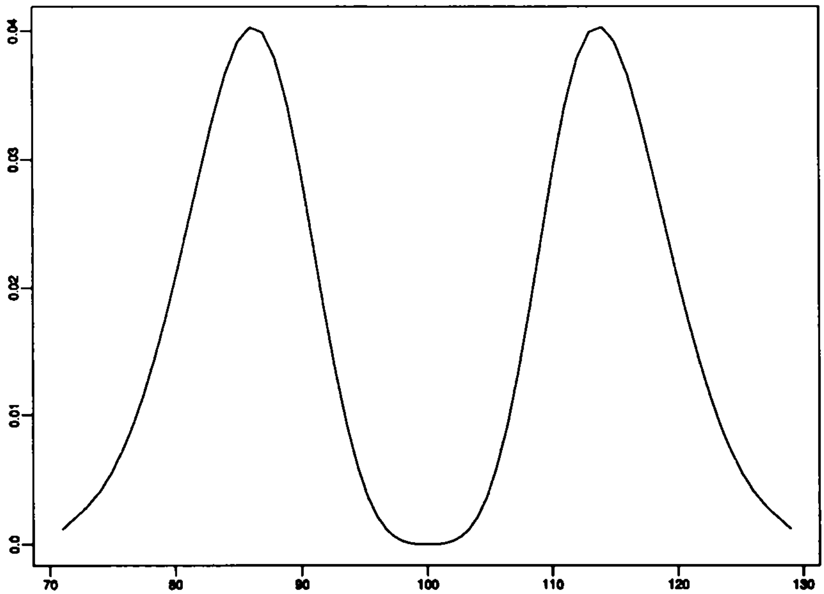

Figure 6
This is the conditional density of v^1 given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) =$ -1 and $Y^o(\frac{1}{2}) = 2$, assuming that $\sigma^{ss} = \sigma^{oo} = 1$, $\sigma^{so} = -\frac{1}{2}$, $K = 100$, and the unconditional variance of v^1 is 100. The stock price in this circumstance is 100, and the option price is 7.41. The conditional variance of v^1 is approximately 243.

tion, so the existence of the option implies that a richer class of signals can be received by the market. This enlargement of the local state space precludes spanning of the option by the underlying asset. The change in the information flow due to the existence of the option also causes the underlying asset to have stochastic volatility.

The new assumptions introduced here, relative to the single-asset Kyle model, concern liquidity trades in the option. It is probably unrealistic to assume the variance of liquidity trades in the option is constant over time. One might expect it to vary depending on the degree the option is in or out of the money and on the time to maturity. Ideally, this would be derived endogenously in a more complete model. This comment applies equally well to the assumption of imperfect correlation of liquidity trades across the two markets.

Appendix A

Define $\pi^s_O = F_O^{-1} \circ G_O$ and $\pi^s_I = F_I^{-1} \circ G_I$. Then we have

$$\pi^s(y) = \begin{cases} \pi^s_O(y^s) & \text{if } y \in \mathcal{O}, \\ \pi^s_I(y^s + y^o) & \text{if } y \in \mathcal{I}. \end{cases} \tag{A1}$$

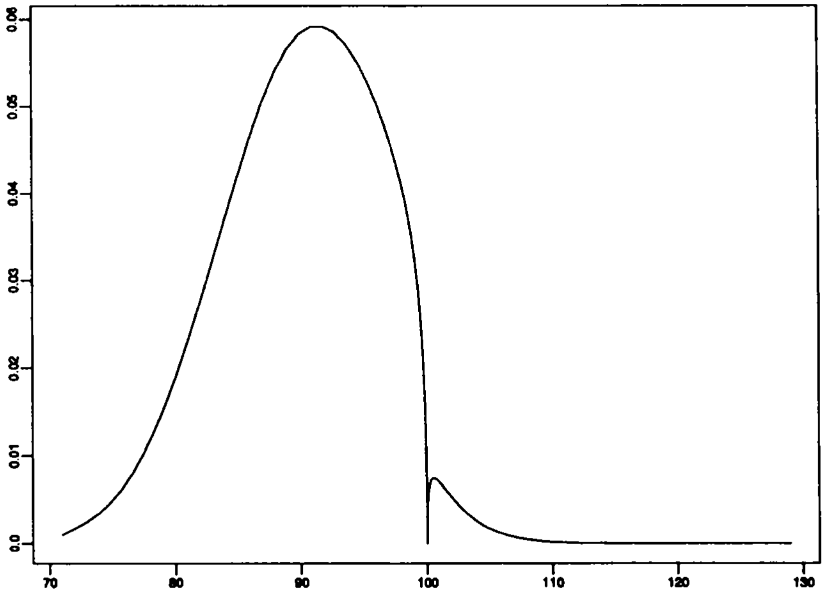

Figure 7
This is the conditional density of v^s given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) =$ -0.95250 and $Y^o(\frac{1}{2}) = -0.47625$, assuming that $\sigma^{ss} = \sigma^{oo} = 1, \sigma^{so} = -\frac{1}{2}, K = 100$, and the unconditional variance of v^s is 100. The stock price in this circumstance is 90, and the option price is 0.07. The conditional variance of v^s is approximately 39. The conditional probability of the option finishing in the money is 2.6%.

Proof of Lemma 1. The proof is in two steps. First, we prove the existence of a nonnegative function $J(v, t, y)$ that has certain smoothness properties and satisfies (where the subscripts denote partial derivatives, s denoting the derivative with respect to y^s and o the derivative with respect to y^o)

$$v^s - P^s + J_s = 0, \tag{A2}$$

$$v^o - P^o + J_o = 0, \tag{A3}$$

$$J_t + \tfrac{1}{2}(\sigma^{ss}J_{ss} + 2\sigma^{so}J_{so} + \sigma^{oo}J_{oo}) = 0, \tag{A4}$$

$$J(v, 1, y) = 0 \qquad \text{if } \pi(y) = v. \tag{A5}$$

Second, we use the function J to prove the lemma. The first step can be seen as solving the Bellman equation and the second as proving a verification theorem. However, the argument is entirely self-contained and does not refer to any results from dynamic programming [for the connection to the Bellman equation, see Back (1992a)].

