A Theory of the Nominal Term Structure of Interest Rates

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A model of the nominal term structure of interest rates is developed that has a positive and stationary process for the interest rate and delivers closed form expressions for the prices of discount bonds and European options on bonds. Unlike the one-state-variable version of the Cox, Ingersoll, and Ross (1985) model this model—even in its one-state-variable version—allows the term premium to change sign as a function of the state and the term to maturity, and also allows for shapes of the yield curve that are observed in the U.S. data but that are disallowed in the Cox, Ingersoll, and Ross model.

The premier model of the term structure of interest rates by Cox, Ingersoll, and Ross (1985; hereafter CIR) has a number of appealing features. It is parsimonious in the number of state variables (one) and in the number of parameters (four). The (short-term) interest rate is a positive and stationary process. The prices of discount bonds and European options on bonds are obtained in closed form. Finally, the conditional

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distribution and its moments of the yield to maturity, forward rates, and bond returns are obtained in closed form, leading to straightforward estimation and testing of the model.[1]

When interpreted as a one-state-variable model of the nominal term structure, the CIR model has, however, three restrictive properties. First, it implies that the term premium, defined as the instantaneous arithmetic expected rate of return minus the interest rate, of a given maturity discount bond is of uniform sign for all states of nature. Second, it implies that the term premium is of uniform sign for all maturities of discount bonds. Third, it implies that the yield curve attains only three shapes: uniformly rising, humped, and uniformly declining. Perusal of the yield curves in the last 40 years reveals inverted-humped yield curves as well.

The translated version of the one-state-variable CIR model leaves the term premiums unchanged and therefore inherits the first two restrictive properties of the untranslated model.[2] It also shifts the yield curve uniformly by adding a constant to the yield to maturity, leaving unchanged the third restriction on the shape of yield curves.

The multivariate extension of the CIR model is free from the above implications.[3] However, the interpretation of empirical evidence that the one-state-variable model is rejected against the nesting two-state-variable model is problematic. The one-state-variable model may be rejected either because there is indeed more than one state variable driving the term structure or because of the restrictive features of the particular one-state-variable model and, in particular, the uniform sign of the term premiums.

In an attempt to address some of these issues, Longstaff (1989) introduced the double-square-root interest rate process. Unfortunately, Longstaff's bond pricing equation is the wrong solution of the problem that he sets up.[4]

[1] In the early and important paper by Vasicek (1977), the interest rate attains negative values and must be interpreted as the real but not as the nominal rate of interest. Nominal term structure models with nonnegative interest rate (or real models with zero inflation rate) were developed by Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1985), Dothan (1978), Heath, Jarrow, and Morton (1992), Longstaff and Schwartz (1992), and Richard (1978), among others. Aspects of the literature are reviewed in Cox, Ingersoll, and Ross (1981) and Singleton (1989).

[2] The translated version of the CIR model was discussed in the original CIR article and also in Marsh (1980) and Pearson and Sun (1990). Essentially, a positive constant is added to the original interest rate process, maintaining the positivity of the interest rate.

[3] The multivariate extension of the CIR model was discussed in the original CIR article and also in Chen and Scott (1990), Dybvig (1989), Heston (1989), and Oldfield and Rogalski (1987).

[4] This observation was first made by Beaglehole and Tenney (1990). In Longstaff's model, the term structure is driven by a state variable $X(t)$, which is reflected at the origin and which follows the stochastic differential equation $dX(t) = m\,dt + s\,dZ(t)$, for $X(t) > 0$, where m and s are constants, $m < 0$, and $Z(t)$ is a Wiener process. The simplest way to see the problem is to recognize that the price of a discount bond, $B(t, T)$, with maturity $T - t$ should satisfy the condition $\partial B(t, T)/\partial X = 0$ at $X = 0$. This follows from the fact that the expected change of the bond price at $X = 0$ is $E[dB(t, T)] \sim s(\partial B(t, T)/\partial X)(dt)^{1/2}$ and the expected rate of return of the bond is infinite unless $\partial B(t, T)/\partial X = 0$. Longstaff's bond pricing equation does not satisfy this boundary condition.

Motivated by the above considerations, we develop a model of the term structure that, even in its one-state-variable special case, is free from the restrictive features of the one-state-variable CIR model. First, the term premium may be positive at some states and negative at other ones. Second, the term premium may be positive for some bonds and negative for others at a given state. Third, the yield curve may have an inverted hump. This shape is permissible along with the more common shapes of uniformly rising, declining, or humped yield curves.

The starting point is the specification of a positive nominal state-price density process or pricing kernel guaranteed to exist by the absence of arbitrage in a frictionless market. The variables forcing the pricing kernel are modeled as squared autoregressive AR(1) independent processes. We refer to the squared-autoregressive-independent-variable nominal term structure model as SAINTS.

The model shares the appealing features of the CIR model. The interest rate is a positive and stationary process. The prices of discount bonds and European options on bonds are obtained in closed form. The conditional mean and autocovariance of the yield to maturity, forward rate, and bond return are obtained in closed form. The one-state-variable version of the model is parsimonious in the number of parameters (four), and each additional state variable introduces three parameters. In addition, the model makes no assumptions about the statistical relation between the real economic variables and the price level. Since the available term structure data in the United States are on nominal bond prices of nominal bonds, the model may be estimated and tested without a maintained hypothesis on the statistical relation between real variables and the price level.[5]

The article is organized as follows. In Section 1, the model is defined and nominal discount bonds are priced. The yield to maturity and the interest rate are discussed in Section 2. In Section 3, I discuss the term premium and its relation to the forward premium. In Section 4, I calibrate the one-state-variable version of the model and contrast its implications to those of the one-state-variable version of the CIR model. Options on bonds are priced in Section 5. Section 6 concludes the article.

1. The Model

We assert the existence of a positive state-price density process or pricing kernel, $M(t)$, such that the nominal price at time t of a claim

[5] In testing the CIR model, Gibbons and Ramaswamy (1987) and Heston (1989) had maintained the hypothesis that the price level deflator is conditionally independent of the marginal rates of substitution. Brown and Dybvig (1986). Chen and Scott (1990), and Pearson and Sun (1990) interpreted the CIR model as a no-arbitrage model of the nominal term structure and avoided this maintained hypothesis. Our approach of modeling directly the pricing kernel of the nominal term structure is in the spirit of Brown and Dybvig, Chen and Scott, and Pearson and Sun.

to a nominal payoff $X(T)$ at some future date T is given by

$$P(t) = E_t[X(T)M(T)]/M(t), \tag{1}$$

where $E_t[\cdot]$ denotes the expectation conditional on the information at time t. Harrison and Kreps (1979) derived the conditions under which $M(t)$ is unique.

