Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World

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Two homogeneous stocks of physical capital are located in two different countries, separated by an "ocean." They are consumed by local residents, invested in a random production process yielding real returns, or transferred abroad Under proportional transfer costs, trade, consumption and capital imbalances* are shown to be persistent. *The heteroskedastic process for the relative price of capital in the two countries has a nonlinear, mean-reverting drift. Nevertheless,* the conditional probability of the price moving from the parity value of unity is greater than the probability of it moving toward parity. *The real interest-rate differential incorporates a simple risk premium.

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The exchange rate is determined in the stock market.

-R. Dornbusch, *Devaluation, Money and Non-traded Goods*

Many goods have an asset character in that some are storable, some can be invested in a production process, and others are consumer durables. When a good has asset characteristics, the dynamic behavior of its price is not simply driven by current flow supply and demand. Instead, it tends to follow a quasimartingale process, like the prices of securities in an efficient market. which is the by-product of the equilibrium in the capital market. In Roll's (1979) "ex ante PPP" theory, the martingale principle applies to the relative price of storable goods located in two different countries. By extension, it applies also to physical capital and to any commodity that is a close substitute for physical capital. The implications of this insight for equilibrium models of the international capital market have not been explored previously.

The streamlined "real" capital-market equilibrium model of this article contains the following features. Two countries are endowed with identical technologies, but uncorrelated productivity shocks affect the two outputs. The two populations, although possessing identical utility functions, are placed in different circumstances because the consumers of each country only have access to goods physically available at home. Transferring goods and/or physical capital from one country to the other is costly.[1] I solve for the joint equilibrium in the goods and capital market and obtain the behavior of capital transfers between the countries (the balance of trade), consumption, investment, the "real exchange rate" (defined as the relative price of physical capital between the two countries), and the real interest-rate differential.

The conditional expected change in the real exchange rate is shown to exhibit mean reversion toward the parity value of unity. Nonetheless, except at some boundary points, the conditional probability of the deviation from parity widening is always larger than the probability of it narrowing. The exchange rate spends most of the time away from parity, close to the boundary situation where a transfer of capital from one country to the other occurs. This means that deviations from parity, even though they do not last forever, nonetheless last *a very long time*. The theory helps to explain why econometric attempts at discerning mean reversion in real exchange rates have been largely unsuccessful.

[1] Black's (1973) model, which is deterministic, is a forerunner of this approach; it is expressed in terms of continuous-time dynamics as will be my model.

No distinction is made within each country between investment goods and consumption goods. Furthermore, I assume constant returns to scale (no rents). Hence, by construction, the relative price of goods is also the relative price of a unit of equity issued by home versus foreign firms. Like earlier models of international asset pricing [see Solnik (1974)) Stulz (1981a), Adler and Dumas (1983)) and Benninga and Protopapadakis (1988)], this one incorporates the fact that investors of different countries *consume goods available in their country of residence* and, therefore, when purchasing power parity does not hold (at the level of consumer prices), evaluate real returns from their investments differently. Earlier models were partial equilibrium models in that the stochastic processes for the prices of goods were given. They also did not incorporate explicitly variations in the several national rates of interest and in the distribution of wealth across countries.[2]

Much of the methodology of the present article is borrowed from the literature dealing with portfolio choice under transaction costs. Grossman and Laroque (1990) considered fixed transactions costs. Constantinides (1986) studied a problem of proportional costs, but setting consumption as a fixed fraction of one kind of asset. Dumas and Luciano (1991) also studied a problem with proportional costs, but allowing for terminal consumption only. In the present model, consumption is chosen optimally as a variable fraction and drawn out of the two stocks of goods. The optimal choice of consumption plays a central role in our model.

The outline of the article is as follows. In Sections 1 and 2, I state and solve the mathematical programming problem that leads to the linchpin of the model-namely, the indirect welfare function for various types of wealth. In Section 3, I describe how the goods move across the world in general equilibrium. The main result-namely, the process for deviations from the Law of One Price-is featured in Section 4. In Section 5, I examine the consequences of deviations from price parity for differences in real interest rates between countries.

