The Effect of Public Information and Competition on Trading Volume and Price Volatility

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In a one-period model of market making with many exogenously informed traders, we first show that the variance of prices and expected trading volume depend on the public information released at the start of trading. This is accomplished by representing beliefs with elliptically contoured distributions, for which the form of optimal decision rules does not depend on the specific distribution used. Second, if the model is altered so that the decision to become informed is made endogenous, then the decision rules of the market-maker and informed traders depend on the public information. Third, in a multiperiod model with many informed traders and long-lived private information, recursion formulas similar to those of Kyle (1985) hold for all elliptically contoured distributions, trading volume is autocorrelated and, unless per period liquidity trading is bounded away from zero as new trading periods are added, informed traders' profits vanish.

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Recent experience suggests that the announcement of unanticipated events can cause large price changes and surges in trading activity. For example, the announcement of unexpectedly low earnings by a firm may lead to a burst of trading in the shares of the firm, the share price may drop quickly, and the share price may be more volatile thereafter. Of course, if the low earnings report was anticipated by investors, we would expect to see no dramatic reaction to the announcement. As another example, consider the reaction of oil futures traders after Iraq's invasion of Kuwait and the U.S. decision to respond by sending troops. Throughout this sequence of events, there was a dramatic increase in oil prices, prices were more volatile, and trading was very active. The announcement of Iraq's invasion of Kuwait led to one of the most volatile days in the history of trading in the oil futures pits at the Chicago Mercantile Exchange.[1] Aside from this anecdotal evidence, the empirical works of Gallant, Rossi, and Tauchen (1992) and Hsieh (1991) use daily data to show that stock price variances and volumes are serially dependent. It is this relation between public news and the serial dependence in trading volume and price volatility that we are interested in modeling.[2]

The dependence of price variability and volume on the arrival of news (or rather the arrival of *unexpected* news) cannot be explained with current market microstructure models. Models such as those offered by Kyle (1984, 1985, 1986) and Admati and Pfleiderer (1988) use the normal distribution to characterize beliefs. As a consequence, their models have conditional variance-covariance matrices that change deterministically. Therefore, these models result in expected trading volume and price volatility that are unaffected by the type of public information released. In other words, these models do not take into account the effect of differences between actual, announced information and the news that traders expected on volume and volatility. At the same time, the use of the normal distribution to represent beliefs yields a simple pricing rule where price changes are linearly related to net order flow.

We use a more general representation of beliefs that allows volume and volatility to depend on the public information that is revealed. By using distribution functions from the elliptically contoured class (ECC), we maintain the linear form of the decision rules found in Kyle (1985) and are also able to have price volatility and trading volume depend on public information in a richer way. In this setting, if the public information is not fully revealing, yet is close to what

[1] See the *Wall Street Journal,* August 3, 1990.

[2] While Gallant, Rossi. and Tauchen (1992) and Hsieh (1991) use daily data to document persistence in price volatility and trading volume, the implications of the model presented in this article are for transaction level (or some other short time interval) data.

investors expect, there is little incremental effect on price volatility and trading volume. On the other hand, if public information is substantially different from what investors expect, then an increase in price volatility and trading volume occurs.

Other researchers have used the ECC in financial economics. Chamberlain (1983) proves that the class of distributions consistent with mean-variance analysis is from the ECC, an insight used by Owen and Rabinovitch (1983) and Ingersoll (1987).[3] In addition, we know that the ECC is very large; Cambanis, Huang, and Simons (1981) show that any univariate distribution can be used to generate an elliptically contoured distribution.[4] The term *elliptically contoured* comes from the fact that contour lines of the density functions in this class are ellipses. While the ECC includes the normal distribution, it contains many other distributions of interest to financial researchers: the multivariate *t,* mixture of normals, and multivariate double exponential. We use ECC distributions because all distributions in this class have linear conditional expectations [see Vershik (1964), Blake and Thomas (1968), and Kelker (1970)], which allow us to maintain the linear form of the price adjustment rule in Kyle (1985).[5]

In Section 1, we present a one-period model and prove that most of the results derived by Kyle (1985) hold for the ECC. In particular, if the number of informed traders is fixed, then decision rules of informed traders and the market-maker have the same form as in Kyle (1985). However, with the ECC, we are able to show that trading volume and price variances can depend on the realized public information, a result that does not hold with a normal distribution representation of beliefs. We also give an example where we use the multivariate t and demonstrate that a greater difference between the realized public information and its expected value leads to higher volume and more informative prices.”

In Section 2, we enrich the model so that some traders decide whether to pay a fixed cost to acquire private information after observing the public information. Traders acquire information simultaneously, and the number of informed traders is announced before

[3] See Harvey (1991) for an application of these distributions to asset pricing. These distributions also belong to the set of separating distributions studied by Ross (1978).

[4] We restrict our analysis to distributions that have second moments; this excludes, for example, stable distributions with characteristic exponents less than 2.

[5] Blake and Thomas (1968) characterize distributions in the elliptically contoured class in terms of the following linear conditional expectation property. Given a random vector $x = (x_1, x_2)'$, let $y = \alpha' x$ and $z = \gamma' x$. Then $E[y|z] = \lambda z$ for all α and *g* if and only if *x* is a member of the elliptically contoured class. It is in this sense that the elliptically contoured class contains all distributions that have linear conditional expectations. See Hardin (1982) for a further discussion of this property.

[6] There are distributions in the ECC, like the uniform distribution on the unit circle or sphere, that give lower volume and less informative prices when the difference between realized public information and its mean is large.

trading commences. The decision rules of the market-maker and informed traders depend on the number of traders who acquire information. With this model of endogenous information acquisition, conditional price volatility is a function of public information. Consequently, the acquisition of information, and hence the equilibrium number of informed traders, depends on the public information; trading volume, price variances, and the equilibrium decision rules of the market-maker and informed traders are all functions of the public information.

In Section 3, we consider multiperiod trading where a fixed number of informed traders have long-lived private information and where information from past prices is publicly available. This extends Kyle's discrete multiperiod model to many traders and a larger class of distributions. As before, the informed traders' decision rules do not change when we use distributions from the ECC; however, price variances are functions of past orders, while volumes are serially correlated. If a fixed amount of liquidity trading (measured as a variance) is split equally over all periods (so that the liquidity trading per period declines as new periods are added), then informed traders' profits vanish as new trading periods are added; this result contrasts with that obtained in Kyle (1985) when there is a monopolistic informed trader.[7] If liquidity trading is the same each trading period (regardless of how many new periods are added, so that the total liquidity trading increases as new periods are added), then informed traders' profits are bounded when new trading periods are added (the profits of a monopolist informed trader become infinite *as* new trading periods are added).

In Section 4, we provide a summary. All proofs and a description of the ECC are in the appendices.

1. The Basic Model

In this section, we introduce the market participants and the assumptions used to solve the model. We prove that there is a unique linear Nash equilibrium (Proposition l), that the conditional variance of the value of the security given the public information is heteroskedastic (Proposition 2), that trading volume depends on public information about the asset's value (Proposition 3), and that, for some elliptically contoured distributions, if actual public information is very different from expected information (whether it is unexpectedly good or bad news does not matter), trading volume is higher and prices are more volatile (Proposition 4).

[7]Similar results are found independently in Holden and Subrahmanyam (1992) and Michener and Tighe (1991).

We consider a one-period model with I risk-neutral informed traders, a market-maker, and a number of liquidity traders. All participants trade an asset with a terminal value, v, that is known only by informed traders. Before trading, public information, s, that is correlated with the terminal value of the asset is observed by all traders. We often refer to public information as the public signal. A typical informed trader, say the ith informed trader, submits an order, $x_i(I, s, v)$, based on the number of informed traders, public information, and knowing the terminal value of the asset. Liquidity traders submit orders summing to l that are uncorrelated with the terminal value of the asset and the public signal. The market-maker is risk-neutral and sets prices to make zero expected profits, given the public information and the net order flow, $y = \Sigma_{i=1}^{I} x_i(I, s, v) + l$. This model has a structure that is similar to those in Kyle (1984, 1985, 1986) and Admati and Pfleiderer (1988).

The timing of events in the model is as follows. First, public information is observed. Then informed and liquidity traders submit orders to the market-maker, who determines the price knowing the public information and the net order flow. The market-maker only knows the net order flow, the number of informed traders, and the public signal; individual trader order sizes or the identity of the informed is not known. After the market-maker sets the price, trade occurs and the terminal value is revealed.

We assume that the terminal value of the asset, the public information, and the liquidity trades are jointly distributed according to some multivariate elliptically contoured distribution, where unconditional expected values of the public information, the full information value of the asset, and the amount of liquidity trading are s_0, p_0, and 0, respectively, and are known to all traders:

$$\begin{pmatrix} s \\ v \\ l \end{pmatrix} \sim \mathrm{ECC}\left(\begin{pmatrix} s_0 \\ p_0 \\ 0 \end{pmatrix}, \Sigma, \psi(\cdot) \right); \tag{1}$$

here $\psi(\cdot)$ is a function that determines the shape of the density. In Appendix A, we describe the characteristic function of a density in the ECC and show that the mean, variance, and function $\psi(\cdot)$ completely characterize the distribution. The function $\psi(\cdot)$ can be any continuous function that satisfies the usual restrictions of a characteristic function.[8] The unconditional variance-covariance matrix, Σ, is

[8]Elliptically contoured distributions in this setup have the property that liquidity trading is semi-independent of $s - s_0$ and $v - p_0$: they are independent only for the normal distribution (see *P7* in Appendix A). While $E[l \mid s, v] = 0$, the conditional variance of the liquidity trades depends on the realized public information. (Two random variables a and b are semi-independent if $E[a \mid b] =$

$$\Sigma = \begin{pmatrix} \sigma_s^2 & \sigma_{sv} & 0 \\ \sigma_{sv} & \sigma_v^2 & 0 \\ 0 & 0 & \sigma_l^2 \end{pmatrix}. \tag{2}$$

The conditional distribution of $v - p_0$ and l, given the realized public information, is an elliptically contoured distribution of a form similar, but not identical, to the original multivariate joint distribution (see *P7* in Appendix A):

$$\begin{pmatrix} v - p_0 \mid s \\ l \mid s \end{pmatrix} \sim \mathrm{ECC}\left(\begin{pmatrix} (\sigma_{sv}/\sigma_s^2)(s - s_0) \\ 0 \end{pmatrix},\right.$$
$$\left. g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\begin{pmatrix} \sigma_v^2 - (\sigma_{sv}^2/\sigma_s^2) & 0 \\ 0 & \sigma_l^2 \end{pmatrix}, \ \psi(\cdot)\right), \tag{3}$$

where $g(\cdot)$ is a function that maps from $\Re$ to $\Re^+$, whose form depends on $\psi(\cdot)$.[9] The conditional variance-covariance matrix depends on the public information through the function $g(\cdot)$ which has as its argument $(s - s_0)^2/\sigma_s^2$. In the case of the normal distribution, the conditional covariance matrix is deterministic and $g(\cdot)$ is equal to unity for all values of the public information, s. In what follows, we use the notation $\Lambda = \sigma_v^2 - (\sigma_{sv}^2/\sigma_s^2)$ and represent conditional variances as

$$\begin{pmatrix} \Lambda \mid s \\ \sigma_l^2 \mid s \end{pmatrix} = g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\begin{pmatrix} \Lambda \\ \sigma_l^2 \end{pmatrix}. \tag{4}$$