We illustrate the meaning of the pricing kernel in a representative-consumer economy in which the consumer has Von Neumann-Morgenstern preferences U defined over the consumption path $\{C(t): 0 \le t\}$. In equilibrium, the first-order condition states

$$E_t\left[-\frac{\partial U}{\partial C(t)}\frac{P(t)}{\Pi(t)} + \frac{\partial U}{\partial C(T)}\frac{X(T)}{\Pi(T)}\right] = 0, \tag{2}$$

where $\Pi(t)$ is the price level. Defining $M(t)$ as

$$M(t) = \Pi^{-1}(t)E_t\left[\frac{\partial U}{\partial C(t)}\right], \tag{3}$$

the first-order condition yields (1). One approach would be to specify a functional form of the utility function and the time-series processes for the per capita consumption and price level and obtain the induced time-series process for the pricing kernel via (3). The alternative approach taken here is to explore directly the time-series process of $M(t)$, which yields plausible implications about the nominal term structure of interest rates. We stress that in this alternative approach it is unnecessary to assume a representative-consumer economy in which the consumer has Von Neumann-Morgenstern preferences.

The pricing kernel may be obtained as the equilibrium implication in a representative-consumer real production economy as in CIR by assuming that preferences are time-separable and logarithmic, $U = \int_0^\infty e^{-\rho t} \ln C(t)\, dt$, and that there is one constant-returns-to-scale technology with return on the capital $K(t)$ over $[t, t + dt]$, given by $K(t) \times [1 + M(t)\, d(M(t)^{-1})]$. The logarithmic preferences imply that the consumption rate equals $C(t) = r K(t)$. The dynamics of the capital is $dK(t) = K(t)M(t)\, d(M(t)^{-1}) - C(t)\, dt$ and the dynamics of consumption is $dC(t) = C(t)M(t)\, d(M(t)^{-1}) - r\, C(t)\, dt$. Hence $C(t) = e^{-\rho t}C(0)M(0)M(t)^{-1}$. The marginal utility of consumption at t is $e^{-\rho t}C(t)^{-1} = C(0)^{-1}M(0)^{-1}M(t)$ and the first-order condition (2) for real cash flows implies (1). We stress that in the alternative approach taken in this article it is unnecessary to assume a representative-consumer real production economy. Instead, we explore directly the time-series process of $M(t)$.

We define the process $M(t)$ in terms of $N + 1$ independent processes, $x_i(t)$, $i = 0, 1, \dots, N$, and constants g, σ_0^2, and a_i, $i = 1, \dots, N$, as

$$M(t) = \exp\left\{-\left(g + \frac{\sigma_0^2}{2}\right)t + x_0(t) + \sum_{i=1}^{N} (x_i(t) - \alpha_i)^2\right\}. \qquad (4)$$

This definition guarantees that $M(t)$ is positive. The rationale for this particular definition of the pricing kernel will become apparent later in this section.

The process $x_0(t)$ is defined as a Wiener process with variance σ_0^2 [i.e., we set $x_0(t) = W_0(t)$]. The expectation and variance of $x_0(t)$ conditional on $x_0(0)$ are given by

$$E_0[x_0(t)] = x_0(0) \qquad (5)$$

and

$$\mathrm{var}_0(x_0(t)) = \sigma_0^2 t. \qquad (6)$$

The process $x_i(t)$, $i = 1, \dots, N$, is a continuous-time AR(1) process (Ornstein-Uhlenbeck diffusion process) defined as the stochastic integral

$$x_i(t) = x_i(0)e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-\tau)}\, dW_i(\tau), \qquad i = 1, \dots, N, \qquad (7)$$

where $\lambda_i > 0$ and $W_i(t)$ is a Wiener process with variance σ_i^2. The $N + 1$ Wiener processes $W_i(t)$, $i = 0, \dots, N$, are assumed to be mutually independent. The mean and variance of $x_i(t)$ conditional on $x_i(0)$ are given by

$$E_0[x_i(t)] = x_i(0)e^{-\lambda_i t} + E_0\left[\int_0^t e^{-\lambda_i(t-\tau)}\, dW_i(\tau)\right] = x_i(0)e^{-\lambda_i t} \qquad (8)$$

and

$$\begin{aligned}\mathrm{var}_0(x_i(t)) &= \int_0^t \mathrm{var}_0\left\{e^{-\lambda_i(t-\tau)}\, dW_i(\tau)\right\} \\ &= \sigma_i^2 \int_0^t e^{-2\lambda_i(t-\tau)}\, d\tau \\ &= \frac{\sigma_i^2}{2\lambda_i}(1 - e^{-2\lambda_i t}).\end{aligned} \qquad (9)$$

We want the pricing kernel to imply that the nominal interest rate is positive. We restrict the model parameters as

$$\sigma_i^2 < \lambda_i, \qquad i = 1, \dots, N, \qquad (10)$$

and

$$g - \sum_{i=1}^{N}\left(\sigma_i^2 + \frac{\lambda_i\alpha_i^2}{2(1 - \sigma_i^2/\lambda_i)}\right) > 0. \tag{11}$$

We show in Section 2 that these conditions guarantee that the nominal interest rate is positive.

In pricing a nominal bond, we need to evaluate the time-r expectation of exp $\{x_0(t+\tau)\}$ and $\exp\{(x_i(t+\tau) - \alpha_i)^2\}$, $i = 1,\dots,N$. Since $x_0(t+\tau)$ is normally distributed with mean and variance given by (5) and (6), we derive in Appendix A the result

$$E_t[\exp\{x_0(t+\tau)\}] = \exp\{x_0(t) + \sigma_0^2\tau/2\}. \tag{12}$$

Since $(x_i(t+\tau) - \alpha_i)^2$ has a noncentral x^2 distribution with one degree of freedom, we derive also in Appendix A the result

$$E_t[\exp\{(x_i(t+\tau) - \alpha_i)^2\} = H_i^{-1/2}(\tau)\exp\{\lambda_i\tau + H_i^{-1}(\tau)(x_i(t) - \alpha_i e^{\lambda_i\tau})^2\},$$
$$i = 1,\dots,N, \tag{13}$$

where

$$H_i(\tau) = \sigma_i^2/\lambda_i + (1 - \sigma_i^2/\lambda_i)e^{2\lambda_i\tau} \geq 1. \tag{14}$$