1. The Model

Consider an economy populated with consumers who are identical, except that they reside (in equal numbers) in two different geograph-

[2] In addition, Solnik (1974) and Adler and Dumas (1983) admit only a constant rate of change for asset prices and real exchange rates. This does not incorporate the current setting, although their focus on national investor classes is identical to mine. Stulz (1981a) and Benninga and Protopapadakis (1988) include the current model as a special case; however. my model is solved out for asset price behavior over time, whereas their CAPMs are not.

ical locations (countries) and can only consume goods physically available in their country of residence. However, they can own and trade stocks of goods physically located abroad. There are two varieties of a single good differentiated by their current physical location at any given time. In both locations, the good in question can be consumed, invested in a random constant-return-to-scale production process, or shipped. The stocks located "at home" and "abroad" are denoted K and K^*, respectively.

The world is symmetrical. The consumer-investors of both countries have the same constant relative risk-aversion coefficient. The production processes of both countries have the same expected rate of return and standard deviation of rate of return, but the output shocks in the two countries are uncorrelated.

Transferring physical capital abroad consumes resources. This assumption is made to produce deviations from price parity and to explore their dynamic behavior in financial-market equilibrium. Transfer costs are assumed proportional.

For reasons of portfolio diversification, consumer-investors of both countries would like the two countries' stocks of goods to be equal. Nonetheless, an imbalance may develop and persist as a result of cumulated random output shocks. As a result of the transactions costs, it need not be optimal to correct the imbalance immediately by shipping physical resources from the country where they are more abundant. I first determine the range of tolerated physical imbalance within which no shipping takes place.

The general equilibrium of this economy is tractable under the assumption that consumer-investors can achieve a Pareto-optimal allocation of consumption because, under this assumption, the capital-market and goods-market equilibrium can be replicated by an appropriate central-planning problem. The Pareto optimum takes into account the costly character of trade between the two countries. The welfare function can be constructed as a weighted average of the individual lifetime utility functions. Implicit prices, which would prevail explicitly in decentralized markets,[3] can then be obtained from the derivatives of the appropriate indirect welfare function for (the various forms of) wealth. A justification for the assumption that the Pareto optimum is achievable is that the model includes no hindrance to the exchange of securities or goods between individuals,[4] only to the movements of goods between locations. If a person ships goods from one place to the other, he suffers the cost of shipping,

[3] Uppal (1991) contains a decentralized version of the present model with derivations of portfolio holdings and financial-market equilibrium.

[4] This is in contrast to the models of Black (1974) or Stulz (1981b).

whether or not he retains ownership of the goods; whereas if a person sells a stock of goods to another without moving them, no cost is borne, regardless of the country of residence of the buyer and seller.

In this context, we can define an admissible policy as a pair of functions, $c(K, K^*)$ and $c^*(K, K^*)$, representing the consumption flows at home and abroad, an open set Ω, and a pair of adapted[5] processes X and X^*, which stand for the cumulative amounts of physical capital ever to have been shipped from home and from abroad. These processes serve to regulate the joint process for K and K^*.[6] The boundary of Ω should be viewed as a barrier or trigger point. When (K, K^*) hits it (from the inside of Ω), a shipment,[7] dX or dX^*, adjusts the two capital stocks and brings them back instantaneously inside Ω. We seek the optimal positioning of the barrier. Inside the Ω region, $dX = dX^* = 0$; no shipping takes place. That does not mean, however, that the two stocks of capital move of their own accord, because consumption in each country is optimized, given the available stocks of goods.

Assume that all consumers originally start their lives with endowments of goods such that the appropriate central-planning welfare function devotes equal weights to the utility levels of the households of both countries. Accordingly, the central optimization problem is

$$V(K, K^*) \equiv \underset{\substack{c,\, c^* \\ \Omega}}{\mathrm{Max}}\, E_t \int_t^{\infty} e^{-\rho(u-t)}\left[\frac{1}{\gamma}c_u^{\gamma} + \frac{1}{\gamma}(c_u^*)^{\gamma}\right] du, \qquad \gamma < 1, \quad (1)$$

subject to

$$dK_t = (\alpha K_t - c_t)\,dt + \sigma K_t\,dz_t - dX_t + s\,dX_t^*, \qquad s \in (0, 1), \quad (2)$$

$$dK_t^* = (\alpha K_t^* - c_t^*)\,dt + \sigma K_t^*\,dz_t^* + s\,dX_t - dX_t^*, \quad (3)$$

where p is the discount rate of utilities common to all investors; 1 - γ is the degree of risk aversion common to all investors; α and σ are the expected value and the standard deviation, respectively, of the physical rates of return in the constant-return-to-scale production processes; $s \in (0, 1)$ is a shipping-loss factor; dz and dz^* are two standard white-noise output (or productivity) shocks, which, for simplicity, we take to be independent of each other; Ω is an open region

[5] Adapted with respect to the filtration generated by $(K_n\ K_t^*)$.