As an example, consider the multivariate t distribution with k degrees of freedom and $m = 3$ variables ($k > 2$ is needed for the variances to be well defined). In this case, the unconditional density is

$$f(s, v, l) = \frac{\Gamma\left[\frac{1}{2}(k + m)\right]}{\Gamma\left[\frac{1}{2}k\right]((k - 2)\pi)^{m/2}}(\det \Sigma)^{-1/2}$$
$$\times \left[1 + \frac{1}{k - 2}(s - s_0 \quad v - p_0 \quad l)\Sigma^{-1}\begin{pmatrix} s - s_0 \\ v - p_0 \\ l \end{pmatrix}\right]^{-(k+m)/2}. \tag{5}$$

The conditional distribution, given the public information, is a multivariate t with k degrees of freedom and $m = 2$ variables. The conditional variance of the difference between the terminal value and the unconditional expected price is

0 and $E[b \mid a] = 0$. (Semi-independence is a weaker condition than independence and a stronger condition than zero correlation.)

[9] If we condition on k variables, then $g(\cdot)$ is a function that maps from $\Re^k$ to $\Re^+$.

$$\operatorname{Var}[v - p_0 \mid s] = g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\Lambda$$

$$= \frac{k-2}{k}\frac{1}{k-1}\left[k + \frac{(s - s_0)^2}{\sigma_s^2}\frac{k}{k-2}\right]\Lambda. \quad (6)$$

Notice that, unlike the normal distribution, if public information has a very large deviation from its mean, there is a large conditional price variance.

Having described the uncertainty in the model, we now turn to the strategies of the participants. Let $p(y, s)$ be the price that the market-maker sets knowing the net order flow and the realized public information. The market-maker sets $p(y, s)$ so that

$$p(y, s) = E[v \mid y, s], \quad (7)$$

while each informed trader chooses $x_i(I, s, v)$ to maximize

$$E[(v - p(y, s))x_i(I, s, v) \mid v, s]. \quad (8)$$

Proposition **1** characterizes the unique linear equilibrium of this game; note that the form of the equilibrium decision rules does not depend on the elliptically contoured distribution used.

Proposition 1. *The unique linear Nash equilibrium is symmetric and is of the form*

$$x_i(I, s, v) = \beta(v - E[v \mid s]),$$
$$p(y, s) = E[v \mid s] + \lambda y,$$

where

$$\lambda = \frac{\sqrt{I}}{I+1}\sqrt{\frac{\Lambda}{\sigma_l^2}},$$

$$\beta = \frac{1}{\sqrt{I}}\sqrt{\frac{\sigma_l^2}{\Lambda}}.$$

l and b are independent of the public information, and the equilibrium decision rules depend only on the unconditional variance-covariance matrix, not on the specific elliptically contoured distribution used.

With the ECC representation of beliefs, the form of the equilibrium decision rules is independent of the elliptically contoured distribution chosen. Also, the equilibrium values of λ and β are independent of the public signal. This property of λ and β follows from the fact that the conditional variance function for elliptically contoured distributions has a very specific form. As demonstrated in Equations (3) and (4) public information affects the conditional variance-covariance matrix via a proportionality factor. In equilibrium, λ and β are functions of the ratio of the conditional variance of information that the informed traders have, $\Lambda | s$, to the conditional variance of the liquidity trading, $\sigma_l^2 | s$. Because this ratio is independent of the public information, λ and β are independent of s. Further, because conditional expectations are independent of $\psi(\cdot)$ [as demonstrated in Equations (1) and (3)] we know that the form of λ and β is independent of the particular elliptically contoured distribution used, for a fixed unconditional variance-covariance matrix.

While the functional form of λ and β is independent of public information and the specific elliptically contoured distribution used, the same is not true for the equilibrium properties of trading volume and the informativeness of prices. For a fixed unconditional variance-covariance matrix, the linearity of conditional expectations implies that the conditional first moment is the same for all distributions in the elliptically contoured class. However, for higher-order moments, the conditional values are different for each distribution in the elliptically contoured class. In the remainder of this section, we explore the implications of this feature of distributions in the elliptically contoured class when we examine the dependence of price volatility and trading volume (which depend on higher-order moments) on the public signal and the order flow (our conditioning variables).

Proposition 2. *For elliptically contoured distributions other than the normal,* $Var(v - p(y, s) | s, y)$ *is conditionally heteroskedastic.*[10]

As an illustration of Proposition 2, consider using the multivariate t to represent beliefs. The conditional variance of the terminal value less the expected price, given s and y, is

$$\begin{aligned}\mathrm{Var}[v - p(y, s) \mid s, y]\\ &= g\left(\frac{y^2}{I^2\beta^2\Lambda + \sigma_l^2} + \frac{(s - s_0)^2}{\sigma_s^2}\right)\frac{\sigma_l^2}{I^2\beta^2\Lambda + \sigma_l^2}\Lambda\\ &= \frac{k-2}{k}\frac{1}{k}\left[k + \frac{(s - s_0)^2}{\sigma_s^2}\frac{k}{k-2} + \frac{y^2}{I^2\beta^2\Lambda + \sigma_l^2}\frac{k}{k-2}\right]\frac{\sigma_l^2}{I^2\beta^2\Lambda + \sigma_l^2}\Lambda. \end{aligned} \tag{9}$$

From expression (9) it is easy to see that a higher value of $|s - s_0|$

[10] While conditional heteroskedasticity is a time-series property. we have shown the dependence of the conditional variance function on past information in our static setup. If we repeat our model to obtain an infinite horizon model, this dependence gives conditional heteroskedasticity. In the rest of the article. we use the phrase conditional heteroskedasticity to mean dependence of the conditional variance on past information.

or the absolute net order flow $|y|$ leads to a higher conditional variance.

Next we examine the implications for trading volume of using distributions from the ECC. The expectedvolume [as defined by Admati and Pfleiderer (1988)], given the public information, and the unconditional expected volume are derived as follows. If $b_1, b_2, \ldots, b_M$ are the trades from M groups of traders, then the expected total volume is[11]

$$V = E[\max\{B^+, B^-\}], \tag{10}$$

where $B^+ = \Sigma_{j=1}^{M} \max\{b_j, 0\}$ and $B^- = \Sigma_{j=1}^{M} \max\{-b_j, 0\}$. This simplifies to

$$V = \frac{1}{2} \sum_{j=1}^{M} E[|b_j|] + \frac{1}{2} E\left[\left|\sum_{j=1}^{M} b_j\right|\right]. \tag{11}$$

The first term of expression (11) consists of the orders from the informed and liquidity traders, while the last term represents the securities traded with the market-maker.

To complete our characterization of trading volume, we need to compute the expected value of the absolute value of an elliptically contoured random variable (which we state as a lemma). First, however, we refer to a result from Chu (1973) that allows us to write an elliptically contoured density function as the product of a normal density function and a weighting function (see *P9* in Appendix A).

Chu's (1973) lemma characterizes an elliptically contoured distribution as "nearly" a randomization over a normal distribution. Thus, draws from an elliptically contoured distribution are "nearly" draws from a normal distribution with a randomly chosen variance. We use the word "nearly" because the "distribution" from which we pick the variance (the weighting measure) can take negative values; hence, it is not necessarily a probability measure. When the weighting measure is a probability measure, the elliptically contoured distribution is a randomization over the normal distribution; these distributions comprise the compound normal class. For example, a draw from the *t* distribution can be viewed as a draw from the normal distribution where the variance is chosen from an inverted gamma distribution (here the weighting measure is an inverted gamma distribution).

Chu (1973) gives another representation for densities in the compound normal class. A draw from a compound normal distribution can also be viewed as the product of a draw from a distribution with a nonnegative support and a draw from an independent standard normal distribution. As an example, the t distribution can be gener-

[11] Foster and Viswanathan (1990) use a similar definition of trading volume.

ated from independent x^2 and normal distributions. Here, the positive random variable that multiplies the unit normal is $\sqrt{Q}/\sqrt{k}$, where Q is drawn from a x^2 distribution with k degrees of freedom.

We use Chu's (1973) lemma to compute the expected value of the absolute value of a random variable in the ECC. Let z represent the "random variable" that determines the variance of the independent normal random variable in Chu's (1973) first characterization (for all distributions in the ECC) and let $W(\cdot)$ be the weighting measure for z. Also, let $\sqrt{Z}$ represent the random variable that multiplies the unit normal in Chu's (1973) second characterization (for the compound normal class). We state the expected value of the absolute value of an elliptically contoured random variable in the following lemma.

Lemma. *If e is from an elliptically contoured distribution with mean zero, $E[|e|]$ is a monotonic function of its variance, σ_e^2, and*

$$E[|e|] = \frac{2}{\sqrt{2\pi}} \frac{\sigma_e}{\left[\int_0^\infty z^2 \, dW(z)\right]^{1/2}} \int_0^\infty z \, dW(z).$$

If e is from a compound normal distribution, then $dW(\cdot)$ is a probability density function and using Chu's (1973) second characterization we obtain

$$E[|e|] = \frac{2}{\sqrt{2\pi}} \frac{\sigma_e}{[E[Z]]^{1/2}} E[\sqrt{Z}].$$

For general elliptically contoured distributions, the weighting measure $dW(\cdot)$ is not a probability density and our characterization is in terms of integrals over the weighting measure. For the compound normal class, the weighting measure is a probability measure and the integrals are expectations.

Proposition 3. *For nonnormal elliptically contoured distributions, informed, liquidity, market-maker, and total expected trading volumes depend on the public information. For compound normal distributions, the portion of total expected volume from each source is constant.*

To understand how the expected trading volume depends on the public signal, consider our example with the multivariate t distribution. If we use Chu's (1973) second characterization of the multivariate t, we can write it as the product of a multivariate standard normal random variable and the square root of a χ_k^2 distributed random variable divided by its degrees of freedom. The public signal gives information about the value of the χ_k^2 random variable, and it is the revision in beliefs about the χ_k^2 random variable that affects volume. With Chu's (1973) first characterization of an elliptically contoured random variable, the public information causes a revision in prior beliefs about the inverted gamma distribution from which the variance of the normal is drawn, With either representation, the public signal affects volume and is the source of conditional heteroskedasticity.