The nominal price at time t of a default-free bond that pays one unit of account at date T is given by

$$\begin{aligned}
B(t, T) &= E_t[M(T)]/M(t) \\
&= E_t\Bigg[\exp\Bigg\{-(g + \sigma_0^2/2)(T - t) + x_0(T) - x_0(t) \\
&\qquad + \sum_{i=1}^{N}(x_i(T) - \alpha_i)^2 - \sum_{i=1}^{N}(x_i(t) - \alpha_i)^2\Bigg\}\Bigg] \\
&= \exp\{-(g + \sigma_0^2/2)(T - t)\}E_t[\exp(x_0(T) - x_0(t))] \\
&\quad \times \prod_{i=1}^{N} E_t[\exp\{(x_i(T) - \alpha_i)^2 - (x_i(t) - \alpha_i)^2\}] \\
&\quad [\text{by the independence of } x_i(T),\ i = 0,1,\dots,N] \\
&= \left\{\prod_{i=1}^{N} H_i(T - t)\right\}^{-1/2} \exp\Bigg\{\left(-g + \sum_{i=1}^{N}\lambda_i\right)(T - t) \\
&\qquad + \sum_{i=1}^{N} H_i^{-1}(T - t)\left(x_i(t) - \alpha_i e^{\lambda_i(T-t)}\right)^2 \\
&\qquad - \sum_{i=1}^{N}(x_i(t) - \alpha_i)^2\Bigg\}
\end{aligned} \tag{15}$$

[by (12)-(14)]. The economic implications of this term structure model are the focus of the next sections.

In pricing options on bonds, it is desirable to employ a term structure model that is sufficiently flexible to fit exactly the bond prices, at least on the date at which the option is being priced. CIR explain how this can be accomplished by making one of their model parameters a deterministic function of time. Related work includes Black, Derman, and Toy (1990), Dybvig (1989), Heath, Jarrow, and Morton (1992), Ho and Lee (1986), Hull and White (1990), and Jamshidian (1989). In the context of our model, this generalization may be implemented by redefining the pricing kernel as $M(t) = M(t) \exp\{-f(t)\}$, where $M(t)$ is defined in (4). Then the bond price is $B(t, T) = \exp\{-f(T) + f(t)\}\, B(t, T)$, where $B(t, T)$ is defined in (15). The interest rate is $\hat{r}(t) = r(t) + f'(t)$, where $r(t)$ is defined in (19) and is guaranteed to be positive at all times and states provided $\min_t[f'(t)] + k > 0$, where k is the left-hand side of (11).

It is useful to stress that throughout the article the processes $x_i(t)$ and expectations are under the true probability measure and not under the equivalent martingale measure. Under the true probability measure, Equation (5) implies $E_t[dx_0(t)] = 0$ and Equation (7) implies $E_t[dx_i(t)] = -\lambda_i x_i(t)\, dt$, $i = 1, \ldots, N$. For completeness, we present the equivalent martingale measure under which pricing is risk neutral. Consider an equivalent probability measure and denote by an asterisk the expectation under this measure. Let the expectation of $dx_i(t)$ under the equivalent measure be $E_t^*[dx_0(t)] = 0$ and $E_t^*[dx_i(t)] = \{-\lambda_i x_i(t) + 2\sigma_i^2(x_i - \alpha_i)\}dt$, $i = 1, \ldots, N$. Employing Equations (15) and (21), we may show that $E_t^*[dB(t, T)/B(t, T)] = r(t)\, dt$ and, therefore, $B(t, T) = E_t^*[\exp\{-\int_t^T r(s)\, ds\}]$. This illustrates that under the equivalent measure pricing is risk neutral.

We conclude this section by explaining why we model the pricing kernel as in (4). First, note that for $N = 0$ the bond price becomes $B(t\ T) = e^{-g(T-t)}$, implying constant interest rate and an unrealistic term structure. Thus, a richer specification of the pricing kernel is required.

Second, note that the state variable $x_0(t)$ does not appear in the bond pricing equation (15). For purposes of developing a theory of the nominal term structure of interest rates, we might just as well have set $\sigma_0^2 = 0$. However, the assumption $\sigma_0^2 = 0$ implies that the pricing kernel is stationary and the consumption and price level processes are stationary as well [unless the nonstationary components of the consumption and price level processes cancel fortuitously in (3)]. We set $\sigma_0^2 \geq 0$ to avoid introducing unnecessary assumptions about the stationarity of the consumption and price level processes.

Third, note that we do not allow for a squared term $-\gamma(x_0(t) -$

$a_o)2$, with γ, a_o constants, in the definition of $M(t)$. The reader may show that, for $-\gamma > 0$, the expectation $E_t[\exp\{-\gamma(x_0(T) - \alpha_0)^2\}]$ does not exist for $T - t > -1/2\gamma\sigma_0^2$; for $\gamma > 0$, the above expectation exists but the resulting bond price is an increasing function of $x_0^2(t)$. Thus, for a sufficiently large value of the state variable, $x_0^2(t)$, the bond price exceeds 1, implying negative interest rate. By contrast, the specification of the pricing kernel as in (4) with the assumption (10) guarantees that the expectation exists and the interest rate is bounded from below by the left-hand side of (11). Equation (11) guarantees that the interest rate is positive with probability 1. These arguments are made transparent in Section 2.

Fourth, we do not allow for a squared term $\gamma(x_i(t) - \alpha_i)^2$, with $\lambda_i > 0$ and a positive value of γ other than 1 in the definition of $M(t)$. The term $\gamma(x_i(t) - \alpha_i)^2$ with $\gamma > 0$ may always be written in the form $(x_j(t) - \alpha_j)^2$, where $\alpha_j = \gamma^{1/2}\alpha_i$, $\lambda_j = \lambda_i$, $\sigma_j^2 = \gamma\sigma_i^2$, and $x_j(t) = \gamma^{1/2}x_i(t)$. Therefore, the case with positive γ, other than 1, does not represent a generalization of our definition of M(t).

Fifth, we do not allow for a term $-\gamma(x_i(t) - \alpha_i)^2$, with $\lambda_i > 0$ and $\gamma > 0$. The reader may retrace the proof of Equation (13) in Appendix A and derive the result that the bond price $B(t, T)$ is proportional to

$$\exp\left\{\gamma(x_i(t) - \alpha_i)^2 - \frac{\gamma(x_i(t)e^{-\lambda_i(T-t)} - \alpha_i)^2}{1 + 2\gamma s^2}\right\},$$

where

$$s^2 = (\sigma_i^2/2\lambda_i)(1 - e^{-2\lambda_i(T-t)}).$$

For sufficiently large values for $x_i(t)$ and term to maturity, $T - t$, the bond price exceeds 1 and implies negative interest rate with positive probability.

Finally, we do not allow for a linear term $x_i(t)$ in the exponent of the pricing kernel, with $\lambda_i > 0$. This specification again implies that the interest rate attains negative values with positive probability.[6] In our specification of the pricing kernel, a linear term does appear as part of the quadratic form $(x_i(t) - a_i)2$ but in this case the interest rate is positive with probability 1.