[6] "Regulated Brownian motion" is the terminology introduced by Harrison (1985)

[7] This shipment can conceivably be either an impulse or an instantaneous control [see Bensoussan and Lyons (1975); Bensoussan, Crouhy. and Proth (1983); Harrison, Sellke, and Taylor (1983); and Harrison and Taksar (1983)]. Both would occur instantaneously (Le.. at infinite speed). However, impulse-control processes would be discontinuous and undergo finite jumps, while an instantaneous-control process is continuous [and singular; see Karatzas (1983)].

of (K, K^*) space; and X and X^* are two adapted, nonnegative, nondecreasing, and right-continuous stochastic processes that increase only when (K, K^*) exits from Ω (enters its complement, which is a closed set).[8]

Considering the linear nature of the constraints and the isoelasticity of the period utility function, the solution for $V(K, K^*)$ must be homogeneous of degree γ and the solutions for X and X^* have the property that, if policy (X, X^*) is optimal for initial conditions (K, K^*), then policy $(2X, 2X^*)$ is optimal for initial conditions $(2K, 2K^*)$.

Furthermore, because the transfer costs considered here are strictly proportional, one could prove-but we will simply assume-that it is optimal for shipments (the increments dX and dX^*) to be infinitesimal in size (i.e., it is optimal for X and X^* to be continuous processes). This is an example of what Harrison and Taksar (1983) call "instantaneous control."

The "homogeneity" and continuity properties, in addition to the symmetry of the problem, imply that the set Ω and the optimal transfer policy can be described by a single number $\lambda > 1$ with the interpretation that

$$X \text{ increases only when } K/K^* = \lambda,$$

and

$$X^* \text{ increases only when } K^*/K = \lambda.$$

In the (K, K^*) space, Ω is a cone delineated by two rays of slopes λ and $1/\lambda$ (see Figure 1). If the existing capital mix places the international economy on the ray of slope λ, a decision is made instantaneously to transfer capital from the home to the foreign country ($dX > 0$). The amount of capital transferred is very small and is such as to directionally push the economy back inside the cone [along a direction of slope $-1/s$ in the (K, K^*) plane].[9] The other ray of slope $1/\lambda$ triggers the reverse transfer of capital (along a direction of slope $-s$).

In the next section, we will optimize the number λ and obtain a numerical solution for the function V.

[8] Harrison (1985) shows that these processes have finite variation

[9] Initially, the economy may start outside the cone, in which case it is moved back inside by a finite. size shipment. Thereafter, given the nature of the control, it is impossible for the economy to be outside. The production set generated by admissible policies is convex, which will buttress the assumption of Pareto optimality. Imagine two allocations achieved by using different shipping cones. Since the two cones in question are inside one another, a convex combination of these two shipping policies is identical to the shipping policy achieved by the smaller cone. Hence, that convex combination is also admissible.

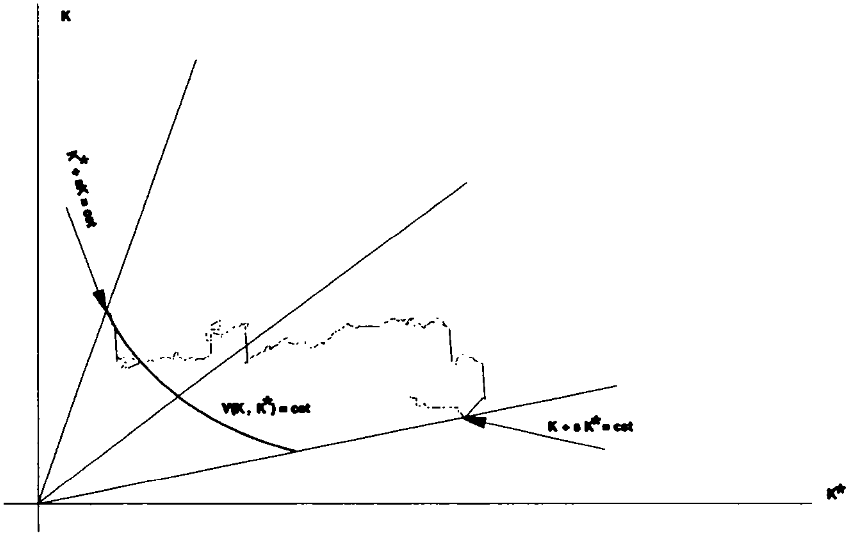

Figure 1
The cone of no shipping
When the stochastic process of capital stocks (K, K^*) reaches the frontiers of a cone, it receives a directional impulse of direction $-s$ or $-1/s$, where s is the shipping loss factor. On the edges, the slopes of the indifference curves of the value function of the programming problem, $V(K, K^*)$, are also equal to $-s$ or $-1/s$.