Next we consider elliptically contoured distributions where the conditional variance and volume, given the public information, are monotonic in the absolute difference between the realized public information and its mean. Examples of these distributions include the multivariate *t*, the discrete mixture of normals, the double exponential, and all other distributions in the compound normal class. For all the distributions in the compound normal class, Chu's (1973) lemma gives an interesting interpretation of the $g(\cdot)$ function introduced in Equations (3) and (4). This function is the conditional variance of the random variable *z*, given the public information. The monotonicity property follows from the fact that larger values of $|s - s_0|$ lead to an upward revision of our prior beliefs about this variance.

Proposition 4. *Suppose $g(\cdot)$ is monotonic in $|s - s_o|$, with strict monotonicity on a positive measure [this is true for the whole compound normal class where $g(\cdot)$ is strictly monotone increasing in $|s - s_o|$]. Then the conditional variance given the public information, $Var(v|s)$, and the volume given the public information, V(s), are monotonic in $|s - s_o|$.*

Under the conditions of Proposition 4, for any distribution in the compound normal class, a large unanticipated realization of the public information leads to higher volatility and higher volume. Unanticipated earnings reports, merger announcements, and major political events should lead to higher trading volume and volatile share prices-these predictions cannot be generated by models with normally distributed beliefs. However, these effects are easily captured by models that represent beliefs with distributions from the ECC.

Throughout this section, we have specified exogenously the number of informed traders. A more realistic model may allow some traders to decide whether to purchase information. Of course, the decision to acquire information depends on others' actions as well as the kind of public information that is revealed. Public information that is much different than expected may induce more traders to choose to become privately informed. This decision is the subject of our next section.

2. Endogenous Information Acquisition

We extend the model of Section 1 to allow endogenous information acquisition by some traders. We consider a model where traders decide whether to become informed by paying a cost, c, after they observe the public information. For example, if public news reduces volatility, fewer traders may acquire information. If the news increases volatility, more traders may acquire information.[12]

In our model of information acquisition, traders simultaneously decide whether to become informed. In our setup, the public information is revealed, then traders simultaneously decide whether to acquire information, the number of informed traders is announced, trades are submitted, and the market-maker sets the price for the orders received.[13] Because the number of informed traders is announced prior to trading, the market-maker and other traders can adjust their decision rules according to the number of traders who become informed. This structure is similar to Admati and Pfleiderer's (1988) second model of information acquisition and yields a unique linear equilibrium.[14]

Traders who can acquire information do so only if their expected profit when they are informed (net of the costs of acquiring information) exceeds their expected profit if they remain uninformed. The expected profit that accrues to an informed trader when there are I informed traders in total is

$$\Pi(I, s) = \frac{1}{(I + 1)^2\lambda(s)} \Lambda \mid s = g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\frac{(\sigma_I^2\Lambda)^{1/2}}{\sqrt{I}(I + 1)} . \tag{12}$$

From expression (12) we can see that the equilibrium number of informed traders is unique and is given by I, where

$$\Pi(I + 1, s) \leq c < \Pi(I, s). \tag{13}$$

In what follows, we characterize the equilibrium that results from

[12] The fact that informed traders acquire information after they observe the public signal is important. If information acquisition occurs before the public signal is observed, given the unconditional variance-covariance matrix. all elliptically contoured distributions lead to the same number of informed traders. The reader may also view our model as a sequence of repeated games where private information is short-lived and public information also contains past trades and public signals. Such a model would be an extension of Admati and Pfleiderer (1988); we do not work out this case because of the complexity of the notation.

[13] Another approach that gives the same equilibrium is to allow for sequential information acquisition by the traders. In this case, traders decide one at a time whether to acquire information, and everyone knows the current number of traders who have already become informed. Traders continue to acquire information until the last trader does not find it profitable to pay for information. At this point, the market-maker accepts the orders from all traders and announces the terms of trade.

[14] For other earlier work on endogenous information acquisition and the volume-volatility relationship, see Pfleidercr (1984).

this model of endogenous information acquisition (Proposition 5).[15] We are able to make more specific claims about these equilibrium properties when beliefs are represented with distributions from the compound normal class (Proposition 6).

Proposition 5. *For all elliptically contoured distributions except the normal, there is a unique number of informed traders, I(s), that is a nontrivial function of the public information. Consequently, the market-maker's sensitivity to trades,* $\lambda(s)$*; the informed trader's intensity of trading,* $\beta(s)$*; the conditional variance of the terminal value given the public information,* $Var(v|s)$*; and the volume given the public information, V(s), are (discontinuous) functions of the public information.*

From Proposition 5, we find that the result of Section 1 which shows that the parameters $\lambda(s)$ and $\beta(s)$ are independent of the elliptically contoured distribution no longer holds. Information acquisition depends on the public signal in a way that varies with the distribution used to represent beliefs. With endogenous information acquisition, changes in $\lambda(s)$ and $\beta(s)$ only occur when more traders become informed.

Proposition 6. *Suppose* $g(\cdot)$ *is monotone increasing in* $|s - s_0|$, *with strict monotonicity on a set of positive measure (the compound normal class satisfies this restriction). With endogenous information acquisition, if* $g(0)(\sqrt{\sigma_i^2 \Lambda}/2)$, *there exist* $\Delta_1, \Delta_2, \ldots,$ *and* $I_1 < I_2 < \ldots$, *such that*

$$
\begin{array}{rcccl}
\text{if } 0 < & |s - s_0| & < \Delta_1, & \text{then } I & = I_1, \\
\text{if } \Delta_1 < & |s - s_0| & < \Delta_2, & \text{then } I & = I_2, \\
\vdots < & \vdots & < \vdots, & & \vdots;
\end{array}
$$

hence, $I(s)$ *is a monotonic decreasing function of* $|s - s_0|$ *discontinuous at* Δ_i, $i = 1,2,\ldots$.

The conditional variance of the terminal value given the public information, $Var(v|s)$*, and the volume given the public information, V(s), are discontinuous increasing functions of* $|s - s_0|$*, where the discontinuities occur at* Δ_i, $i = 1,2,\ldots$.

[15] Because the number of informed traders is discrete. there are discontinuities in the model's parameters. In an earlier version of this article, we considered a model of simultaneous Information acquisition, where traders chose a probability of acquiring information. and then according to that probability information was revealed to some traders. In this case, the model's parameters were continuous; otherwise, this model had similar implications to those given here.

As previously discussed, the requirement in Proposition 6 that $g(\cdot)$ be monotonic is satisfied by a large number of elliptically contoured distributions; in particular, it contains all distributions in the compound normal class. For these distributions, the discontinuities in the decision rules mean that there can be large changes in price volatility and trading volume for relatively small changes in the public signal. In other words, small pieces of news can cause large changes in market behavior when more traders decide to become informed.

Finally, we are interested in the effects of long-lived information in our model and the effects of competition among the informed traders in such a setting. To introduce these changes, we use a model like that of Section 1, where the number of informed traders is given, but where there are a number of consecutive trading sessions. This analysis is reported in Section 3.

3. Multiperiod Trading with Long-Lived Information

In this section, we are interested in how the model's participants act in a multiperiod setting where the information is about the value of a security a number of periods in the future (we refer to this as long-lived information). To focus on this effect, we analyze a model where there are many trading periods. Of particular interest to us is how the informed traders choose to release their information through trading, and what effect competition among informed traders has on their profits.

Before the first round of trading, a fixed (and known) number of informed traders see the terminal value of the asset, v. In each subsequent trading period, liquidity traders and informed traders place their orders, the market-maker observes the sum of these orders and adjusts the market price, and trade occurs. Thus, past prices and volumes become part of the current publicly available information.

We are principally concerned with how the informed traders' and market-maker's strategies change in this dynamic environment. Related to this, we want to learn about the time-series properties of trading volume, and whether there are time dependencies in price volatility. In order to compute the effects of adding more time periods, we need to know how liquidity trades are partitioned as new trading intervals are added. We analyze two distinct environments. In the first, existing liquidity is split into equal and declining amounts as new periods are added. Our second representation has an increase in liquidity as new periods are added, so that the liquidity per period is constant.

Consider a T-period model with a fixed number of informed traders, where beliefs are given by the elliptically contoured distribution

$$\begin{pmatrix} v \\ \mathbf{l} \end{pmatrix} \sim \mathrm{ECC}\left(\begin{pmatrix} p_0 \\ 0 \end{pmatrix},\ \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \sigma_l^2 \mathbf{I}_T \end{pmatrix},\ \psi(\cdot)\right). \tag{14}$$

We assume that there is no public information other than past prices and net trades until time *T*, at which time the value of the asset is announced. Thus, the model is identical to the discrete multiperiod model of Kyle (1985), except that we allow for many informed traders and a more general representation of the information structure of the model.

Next we consider how competition among the informed traders and the availability of liquidity affect the profits and trading intensities of informed traders. We are interested in how the limiting behavior of the model changes with competition among informed traders (the limit in this section refers to adding more trading periods with the same long-lived information). We show that the profits of a monopolist informed trader depend on the total liquidity variance in all periods. In contrast, for competitive informed traders, profits depend on the liquidity variance in each period. More specifically, we find the unique linear solution to the model and show that the decision rules are invariant to the elliptically contoured distribution used (Proposition 7); that price variances are functions of past orders and prices; and that trading volume is autocorrelated (Proposition 8). We provide a recursion that can be used to solve the model explicitly (Proposition 9), to examine limits when a fixed amount of liquidity trading is spread over a larger number of time periods (Proposition 10), and to discuss limits when there is a constant amount of liquidity trading per period as the number of trading periods is increased (Proposition 11).

Proposition 1 states that public information does not affect the equilibrium decision rules when the number of informed traders is fixed. Similarly, in this multiperiod, discrete-time model, information in previous order flows becomes a public signal, and decision rules are invariant to the elliptically contoured distribution used. This is stated as Proposition 7.