2. The Yield to Maturity

The nominal yield to maturity is defined as

$$y(t, T) = -(T - t)^{-1} \ln B(t, T). \tag{16}$$

[6] In the context of a real term structure model, Campbell (1986) considered an ARIMA specification of the real consumption process that amounts to the specification of a pricing operator of the form $M(t) = \exp\{x_i(t)\}$ with $x_i(t)$ being an ARIMA process. The real spot rate attains negative values with positive probability. Clearly, this model does not have a useful interpretation as a model of the nominal term structure. The model also implies that the short-term expected rate of return of a bond in excess of the interest rate is independent of the *state*.

With the aid of (15), we express the yield to maturity in terms of the state variables as

$$y(t, T) = g - \sum_{i=1}^{N} \lambda_i + (T - t)^{-1} \times \sum_{i=1}^{N} \left\{ \frac{1}{2} \ln H_i(T - t) - H_i^{-1}(T - t)(x_i(t) - \alpha_i e^{\lambda_i(T - t)})^2 + (x_i(t) - \alpha_i)^2 \right\}. \tag{17}$$

Since the state variables are unobservable, one way to estimate the model parameters and test the model is to fit to the data the unconditional moments of the yield to maturity. This general approach was initiated by Gibbons and Ramaswamy (1987) in estimating and testing the CIR model. In Appendix A, we present the unconditional mean and autocovariance of the yield to maturity implied by (17).

As the term to maturity tends to infinity, the bond yield tends to the constant g irrespective of the state. Thus, the parameter g has the economic interpretation as the yield of a long-maturity discount bond. The yield curves implied by our model for $N = 1$ are discussed in Section 4. In particular, Figure 2 shows that the yield curve may be inverted humped.

The nominal interest rate is defined as

$$r(t) = \lim_{T \to t} y(t, T). \tag{18}$$

Evaluating the above expression we obtain

$$r(t) = g + \sum_{i=1}^{N} \left\{ -\sigma_i^2 - \frac{\lambda_i \alpha_i^2}{2(1 - \sigma_i^2/\lambda_i)} + 2\lambda_i \left(1 - \frac{\sigma_i^2}{\lambda_i}\right) \times \left(x_i(t) - \alpha_i + \frac{\alpha_i}{2(1 - \sigma_i^2/\lambda_i)}\right)^2 \right\}. \tag{19}$$

The minimum value of the interest rate occurs at $x_i(t) - \alpha_i + \alpha_i/2(1 - \sigma_i^2/\lambda_i) = 0$, $i =$ 1,...,N, and is guaranteed to be positive by (11). The distribution of the interest rate is that of the sum of squared normally distributed variables with different variances.

We compare the interest rate process for $N = 1$ with the interest rate process implied by the one-state-variable version of the translated CIR model. For $N = 1$, the stochastic differential equation of the interest rate is

$$dr(t) = 2\lambda\left(1 - \frac{\sigma^2}{\lambda}\right)\left\{\sigma^2 - 2\lambda x\left(x - \alpha + \frac{\alpha}{2(1 - \sigma^2/\lambda)}\right)\right\} dt$$
$$+ 4\lambda\left(1 - \frac{\sigma^2}{\lambda}\right)\left(x - \alpha + \frac{\alpha}{2(1 - \sigma^2/\lambda)}\right) dW(t), \quad N = 1. \quad (20)$$

This interest rate process neither nests nor is nested by the interest rate process of the translated CIR model. Also, in our model, knowledge of $r(t)$ determines the absolute value, but not the sign of $x - \alpha + \alpha/2(1 - \sigma^2/\lambda)$. Therefore, knowledge of $r(t)$ does not determine unambiguously the drift of the interest rate process and $r(t)$ is not a sufficient statistic of the state, except in the special case ($a = 0$.

3. The Term Premium

The instantaneous, arithmetic expected bond return is defined as $R(t, T) = E_t[dB(t, T)]/B(t, T)\, dt$. The arithmetic term premium is defined as $R(t, T) - r(t)$ and is given by

$$R(t, T) - r(t)$$
$$= 4 \sum_{i=1}^{N} [\sigma_i^2(x_i - \alpha_i)\{(1 - H_i^{-1}(T - t))x_i$$
$$- \alpha_i(1 - H_i^{-1}(T - t)e^{\lambda_i(T-t)})\}]. \quad (21)$$

The term premium may be positive at some states and negative at other ones. Also, for a given state, the term premium may be positive for some bonds and negative for others. This statement is true for $N \geq 1$. To see this, write (21) for $N = 1$ as

$$R(t, T) - r(t) = 4\sigma^2\alpha^2(1 - H^{-1}(T - t))\left(\frac{x}{\alpha} - 1\right)$$
$$\times \left(\frac{x}{\alpha} - \frac{1 - H^{-1}(T - t)e^{\lambda(T-t)}}{1 - H^{-1}(T - t)}\right), \qquad N = 1. \quad (22)$$

Note that $R(t, T) - r(t)$ is negative for

$$\frac{1 - H^{-1}(T - t)e^{\lambda(T-t)}}{1 - H^{-1}(T - t)} < \frac{x}{\alpha} < 1, \qquad (23)$$

and positive otherwise.

For a given maturity bond, if the state satisfies (23), the term premium is negative; otherwise, it is positive. Thus, for a given maturity bond, the sign of the term premium may switch, depending on the state. This contrasts with the implication of the translated CIR model that the term premium is of uniform sign at all states.

If the state satisfies $x/\alpha > 1$ or $x/\alpha < 1 - 1/2(1 - \sigma^2/\lambda)$, then (23) is not satisfied for any bond maturity and the term premium is positive for all maturities. However, if the state satisfies $1 - 1/2(1 - \sigma^2/\lambda) < x/\alpha < 1$, then short-maturity bonds have a negative term premium and long maturity bonds have a positive premium. This contrasts with the implication of the translated CIR model that for any given state the term premium of all maturity bonds is of uniform sign.

We define the instantaneous, geometric expected bond return as $R_G(t, T) = E_t[d \ln B(t, T)]/dt$; and the geometric term premium as $R_G(t, T) - r(t)$. As before, we may show that for $N \geq 1$, the geometric term premium may be positive or negative depending on the state and on the term to maturity.

We also define the forward rate as $F(t, T) = -\partial \ln B(t, T)/\partial T$. We may write

$$
\begin{aligned}
&R_G(t, T) - F(t, T) \\
&\quad = -\sum_{i=1}^{n} [2\lambda_i x_i\{-H_i^{-1}(T - t)(x_i(t) - \alpha_i e^{\lambda_i(T-t)}) + x_i(t) - \alpha_i\} \\
&\qquad\qquad + \sigma_i^2(H_i^{-1}(T - t) - 1)]. \qquad (24)
\end{aligned}
$$

Since the right-hand side of (24) is not necessarily equal to zero, we conclude that the forward rate does not equal the expected geometric return of the matching-maturity bond. This property of the model is supported by the empirical evidence in Stambaugh (1988).