2. The Solution of the Central Planning Problem

Once the solutions c, c^* of the first-order conditions for consumption have been substituted in, the Hamilton-Jacobi partial differential equation, for values of K and K^* in the interior of the cone, can be written as follows:[10]

$$\begin{aligned} 0 \equiv & \left[\frac{1}{\gamma} - 1\right] V_1^{\gamma/(\gamma-1)} + \left[\frac{1}{\gamma} - 1\right] V_2^{\gamma/(\gamma-1)} - \rho V \\ & + V_1 \alpha K + V_2 \alpha K^* + \frac{1}{2} V_{11} \sigma^2 K^2 + \frac{1}{2} V_{22} \sigma^2 K^{*2}, \\ & K^*/\lambda < K < \lambda K^*. \end{aligned} \tag{4}$$

An essential property of the process followed by the expected discounted utility (1) over time is that its sample path must be continuous with probability unity. This is the mechanical result of the fact that no instantaneous (or impulse) utility cost or benefit ever accrues and information arrives continuously. When a transfer of capital occurs,

[10] See, for example, Merton (1969, 1971). We *assume* that V is twice continuously differentiable over $K^*/\lambda \leq K \leq \lambda K^*$. Because we optimize consumption, the partial differential Equation (5) differs from the one obtained by Constantinides (1986), which was linear and admitted a "closed-form" solution.

the movement to the target position is instantaneous. Hence, the value of the discounted utility at the arrival (target) point must "match the value" at the departure (trigger) point: for $K/K^* = \lambda$,

$$V(K, K^*) = V(K - dX, K^* + s\,dX),$$

or, after expansion,[11]

$$0 = -V_1(K, K^*) + sV_2(K, K^*). \tag{5a}$$

Value matching must hold also when transfer is triggered in the opposite direction, at the symmetric point: for $K^*/K = \lambda$,

$$0 = s\,V_1(K, K^*) - V_2(K, K^*). \tag{5b}$$

The partial differential equation (4) and the boundary conditions (5) must hold even if the positioning of the two transfer barriers were arbitrary, as long as the regulators involve infinitesimal shipments. The set of equations developed so far (in addition to the homogeneity property of V) would determine a function $V(K, K^*; \lambda)$. In order for the barriers of slopes λ and $1/\lambda$ to be optimal, a first-order condition known as "smooth pasting" must be satisfied. This requires that marginal utilities, in addition to utility itself, take the same values before and after the transfer of capital. We state that, for $K/K^* = \lambda$,

$$V_1(K, K^*) = V_1(K - dX, K^* + s\,dX),$$

$$V_2(K, K^*) = V_2(K - dX, K^* + s\,dX),$$

which, after expansion, imply

$$0 = -V_{11}(K, K^*) + sV_{12}(K, K^*), \tag{6a}$$

$$0 = -V_{21}(K, K^*) + sV_{22}(K, K^*), \tag{6b}$$

and, for $K^*/K = \lambda$,

$$0 = sV_{11}(K, K^*) - V_{12}(K, K^*), \tag{6c}$$

$$0 = sV_{21}(K, K^*) - V_{22}(K, K^*). \tag{6d}$$

In Dumas (1988, 1991), these conditions, which involve the *second* derivatives of the value function, were referred to as "super-contact" conditions.[12]

[11] Conditions (6a) and (6b) are identical to those used by Constantinides (1986).