Proposition 7. *The unique recursive linear equilibrium is symmetric and has decision rules that are invariant to the elliptically contoured distribution chosen. The parameters* β_t, λ_t, α_t, δ_t ***and*** Λ_t *are constant and satisfy*

$$x_t = \beta_t(v - p_{t-1}),$$

$$p_t = p_{t-1} + \lambda_t(Ix_t + l_t),$$

$$\Lambda_t = E[\mathrm{Var}(v \mid Ix_1 + l_1, \ldots, Ix_t + l_t)],$$

$$E[\Pi_t \mid p_1, \ldots, p_{t-1}, v] = \alpha_{t-1}(v - p_{t-1})^2 + \delta_{t-1}(l_1, \ldots, l_{t-1}, v),$$

$$\delta_{t-1} = E[\delta_{t-1}(l_1, \ldots, l_{t-1}, v)]; \qquad \text{for } t = 1,\ldots,T.$$

Given Λ_0, the parameters β_t, λ_t, α_t, δ_t, and Λ_t are solutions to the difference equation system

$$\beta_t = \frac{1 - 2\alpha_t\lambda_t}{\lambda_t[1 + I(1 - 2\alpha_t\lambda_t)]},$$

$$\alpha_{t-1} = \frac{1 - \alpha_t\lambda_t}{\lambda_t[1 + I(1 - 2\alpha_t\lambda_t)]^2},$$

$$\delta_{t-1} = \alpha_t\lambda_t^2\sigma_l^2 + \delta_t,$$

$$\lambda_t = (I\beta_t/\sigma_l^2)\Lambda_t,$$

$$\Lambda_t = (1 - I\lambda_t\beta_t)\Lambda_{t-1},$$

subject to $\alpha_T = \delta_T = 0$ and the second-order condition

$$\lambda_t(1 - \alpha_t\lambda_t) > 0.$$

The conditional variance of the terminal value, v, given the order flow, $\Lambda_t(y_1, \ldots, y_t)$, and the time-varying "constant" in the value function of informed agents, $\delta_t(l_1, \ldots, l_t, v)$, evolve according to

$$\Lambda_t(y_1, \ldots, y_t) = \Lambda_t b_t\left(\sum_{k=1}^{t} \frac{y_k^2}{I^2\beta_k^2\Lambda_{k-1} + \sigma_k^2}\right),$$

$$\delta_{t-1}(l_1, \ldots, l_{t-1}, v) = \delta_{t-1} g_{t-1}\left(\frac{1}{\sigma_l^2}\sum_{t=1}^{t-1} l_t^2 + \frac{(v - p_0)^2}{\Lambda_0}\right),$$

where $b_t(\cdot)$ and $g_{t-1}(\cdot)$ depend on the particular elliptically contoured distribution used.

In this multiperiod model, information from past prices and orders does not affect the market-maker's sensitivity to trades and the informed traders' trading intensities. As in the single-period model, the information in past prices and order flows affects the conditional variance of the unknown information and the conditional expectation of the volume. Because of the influence of past events on the conditional variances, trading volume is correlated across periods.[16]

Proposition 8. *For all elliptically contoured distributions other than the normal, $Var(v - p_t \mid p_0, \ldots, p_{t-1})$ is a function of the price and*

[16] This multiperiod model, like Kyle's (1985) discrete model, is nonstationary. Consequently, we refrain from using the term conditional heteroskedasticity. One can obtain similar autocorrelation results from a stationary model obtained by repeating a one-period model infinitely often.

order flow history. For all elliptically contoured distributions (excluding the normal), trading volume is a function of the price and order flow history and is autocorrelated; when beliefs are represented with compound normal distributions, trading volume is positively autocorrelated.

Corollary. ***For the compound normal class,*** $Cov(V_t, V_{t-1}) > Cov(V_t, V_{t-2}) > 0$. ***Also,*** $Cov(V_t, V_{t-1}) > Cov(V_{t-1}, V_{t-2}) > 0$.[17]

From Proposition 8, we see that the autocorrelation in volume is intimately related to the random variable that drives the variance-covariance matrix of the normal distribution in Chu's (1973) characterization. For example, for the multivariate *t* distribution, the random variable that determines the variance-covariance matrix is the inverted gamma distribution. The covariance between trading volume in any two periods is related to the conditional variance given the past order flow of this inverted gamma random variable (which is the source of second moment variation over time in the multivariate *t* distribution).

The recursion listed in Proposition 7 cannot be used easily to compute the equilibrium as it has two constraints, one at $t = 0$ (on the initial variance Λ_0) and the other at $t = T$ (that $\alpha_T = 0$). Proposition 9 provides recursive formulas in a transformed variable that has only one endpoint constraint at $t = T$, and can be used to solve for the equilibrium.[18] We use Proposition 9 to understand how the equilibrium decision rules change as we vary the liquidity trading in each period and as we change the number of periods.

Proposition 9. ***Let*** $\theta_t = I\alpha_t^2\Lambda_t/\sigma_t^2$, $\zeta_t = \alpha_t\lambda_t$. ***Solve for*** θ_t ***and*** ζ_t ***as follows. First find*** ζ_t ***given*** θ_t ***using***

$$\theta_t = \zeta_t^2 \frac{1 + I(1 - 2\zeta_t)}{1 - 2\zeta_t}$$

(where $0 < \zeta_t < \frac{1}{2}$***), and then solve for*** θ_{t-1} ***given*** ζ_t ***using***

$$\theta_{t-1} = \frac{(1 - \zeta_t)^2}{(1 - 2\zeta_t)[1 + I(1 - 2\zeta_t)]^2},$$

with the endpoint constraint $\theta_T = 0$.

Then the recursions

[17] Because the model is nonstationary. the first-order autocovariance function is declining with the number of lags used [i.e., $Cov(V_t, V_{t-1}) > Cov(V_{t-1}, V_{t-2}) > 0$].

[18] Kyle (1985) also provides a recursive solution method that is specific to the monopolist informed trader case.

$$\alpha_t = \alpha_{t-1} \frac{\zeta_t[1 + I(1 - 2\zeta_t)]^2}{1 - \zeta_t},$$

$$\Lambda_t = \frac{\Lambda_{t-1}}{1 + I(1 - 2\zeta_t)},$$

with the endpoint constraints Λ_0 and α_T (= 0) yield the solution.[19]

We use Proposition 9 to determine how informed traders' profits and other model parameters change as the number of time periods is increased, while keeping the total liquidity trading variance fixed (for T periods, the liquidity variance in each period is σ_l^2/T). Kyle (1985), in a model with a single informed trader, finds that spreading liquidity trading over many periods causes the informed trader's profits to increase, and that in the continuous time limit these profits are the highest. However, when there are at least two informed traders, we find that as the number of periods increases, the expected profits of informed traders (calculated at time 0) vanish.[20]

Proposition 10. *Suppose the liquidity trading variance in each of T periods is σ_l^2/T, that is, the total liquidity variance is held constant. If the number of informed traders is greater than or equal to 2, as $T \to \infty$, for the mth trading period, the equilibrium parameters converge as follows:*

$$\begin{aligned} E[\Pi_m^T] &\to 0, & E[\Pi_m^T \mid v] &\to 0, \\ \lambda_m^T &\to \infty, & \beta_m^T &\to 0, \\ T\beta_m^T &\to \infty, & T(\Lambda_m^T - \Lambda_{m-1}^T) &\to -\infty. \end{aligned}$$

When the number of trading periods is very large, informed traders' total expected profits go to zero, the market-maker's sensitivity to trades in the initial periods is infinite, and the size of the informed traders' orders converges to zero. With competition, informed traders reduce their order size at a rate greater than $1/T$ (this is reflected in the result that $T\beta_m^T$ limits to infinity). Thus, the reduction in the size of informed orders occurs at a slower rate than the reduction in liquidity trading variance in each period. As the per-period liquidity trading variance falls with the increase in the number of trading periods, informed traders trade too intensely and drive their profits to zero. This higher intensity of trade implies that the market-maker

[19] For the last period, we have $\alpha_T = 0$ and $\zeta_T = 0$; hence, the recursion on α_t makes no sense. If we solve for the time series of Λ_t, and use Equations (B32) and (B25) to solve for λ_T and α_{T-1}, the recursion gives other values of α_t.

[20] In independent work, Holden and Subrahmanyam (1992) obtain a similar limit. However, the-y do not show that the expected profits of informed traders vanish. See also Michener and Tighe (1991).

learns efficiently the true value of the asset, and that informed traders immediately release all of their private information to market through their trades.

These results are in contrast with those obtained by Kyle (1985) in his analysis of the same model with a monopolistic informed trader. In Kyle's (1985) model, the informed trader reduces the size of trade so that it is of order $1/T$; the absence of competition allows the monopolist to capture greater rents as liquidity trading variance is spread over many periods. In fact, a monopolist informed trader makes the greatest profits when liquidity trading variance is spread over many periods.[21]

Proposition 11 shows that when a constant liquidity trading variance is spread over an increasing number of trading periods, intense competition among the informed traders causes their trading profits to vanish. This intense competition can be found in other environments where the total liquidity trading variance is not constant. In general, as the liquidity trading variance per period goes to zero as the number of trading periods limits to infinity and the number of informed traders is greater than unity, the profits of informed traders vanish. Conversely, if the liquidity trading variance per period is nonzero and finite, then profits of competitive informed traders are finite as the number of periods increases. With competition, the important variable becomes liquidity trading variance per period; with a monopolist informed trader, it is the total amount of liquidity trading variance across all periods that matters.

Proposition 11. *When there are T trading periods, let the liquidity trading variance in eachperiod be h_T. The limitingparameter values in the mtb trading period as $T \to \infty$ are, if $I = 1$ and $Th_T \to \infty$, then*

$$E[\Pi_m^T] \to \infty, \qquad \lambda_m^T \to 0, \qquad \beta_m^T \to 0, \qquad b_T^{-1}(\Lambda_1^T - \Lambda_0^T) \to 0;$$

if $I = 1$ and $Th_T \to \kappa^* > 0$, then

$$E[\Pi_m^T] \to E_m^*, \qquad \lambda_m^T \to \lambda_m^*, \qquad \beta_m^T \to 0, \qquad b_T^{-1}(\Lambda_1^T - \Lambda_0^T) \to -b^*;$$

if $I \geq 2$ and $h^T \to \bar{\kappa} > 0$, then

$$E[\Pi_m^T] \to \bar{E}_m, \lambda_m^T \to \bar{\lambda}_m, \beta_m^T \to \bar{\beta}_m, b_T^{-1} (\Lambda_1^T - \Lambda_0^T) \to -\bar{b};$$

and, if $I \geq 2$ and $h^T \downarrow 0$, then

$$E[\Pi_m^T] \to 0, \lambda_m^T \to \infty, \beta_m^T \to 0, b_T^{-1} (\Lambda_1^T - \Lambda_0^T) \to \infty.$$

[21] In contrast to Kyle (1985). there is no well-defined continuous-time limit if there are more than two informed traders. The usual assumption that informed traders make infinitesimal trades of order *dt* yields infinite trading intensities. A previous version of our article discussed the continuous-time case. Also, see Back (1992) for a more detailed discussion of insider trading in continuous time.