Step 1. The intuition behind the definition of $J(v, t, Y(t))$ at any time t is to consider not trading after time t until "an instant" before time

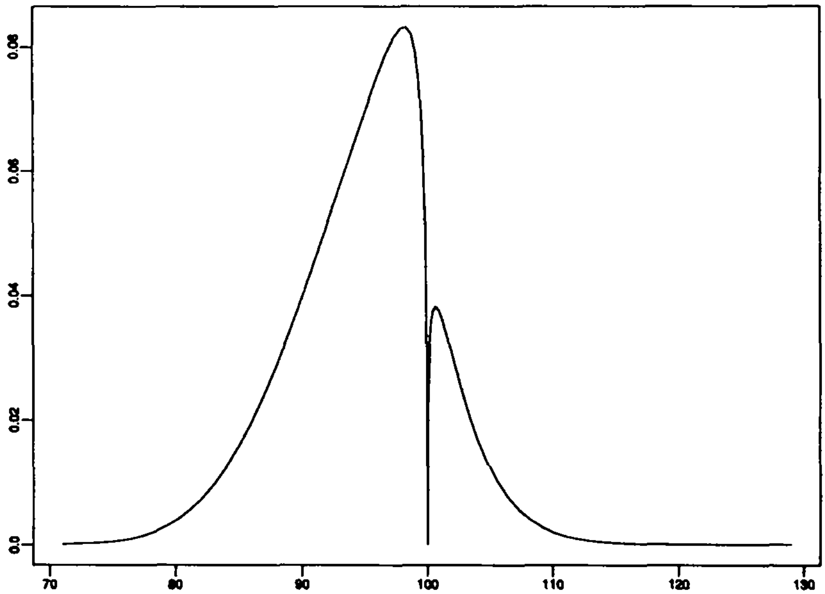

Figure 8
This is the conditional density of v^s given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) = -0.36705$ and $Y^o(\frac{1}{2}) = -0.47625$ assuming that $\sigma^{ss} = \sigma^{oo} = 1, \sigma^{so} = -\frac{1}{2}, K = 100$, and the unconditional variance of v^s is 100. The stock price in this circumstance is 95, and the option price is 0.52. The conditional variance of v^s is approximately 33. The conditional probability of the option finishing in the money is 16.2%.

1. The prices then will be given by $\pi^s(y)$ and $\pi^o(y)$, where $y = Y(t) + Z(1) - Z(t)$. At that point, consider trading against the pricing rule $\pi = (\pi^s, \pi^o)$ until the price vector is moved to v. The value $J(v, t, Y(t))$ will be defined as the expected profits earned from this "strategy." This discussion is intended only to motivate the following construction; it is not implied that there is an actual strategy of this form.

Given y, consider a smooth curve $\{w(a) \mid 0 \le a \le 1\}$ with the property that $w(0) = y$ and $\pi(w(1)) = v$. Take the curve to lie strictly within $\mathcal{O}$ (respectively, $\mathcal{J}$) if $v^s < K$ and $y \in \mathcal{O}$ (respectively, $v^s > K$ and $y \in \mathcal{J}$). Otherwise, let the curve intersect the line Λ exactly once. Denote the point of intersection by η. Define

$$j(v, y) = \int_0^1 \{[v^s - \pi^s(w(a))]\dot{w}^s(a) + [v^o - \pi^o(w(a))]\dot{w}^o(a)\}\, da, \tag{A6}$$

where $\dot{w}^i \equiv dw^i/da$. In order for j to be well defined, we have to ensure that the integral depends on the curve w only through v and y. The integral depends at most on the endpoints $w(0) = y$ and $w(1)$

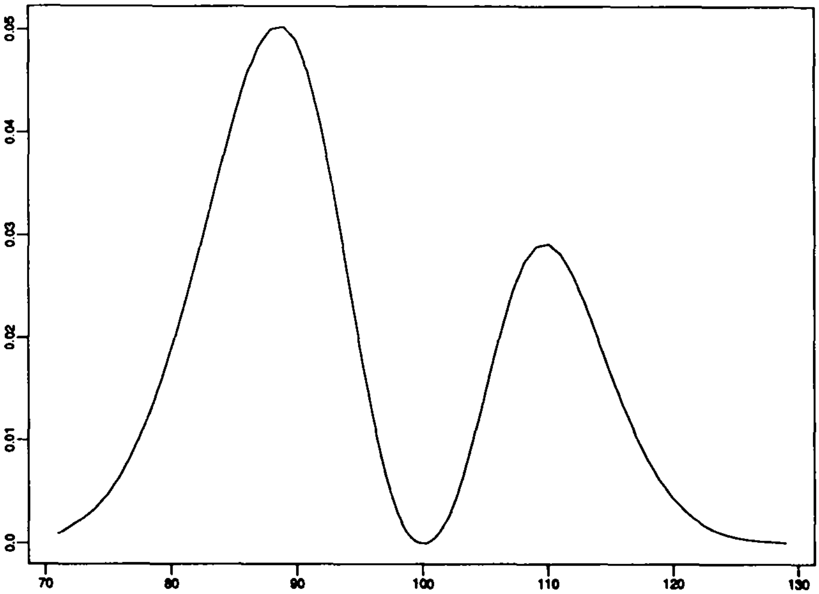

Figure 9
This is the conditional density of v^s given the market makers' information at time $\frac{1}{2}$, when $Y^s(\frac{1}{2}) = -0.95250$ and $Y^o(\frac{1}{2}) = 1.36316$, assuming that $\sigma^{ss} = \sigma^{oo} = 1, \sigma^{so} = -\frac{1}{2}, K = 100$, and the unconditional variance of v^s is 100. The stock price in this circumstance is 95, and the option price is 3.62. The conditional variance of v^s is approximately 150. The conditional probability of the option finishing in the money is 33.0%.

and on h, because $\partial\pi^s/\partial y^o = \partial\pi^o/\partial y$ in each of the regions $\mathfrak{I}$ and O. So we need to show that it does not depend on η and that it is invariant with respect to $w(1)$ in $\{w(1) \mid \pi(w(1)) = v\}$.