[12] For an exposition in a simpler setting, see Dumas (1991). Some of the optimality conditions used there (i.e., the "smooth-pasting" conditions) were encountered first when solving the optimal early-exercise problem for American-type options [see, e.g., Merton (1973)]. In the current context, they take a novel (higher-contact) form baptized "super-contact condition." The original mathematical derivation of these conditions (albeit in a one-dimensional case) is to be found in Benes, Shepp, and Witsenhausen (1980), Karatzas (1983), Karatzas and Shreve (1984), and, especially,

The optimal solution of the central planning problem, including the optimal value for the intervention point λ, is obtained by solving the partial differential equation (4) subject to boundary conditions (5) and (6)-a number of which are redundant, because of homogeneity and symmetry. The solution is calculated by first implementing the following change of variable (from K, K^* to ω) and unknown function [from $V(K, K^*)$ to $I(\omega)$]:[13]

$$\omega = \ln K - \ln K^*$$

$$I(\omega) \equiv -\gamma \ln K^* + \ln[V(K, K^*)], \tag{7}$$

$$I(-\omega) \equiv -\gamma \ln K + \ln[V(K, K^*)],$$

which exploit the homogeneity property. Then, we can apply a numerical technique.[14]

We now turn to the results. They pertain to the behavior of consumption and investment in both countries and to the behavior of the relative price of capital in both countries.

3. Equilibrium Behavior of the Allocation of Physical Capital

The equilibrium behavior of the economy can be derived from the knowledge of the function $V(K, K^*)$, or $I(\omega)$. The dynamics of the physical stocks of goods, specifically the size of the cone of no shipping and the behavior of quantities inside the cone, are examined in this section. While no shipping takes place inside the cone, endogenous consumption rates serve to drive the stocks of goods.

3.1 The tolerated imbalance-comparative dynamics

The size of the cone (i.e., the value of A, the tolerated imbalance) has been obtained for varying degrees of risk aversion $(1 - \gamma)$ and varying degrees of risk (σ). The results are displayed in Table 1, which

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Harrison and Taksar (1983) [proposition 5.11. p. 449 (as well as the discussion that precedes it)]. Constantinides (1986) did not make use of these optimality conditions; he directly maximized with respect to λ his analog of our function $V(K, K^*; \lambda)$ which, in his case, was available in closed form.

[13] I have a specific purpose in mind in defining the physical imbalance as $\omega = \ln K - \ln K^*$, rather than, for example, $K/(K + K^*)$, or some other way. The conditional variance of ω is constant [see stochastic differential Equation (8)] which is a great simplification in interpreting results.

[14] Briefly, the numerical technique is a "shooting" one. Pick a trial value for λ and apply (5) and (6) at the extreme point ω - 1n λ, to get the value of I' and I'' there. Given these initial conditions, the partial differential Equation (4) [rewritten as an ordinary differential equation for $I(\omega)$] can be continued until the central point, using, for instance, the Runge-Kutta method of order 4 [see Abramowitz and Stegun (1972), p. 897]. By symmetry of the V function, at the central point ω - 0, we must have $I'(0) = \gamma/2$. If this condition is satisfied, the solution is found; otherwise, choose a new trial value for λ and go back to the first step. A similar procedure is described in Davis and Norman (1990).

Table 1
Opening of the cone of no shipping as a function of risk and risk aversion

Risk aversion $1-\gamma$	Risk (σ) 0.02	0.1	0.45	0.5
0 (neutral)	∞	∞ (∞)	∞	∞
0.1	NA	9.63285 (97.96805)	14.44122 (12.63541)	14.91199 (10.66747)
1/2	1.71928 (150.04355)	2.47011 (19.74222)	4.09539 (3.88466)	4.18308 (3.32992)
1 (log)	1.45097 (64.28322)	2.15732 (14.50696)	3.22199 (2.95456)	3.26079 (2.49653)
2	1.34469 (33.89208)	1.98135 (12.44918)	2.64698 (2.22487)	2.66438 (1.85691)

Numerical values of the parameter, other than γ and σ, are $\rho = 0.15$, $s = 1/1.22$, $\alpha = 0.11$. The body of the table gives the value of λ and, in parentheses, the expected time it takes for a shipment to take place starting from a balanced position. The values in this table are obtained by a numerical procedure (see description in note 14). If the "rate" parameters σ, ρ, and α are interpreted as being measured on a per year basis (e.g., α = 11%/year), then the expected time till shipment occurs is measured in years.

also shows the expected time till a shipment takes place when starting the economy at the central point (a rough measure of the frequency of shipments).[15] Table 1 is meant to indicate the direction of the effects of the various parameters; it is not meant to quantify these effects, as no calibration is attempted.