From Proposition 11 we can see that the differences between competitive and monopolistic settings are most striking when there is an infinite amount of total liquidity trading. In such a case, the monopolist makes infinite profits as he trades infinitesimal amounts in each period. However, infinite total liquidity trading variance can lead to zero profits with competition. To see this, let the liquidity trading variance in each period be $1/\sqrt{T}$ when there are T trading periods. Then informed traders' profits vanish as T increases, even though the total liquidity trading variance, $T(1/\sqrt{T})$, becomes infinite. All of this serves to emphasize the importance of understanding the competitive environment when modeling long-lived private information.

4. Summary

We consider a model where public information is released prior to trading and the relevant random variables are represented by elliptically contoured distributions. In a model with exogenously informed traders, we show that the equilibrium decision rules of the market-maker and informed traders are independent of the specific distribution used. In addition, the market-maker's sensitivity to trades and the informed traders' intensity of trading do not depend on the value of the public information. However, the informativeness of prices and the volume of trading depend on the public information. Distributions like the multivariate t and other members of the compound normal class have the property that a higher absolute difference between the realized public information and its mean leads to greater price volatility and greater trading volume.

When the decision to acquire information is made endogenous and occurs after the realization of the public information, this decision depends on the realized value of public information, as do the decision rules of the market-maker and the trading intensities of the informed traders.

In a multiperiod model with many informed traders, representing beliefs with elliptically contoured distributions, we show that trading volumes are autocorrelated, and, for the compound normal class, trading volume is positively autocorrelated. We present a recursive method to solve the multiperiod model and show that with competition, when the total liquidity trading variance is fixed and allocated equally over T periods, the profits of informed traders go to zero as T increases. This finding is in contrast to the result of Kyle (1985) that profits are positive for the monopolist informed trader. Also, if the liquidity trading variance is fixed in each period, as the number of trading periods increases, informed traders' profits are finite with competition, even though the total liquidity trading variance becomes infinite.

Appendix A: Some Properties of Elliptically Contoured Distributions

What follows is a summary of definitions and properties of elliptically contoured distributions. For more detail, see Blake and Thomas (1968), Cambanis, Huang, and Simons (1981), Chu (1973), Devlin, Gnanadesikan, and Kettenring (1976), Kelker (1970), and Johnson (1987).

P1: A random variable x is said to have an elliptically contoured distribution if its characteristic function is of the form

$$E[e^{it'x}] = e^{it'\mu} \times \phi(t'\Sigma^{-1}t), \tag{A1}$$

where μ is the mean vector of the random variable x. $\phi(\cdot)$ is a function that satisfies the necessary conditions for a characteristic function: $\phi(0) = 1$, $|\phi(t)| < 1$ for all $t \neq 0$, and $\phi(t)$ is continuous on the real

P2: If the variance-covariance matrix exists, it is equal to $-2\phi'(0)\Sigma$. Since we can scale the representation so that $-2\phi'(0) = 1$, Σ is the variance-covariance matrix.

P3: An elliptically contoured random variable x whose measure is absolutely continuous (with respect to the Bore1 measure) has the density

$$f(x) = c_n(\det\Sigma)^{-1/2}b_n((x-\mu)'\Sigma^{-1}(x-\mu)), \tag{A2}$$

where c_n is a scaling function that ensures that we have a valid density. The function $b_n(\cdot)$ depends on n, unlike $\phi(\cdot)$, which is independent of n.

P4: Suppose the random variable x has $\Sigma^{-1} = I$ and the variance-covariance matrix exists. The elements of x are mutually independent if and only if $x \sim N(0, I)$. The normal is the only elliptically contoured distribution with finite second moments that has the independence property.

P5: If $x \sim \mathrm{ECC}(\mu, \Sigma, \phi)$, then $Ax + \alpha$ is also an elliptically contoured distribution in the same class [i.e., it is distributed $\mathrm{ECC}(A\mu + \alpha, A\Sigma A', \phi)$].

P6: If $x \sim \mathrm{ECC}(\mu, \Sigma, \phi)$ and we partition $x' = (x_1', x_2')$,

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{A3}$$

Then

$$E[x_1 \mid x_2] = \Sigma_{12}\Sigma_{22}^{-1}x_2. \tag{A4}$$

In particular, if $E[x_1 \mid x_2] = 0$, then x_1 and x_2 are semi-independent. Elliptically contoured distributions are the only distributions with linear conditional expectations.

P7: From *P4*, $\mathrm{Var}(x_1 | x_2)$ is independent of x_2 if and only if $x \sim N(0, \Sigma)$. More generally,

$$\mathrm{Var}(x_1 | x_2) = g(x_2' \Sigma_{22}^{-1} x_2)\ (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}). \tag{A5}$$

P8: If $x = \sqrt{Z}y$, where $\sqrt{Z}$ and y are independent, with $y \sim N(0, S)$, then x is from a compound normal distribution, which is a particular type of elliptically contoured distribution.

P9: If e has an elliptically contoured multivariate distribution with mean zero (Chu, 1973),

$$f(e) = \int_0^\infty n(e, z)\ dW(z), \tag{A6}$$

where $dW(\cdot)$ is a weighting function on $[0, \infty)$ that can assume negative values, $\int_0^\infty dW(z) = 1$, and, given z, $n(\cdot, z)$ is the normal density, with zero mean and variance $z^2\Sigma$. If the weighting function takes only positive values, it is itself a density and $f(e)$ is in the compound normal class. In this case, a second characterization is that $e = \sqrt{Z} \times u$, where Z is a positive random variable independent of the normal random variable u.

Appendix B: Proofs of Propositions 1-11

Proof of Proposition 1

If the market-maker uses a linear pricing rule of the form $p(y, s) = a(s) + \lambda(s)y$, then the ith informed trader maximizes

$$E[(v - a(s) - \lambda(s)y)x_i(I, s, v) \mid v, s]$$

$$= \left[v - a(s) - \lambda(s) \sum_{j \neq i} x_j(I, s, v)\right] x_i(I, s, v) - \lambda(s)[x_i(I, s, v)]^2, \tag{B1}$$

where we use $y = \Sigma_j x_j(I, s, v) + l$ and simplify. The first-order condition is

$$v - a(s) - \lambda(s) \sum_{j \neq i} x_j(I, s, v) - 2\lambda(s)x_i(I, s, v) = 0, \tag{B2}$$

and the second-order condition is $-2\lambda(s) < 0$, or $\lambda(s) > 0$. The first-order condition becomes

$$x_i(I, s, v) = \frac{1}{\lambda(s)}\left[v - a(s) - \lambda(s) \sum_{j} x_j(I, s, v)\right]. \tag{B3}$$

Thus, $x_i(I, s, v)$ is independent of i and

$$x(I, s, v) = \frac{1}{(I+1)\lambda(s)}[v - a(s)]. \tag{B4}$$

Hence, $\beta(s) = 1/[(I+1)\lambda(s)]$.

The market-maker takes the linear strategy of the informed traders as given and sets a price $p(y, s) = E[v \mid s, y]$. Because elliptically contoured distributions have linear conditional expectations and are closed under addition, we obtain

$$E[v - E[v \mid s] \mid s, y]$$
$$= \frac{I\beta(s)\Lambda \mid s}{I^2\beta(s)^2\Lambda \mid s + \sigma_l^2 \mid s}\left[y - \frac{I}{(I+1)\lambda(s)}(E[v \mid s] - a(s))\right]. \tag{B5}$$

Thus,

$$\lambda(s) = \frac{I\beta(s)\Lambda \mid s}{I^2\beta(s)^2\Lambda \mid s + \sigma_l^2 \mid s}, \tag{B6}$$

and

$$a(s) = E[v \mid s] - \lambda(s)\left[\frac{I}{(I+1)\lambda(s)}(E[v \mid s] - a(s))\right]. \tag{B7}$$

This yields $a(s) = E[v \mid s]$. Equation (B6) rearranges to $\lambda(s)[I^2\beta(s)^2\Lambda \mid s + \sigma_l^2 \mid s] = I\beta(s)\Lambda \mid s$. The formula for the conditional covariance matrix [Equation (4)] and $\beta(s) = 1/[(I+1)\lambda(s)]$ yield

$$\lambda(s)\left[\frac{I^2}{(I+1)^2\lambda(s)^2}\Lambda + \sigma_l^2\right] = \frac{I}{(I+1)\lambda(s)}\Lambda. \tag{B8}$$

This gives

$$\lambda(s) = \lambda = \frac{\sqrt{I}}{I+1}\left[\frac{\Lambda}{\sigma_l^2}\right]^{1/2} \quad \text{and} \quad \beta(s) = \beta = \frac{1}{\sqrt{I}}\left[\frac{\sigma_l^2}{\Lambda}\right]^{1/2}. \tag{B9}$$

■

Proof of Proposition 2

Using property *P7* in Appendix A for elliptically contoured distributions and the fact that $s - s_0$ and y are uncorrelated, we find

$$\mathrm{Var}[v - p(y, s) \mid s, y]$$
$$= b\left(\frac{y^2}{I^2\beta^2\Lambda + \sigma_l^2} + \frac{(s - s_0)^2}{\sigma_s^2}\right)\frac{\sigma_l^2}{I^2\beta^2\Lambda + \sigma_l^2}\Lambda, \tag{B10}$$

which is conditionally heteroskedastic. ■

Proof of Lemma

Using Chu's (1973) characterization of elliptically contoured distributions and Tonelli's theorem [see Royden (1968, p. 270)] $-dW(z)$

is a finite measure and $|e|n(e, z)$ is a positive function-we have

$$E[|e|] = \int_{-\infty}^{\infty} |e| f(e)\, de = \int_{-\infty}^{\infty} |e| \int_0^{\infty} n(e, z)\, dW(z)\, de$$

$$= \int_0^{\infty} \int_{-\infty}^{\infty} |e|\, n(e, z)\, de\, dW(z)$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sigma_e}{(\int_0^{\infty} z^2\, dW(z))^{1/2}} z\, dW(z). \tag{B11}$$

The monotonicity of $E[|e|]$ in σ_e follows immediately.