Taking the unconditional expectation of $RG(t, T) - F(t, T)$, we find that

$$E[R_G(t, T) - F(t, T)] = 0. \qquad (25)$$

Therefore, the matching-maturity forward rate is an unbiased estimator of the expected rate of bond return. The reader may verify that the unbiasedness property in (25) is also satisfied in the CIR and Vasicek (1977) models of the term structure and possibly in other models as well.

4. Calibration

We calibrate the one-state-variable version of the model to match the minimum interest rate, the unconditional mean and standard deviation of the interest rate, and the unconditional mean of the yield and of the term premium of a long maturity bond. The model implies the

Table 1
Calibration of the one-state-variable version of the model minimizing the sum of absolute errors of the five theoretical population moments from their sample counterparts

Moment	Sample value (per year)	Fitted value (per year)
$\min[r]$	.000	.000
$E[r]$	.036	.041
$SD(r)$	.033	.033
$E[y(t, \infty)]$	.051	.046 (.047)[1]
$E[R(t, \infty) - r(t)]$	.011	.011 (.010)[1]

The fitted parameter values are g = 0.0462/year, λ = 0.2169/year, s = 0.0873/(year)$^{1/2}$, and a = 0.5858.
[1] Fitted value for a 10-year discount bond.

following values for these targets:

$$\min[r] = g - \sigma^2 - \frac{\lambda\alpha^2}{2(1 - \sigma^2/\lambda)}, \tag{26}$$

$$E[r] = g - \sigma^2 - \frac{\lambda\alpha^2}{2(1 - \sigma^2/\lambda)} + 2(\lambda - \sigma^2)\left\{\frac{\sigma^2}{2\lambda} + \alpha^2\left(1 - \frac{1}{2(1 - \sigma^2/\lambda)}\right)^2\right\}, \tag{27}$$

$$SD(r) = 2(\lambda - \sigma^2)\left\{\frac{\sigma^4}{2\lambda^2} + \frac{2\alpha^2\sigma^2}{\lambda}\left(1 - \frac{1}{2(1 - \sigma^2/\lambda)}\right)^2\right\}^{1/2}, \tag{28}$$

$$E[y(t, \infty)] = g, \tag{29}$$

and

$$E[R(t, \infty) - r(t)] = 4\sigma^2\left(\frac{\sigma^2}{2\lambda} + \alpha^2\right). \tag{30}$$

As proxies for the population moments, we use the sample moments in the period 1926-1988 taken from the *SBBI 1989 Yearbook* by Ibbotson Associates. Whereas structural changes in the term structure may well have occurred in the early 1950s, we use the entire 1926-1988 sample in order to have a longer sample. Since we have fewer parameters (four) than targets (five), we minimize the sum of the absolute errors of the five theoretical population moments from their sample counterparts. The results are presented in Table 1. Given the fitted parameter values, I also calculated the unconditional mean of the yield to maturity and of the term premium of a 10-year discount bond and found that these moments are practically the same as those calculated for the infinite-maturity bond.

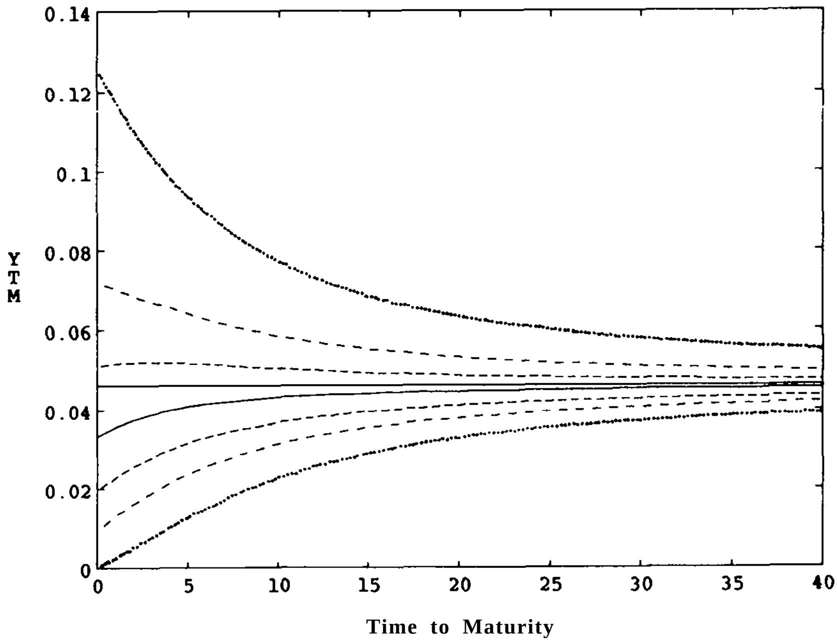

Figure 1
Yield curves implied by the one-state-variable version of the model at values of the state variable equal to its unconditional mean (0), half a standard deviation (±0.07), one standard deviation (±0.13), and two standard deviations (±0.27) away from the mean
The fitted parameter values are g = 0.0462/year, λ = 0.2169/year, (σ = 0.0873/(year)$^{1/2}$, and a = 0.5858.

Using the fitted parameter values and Equation (19), we find that the interest rate attains its minimum value of zero at the state $x = \alpha(1 - 1/2(1 - \sigma^2/\lambda)) = .3032$. The unconditional mean of x is zero and its standard deviation is $(\sigma^2/2\lambda)^{1/2} = .13$, implying that the interest rate is zero when the state variable is about two standard deviations away from its mean. Thus, the occurrence of interest rates close to zero between the mid 1930s and mid 1940s is a plausible event.

Of greatest interest is the fitted value of the parameter λ, which states that the half-life of the state variable is $\lambda^{-1} \ln 2 = 3.2$ years and is of the same order of magnitude as the length of the business cycle.

In Figure 1, I present the yield curves at the calibrated parameter values and values of the state variable equal to its unconditional mean of zero, one-half, one, and two standard deviations away from the unconditional mean. We observe that the common shapes of the yield curve are uniformly rising or declining. In Figure 2, I present two yield curves at the calibrated parameter values. At $x = .65$, the yield curve is inverted humped, a pattern not allowed in the CIR one-state-variable model. At $x = -.04$, the yield curve is humped.