Observe that, under any level of risk (σ), increasing $1 - \gamma$ reduces the opening of the cone, increases the frequency of shipment, and tightens, therefore, the control exercised by the shipping activity on the balance of capital stocks. Several reasons are present, all of which act in the same direction. First, a larger value of $1 - \gamma$ implies greater risk aversion, which increases the benefits of diversification and makes it more worthwhile to keep the capital stocks in balance. Second, an increase in $1 - \gamma$ is also a decrease in the elasticities of intertemporal and interpersonal substitutions. The latter, for instance, implies that more social welfare is lost from unbalanced consumption, so the cone narrows.[16]

As one approaches risk neutrality, the cone of no shipping widens indefinitely and, in the symmetric setup considered here, shipping occurs almost never, as long as shipping costs are positive. The ex ante benefits of diversification procured by shipping no longer matter.[17]

[15] I am grateful to the editor, Richard Green, for suggesting this measure.

[16] I am grateful to a referee for suggesting this line of interpretation.

[17] Roll (1979) has developed a partial-equilibrium framework in which deviations from PPP must follow a martingale; this is the "ex ante PPP" hypothesis. The Roll situation can be viewed as a

The manner in which increasing risk (σ), in Table 1, affects the size of the cone is somewhat surprising; it widens it. Increasing risk has an effect opposite to that of increasing risk aversion.[18] An increase in the degree of risk, like an increase in risk aversion, does increase the benefits of diversification and makes rebalancing of physical capital more desirable. However, observe that a wider cone, when risk increases, does not automatically translate into less frequent shipping. The greater volatility makes reaching the boundaries of a given cone more likely; the increase in risk by itself-without a change in the cone-would increase the frequency of shipping.[19] The implication of the numerical result is that it increases it too much. In order to counter that increase, the optimal cone widens. The numbers for the expected time till a shipment takes place, starting from the central point, are consistent with the interpretation just given. On the basis of the numerical evidence, we conclude that an increase in risk aversion and an increase in risk act in the same direction on the frequency of shipments; they both increase it.

3.2 The process for the physical imbalance

The behavior of $\omega = \ln K - \ln K^*$ (the allocation of the goods between the two locations) inside the cone can be obtained easily from the knowledge of the value function. Based on (2) and (3), we have

$$d\omega = \left[-\frac{c}{K} + \frac{c^*}{K^*} \right] dt + \sigma\sqrt{2}\, dz', \tag{8}$$

where $dz' = (dz - dz^*)/\sqrt{2}$ is a standardized white noise. Furthermore, based on (7),

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limit of the present model as one lets the degree of risk aversion tend to zero. The rarefaction of shipping in the limit rationalizes the Roll assumption of a total absence of international trade. In the neat section, we verify that Roll's martingale result regarding price behavior holds in the limit.

[18] This result is surprising for another reason. In the context of portfolio choice under transactions costs, Constantinides (1986) finds that increasing risk decreases the opening of the cone of no transactions (see his table 3. p. 854). But there are several differences between Constantinides' problem and this one. The present problem is symmetric as far as the choice of assets is concerned; investors choose between two physical investments of equal volatility but located in different places. His investor faces a choice between a risky and a riskless asset, so that increasing risk, in his context, means increasing the risk of one asset only, not both as we do here. Furthermore, our problem is symmetric as far as the origin of consumption is concerned; people of each country consume (possibly different amounts) from the stock of goods available locally, whereas Constantinides' investors consume only out of the wealth invested in the riskless asset. Finally, Constantinides utilized an approximate optimization procedure based on suboptimal consumption behavior, while our results are optimal. In an earlier draft of this article [Dumas (1987)], I reported the results of applying the Constantinides approximate procedure to the present problem; it was verified that the differences between the two settings, not the difference in the technique, account for the difference in the comparative dynamic result.

[19] I am grateful to a referee for suggesting this line of interpretation.

$$V_1 = V(I'/K), \tag{9a}$$

$$V_2 = V[(-I' + \gamma)/K^*], \tag{9b}$$

and thus

$$\frac{c}{K}(\omega) = [e^{I(\omega)} I'(\omega)]^{1/(1-\gamma)}, \tag{10a}$$

$$\frac{c^*}{K^*}(\omega) = [e^{I(\omega)} I'(-\omega)]^{1/(1-\gamma)}. \tag{10b}$$

Substituting (10a) and (10b) into (8), the I (ω) function fully determines the stochastic differential equation for ω inside the cone.