In the case of the compound normal distribution, the second characterization of Chu (1973) gives

$$E[|e|] = \frac{2}{\sqrt{2\pi}} E[\sqrt{Z}] \frac{\sigma_e}{(E[Z])^{1/2}}. \tag{B12}$$

■

Proof of Proposition 3

The three components of volume are given by

$$V_I = f\left(g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\beta^2 I^2 \Lambda\right), \qquad V_L = f\left(g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\sigma_l^2\right),$$
$$V_M = f\left(g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)[\sigma_l^2 + \beta^2 I^2 \Lambda]\right), \tag{B13}$$

where $f(\cdot)$ is the monotone function referred to in the lemma proved above. From these expressions, the dependence on the public information is clear. When the distributions belong to the compound normal class, conditional expectations are well defined and

$$V_I = E[\sqrt{Z}|s]\frac{1}{\sqrt{2\pi}} \frac{I\beta(\Lambda)^{1/2}}{(E[Z])^{1/2}},$$

$$V_L = E[\sqrt{Z}|s]\frac{1}{\sqrt{2\pi}} \frac{\sigma_l}{(E[Z])^{1/2}}, \tag{B14}$$

$$V_M = E[\sqrt{Z}|s]\frac{1}{\sqrt{2\pi}} \frac{[\sigma_l^2 + \beta^2 I^2 \Lambda]^{1/2}}{(E[Z])^{1/2}}.$$

The proportionality stated in the proposition is obvious. ■

Proof of Proposition 4

The only claim that we need to prove is that for the compound normal class, $g((s - s_0)^2/\sigma_s^2)$ is strictly monotone increasing in $|s - s_0|$ For the compound normal class, we know that $s - s_0 = \sqrt{Z}u$. The random variable Z drives the conditional variance updating, and it is easy to show using Chu's (1973) lemma that $g((s - s_0)^2/\sigma_s^2) = E[Z|s]$. But

$$E[Z \mid s] = \frac{\int_0^\infty Zr(Z)b(|s - s_0|/\sqrt{Z})\, dz}{\int_0^\infty r(Z)b(|s - s_0|/\sqrt{Z})\, dz}, \tag{B15}$$

where $r(\cdot)$ is the density function of Z, and $b(\cdot)$ is the density function of $|u|$, where u is the normal random variable referred to in Chu's (1973) second characterization.

Differentiating this with respect to $|s - s_0|$ yields

$$\begin{aligned}
\frac{dE[Z \mid s]}{d|s - s_0|} > 0 &\Leftrightarrow \frac{dE[Z \mid s]}{d(|s - s_0|^2)} > 0 \\
&\Leftrightarrow -\left[\int_0^\infty r(Z)\, b\left(\frac{|s - s_0|}{\sqrt{Z}}\right) dz\right]^2 \\
&\quad + \left[\int_0^\infty Zr(Z)\, b\left(\frac{|s - s_0|}{\sqrt{Z}}\right) dz\right] \\
&\quad \times \left[\int_0^\infty \frac{1}{Z} r(Z)\, b\left(\frac{|s - s_0|}{\sqrt{Z}}\right) dz\right] > 0 \\
&\Leftrightarrow E[Z \mid s]\cdot E[1/Z \mid s] - 1 > 0 \\
&\Leftrightarrow \mathrm{Cov}[Z, 1/Z \mid s] < 0. \qquad \text{(B16)}
\end{aligned}$$

This conditional covariance is negative, using the Cauchy-Schwarz inequality, and therefore the claim follows. ∎

Proof of Proposition *5*

With endogenous information acquisition, the number of informed traders depends on

$$\Pi(I, s) = g\left(\frac{(s - s_0)^2}{\sigma_s^2}\right)\frac{(\sigma_I^2\Lambda)^{1/2}}{\sqrt{I}\,(I + 1)}. \tag{B17}$$

For fixed s, expression (B17) is a decreasing function of Z and therefore a unique $I(s)$ exists. If the elliptically contoured distribution has a smooth density, $g((s - s_0)^2/\sigma_s^2)$ is a smooth function of s. Let $0 < g_1 < g_2 < \ldots$ be the values of $g(\cdot)$ in its range where it is just optimal for the next trader to become informed. Let $S_k = \{s|g((s - s_0)^2/\sigma_s^2) = g_k\}$, which is a nonempty set. Since g_{k-1} and g_k are in the range of the continuous function $g(\cdot)$ there must be a sequence of s^n converging to some $s \in S_k$ and $g((s^n - s_0)^2/\sigma_s^2) < g_k$. For any member of the sequence, s^n, $I(s_n) = k - 1$ while at s, $I(s) = k$. Consequently, $\lambda(s)$ and $\beta(s)$ are discontinuous at s, as are the conditional variance and volume given the public information. ∎

Proof of Proposition 6
For endogenous information acquisition, if $g(\cdot)$ is a monotonic function of $|s - s_0|$, then $\Pi(I, s)$ is monotonically increasing in $|s - s_0|$. The assumption that at least one trader becomes informed when $s = s_0$ completes the proof. ■

Proof of Proposition 7
We conjecture a value function for each informed agent of the form

$$\Psi_t^i = \psi_t^i(p_1, \ldots, p_t, l_1, \ldots, l_t, v)$$
$$= \alpha_t(v - p_t)^2 + \delta_t g_t\left(\frac{1}{\sigma_l^2}\sum_{j=1}^{t} l_j^2 + \frac{(v - p_0)^2}{\Lambda_0}\right). \quad \text{(B18)}$$

The function $g_t(\cdot)$ depends on the particular distribution used. Our proof is by induction. For each t, each informed agent maximizes

$$\Psi_{t-1}^i = \max_{x_t^i} E[(v - p_t)x_t^i + \alpha_t(v - p_t)^2$$
$$+ \delta_t(l_1, \ldots, l_t, v) \mid p_1, \ldots, p_{t-1}, v], \quad \text{(B19)}$$

where $\delta_t(l_1, \ldots, l_t, v)$ is the last term in Equation (B18). For a price function of the form $p_t = p_{t-1} + \lambda_t[x_t^i + \Sigma_{j\neq i} x_t^j + l_t]$, this maximization reduces to

$$\max_{x_t^i}\Bigg[\Bigg[v - p_{t-1} - \lambda_t x_t^i - \lambda_t \sum_{j\neq i} x_t^j\Bigg]x_t^i$$
$$+ \alpha_t\Bigg(v - p_{t-1} - \lambda_t x_t^i - \lambda_t \sum_{j\neq i} x_t^j\Bigg)^2$$
$$+ \alpha_t \lambda_t^2 E[l_t^2 \mid p_1, \ldots, p_{t-1}, v]$$
$$+ E[\delta_t(l_1, \ldots, l_t, v) \mid p_1, \ldots, p_{t-1}, v]\Bigg]. \quad \text{(B20)}$$

The first-order condition is

$$\Bigg(v - p_{t-1} - \lambda_t \sum_{j\neq i} x_t^j\Bigg) - 2\lambda_t x_t^i - 2\alpha_t\lambda_t\Bigg(v - p_{t-1} - \lambda_t x_t^i - \lambda_t \sum_{j\neq i} x_t^j\Bigg)$$
$$= 0, \quad \text{(B21)}$$

and the second-order condition is $-2\lambda_t + 2\lambda_t^2\alpha_t < 0$. The second-order condition simplifies to $\lambda_t(1 - \lambda_t\alpha_t) > 0$, while the first-order condition rearranges to

$$x_t^i = \frac{1}{\lambda_t}\Bigg[\Bigg(v - p_{t-1} - \lambda_t \sum_{\forall j} x_t^j\Bigg)$$

$$- 2\alpha_t\lambda_t\left(v - p_{t-1} - \lambda_t \sum_{\forall j} x_t^j\right)\Bigg]. \qquad \text{(B22)}$$

Because the right-hand side of expression (B22) is independent of x_t^i, $x_t^i = x_t$ for all i. Rearranging Equation (B22) yields

$$x_t = \frac{1 - 2\alpha_t\lambda_t}{\lambda_t[1 + I(1 - 2\alpha_t\lambda_t)]}(v - p_{t-1}), \qquad \text{(B23a)}$$

$$\beta_t = \frac{1 - 2\alpha_t\lambda_t}{\lambda_t[1 + I(1 - 2\alpha_t\lambda_t)]}. \qquad \text{(B23b)}$$

Equation (B23b) leads to

$$1 - I\lambda_t\beta_t = \frac{1}{1 + I(1 - 2\alpha_t\lambda_t)}. \qquad \text{(B24)}$$

The value function recursion then yields

$$\begin{aligned}\alpha_{t-1} &= \beta_t[1 - I\lambda_t\beta_t] + \alpha_t[1 - I\lambda_t\beta_t]^2 \\ &= \frac{1 - \alpha_t\lambda_t}{\lambda_t[1 + I(1 - 2\alpha_t\lambda_t)]^2} \quad \text{[using (B24)].} \qquad \text{(B25)}\end{aligned}$$

Also, the value function recursion [Equation (B19)] implies that

$$\begin{aligned}\delta_t(l_1, \ldots, l_{t-1}, v) &= \alpha_t\lambda_t^2 E[l_t^2 \mid l_1, \ldots, l_{t-1}, v] \\ &\quad + E[\delta_t(l_1, \ldots, l_t, v) \mid l_1, \ldots, l_{t-1}, v]. \qquad \text{(B26)}\end{aligned}$$

Let $g(\cdot)$ be the functions that determine the conditional variance for the informed traders, so

$$\begin{aligned}&g_{t-1}\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t-1} l_k^2 + \frac{(v - p_0)^2}{\Lambda_0}\right) \\ &= E\left[g_t\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t} l_k^2 + \frac{(v - p_0)^2}{\Lambda_0}\right) \mid l_1, \ldots, l_{t-1}, v\right]. \qquad \text{(B27)}\end{aligned}$$

Given our conjecture that $\delta_t(l_1, \ldots, l_t, v)$ is of the form

$$\delta_t g_t\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t} l_k^2 + \frac{(v - p_0)^2}{\Lambda_0}\right),$$

induction and Equation (B27) lead to

$$\begin{aligned}\delta_{t-1}(l_1, \ldots, l_{t-1}, v) &= \alpha_t\lambda_t^2\sigma_l^2 g_{t-1}\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t-1} l_k^2 + \frac{(v-p_0)^2}{\Lambda_0}\right) \\ &\quad + E\left[\delta_t g_t\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t} l_k^2 + \frac{(v-p_0)^2}{\Lambda_0}\right) \mid l_1, \ldots, l_{t-1}, v\right]\end{aligned}$$

$$= [\alpha_t\lambda_t^2\sigma_l^2 + \delta_t]g_{t-1}\left(\frac{1}{\sigma_l^2}\sum_{k=1}^{t-1} l_k^2 + \frac{(v-p_0)^2}{\Lambda_0}\right)$$

$$\text{[using (B27)].} \tag{B28}$$

Therefore, $\delta_{t-1} = [\alpha_t\lambda_t^2\sigma_l^2 + \delta_t]$.