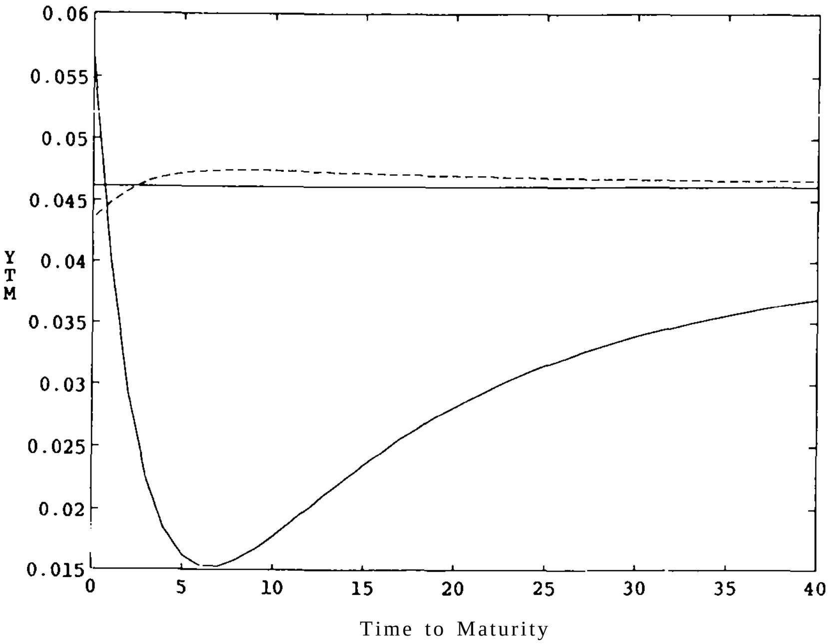

Figure 2
Yield curves implied by the one-state-variable version of the model
At $x = -0.04$, the yield curve is humped. At $x = 0.65$, the yield curve is inverted humped, illustrating a shape of the yield curve that is disallowed by the onestate-variable version of the CIR model. The fitted parameter values are $g = 0.0462$/year, $\lambda = 0.2169$/year, $\sigma = 0.0873/(\text{year})^{1/2}$, and $a = 0.5858$.

5. Options on Bonds

Define by $P(t, T)$ the nominal price at time zero of a European call option that matures at time t, has exercise price K, and is written on a default-free discount bond that matures at time T, $0 \le t \le T$. The option price at its maturity is $\max[0, B(t, T) - K]$, and the option price at time zero is

$$P(t, T) = E_0[\{M(t)/M(0)\}\max[0, B(t, T) - K]]. \qquad (31)$$

The random variables that appear on the right-hand side are $x_i(t)$, $i = 0,1,\ldots, N$. They are independently and normally distributed with known mean and variance conditional on $x_i(0)$, $i = 0,1,\ldots, N$.

In the special case $N = 1$, ($a = 0$, we express the option price in terms of the standard normal distribution function. We write (31) as

$$\begin{aligned} P(t, T) = E_0[&\exp\{-gt + x^2(t) - x^2(0)\} \\ &\times \max[0, -K + H^{-1/2}(T - t) \\ &\qquad \times \exp\{(\lambda - g)(T - t) - (1 - H^{-1}(T - t))x^2(t)\}]]. \end{aligned} \qquad (32)$$

The bond price $B(t, T)$ attains its maximum value at $x(t) = 0$. If $H^{-1/2}(T-t)\exp\{(\lambda - g)(T-t)\} \leq K$, the option is out-of-the-money at time t for all values of $x(t)$, and the option is worthless at time zero. Consider then the nontrivial case

$$H^{-1/2}(T-t)\exp\{(\lambda - g)(T-t)\} > K \tag{33}$$

and define the parameter $\hat{x}^2$ by

$$H^{-1/2}(T-t)\exp\{(\lambda - g)(T-t) - (1 - H^{-1}(T-t))\hat{x}^2\} = K. \tag{34}$$

Equation (33) guarantees that $\hat{x}^2 > 0$. Let $\hat{x}$ denote the positive square root of $\hat{x}^2$, that is,

$$\hat{x} = \left\{\frac{(\lambda - g)(T-t) - \frac{1}{2}\ln H(T-t) - \ln K}{1 - H^{-1}(T-t)}\right\}^{1/2} > 0. \tag{35}$$

The option price is

$$\begin{aligned} P(t, T) = {} & H^{-1/2}(T-t)\exp\{\lambda(T-t) - gT - x^2(0)\} \\ & \times \int_{-\hat{x}}^{\hat{x}} \exp\{H^{-1}(T-t)x^2\} f(x)\, dx \\ & - K\exp\{-gt - x^2(0)\} \int_{-\hat{x}}^{\hat{x}} \exp(x^2) f(x)\, dx, \end{aligned} \tag{36}$$

where $f(\cdot)$ is the probability density of $x(t)$ conditional on $x(0)$. Recall that $x(t)$ is normally distributed with mean $\mu = x(0)e^{-\lambda t}$ and variance $s^2 = (\sigma^2/2\lambda)(1 - e^{-2\lambda t})$. The integrals are evaluated in Appendix B in terms of the standard normal distribution function,

$$N(\hat{z}) \equiv (2\pi)^{-1/2}\int_{-\infty}^{\hat{z}} \exp\left(\frac{-z^2}{2}\right) dz.$$

The option price is

$$\begin{aligned} P(t, T) = {} & B(0, T)\left[N\left(\frac{\hat{x} - a}{b}\right) - N\left(\frac{-\hat{x} - a}{b}\right)\right] \\ & - B(0, t)\left[N\left(\frac{\hat{x} - a'}{b'}\right) - N\left(\frac{-\hat{x} - a'}{b'}\right)\right], \end{aligned} \tag{37}$$

where

$$a = \frac{x(0)e^{-\lambda t}}{1 - H^{-1}(T-t)(\sigma^2/\lambda)(1 - e^{-2\lambda t})},$$

$$b = \left[\frac{(\sigma^2/2\lambda)(1 - e^{-2\lambda t})}{1 - H^{-1}(T-t)(\sigma^2/\lambda)(1 - e^{-2\lambda t})}\right]^{1/2}, \tag{38}$$

$$a' = \frac{x(0)e^{-\lambda t}}{1 - (\sigma^2/\lambda)(1 - e^{-2\lambda t})},$$

$$b' = \left[\frac{(\sigma^2/2\lambda)(1 - e^{-2\lambda t})}{1 - (\sigma^2/\lambda)(1 - e^{-2\lambda t})}\right]^{1/2}.$$

6. Directions for Future Research

Since the model yields closed-form expressions for the unconditional moments and autocovariances of the forward rates and bond returns, it may be estimated by the general method of moments as in Gibbons and Ramaswamy (1987) and Heston (1989). Since the model also yields a closed-form expression for the likelihood function of bond prices, it may be estimated by the maximum likelihood method as in Brown and Dybvig (1986), Chen and Scott (1990), and Pearson and Sun (1990).