The diffusion coefficient in (8) is constant. This has one useful consequence. The instantaneous-drift function, which we are about to examine, is, by itself, representative of the process and suffices to characterize the behavior of the variable within the cone. The diffusion does not in any way "counteract" the drift function.

Figure 2 shows the conditional expected change (drift) of the ω process as a function of the current level of ω for the set of numerical values indicated in the legend. The slope of the drift function increases as one approaches the edges of the cone. The process is nonlinear AR(1), with the drift equal to the imbalance in the rates of local consumption relative to local capital [see Equation (8)]. It is zero at the central point ($\omega = 0$) when the two stocks of goods are balanced. Otherwise, the drift increases as a function of the imbalance, causing the process to be *centrifugal* or *divergent.* Its sign is always such as to pull ω away from the central point or toward the boundaries of the cone. Even though consumers in the capital-rich country consume more, they do so less than proportionately to the capital stock:

$$\text{when } K > K^*, \qquad c > c^* \qquad \text{but } c/K < c^*/K^*. \tag{11}$$

The difference in consumption rates is less than what would be needed to keep the mix of K versus K^* at an unchanged level over time. The centrifugal tendency of ω follows from this fact.

The economic rationale that explains this tendency is very basic and very general. Given the isoelasticity of utility functions and the constant returns to scale on the production side, under autarky each country's consumption would be strictly proportional to the local stock of capital. A 1 percent increase in the local stock of capital would produce a 1 percent increase in local consumption. Domestic capital would have no impact on foreign consumption. When goods are transferable but immediate shipping is not optimal, a 1 percent increase in one country's capital stock induces a smaller than 1 percent increase in that country's consumption *because foreigners also*

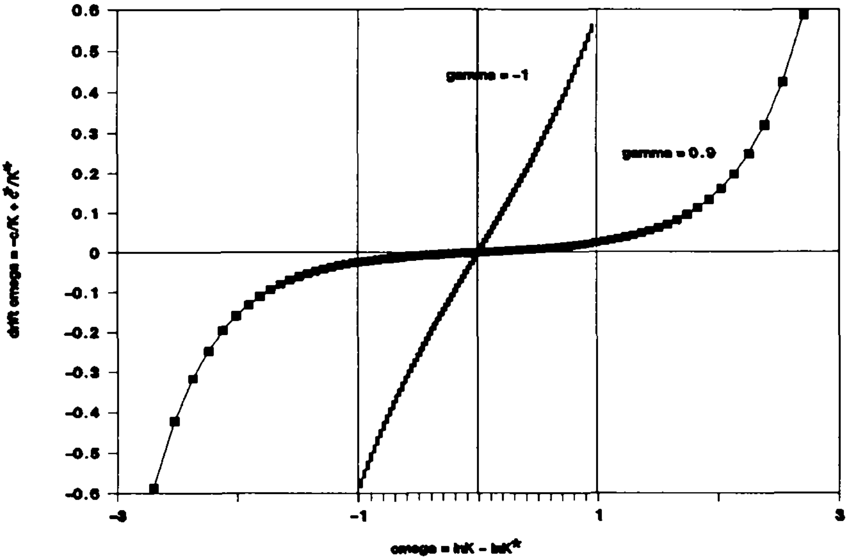

Figure 2
Drift of omega process
$\omega = \ln K - \ln K^*$ measures physical imbalance. The conditional variance of this process is constant. The figure shows the conditional expected change in ω for the following numerical values of parameters: $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$, $\gamma = -1$ or 0.9. The figure demonstrates that the process for the physical imbalance is centrifugal.

have a consumption claim on that increased stock. Foreigners immediately consume more out of the capital located in their country but they physically collect their claim on the occasion of the next shipment only. In the meantime, the goods are accumulated in the production process of the country where the positive shock has occurred. This accumulation, as well as the increased foreign consumption, marginally contribute to making the next shipment nearer in time and more probable.[20]

When one of the two countries, after a series of random shocks, has accumulated a larger capital stock than the other, it tends to accumulate proportionately even more capital and to remain in that position for a long period of time, during which it is a recurrent excess producer and exporter. The capital mix ω spends most of the time near the boundaries where it exhibits a high degree of persistence; crossovers are rare. The stationary probability measure of ω

[20] This rationale can be couched in mathematical terms, thus providing a "proof' of the assertion. The function *c(K, K*)* is homogeneous of degree 1. Presuming that ¶ *c/* ¶ *K** > 0-more foreign capital can only be of some benefit to domestic consumers--we have ¶ *c/* ¶ *K= (c - K** ¶ c/ ∂ *K*)/K* < *c/K.* But $\partial c/\partial K < c/K$ means $\partial(c/K)/\partial K < 0$. Starting from the balanced situation *(K = K*) in* which *c - c**, this last inequality applied throughout implies result (11). An alternative and equivalent rationale, in *terms* of prices, will be provided in Section 4.1.