To obtain λ_t, the market-maker's sensitivity to order flows, notice that $\beta_t(v - p_{t-1})$ and l_t are jointly elliptically distributed, given the information $p_0, \ldots, p_{t-1}$, with zero means, and a variance-covariance matrix of

$$h_{t-1}\left(\sum_{k=1}^{t-1} \frac{y_k^2}{I^2\beta_k^2\Lambda_{k-1} + \sigma_k^2}\right)\begin{pmatrix}\Lambda_{t-1} & 0 \\ 0 & \sigma_l^2\end{pmatrix}. \tag{B29}$$

The conditional expectation of $v - p_{t-1}$, given $I\beta_t(v - p_{t-1}) + l_t$, linear and equal to $\lambda_t[I\beta_t(v - p_{t-1}) + l_t]$, where

$$\lambda_t = \frac{I\beta_t\Lambda_{t-1}h(\cdot)}{I^2\beta^2\Lambda_{t-1}h(\cdot) + \sigma_l^2 h(\cdot)} = \frac{I\beta_t\Lambda_{t-1}}{I^2\beta^2\Lambda_{t-1} + \sigma_l^2}. \tag{B30}$$

The function $h(\cdot)$ is the part of the conditional variance function that depends on past prices. From Equation (B30), λ_t is constant.

The recursion of the unconditional expectation of the conditional variances is

$$\begin{aligned}\Lambda_t &= \Lambda_{t-1}\left[1 - \frac{[\mathrm{Cov}(v - p_{t-1}, y_t)]^2}{\Lambda_{t-1}\,\mathrm{Var}(y_t)}\right] \\ &= \Lambda_{t-1}\left[1 - \frac{(I\beta_t\Lambda_{t-1})^2}{\Lambda_{t-1}(I^2\beta_t^2\Lambda_{t-1} + \sigma_l^2)}\right] \\ &= \Lambda_{t-1}\left[1 - \frac{I\lambda_t\beta_t\Lambda_{t-1}}{\Lambda_{t-1}}\right] = \Lambda_{t-1}[1 - I\lambda_t\beta_t].\end{aligned} \tag{B31}$$

Expressions (B30) and (B31) yield $\lambda_t = (I\beta_t/\sigma_l^2)\Lambda_t$, and complete our derivation of the recursive formulas.

To solve the recursions equate $\beta_t = (\lambda_t\sigma_l^2)/(I\Lambda_t)$ to expression (B23b) to obtain

$$\frac{\lambda_t^2\sigma_l^2}{I\Lambda_t} = \frac{1 - 2\alpha_t\lambda_t}{1 + I(1 - 2\alpha_t\lambda_t)}. \tag{B32}$$

If we fix $\alpha_t \geq 0$ and $\Lambda_t > 0$, then the left-hand side of Equation (B32) is an increasing convex function in λ_t that is zero at $\lambda_t = 0$. The right-hand side of Equation (B32) is a decreasing concave function that is positive at $\lambda_t = 0$ and is zero at $\lambda_t = 1/2\alpha_t$; consequently, there is a unique value of A, between 0 and $1/2\alpha_t$ that solves Equation (B32). We leave it to the reader to check that this is the only solution of Equation (B32) that satisfies the second-order conditions and ensures a positive Λ_t. A related uniqueness proof is shown in the proof of Proposition 9.

We solve Equation (B32) recursively to obtain the unique solution given the endpoint constraints $\alpha_T = 0$ and $\Lambda_T > 0$. To show uniqueness given an initial Λ_0, change Λ_T to $\Lambda'_T = v^2\Lambda_t$. The new solution has $\lambda'_t = v\lambda_t, \alpha'_t = v^{-1}\alpha_t, \beta'_t = v^{-1}\beta_t$, and $\delta'_t = v\delta_t$, so there is unique correspondence between Λ_0 and Λ_T and hence a unique solution. ∎

Proof of Proposition 8

The dependence of price variances and volumes on past order flows and past prices follows from the one-period model. For the compound normal class, write the random variables $x_t = \sqrt{Z}r_t$, and $l_t = \sqrt{Z}w_t$, where $r_1,\ldots,r_T$, $w_1,\ldots,w_T$ are jointly normally distributed and Z is an independent positive random variable. With this transformation we have

$$\begin{aligned}
&\mathrm{Cov}(V_t, V_{t-1}) \\
&= E\Bigg[\frac{(\sqrt{Z})^2}{4}(|Ir_t| + |w_t| + |Ir_t + w_t|) \\
&\qquad \times (|Ir_{t-1}| + |w_{t-1}| + |Ir_{t-1} + w_{t-1}|)\Bigg] \\
&\quad -\Bigg(E\left[\frac{\sqrt{Z}}{2}(|Ir_t| + |w_t| + |Ir_t + w_t|)\right] \\
&\qquad \times E\left[\frac{\sqrt{Z}}{2}(|Ir_{t-1}| + |w_{t-1}| + |Ir_{t-1} + w_{t-1}|)\right]\Bigg) \\
&= \tfrac{1}{4}\left[E(Z) - \left(E\sqrt{Z}\right)^2\right] \times E[(|Ir_t| + |w_t| + |Ir_t + w_t|)] \\
&\qquad \times E[(|Ir_{t-1}| + |w_{t-1}| + |Ir_{t-1} + w_{t-1}|)] > 0. \qquad \text{(B33)}
\end{aligned}$$

The last inequality in Equation (B33) that follows because $\sqrt{Z}$ has positive variance. Thus, trading volume is positively autocorrelated.

For general elliptical distributions, Chu's (1973) characterization leads to the covariance between V_t and V_{t-1} being given by $K_{t,t-1}$ times $[\int z^2\, dW(z) - (\int z\, dW(z))^2]$, where $K_{t,t-1}$ is a positive number. The term in brackets in this expression cannot be signed because $dW(z)$ is a weighting measure that can be negative; to use Jensen's inequality, it must be a probability measure. ∎

Proof of Corollary to Proposition 8

After some manipulation, one can show that $\beta_t^2\Lambda_t = \sigma_l^2 \times (1 - 2\alpha_t\lambda_t)/(1 + I(1 - 2\alpha_t\lambda_t))$. Because $\alpha_t\lambda_t$ is a sequence that decreases over time (see the proof of Proposition 10), $\beta_t^2\Lambda_t$ increases over time. Since liquidity trading is constant in each period, any differences in autocorrelations are determined by differences in the expected volume of informed trading. As $\beta_t^2\Lambda_t$ increases, so do expected orders from informed traders. ■

Proof of Proposition 9
First use the expressions for β_t and α_{t-1} from Equations (B23b) and (B25) to get

$$\beta_t^2 = \frac{\alpha_{t-1}}{\lambda_t}\frac{(1-2\alpha_t\lambda_t)^2}{1-\alpha_t\lambda_t}. \tag{B34}$$

Also Equation (B32) and $\lambda_t = (I\beta_t/\sigma_l^2)\Lambda_t$ lead to $\beta_t^2 = \sigma_l^2(1-2\alpha_t\lambda_t)/(I\Lambda_{t-1})$ Equating these two expressions and rearranging yield

$$\frac{I\Lambda_{t-1}\alpha_{t-1}}{\sigma_l^2} = \frac{\lambda_t(1-\alpha_t\lambda_t)}{1-2\alpha_t\lambda_t}. \tag{B35}$$

Equation (B32) and $\Lambda_t = \Lambda_{t-1}/(1+I(1-2\lambda_t\alpha_t))$ yield

$$\lambda_t^2 = \frac{I\Lambda_{t-1}}{\sigma_l^2}\frac{(1-2\lambda_t\alpha_t)}{[1+I(1-2\lambda_t\alpha_t)]^2}. \tag{B36}$$

Thus,

$$\frac{I^2\Lambda_{t-1}^2\alpha_{t-1}^2}{\sigma_l^4} = \frac{(1-\alpha_t\lambda_t)^2}{(1-2\alpha_t\lambda_t)^2}\frac{I\Lambda_{t-1}}{\sigma_l^2}\frac{(1-2\lambda_t\alpha_t)}{[1+I(1-2\lambda_t\alpha_t)]^2}, \tag{B37}$$

which simplifies to

$$\theta_{t-1} = \frac{I\Lambda_{t-1}\alpha_{t-1}^2}{\sigma_l^2} = \frac{(1-\alpha_t\lambda_t)^2}{(1-2\alpha_t\lambda_t)[1+I(1-2\alpha_t\lambda_t)]^2}, \tag{B38}$$

which defines θ_{t-1}. Equations (B25) and (B31) yield the relationship

$$\theta_t = \theta_{t-1}\frac{\alpha_t^2\lambda_t^2[1+I(1-2\alpha_t\lambda_t)]^3}{(1-\alpha_t\lambda_t)^2}. \tag{B39}$$

Substituting for θ_{t-1} from Equation (B38) and letting $\zeta_t = \alpha_t\lambda_t$ yield

$$\theta_t = \zeta_t^2\frac{1+I(1-2\zeta_t)}{1-2\zeta_t}, \quad \theta_{t-1} = \frac{(1-\zeta_t)^2}{(1-2\zeta_t)[1+I(1-2\zeta_t)]^2}. \tag{B40}$$

The two equations in expression (B40) can be used to solve recursively for q_t, and ζ_t. Start with the endpoint constraint $\theta_T = 0$, which implies $\zeta_T = 0$. Use the second equation of (B40) to solve for θ_{T-1}. The first equation of (B40) is then used to solve for ζ_{T-1} Plug this value back into the second equation to get θ_{T-2} and so on.

Consider the first equation of (B40). The right-hand side of the equation starts at 0, increases, and limits to infinity at $\zeta_t = \frac{1}{2}$. Then, at

$\zeta_t = \frac{1}{2}$, the right-hand side starts at $-\infty$ and is strictly increasing. The right-hand side reaches zero at $(I+1)/2I$ and then increases to infinity as ζ_t limits to infinity. Given an arbitrary $\theta_t > 0$, there is one solution such that $0 < \zeta_t < \frac{1}{2}$. There is another solution such that $(I+1)/2I < \zeta_t$. However, this solution implies that $1 + I(1 - 2\zeta_t) < 0$ and thus, by Equations (B24) and (B31), either $\Lambda_t < 0$ or $\Lambda_{t-1} < 0$. In either case, we violate the constraint that Λ_t is positive for all t. Hence, we need to consider the values of these equations over the interval $0 < \zeta_t < \frac{1}{2}$, from which we get our uniqueness result. ■

Proof of Proposition 10

First, note that the parameters θ_t and ζ_t in Equation (B40) are independent of Λ_0 and σ_I^2. Second, in the region $0 < \zeta_t < \frac{1}{2}$, θ_t is strictly monotone increasing and one-to-one in ζ_t, and θ_{t-1} is strictly monotone increasing in ζ_t. Thus, θ_{t-1} is a strictly monotone increasing function of θ_t; call this map $T(\cdot)$. Then $\theta_{t-1} = T(\theta_t)$. Since $\theta_T = 0$, $\zeta_T = 0$, we have $\theta_{T-1} = T(\theta_T) = T(0) > 0$. Assume that $\theta_t > \theta_{t+1}$. Then $\theta_{t-1} > \theta_t \Leftrightarrow T(\theta_t) > T(\theta_{t+1}) \Leftrightarrow \theta_t > \theta_{t+1}$ because $T(\cdot)$ is a strictly monotone map. Since $\theta_t > \theta_{t+1}$ by the induction assumption, we have an increasing sequence.