There are four cases to consider: (1) $v^s < K$ and $y \in \mathrm{O}$, (ii) $v^s > K$ and $y \in \mathfrak{I}$, (iii) $v^s < K$ and $y \in \mathfrak{I}$, and (iv) $v^s > K$ and $y \in \mathrm{O}$. By substituting $v^o = (v^s - K)^+$ and $\pi^o = (\pi^s - K)^+$ and making a change of variables, we can rewrite the right-hand side of (A6) in these cases as follows:

$$\text{(i)} \qquad \int_{y^s}^{w^s(1)} [v^s - \pi^s_O(b)]\, db,$$

$$\text{(ii)} \qquad \int_{y^s+y^o}^{w^s(1)+w^o(1)} [v^s - \pi^s_I(b)]\, db,$$

$$\text{(iii)} \qquad \int_{y^s+y^o}^{\eta^s+\eta^o} [v^s - \pi^s_I(b)]\, db + \int_{\eta^s}^{w^s(1)} [v^s - \pi^s_O(b)]\, db$$
$$+ (K - v^s)(\eta^o - y^o),$$

$$\text{(iv)} \qquad \int_{y^s}^{\eta^s} [v^s - \pi^s_O(b)]\, db + \int_{\eta^s+\eta^o}^{w^s(1)+w^o(1)} [v^s - \pi^s_I(b)]\, db$$
$$+ (v^s - K)(\eta^o - y^o).$$

Expressions (i) and (iii) depend on $w(1)$ only through $w^s(1)$, which is uniquely defined by v in these cases. Expressions (ii) and (iv) depend on $w(1)$ only through $w^s(1) + w^o(1)$, which is uniquely defined by v. It remains to show that (iii) and (iv) do not depend on η. It suffices to show that the gradients of (iii) and (iv) as functions of η are proportional to the vector (2, 1), because that implies that $\mathcal{A}$ is a level curve of (iii) and (iv).

The gradient of (iii) is

$$\begin{pmatrix} \pi^s_O(\eta^s) - \pi^s_I(\eta^s + \eta^o) \\ K - \pi^s_I(\eta^s + \eta^o) \end{pmatrix},$$

and the gradient of (iv) is the negative of this. As noted in the text,

$$\pi^s_O(y^s) + \pi^s_I(y^s + y^o) = 2K$$

whenever $y \in \mathcal{A}$. Therefore,

$$\pi^s_O(\eta^s) - \pi^s_I(\eta^s + \eta^o) = 2(K - \pi^s_I(\eta^s + \eta^o)).$$

This completes the proof that j is well defined as a function of v and y.

Note that j is a nonnegative function. This follows from the fact, which can be straightforwardly verified, that for any curve $\{w(a) \mid 0 \le a \le 1\}$ which lies entirely inside one of the regions $\mathcal{I}$ or O, the integral

$$\int_0^1 \{[\pi^s(w(1)) - \pi^s(w(a))]\dot{w}^s(a)$$
$$+ [\pi^o(w(1)) - \pi^o(w(a))]\dot{w}^o(a)\}\, da$$

is nonnegative. To deduce the nonnegativity of j in cases (iii) and (iv), one can apply this fact twice.

Now define

$$J(v, t, y) = E[j(v, y + Z(1) - Z(t))], \tag{A7}$$

where we are taking the expectation over Z, regarding v as a constant. All expectations in the remainder of this step are to be understood in this way. We need to show that $(\forall\ (v, t, y))\, J(v, t, y) < \infty$. It suffices to show that

$$J(v, 0, 0) \equiv E[j(v, Z(1))] < \infty,$$

because $J(v, t, y) = E[j(v, Z(1))\ ]\ Z_t = y]$ and the conditional expectations are finite if the unconditional expectation is finite. To save on notation, write ζ for Z(1).

We have

$$J(v, 0, 0) = \tfrac{1}{2}E[j(\zeta) \mid \zeta \in \mathcal{O}] + \tfrac{1}{2}E[j(\zeta) \mid \zeta \in \mathcal{I}].$$

Suppose that $v^s < K$. Then, when $\zeta \in \mathcal{O}$ we are in case (i), and when $\zeta \in \mathcal{I}$ we are in case (iii). By the monotonicity of π^s,

$$E[j(\zeta) \mid \zeta \in \mathcal{O}] \leq E[\{w^s(1) - \zeta^s\}\{v^s - \pi^s_{\mathcal{O}}(\zeta^s)\} \mid \zeta \in \mathcal{O}].$$

For this mean to be finite, it suffices that each product have a finite variance. This is clearly true of the first factor [recall that we are regarding $w^s(1) \equiv (\pi^s_{\mathcal{O}})^{-1}(v^s)$ as a constant]. In the second factor, we are treating v^s as a constant, and the distribution function of $\pi^s_{\mathcal{O}}(\zeta^s)$ conditional on $\zeta \in \mathcal{O}$ is $F_{\mathcal{O}}$. This has a finite variance by assumption. Similar reasoning yields the finiteness of $E[j(\zeta) \mid \zeta \in \mathcal{I}]$, and the case $v^s > K$ is handled in the same way.

To verify (A2) and (A3), we begin by differentiating the formulas for j in cases (i)-(iv). In each case, we obtain $j_s(v, y) = \pi^s(y) - v^s$ and $j_o(v, y) = \pi^o(y) - v^o$. Hence,

$$\begin{aligned} P^s(t, y) - v^s &\equiv E[\pi^s(y + Z(1) - Z(t))] - v^s \\ &= E[j_s(v, y + Z(1) - Z(t))], \end{aligned}$$

and similarly for $P^o - v^o$ (note that we are still taking the expectation over Z, regarding v as a constant). To establish (A2) and (A3), it suffices to show that we can pass differentiation through the expectation operator in (A7).