Fama and Bliss (1987), Heston (1989), Huang and Lin (1990), and Stambaugh (1988) investigated the information in the forward rates in predicting bond returns. With the exception of Huang and Lin, these investigations focused on forecasts that are linear functions of the forward rates, as implied by the multivariate extension of the untranslated CIR model. My model implies that the forecasts are linear functions of the forward rates only in the special case $a_i = 0$, $i = 1, \ldots, N$. In the general case, my model implies that the forecasts of bond returns are quadratic functions of the forcing variables and the forward rates are also quadratic functions of the forcing variables. It is of interest then to investigate the nonlinear implications of my model.

Appendix A: Derivation of Various Results Stated in the Main Text

Derivation of Equation (12)

If x is normally distributed with mean μ and variance s^2, its moment generating function is

$$E[e^{\gamma x}] = \exp\{\gamma\mu + \tfrac{1}{2}\gamma^2 s^2\}. \tag{A1}$$

[See Johnson and Kotz (1970, Volume 1, p. 49).] Setting $\gamma = 1$, $\mu = x_0(t)$, and $s^2 = \sigma_0^2\tau$, we obtain (12).

Derivation of Equation (13)

Consider the integral $\int_{-\infty}^{\hat{x}} e^{-\gamma x^2} f(x)\,dx$, where $f(x)$ is the normal density function and x has mean μ and variance s^2. Define

$$a = \mu/(1 + 2\gamma s^2) \tag{A2}$$

and

$$b = s(1 + 2\gamma s^2)^{-1/2}, \qquad \text{provided } 1 + 2\gamma s^2 > 0. \tag{A3}$$

Then

$$-\gamma x^2 - \frac{(x-\mu)^2}{2s^2} = -\frac{(x-a)^2}{2b^2} - \frac{\gamma\mu^2}{1 + 2\gamma s^2}.$$

In terms of $z = (x - a)/b$ the integral is

$$\frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\hat{x}} \exp\left\{-\gamma x^2 - \frac{(x-\mu)^2}{2s^2}\right\} dx$$
$$= (1 + 2\gamma s^2)^{-1/2} \exp\left\{\frac{-\gamma\mu^2}{1 + 2\gamma s^2}\right\} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\hat{x}-a)/b} e^{-z^2/2}\,dz. \tag{A4}$$

We set $\gamma = -1$, $\mu = x_i(t)e^{-\lambda_i\tau} - \alpha_i$, $s^2 = (\sigma_i^2/2\lambda_i)(1 - e^{-2\lambda_i\tau})$, and $\hat{x} = \infty$. The restriction $1 + 2\gamma s^2 > 0$ becomes $1 - (\sigma_i^2/\lambda_i)(1 - e^{-2\lambda_i\tau}) > 0$ and is satisfied for all $\tau \geq 0$ since $\sigma_i^2 < \lambda_i$ by assumption. Therefore, the expectation exists. With the above parameter values (A4) yields (13).

The unconditional yield to maturity

The unconditional mean and variance of $x_i(t)$ are zero and $\sigma_i^2/2\lambda_i$ as given by (8) and (9) in the limit as $t \to \infty$. Expanding the quadratic terms in (17) and taking the expectation, we obtain

$$E[y(t, T)] = g - \sum_{i=1}^{N} \lambda_i + (T - t)^{-1}$$
$$\times \sum_{i=1}^{N} \left\{\frac{1}{2} \ln H_i(T - t) - H_i^{-1}(T - t)\left(\frac{\sigma_i^2}{\lambda_i} + \alpha_i^2 e^{2\lambda_i(T-t)}\right)\right.$$
$$\left. + \frac{\sigma_i^2}{\lambda_i} + \alpha_i^2\right\}. \tag{A5}$$

The conditional moments of $x(t + \tau)$

Conditional on $x(t)$, $x(t + \tau)$ is normally distributed with mean $\mu = x(t)e^{-\lambda\tau}$ and variance $s^2 = (\sigma^2/2\lambda)(1 - e^{-2\lambda\tau})$. See (8) and (9). By

(A1), the moment generating function of $x(t+\tau)$ is

$$E_t[\exp\{\gamma x(t+\tau)\}] = \exp\{\gamma x(t)e^{-\lambda\tau} + \gamma^2(\sigma^2/4\lambda)(1 - e^{-2\lambda\tau})\}. \quad \text{(A6)}$$

Differentiating (A6) twice with respect to γ and setting $\gamma = 0$, we obtain

$$E_t[x^2(t+\tau)] = x^2(t)e^{-2\lambda\tau} + (\sigma^2/2\lambda)(1 - e^{-2\lambda\tau}). \quad \text{(A7)}$$

Differentiating (A6) four times with respect to γ and setting $\gamma = 0$, we obtain

$$\begin{aligned} E_t[x^4(t+\tau)] = {} & x^4(t)e^{-4\lambda\tau} + (3\sigma^2/\lambda)x^2(t)e^{-2\lambda\tau}(1 - e^{-2\lambda\tau}) \\ & + 3(\sigma^2/2\lambda)^2(1 - e^{-2\lambda\tau})^2. \end{aligned} \quad \text{(A8)}$$

The conditional variance of $x^2(t+\tau)$ is

$$\begin{aligned} \operatorname{var}_t(x^2(t+\tau)) &= E_t[x^4(t+\tau)] - \{E_t[x^2(t+\tau)]\}^2 \\ &= 4(\sigma^2/2\lambda)x^2(t)e^{-2\lambda\tau}(1 - e^{-2\lambda\tau})^2 \\ &\quad + 2(\sigma^2/2\lambda)^2(1 - e^{-2\lambda\tau})^2. \end{aligned} \quad \text{(A9)}$$

The process $x^2(t)$ has heteroskedastic increments and the variance of $x^2(t+\tau) - x^2(t)$, conditional on $x^2(t)$, is linear and increasing in $x^2(t)$. The unconditional variance of $x^2(t)$ is $\sigma^4/2\lambda^2$.