Consider the sequence θ_0^T. This sequence is obtained through repeated use of the strictly monotone operator $T(\cdot)$ (remember that $\theta_0^0 = 0$), and thus is an increasing sequence. Suppose $\theta_0^T \to \infty$. Then, the second expression of Equations (B40) relating θ_{t-1} and ζ_t implies that $\zeta_1^T \to \frac{1}{2}$. Now

$$\theta_1^T - \theta_0^T = \frac{(\zeta_1^T)^2[1 + I(1 - 2\zeta_1^T)]^3 - (1 - \zeta_1^T)^2}{(1 - 2\zeta_1^T)[1 + I(1 - 2\zeta_1^T)]^2}. \tag{B41}$$

When $\zeta_1^T \to \frac{1}{2}$, expression (B41) converges to 0/0. Using l'Hôpital's rule, we obtain the limit as $\zeta_1^T \to \frac{1}{2}$ to be $\frac{3}{4}I - 1$.

When $I = 1$, $\theta_1^T - \theta_0^T$ converges to $-\frac{1}{4}$ as in Kyle (1985). When $I \geq 2$, $\theta_1^T - \theta_0^T$ converges to a strictly positive limit. This implies for large enough T, $\theta_1^T - \theta_0^T$ is strictly positive and contradicts the fact that $\theta_1^T < \theta_0^T$. Thus, $\theta_0^T \to \infty$ is contradicted and $\theta_0^T \to \theta^*$, a finite positive number.

But $\theta_0^T = TI(\alpha_0^T)^2\Lambda_0/\sigma_I^2$. Since $T \to \infty$ and $\theta_0^T \to \theta^*$, α_0^T must converge to zero. Second, ζ_1^T is a strictly monotone and continuous map of θ_0^T. Consequently, $\zeta_1^T \to \zeta^*$, a finite positive number less than $\frac{1}{2}$ In fact, we can show that ζ_1^T is a strictly increasing sequence in a way identical to that of θ_0^T (strictly monotone map that starts at 0). Since α_1^T is determined by $\alpha_0^T = (\alpha_1^T(1 - \zeta_1^T))/(\zeta_1^T[1 + I(1 - 2\zeta_1^T)]^2)$ we obtain that $\alpha_1^T \to 0$. Since $\zeta_1^T = \alpha_1^T\lambda_1^T \to \zeta^*$, $\lambda_1^T \to \infty$.

Again, $\lambda_1^T\beta_1^T$ is given by a monotone decreasing and continuous map of ζ_1^T and thus converges to a finite limit; hence, we must have

$\beta_1^T \to 0$. However, $\lambda_1^T = (I/\sigma_1^2) \times (\Lambda_0/(1 + I(1 - 2\zeta_1^T)) \times T\beta_1^T$. Because $\zeta_1^T \to \zeta^*$, $T\beta_1^T \to \infty$ and $\lambda_1^T/T \to 0$. Finally, the fact that $\zeta_1^T \to \zeta^* < \frac{1}{2}$ implies that $T(\Lambda_1^T - \Lambda_0^T) = T\Lambda_0 \times (-I(1 - 2\zeta_1^T))/(1 + I(1 - 2\zeta_1^T)) \to -\infty$.

The results for $\lambda_m^T, \beta_m^T, \alpha_m^T, \lambda_m^T$, for a fixed m, can be obtained in a similar manner.

To complete our proof that profits go to zero, we need to show that δ_0^T limits to zero. First note that for small ϵ_v, there is a T such that for all $T \geq \hat{T}$,

$$\Lambda_1^T = \frac{1}{[1 + I(1 - 2\zeta_1^T)]} \Lambda_0^T < \left(\frac{1}{[1 + I(1 - 2\zeta^*)]} + \epsilon\right)\Lambda_0 = v^2\Lambda_0, \quad \text{(B42)}$$

where $0 < v < 1$ (this defines v). Now note that if we fix the number of trading periods at T and change the parameter Λ_0 to $\bar{\mu}^2\Lambda_0$, δ_0 changes to $\bar{\mu}\delta_0$; similarly, if we change σ_I^2 to $\hat{\mu}^2\sigma_I^2$, δ_0 changes to $\hat{\mu}\sigma_I^2$. Then $\delta_1^{T+1} < v(T/(T+1))^{1/2}\delta_0^T < v\delta_0^T$:

$$\delta_0^{T+1} = \zeta_1^{T+1}\frac{\lambda_1^{T+1}}{T+1}\sigma_I^2 + \delta_1^{T+1} < \zeta_1^{T+1}\frac{\lambda_1^{T+1}}{T+1}\sigma_I^2 + v\delta_0^T. \quad \text{(B43)}$$

Because λ_1^T/T limits to zero, there is $\bar{T}$ such that for T, $\bar{T}$, $\zeta_1^T(1/T)\lambda_1^T\sigma_I^2 < \epsilon$. Then for $T^* = \max\{\hat{T}, \bar{T}\}, \delta_0^{T^*+T} < [\epsilon + v\epsilon + \cdots + v^{T-1}\epsilon] + (v)^T\delta_0^{T^*}$. The right-hand side of this expression converges to $\epsilon/(1 - v)$ and bounds any limit points of the sequence δ_0^T. But ϵ is arbitrary; thus, the limit must be zero. It immediately follows that δ_m^T also converges to zero. Expected profits limit to zero as α_0^T and δ_0^T both converge to zero. ■

Proof of Proposition 11

Consider the case where $I \geq 2$. Then the sequence θ_0^T is a bounded increasing sequence and therefore converges. Now $\theta_0^T = [I(\alpha_0^T)^2\Lambda_0]/h_T$. Thus, if h_T is a decreasing sequence converging to 0, $h_T\theta_0^T \to 0$ thus $\alpha_0^T \to 0$. Then as in the proof of Proposition 10, $\alpha_m^T \to 0, \lambda_m^T \to \infty$, $\beta_m^T \to 0$, $h_T\lambda_m^T \to 0$, and $h_T^{-1}(\Lambda_1^T - \Lambda_0^T) \to -\infty$. To prove that $\delta_0^T \to 0$, we amend the proof in Proposition 10 slightly. By an argument similar to that in Equation (B43),

$$\delta_0^{T+1} = \zeta_1^T\frac{\lambda_1^T}{T} + \delta_1^{T+1} < \zeta_1^T\frac{\lambda_1^T}{T} + v\left(\frac{h_{T+1}}{h_T}\right)^{1/2}\delta_0^T < \zeta_1^T\frac{\lambda_1^T}{T} + v\delta_0^T, \quad \text{(B44)}$$

where we have used the fact that h_T is a decreasing sequence. Then using the same argument as in Proposition 10, it follows that $\delta_0^T \to 0$ and thus $\delta_m^T \to 0$.

On the other hand, if $h_T \to c >$ we obtain that $h^T\theta_0^T$ converges to a finite limit and thus α_0^T converges to a finite limit. Since ζ_1^T is also a bounded increasing sequence in this case, ζ_1^T converges to a finite number; thus, we obtain that α_m^T, λ_m^T, and β_m^T are all finite convergent sequences. Also $b_T^{-1}(\Lambda_1^T - \Lambda_0^T) \to -k^*$, a finite number. To prove that profits are finite, we also need to show that δ_0^T is a bounded sequence. As in Proposition 10, we know, for T sufficiently large,

$$\delta_0^{\hat{T}+1} < \zeta_1^{\hat{T}+1} \lambda_1^{\hat{T}+1} b_{T+1} + v(b_{\hat{T}+1}/b_{\hat{T}})^{1/2}\delta_0^{\hat{T}}, \tag{B45}$$

where $0 < v < 1$. Since $\zeta_1^T \lambda_1^T b_T$ converges to a finite limit, it is bounded by B. Also for $\hat{T}$ large enough, $v^2(b_{\hat{T}+1})/(b_{\hat{T}}) < \gamma^2 < 1$; this follows from the fact that $b_T \to c > 0$. Then $\delta^{\hat{T}+T} < [B + \gamma B + \cdots + \gamma^{T-1} B] + (\gamma)^T \delta_0^{\hat{T}}$. The right-hand side of this expression converges to $B/(1 - \gamma)$, which is finite; thus, δ_0^T and α_0^T converge to finite limits and profits are finite.

When I= 1, we know that $\theta_t - \theta_{t-1} < -1$ [see Kyle (1985, p. 1332)] and thus $\theta_{t-1} > \theta_t + 1$. Hence, θ_0^T converges to infinity. In the case where $b_T = 1/T$ (Kyle's case), we know that $[\theta_0^T]/T = [I(\alpha_0^T)^2\Lambda_0]$. But Kyle (1985) shows that α_0^T is bounded and thus θ_0^T/T is a bounded function.

Then for a general h_T we can write $(b_T T)(\theta_0^T/T) = (I(\alpha_0^T)^2\Lambda_0)$ [the ~ in this equation is different from that in the Kyle (1985) case]. Since θ_0^T/T is bounded, if $h^T T$ converges to ∞, so does α_0^T, and profits are unbounded. Conversely, if $h^T T$ converges to a finite number, so does α_0^T. In either case, ζ_1^T converges to ½ and thus β_1^T converges to zero. Also if Th_T is bounded, so is α_0^T and thus λ_1^T; if Tb_T is unbounded, α_0^T is infinite and thus λ_1^T converges to zero.

Further, $b_T^{-1}(\Lambda_1^T - \Lambda_0^T) = (1/b_T T) \times [T\Lambda_0 \times -(I(1 - \zeta_1^T))/(1 + I(1 - 2\zeta_1^T))]$. Since the term in the square brackets is finite from Kyle (1985), when $h_T T$ is finite, so is $b_T^{-1}(\Lambda_1^T - \Lambda_0^T)$; and when $h_T T$ *is* infinite, it is zero.

To prove the limits for δ_0^T, we note that when $b_T = 1/T$, δ_0^T [Kyle (1985)] converges to a finite limit. By the homogeneity of $\bar{\delta}_0^T$ in the liquidity trading for fixed T (see Proposition 10), $\delta_0^T = (b_T T)^{1/2}\delta_0^T$ (Kyle, 1985). Thus, if $h_T T$ bounded, so is δ_0^T; and if δ_0^T is unbounded, so is $h_T T$. Extensions to the *m*th trading period for all the parameters are obvious. ∎

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