The only complication here is that j' is not continuously differentiable, as the usual theorems for the interchange of differentiation and expectation require. However, it is continuously differentiable on each of the regions $\mathcal{I}$ and $\mathcal{O}$, and everywhere continuous, and these properties are sufficient. Proceeding along the lines of the usual argument, we have to establish the uniform integrability of the difference quotients. Consider the partial derivative with respect to y^s. Given $1 > \epsilon > 0$, if both $b \equiv y + Z(1) - Z(t)$ and $b' \equiv y + Z(1) - Z(t)$ $(\varepsilon, 0)$ belong to $\mathcal{I}$ or both belong to $\mathcal{O}$, then

$$\begin{aligned} (j(b') - j(b)/\epsilon = j_s(b'') &= \pi^s(b'') - v^s \\ &\leq \max\{\pi^s(y + Z(1) - Z(t)), \pi^s(y + Z(1) \\ &\quad - Z(t) + (1, 0))\} - v^s \end{aligned}$$

for some 6” on the line segment joining b and b'. The last inequality follows from the monotonicity of $\pi^s(\cdot)$. The right-hand-side variable is integrable and independent for ε, which establishes the uniform integrability. If one of b and b' belongs to 3 and the other to 0, then the line segment joining b and b' crosses the boundary between 3 and 0 at some point $\eta \equiv y + Z(1) - Z(t) + (\epsilon', 0)$. The difference quotient can be written as

$$\left(\frac{\epsilon - \epsilon'}{\epsilon}\right)\left(\frac{j(b') - j(\eta)}{\epsilon - \epsilon'}\right) + \left(\frac{\epsilon'}{\epsilon}\right)\left(\frac{j(\eta) - j(b)}{\epsilon'}\right).$$

This weighted average of difference quotients can be bounded in the same way as above, completing the argument. The partial derivative with respect to y^o is analogous.

Because $J(v, 1, y) \equiv j(v, y)$, condition (A4) follows directly from the definition of j. Finally, the PDE (A5) follows from the fact that $J(t, Z(t))$ is, by definition, a martingale [Karatzas and Shreve (1988, Theorem 3.6)].

Step 2. Let X be an arbitrary policy and apply Itô's formula to $J(v, t, Y(t))$, using (A5). This yields

$$J(v, t, Y(t)) = J(v, 0, 0) + \int_0^t J_s(v, u, Y(u))\, dY^s(u) + \int_0^t J_o(v, u, Y(u))\, dY^o(u).$$

Now substitute from (A2) and (A3) and rearrange terms to obtain

$$\int_0^t \{[v^s - P^s(u, Y(u))]\alpha^s(u) + [v^o - P^o(u, Y(u))]\alpha^o(u)\}\, du$$
$$= J(0, 0) - J(v, t, Y(t)) + \int_0^t [v^s - P^s(u, Y(u))]\, dZ^s(u) + \int_0^t [v^o - P^o(u, Y(u))]\, dZ^o(u).$$

The left-hand side of this is the profits earned during [0, t]. Because $J(v, \cdot)$ is continuous on $[0, 1] \times \mathfrak{R}$ [Karatzas and Shreve (1988, Problem 4.3.2)], this also holds at $t = 1$. Taking expectations throughout and recalling that J is nonnegative, we conclude that the expected profits are bounded above by $E[J(v, 0, 0)]$. In this step, we use the fact that $\int_0^t [v^i - P^i(u, Y(u))]\, dZ^i(u)$ is a martingale.

If the strategy implies $J(v, 1, Y(1)) = 0$ a.s., then the expected profits actually attain the upper bound. A sufficient condition for this is that $\pi(Y(1)) \equiv P(1, Y(1)) = v$ ■

Proof of Lemma 2. To reduce notation, we will write v for v^s and π for π^s throughout this proof. Given any strategy for the informed trader, the joint distribution of Z and v defines a joint distribution for Y and v [a distribution for Y or Z is a probability measure on $C([0, 1], \Re^2)$]. The distribution of Y given the informed trader's information is this joint distribution conditioned on v. Denote this by $\mu(\cdot \mid v)$. In equilibrium, the market makers are assumed to know the strategy of the informed trader, so they know the joint distribution of Y and v, but, of course, they do not observe v. Therefore, the distribution of Y given their information is the marginal distribution of Y Denote this by μ. Of course, the marginal distribution of v is F.

First, we show that if $\mu(\cdot \mid v)$ is as specified in the lemma [i.e., the distribution of a $(0, \Sigma)$– Brownian motion conditioned on ending in the set $\pi^{-1}(v)$] then μ will be the distribution of a $(0, \Sigma)$– Brownian motion. Then we show that there is a strategy for the informed trader such that Y actually has the distribution $\mu(\cdot \mid v)$ for each v. This will complete the proof.

Let λ denote the distribution of a $(0, \Sigma)$– Brownian motion. All the distributions on $C([0, 1], \Re^2)$ are determined by their finite-dimensional distributions, so what we need to show is that for any finite set of times $t_1 < \ldots < t_N$ and sets $A_n \subset \Re^2$, we have

$$\mu\{Y \mid (\forall\ n) Y(t_n) \in A_n\} = \lambda\{Y \mid (\forall\ n) Y(t_n) \in A_n\}. \tag{A8}$$

By assumption, μ is obtained by conditioning λ on $\pi(Y(1)) = v$ and then integrating over v. Thus, we can write (A8) in terms of density functions as

$$g(y(t_1),\ldots,y(t_n) \mid y(1))h(y(1)) = g(y(t_1),\ldots,y(t_n) \mid y(1))g(y(1)), \tag{A9}$$

where $g(\cdot \mid \cdot)$ is the conditional density function defined by λ, $g(\cdot)$ is the normal $(0, \Sigma)$ density function [the marginal of $y(1)$ under λ], and $h(\cdot)$ is the density obtained by conditioning $g(y(1))$ on $\pi(y(1)) = Y$ and then integrating over v. What we need to show is that $g(y(1)) = h(y(1))$; that is, the marginals of μ and λ on $Y(1)$ coincide.