The autocovariance of $x(t)$

We define h by

$$\eta = x(t+\tau) - x(t)e^{-\lambda\tau}. \quad \text{(A10)}$$

Since $E_t[h] = 0$, then $E[h\,x(t)] = 0$ and cov(h, $x(t)$> = 0. Then

$$\begin{aligned} \operatorname{cov}(x(t+\tau), x(t)) &= \operatorname{cov}(\eta + x(t)e^{-\lambda\tau}, x(t)) \\ &= e^{-\lambda\tau}\operatorname{var}(x(t)) \\ &= (\sigma^2/2\lambda)e^{-\lambda\tau}. \end{aligned} \quad \text{(A11)}$$

The autocovariance of $x^2(t)$

We define h by

$$\eta = x^2(t+\tau) - x^2(t)e^{-2\lambda\tau} - (\sigma^2/2\lambda)(1 - e^{-2\lambda\tau}). \quad \text{(A12)}$$

By (A7), we have $E_t[h] = 0$, $E[h\,x^2(t)] = 0$, and cov(h, $x^2(t)$) = 0. Then

$$\begin{aligned} & \operatorname{cov}(x^2(t+\tau), x^2(t)) \\ & \quad = \operatorname{cov}(\eta + x^2(t)e^{-2\lambda\tau} + (\sigma^2/2\lambda)(1 - e^{-2\lambda\tau}), x^2(t)) \end{aligned}$$

$$= e^{-2\lambda\tau}\operatorname{var}(x^2(t))$$
$$= (\sigma^4/2\lambda^2)e^{-2\lambda\tau}. \tag{A13}$$

The autocovariance of the yield to maturity
We expand the quadratic terms in (17) and apply (A11) and (A13). Note that $\operatorname{cov}(x(t+\tau), x^2(t)) = 0$ since $x(t+\tau)$ is symmetric and mean zero. Likewise, $\operatorname{cov}(x^2(t+\tau), x(t)) = 0$. We readily obtain

$$\begin{aligned}
&\operatorname{cov}(y(t, T), y(t+\tau, T+\tau)) \\
&\quad = (T-t)^{-2}\sum_{i=1}^{N}\left\{2(1 - H_i^{-1}(T-t))^2\left(\frac{\sigma_i^2}{2\lambda_i}\right)^2 e^{-2\lambda_i\tau}\right. \\
&\qquad\qquad \left. + 4(H_i^{-1}(T-t)e^{\lambda_i(T-t)} - 1)^2\left(\frac{\sigma_i^2}{2\lambda_i}\right)\alpha_i^2 e^{-\lambda_i\tau}\right\}.
\end{aligned} \tag{A14}$$

Appendix B: Derivation of the Price of an Option on a Bond

We evaluate the integrals in (36) using (A4) derived in Appendix A. Setting $\gamma = -H^{-1}(T-t)$ in (A4), we obtain

$$\begin{aligned}
&H^{-1/2}(T-t)\exp\{\lambda(T-t) - gT - x^2(0)\} \\
&\quad \times \int_{-\hat{x}}^{\hat{x}} \exp\{H^{-1}(T-t)x^2\}f(x)\,dx \\
&\quad = H^{-1/2}(T-t)(1 - 2H^{-1}(T-t)s^2)^{-1/2} \\
&\qquad \times \exp\left\{\lambda(T-t) - gT - x^2(0) + \frac{H^{-1}(T-t)\mu^2}{1 - 2H^{-1}(T-t)s^2}\right\} \\
&\qquad \times \left[N\left(\frac{\hat{x}-a}{b}\right) - N\left(\frac{-\hat{x}-a}{b}\right)\right] \\
&\quad = B(0, T)\left[N\left(\frac{\hat{x}-a}{b}\right) - N\left(\frac{-\hat{x}-a}{b}\right)\right],
\end{aligned} \tag{B1}$$

where

$$\begin{aligned}
a &= \mu/(1 + 2\gamma s^2) \\
&= \frac{x(0)e^{-\lambda t}}{1 - H^{-1}(T-t)(\sigma^2/\lambda)(1 - e^{-2\lambda t})}
\end{aligned} \tag{B2}$$

and

$$\begin{aligned} b &= s(1 + 2\gamma s^2)^{-1/2} \\ &= \left[\frac{(\sigma^2/2\lambda)(1 - e^{-2\lambda t})}{1 - H^{-1}(T - t)(\sigma^2/\lambda)(1 - e^{-2\lambda t})} \right]^{1/2}. \end{aligned} \tag{B3}$$

The last line of (B1) follows from the fact that

$$\begin{aligned} &H(T - t)(1 - 2H^{-1}(T - t)s^2) \\ &\quad = \sigma^2/\lambda + (1 - \sigma^2/\lambda)e^{2\lambda(T-t)} - (\sigma^2/\lambda)(1 - e^{-2\lambda t}) \\ &\quad = e^{-2\lambda t}\{\sigma^2/\lambda + (1 - \sigma^2/\lambda)e^{2\lambda T}\} \\ &\quad = e^{-2\lambda t}H(T). \end{aligned}$$

Then the last line of (B1) becomes

$$\begin{aligned} &H^{-1/2}(T)\exp\{\lambda t + \lambda(T - t) - gT - x^2(0) + H^{-1}(T)e^{2\lambda t}x^2(0)e^{-2\lambda t}\} \\ &\quad \times \left[N\left(\frac{\hat{x} - a}{b}\right) - N\left(\frac{-\hat{x} - a}{b}\right) \right] \\ &\qquad = B(0, T)\left[N\left(\frac{\hat{x} - a}{b}\right) - N\left(\frac{-\hat{x} - a}{b}\right) \right]. \end{aligned}$$

This gives the first term of (37).

Setting $\gamma = -1$ in (A4), we obtain

$$\begin{aligned} &K\exp\{-gt - x^2(0)\} \int_{-\hat{x}}^{\hat{x}} \exp(x^2)f(x)\,dx \\ &\quad = K(1 - 2s^2)^{-1/2}\exp\left\{-gt - x^2(0) + \frac{\mu^2}{1 - 2s^2}\right\} \\ &\qquad \times \left[N\left(\frac{\hat{x} - a'}{b'}\right) - N\left(\frac{-\hat{x} - a'}{b'}\right) \right] \\ &\quad = B(0, t)K\left[N\left(\frac{\hat{x} - a'}{b'}\right) - N\left(\frac{-\hat{x} - a'}{b'}\right) \right], \end{aligned} \tag{B4}$$

where

$$\begin{aligned} a' &= \mu/(1 - 2s^2) \\ &= \frac{x(0)e^{-\lambda t}}{1 - (\sigma^2/\lambda)(1 - e^{-2\lambda t})} \end{aligned} \tag{B5}$$

and

$$\begin{aligned} b' &= s(1 - 2s^2)^{-1/2} \\ &= \left[\frac{(\sigma^2/\lambda)(1 - e^{-2\lambda t})}{1 - (\sigma^2/\lambda)(1 - e^{-2\lambda t})} \right]^{1/2}. \end{aligned} \tag{B6}$$

The last line of (B4) follows from the fact that

$$\begin{aligned}(1 - 2s^2)^{-1/2}&\exp\left\{-gt - x^2(0) + \frac{\mu^2}{1-2s^2}\right\} \\ &= \{1 - (\sigma^2/\lambda)(1 - e^{-2\lambda t})\}^{-1/2} \\ &\quad \times \exp\left\{-gt - x^2(0) + \frac{x^2(0)e^{-2\lambda t}}{1 - (\sigma^2/\lambda)(1 - e^{2\lambda t})}\right\} \\ &= B(0, t).\end{aligned}$$

This gives the second term of (37).

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