I will write y for $y(1)$ in this paragraph. Consider any $A \subset \Re^2$. We want to show that $\mu(y \in A) = \lambda(y \in A)$. It suffices to consider separately the two cases $A \subset O$ and $A \subset \mathcal{I}$. Suppose that $A \subset O$. If $v > K$, then $\mu(A \mid v) = 0$. If $v < K$, then conditioning on $\pi(y) = v$ is equivalent to conditioning on $y \in O$ and $y^s = \pi_O^{-1}(v)$. Thus,

$$\mu(y \in A \mid v) = \frac{\int_{\{y^o \mid (\pi_O^{-1}(v),\ y^o) \in A\}} g(\pi_O^{-1}(v), y^o)\ dy^o}{\int_{\{y^o \mid (\pi_O^{-1}(v), y^o) \in O\}} g(\pi_O^{-1}(v), y^o)\ dy^o}.$$

Set $y^s = \pi_O^{-1}(v)$. The denominator of the above is $g(y^s \mid y \in O)/2$. Therefore, we have

$$\mu(y \in A) = \mu(y \in A \mid v < K)/2$$
$$= \int_{-\infty}^{K} \frac{\int_{\{y^o \mid (\pi_O^{-1}(v), y^o) \in A\}} g(\pi_O^{-1}(v), y^o)\, dy^o}{g(y^s \mid y \in O)/2} f(v),$$

where f denotes the density function of v. By (A1),

$$\frac{f(v)}{g(y^s \mid y \in O)/2} = \frac{1}{\pi_O'(y^s)}.$$

Hence,

$$\mu(y \in A) = \int_{-\infty}^{\infty} \int_{\{y_o \mid (y^s, y^o) \in A\}} g(y^s, y^o)\, dy^o\, dy^s \equiv \lambda(y \in A).$$

One can show similarly that $\mu(y \in A) = \lambda(y \in A)$ if $A \subset \mathcal{J}$. We conclude that $\mu = \lambda$.

It remains only to show that there is a strategy for the informed trader such that Y is distributed according to $\mu(\cdot \mid v)$ for each v. We consider only the case $v < K$, the other being handled similarly. Define $\hat{Z}^o = Z^o - \beta Z^s$, where $\beta = \sigma^{so}/\sigma^{ss}$. Then Z^s and $\hat{Z}^o$ are independent Brownian motions, and the instantaneous variance of $\hat{Z}$ is $\sigma^{oo}(1 - \rho^2)$ where ρ is the correlation between Z^S and Z^O. By Rogers and Williams (1987, IV.40), there exists a function $\alpha^s(y^s, t)$ such that

$$Y^s(t) \equiv \int_0^t \alpha^s(Y^s(u), u)\, du + Z^s(t)$$

is distributed as a $(0, \sigma^{ss})$-Brownian motion conditional on

$$Y^s(1) = \pi_O^{-1}(v); \tag{A10}$$

in other words, Y^S is a Brownian bridge. Also, by Rogers and Williams (1987, IV.39)) there exists a function $\hat{\alpha}(y^o, t)$ such that

$$\hat{Y}^o(t) \equiv \int_0^t \hat{\alpha}^o(\hat{Y}^o(u), u)\, du + \hat{Z}^o(t)$$

is distributed as a $(0, \sigma^{oo}(1 - \rho^2))$-Brownian motion conditional on

Note that Y^s and $\hat{Y}^o$ are independent, by the independence of Z^S and Z^O.

Define

$$Y^o(t) = \beta Y^s(t) + \hat{Y}^o(t)$$
$$\equiv \int_0^t [\beta\alpha^s(Y^s(u), u) + \hat{\alpha}^o(\hat{Y}^o(u), u)]\, du + \beta Z^s(t) + \hat{Z}^o(t)$$

$$\equiv \int_0^t \alpha^o(Y^s(u), \hat{Y}^o(u), u)\, du + Z^o(t),$$

the last identity being a definition of α^o. Note that the stochastic process $\alpha^o(Y^s(\cdot), \hat{Y}^o(\cdot), \cdot)$ is adapted to Z, and $\equiv (\alpha^s, \alpha^o)$ is therefore an admissible strategy (in the artificial market). This strategy produces a Y that is distributed as a (0, Σ)– Brownian motion conditioned on (A10) and (A11). Given (A10), (A11) is equivalent to $2Y^s(1) + Y^o(1) < 0$; that is, $Y(1) \in O$. Hence, (A10) and (All) are equivalent to $\pi(Y(1)) = v$. This means that the strategy a has the property stated in the lemma.

We still need to show that a satisfies the no-doubling-strategies condition; in other words,

$$\int P(t, Y(t))^{\mathsf{T}}\, dZ(t)$$

is a martingale. A sufficient condition for this is that

$$E\left[\int_0^1 P^i(t, Y(t))^2\, dt\right] < \infty$$

for $i = s, o$. Here we are taking the expectation over v and Z. Because, unconditional on v, each P^i is a martingale, we have

$$E\left[\int_0^1 P^i(t, Y(t))^2\, dt\right] \leq E[P^i(1, Y(1))^2].$$

By (Al), the random variable P^i (1, Y(1)) has the same distribution as v, so the finiteness of the variance follows from the assumed finite variance of v. ■

***Proof of Lemma* 3.** Let the informed trader follow the strategy given in Lemma 2. Lemma 1 ensures that this is an optimal strategy in the artificial market. Given the definitions (8) and (9), the zero-profit condition (2) follows from the properties of the trading strategy, namely that $\pi^s(Y(1)) = v^s$ a.s. and $Y(t) + Z(1) - Z(t) \sim Y(1)$ given the market makers' information at time t. ■

Proof of Theorem 2. Theorem 2 is a consequence of Lemma 3 and the properties of the matrix A established in Theorem 3. The non-singularity and continuity of A imply, by Harrison and Kreps (1979, Proposition 1), that the filtration generated by P is the same as that generated by Y Therefore, the informed trader can infer Y and, hence, Z from P. This implies that the equilibrium of the artificial market is also an equilibrium of the actual market. ■

Proof of Theorem 3. Equations (8) and (9) imply that the pricing rule is smooth enough to apply Itô's formula. Using the fact that the quadratic variation of Y is the same as that of Z, we obtain, for $i = s, o$

$$dP^i = P^i_s\, dY^s + P^i_o\, dY^o + P^i_t\, dt + \tfrac{1}{2}(\sigma^{ss}P^i_{ss} + 2\sigma^{so}P^i_{so} + \sigma^{oo}P^i_{oo})\, dt,$$

where the subscripts denote partial derivatives (s denoting the derivative with respect to y^s and o the derivative with respect to y^o). Equations (8) and (9) also imply that P^i must satisfy the PDE

$$P^i_t + \tfrac{1}{2}(\sigma^{ss}P^i_{ss} + 2\sigma^{so}P^i_{so} + \sigma^{oo}p^i_{oo}) = 0. \tag{A12}$$

Hence the price dynamics are actually given by

$$dP^i = P^i_s\, dY^s + P^i_o\, dY^o.$$

Thus (11) holds with

$$\Lambda(t, y) = \begin{pmatrix} P^s_s(t, y) & P^s_o(t, y) \\ P^o_s(t, y) & P^o_o(t, y) \end{pmatrix}.$$

We will calculate these derivatives by differentiating (8) and (9).

Let ζ denote a normal $(0, \Sigma)$ random vector. Let f_o, f_I, g_o, g_I denote, respectively, the density function of v conditional on v < K, the density function of v conditional on v > K, the density function of ζ^s conditional on $\zeta \in \mathcal{O}$, and the density function of $\zeta^s + \zeta^o$ conditional on $\zeta \in \mathcal{I}$. Consider a fixed t and y. For the duration of the proof, we take $\psi = (\psi^s, \psi^o)$ to denote a normal random vector with mean zero and covariance matrix $(1 - t)$ Set $y^c = y^s + y^o$ and $\psi^c = \psi^s + \psi^o$. Let h denote the density function of $\psi^c + \psi^s$. Define

$$\gamma_O = E\left[\frac{g_O(y^s + \psi^s)}{f_O(\pi^s_O(y^s + \psi^s))}\chi_O(y + \psi)\right],$$

$$\gamma_I = E\left[\frac{g_I(y^c + \psi^c)}{f_I(\pi^s_I(y^c + \psi^c))}\chi_I(y + \psi)\right],$$

$$\beta = E[\pi^s_I(y^c + \psi^c) - K \mid y^c + y^s + \psi^c + \psi^s = 0]\cdot h(-y^c - y^s),$$

where the expectations are all taken over ψ and x denotes the 0-1 indicator function. We will show that

$$\begin{pmatrix} P^s_s & P^o_s \\ P^s_o & P^o_o \end{pmatrix} = \begin{pmatrix} \gamma_O + \gamma_I + 4\beta & \gamma_I + 2\beta \\ \gamma_I + 2\beta & \gamma_I + \beta \end{pmatrix}. \tag{A13}$$

This matrix is positive definite, because γ_I, γ_O, and β are all positive. Also, the ordering of the elements is the same as claimed in the theorem. The positive definiteness of Λ and Σ implies that $\Lambda\Sigma\Lambda$ is positive definite, so establishing (A13) will complete the proof.

We go through the calculation only for P_s^s. The others are similar. To reduce the notation, we write π for the stock price π^s throughout the remainder of this proof. Set $\theta = y + \psi$. Also, for any ϵ, set $\theta_\epsilon = (y^s + \psi^s + \epsilon, y^o + \psi^o)$, $\theta^c = \theta^s + \theta^o$, and $\theta_\epsilon^c = \theta_\epsilon^s + \theta_\epsilon^o$. From (8), we have

$$P_s^s = \lim_{\epsilon \to 0} \frac{1}{\epsilon} E[\pi(\theta_\epsilon) - \pi(\theta)].$$

Using definition (A1), we have

$$\begin{aligned} \pi(\theta_\epsilon) - \pi(\theta) = {} & [\pi_O(\theta_\epsilon^s) - \pi_O(\theta^s)]\chi_O(\theta_\epsilon)\chi_O(\theta) \\ & + [\pi_I(\theta_\epsilon^c) - \pi_I(\theta^c)]\chi_I(\theta_\epsilon)\chi_I(\theta) \\ & + [\pi_O(\theta_\epsilon^s) - \pi_I(\theta^c)]\chi_O(\theta_\epsilon)\chi_I(\theta) \\ & + [\pi_I(\theta_\epsilon^c) - \pi_O(\theta^s)]\chi_I(\theta_\epsilon)\chi_O(\theta). \end{aligned} \tag{A14}$$

For each realization of ψ, there exists by the mean value theorem some $\theta_{O\epsilon}$ and $\theta_{I\epsilon}$, each lying on the line segment between θ and θ_ϵ, such that

$$(1/\epsilon)[\pi_O(\theta_\epsilon^s) - \pi_O(\theta^s)] = \pi_O'(\theta_{O\epsilon}^s),$$

$$(1/\epsilon)[\pi_I(\theta_\epsilon^c) - \pi_I(\theta^c)] = \pi_I'(\theta_{I\epsilon}^c).$$

By (A1), we have

$$\pi_O'(w) = g_O(w)/f_O(\pi_O(w)), \tag{A15}$$

$$\pi_I'(w) = g_I(w)/f_I(\pi_I(w)). \tag{A16}$$

We will defer the proof of the following.

$$\lim_{\epsilon \to 0} E\left[\frac{g_O(\theta_{O\epsilon}^s)}{f_O(\pi_O(\theta_{O\epsilon}^s))}\chi_O(\theta_\epsilon)\chi_O(\theta)\right] = \gamma_O, \tag{A17}$$

$$\lim_{\epsilon \to 0} E\left[\frac{g_I(\theta_{I\epsilon}^c)}{f_I(\pi_I(\theta_{I\epsilon}^c))}\chi_I(\theta_\epsilon)\chi_I(\theta)\right] = \gamma_I. \tag{A18}$$

Consider now the third and fourth terms in (A14). We can consider separately the cases $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$. First, let $\epsilon \downarrow 0$. When $\epsilon > 0$, the third term in (A14) is zero for every realization of ψ. For each realization of ψ such that the fourth term is nonzero, there is some ϵ' between 0 and ε such that $\theta_{\epsilon'} \in \mathcal{A}$. Using the mean value theorem again, we have

$$\frac{1}{\epsilon}[\pi_I(\theta_\epsilon^c) - \pi_O(\theta^s)] = \frac{\epsilon - \epsilon'}{\epsilon}\left(\frac{1}{\epsilon - \epsilon'}\right)[\pi_I(\theta_\epsilon^c) - \pi_I(\theta_{\epsilon'}^c)]$$

$$+ \frac{1}{\epsilon}[\pi_I(\theta^c_{\epsilon'}) + \pi_O(\theta^s_{\epsilon'})] + \frac{\epsilon'}{\epsilon}\left(\frac{1}{\epsilon'}\right)[\pi_O(\theta^s_{\epsilon'}) - \pi_O(\theta^s)]$$

$$= \frac{\epsilon - \epsilon'}{\epsilon}\pi'_I(\theta^c_{k'}) + \frac{1}{\epsilon}[\pi_I(\theta^c_{\epsilon'}) - \pi_O(\theta^s_{\epsilon'})] + \frac{\epsilon'}{\epsilon}\pi'_O(\theta_{\alpha'}),$$

where $\theta_{k'}$ lies on the line segment between θ_ϵ and $\theta_{\epsilon'}$, and $\theta_{\alpha'}$ lies on the line segment between θ and $\theta_{\epsilon'}$. Using (A15) and (A16) and the reasoning underlying (A17) and (A18), noting that $\chi_I(\theta_\epsilon)\chi_O(\theta)$ converges to zero a.s., we obtain

$$\lim_{\epsilon\downarrow 0} E\left[\frac{\epsilon - \epsilon'}{\epsilon}\pi'_I(\theta^c_{k'})\chi_I(\theta_\epsilon)\chi_O(\theta)\right] = 0,$$

$$\lim_{\epsilon\downarrow 0} E\left[\frac{\epsilon'}{\epsilon}\pi'_O(\theta_{\alpha'})\chi_I(\theta_\epsilon)\chi_O(\theta)\right] = 0.$$

We have $\theta_\epsilon \in \mathcal{I}$ and $\theta \in O$ iff $-2\epsilon < \theta^c + \theta^s < 0$. Hence,

$$E[\{\pi_I(\theta^c_{\epsilon'}) - \pi_O(\theta^s_{\epsilon'})\}\chi_I(\theta_\epsilon)\chi_O(\theta)]$$
$$= E[\pi_I(\theta^c_{\epsilon'}) - \pi_O(\theta^s_{\epsilon'}) \mid -2\epsilon < \theta^c + \theta^s < 0]\cdot \text{prob}[-2\epsilon < \theta^c + \theta^s < 0].$$

Moreoever,

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\text{prob}[-2\epsilon < \theta^c + \theta^s < 0] = 2b(-y^c - y^s),$$

and

$$\pi_I(\theta^c_{\epsilon'}) - \pi_O(\theta^s_{\epsilon'}) = 2[\pi_I(\theta^c_{\epsilon'}) - K].$$

Finally, it is clear that

$$\lim_{\epsilon\downarrow 0} E[\pi_I(\theta^c_{\epsilon'}) - K \mid -2\epsilon < \theta^c + \theta^s < 0] = \beta/[b(-y^c - y^s)].$$

This completes the calculation for $\epsilon \downarrow 0$. The calculation for $\epsilon \uparrow 0$ is the same, noting that the fourth term in (A14) is zero whenever $\epsilon < 0$.

It remains now to verify (A17) and (A18). The two are similar, so we go through the proof only for (A17). We have

$$\lim_{\epsilon\to 0} \frac{g_O(\theta^s_{\alpha_\epsilon})}{f_O(\pi_O(\theta^s_{\alpha_\epsilon}))}\chi_O(\theta_\epsilon)\chi_O(\theta) = \frac{g_O(\theta^s)}{f_O(\pi_O(\theta^s))}\chi_O(\theta)$$

a.s., so it suffices, by Lebesgue's convergence theorem, to show that the random variables

$$\frac{g_O(\theta^s_{\alpha_\epsilon})}{f_O(\pi_O(\theta^s_{\alpha_\epsilon}))}\chi_O(\theta_\epsilon)\chi_O(\theta)$$

are uniformly integrable. This will certainly be true if the random variables $g_O(\theta^s_\alpha)/f_O(\pi_O(\theta^s_\alpha))$ are uniformly integrable. Thus, it suffices to establish the existence of a function $k(\cdot)$ such that

$$E[k(\theta^s)] < \infty \tag{A19}$$

and

$$\frac{g_O(\theta^s_\alpha)}{f_O(\pi_O(\theta^s_\alpha))} \leq k(\theta^s)$$

for every realization of ψ and all sufficiently small ε. This last condition will be true if

$$\sup_{|\epsilon|<1} \frac{g_O(w+\epsilon)}{f_O(\pi_O(w+\epsilon))} \leq k(w) \tag{A20}$$

for each real w.

The expectation in (A19) equals

$$E[k(y^s + \psi^s)] = E[k(Z^s(1)) \mid Z^s(t) = y^s],$$

and the conditional expectation is finite if the unconditional expectation is finite. Hence, it suffices to show that $E[k(Z^s(1))] < \infty$. In other words, we want

$$\int_{-\infty}^{\infty} k(w)g(w)\, dw < \infty, \tag{A21}$$

where g denotes the normal $(0, \sigma^{ss})$ density function. Thus, we can take

$$k(w) = \kappa_1 + \kappa_2 \exp(\kappa_3 w^2/2\sigma^{ss}),$$

for any constants $\kappa_1, \kappa_2, \kappa_3$, provided $\kappa_3 < 1$.

By continuity and the fact that

$$\lim_{w\to\infty} \frac{g_O(w+\epsilon)}{f_O(\pi_O(w+\epsilon))} = \frac{0}{f_K(K)} = 0,$$

we have

$$\sup_{w\geq u}\sup_{|\epsilon|<1} \frac{g_O(w+\epsilon)}{f_O(\pi_O(w+\epsilon))} < \infty,$$

for every real u. Hence we can ensure (A20) holds by choosing κ_1 sufficiently large, if

$$\lim_{w\to-\infty}\left\{\sup_{|\epsilon|<1} \frac{g_O(w+\epsilon)}{f_O(\pi_O(w+\epsilon))} - \kappa_2 \exp\left(\frac{\kappa_3 w^2}{2\sigma^{ss}}\right)\right\} \leq 0. \tag{A22}$$

We have $\pi_O = F_O^{-1} \circ G_O$ and $F_O(w) = 2F(w)$ when $w < 0$. Hence,

$$f_O(\pi_O(w)) = 2f(F^{-1}(G_O(w/2)).$$

It is straightforward to check that $G \leq G_O \leq 2G$, where G denotes the normal $(0, \sigma^{ss})$ distribution function. These facts imply

$$\frac{g_O(w)}{f_O(F_O^{-1}(G_O(w)))} \leq \frac{g(w)}{f(F^{-1}(G_O(w)/2))} \leq \frac{g(w)}{f(F^{-1}(G(w))/2)}.$$

Set $\gamma = \sqrt{(1+\delta)\sigma_v^2/\sigma^{ss}}$ for any $0 < \delta < 1$. Using l'Hôpital's rule, we obtain

$$\lim_{w \to -\infty} \frac{G(w)/2}{F(\gamma w)} = \infty.$$

Hence, for any w sufficiently close to $-\infty$, we have

$$\frac{g(w)}{f(F^{-1}(G(w))/2)} \leq \frac{g(w)}{f(\gamma w)} = \kappa_2 \exp\left(\frac{\delta w^2}{\sigma^{ss}}\right),$$

where $\kappa_2 = \sqrt{\sigma_v^2/\sigma^{ss}}$. By choosing κ_3 between δ and 1, we obtain (A22). ■

Proof of Theorem 4. We argue by contradiction. Suppose that P^s is a Markov process. We have $P^s(1) = v$ a.s., so the unique pricing rule for the option consistent with the equilibrium condition (2) is

$$\begin{aligned} P^o(t) &= E[(P^s(1) - K)^+ \mid (Y(u))_{u \leq t}] \\ &= E[(P^s(1) - K)^+ \mid P^s(t)]. \end{aligned} \tag{A23}$$

This shows that P^o is adapted to the filtration generated by P^s.

Let $\lambda(t, y)$ denote the transpose of the first row of $\Lambda(t, y)$. Then (11) implies

$$dP^s(t) = \lambda(t, Y(t))^\top dY(t),$$

so the instantaneous variance of P^s is $\sigma^2 \equiv \lambda^\top \Sigma \lambda$. If P^s is Markov, this variance must be a function of $(t, P^s(t))$. Define

$$W(t) = \int_0^t \frac{1}{\sigma(u, P^s(u))}\, dP^s(u).$$

This stochastic integral exists because $\sigma(\cdot)$ is continuous and, hence, predictable and locally bounded. The instantaneous variance of W is unity, so Wis a standard Brownian motion [Karatzas and Shreve (1988, Theorem 3.3.16)]. Hence any square-integrable martingale on the filtration generated by W can be represented as a stochastic integral *with* respect to W [Karatzas and Shreve (1988, Theorem 3.4.15)]. The filtration generated by W is the same as that generated by P^S [Harrison and Kreps (1979, Proposition 1)]. Therefore, P^o is adapted to (and a square-integrable martingale on) the filtration generated by W, and we conclude that there is a process β adapted to this filtration such that

$$dP^o(t) = \beta(t)\ dW(t) = \frac{\beta(t)}{\sigma(t,\ P^s(t))}\ dP^s(t). \tag{A24}$$

Equation (A24) implies that P^o and P^s are locally perfectly correlated. But Theorem 3 shows that the instantaneous covariance matrix of P^s and P^o is positive definite, which is a contradiction. ■

Appendix B

The conditional distribution of v^s at any time t, given the market makers' information, can be explained with reference to Figure 2. Consider first the calculation of the density at a point $p^L < K$. The conditional probability that $v^s \leq p^L$ is the probability that $y + Z(1) - Z(t) \in L$, given $y = Y(t)$. Recall that p^L and L are related by the condition that the unconditional probability that $Z(1) \in L$ equals $F(p^L)$. Let $a(p^L)$ denote the y^s intercept of the line $\pi^s = p^L$, so the region L is bounded by the lines $y^c + y^s = 0$ and $y^s = a(p^L)$. Then we have

$$\begin{aligned} F(p^L) &= \text{prob}(2Z^s(1) + Z^o(1) \leq 0,\ Z^s(1) \leq a(p^L)) \\ &= \int_{-\infty}^{0} \int_{\sqrt{3}x_1 - 2a(p^L)}^{\infty} n(x_1) n(x_2)\ dx_2\ dx_1, \end{aligned} \tag{B1}$$

where n denotes the standard normal density. The last equality follows from making the change of variables $\hat{Z}^s(t) = (\sqrt{3}/3)(2Z^s(t) + Z^o(t))$, $\hat{Z}^o(t) = Z^o(t)$ and noting that Z(1) has a normal (0, I) distribution.

Let $F(\cdot \mid t)$ denote the conditional distribution function and $f(\cdot \mid t)$ the conditional density of v^s, given the market makers' information at time t. Let $N(x \mid \mu, t)$ denote the normal $(\mu, 1 - t)$ distribution function evaluated at x, and let $n(x \mid \mu, t)$ denote the corresponding density. Analogous to (B1), we have

$$\begin{aligned} &F(p^L \mid t) \\ &= \int_{-\infty}^{0} \int_{\sqrt{3}x_1 - 2a(p^L)}^{\infty} n(x_1 \mid \hat{Z}^s(t),\ t) n(x_2 \mid \hat{Z}^o(t),\ t)\ dx_2\ dx_1. \end{aligned} \tag{B2}$$

We can calculate $f(\cdot \mid t)$ in terms of $a'(p^L)$ from (B2), and we can calculate fin terms of a $a'(p^L)$ from (B1). Substituting out $a'(p^L)$ gives

$$f(p^L \mid t)$$
$$= f(p^L)\frac{\int_{-\infty}^{0} n(x \mid \hat{Z}^s(t), t)n(\sqrt{3}x - 2a(p^L) \mid \hat{Z}^o(t), t)\, dx}{\int_{-\infty}^{0} n(x)n(\sqrt{3}x - 2a(p^L))\, dx}.$$

To calculate $a(p^L)$ we can use (B1).

Similarly, letting $b(p^H)$ denote the y^c intercept of the level curve $\pi^s = p^H$ for $p^H > K$, we obtain

$$f(p^H \mid t)$$
$$= f(p^H)\frac{\int_{0}^{\infty} n(x \mid \hat{Z}^s(t), t)n(2b(p^H) - \sqrt{3}x \mid \hat{Z}^o(t), t)\, dx}{\int_{0}^{\infty} n(x)n(2b(p^H) - \sqrt{3})\, dx},$$

where $b(p^H)$ can be calculated from

$$1 - F(p^H) = \int_0^\infty \int_{2b(p^H)-\sqrt{3}x_1}^{\infty} n(x_1)n(x_2)\, dx_2\, dx_1.$$

In each of Figures 4-9, the density was elevated at 120 points and the resulting graph smoothed. The expectations that define the stock and option prices and the variance were computed as Riemann sums from these 120 points.

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