(not shown) is U-shaped, with most of the probability mass on the sides.

In a decentralized version of this economy, the recurrent flow of export would be sustained by conditional financial claims that pay off in favor of the importing country. At the origin of time, each country would have placed financial bets against itself becoming the capital-rich location, as a way of diversifying the locational advantage or handicap, which acts, in the choice of portfolios, as a nontraded asset.[21]

4. Equilibrium Behavior of Deviations from Parity

Even though the optimization problem has been formulated as a centralized one, one can infer the prices that would prevail in a decentralized market economy, by examining the first derivatives of the value function $V(K, K^*)$. This was, of course, the main object of the determination of this function.

4.1 The deviation from parity

Define p as the price of goods located at home relative to goods located abroad (the price of a unit of K in units of K^*). It is given by

$$p = \frac{V_1(K, K^*)}{V_2(K, K^*)}. \tag{12}$$

Because of the homogeneity of V, p is a function of ω only. This function is [recall (7), (9a), and (9b)]

$$p(\omega) = \frac{I'(\omega)e^{-\omega}}{-I'(\omega) + \gamma} = e^{-\omega}\frac{I'(\omega)}{I'(-\omega)}. \tag{13}$$

The Law of One Price (LOP) prevails when p = 1. The variable p can be viewed as "the real exchange rate."

It is inconvenient to study the stochastic process of p itself, whose definition is asymmetric in nature in that interchanging the two goods changes p into $1/p$, which is a nonlinear transformation. Instead, the symmetry will be preserved by studying the behavior of the relative deviation from parity, defined as the natural logarithm of p. Again, using the $I(\omega)$ function, Equation (13) provides 1n p as a function of ω. Since the process of ω is known, the process for 1n p is obtained.

The function 1n $p(\omega)$ is displayed in Figure 3. Under perfect balance in the quantities, 1n $p = 0$ and the LOP prevails. The boundary con-

[21] If, historically, the process was started neat one boundary, the capital-poor country had to start with more financial wealth (i.e.. with some claims on the capital-rich country). Otherwise, the welfare weights could not have been set equal to each other in the associated central-planning problem.

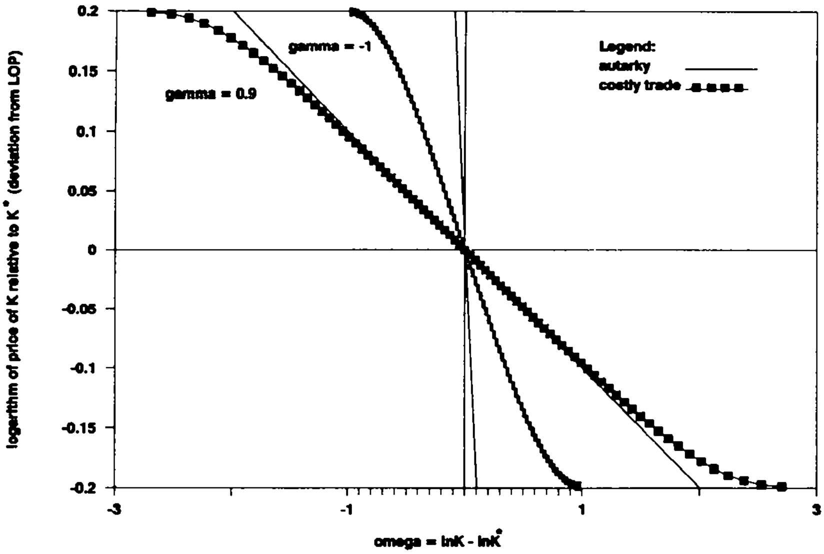

Figure 3
The function $\ln p(\omega)$
p is the price of physical capital K, loaded at home, relative to physical capital K^*, located abroad. It is a decreasing function of the relative amounts of K versus K^*; $\omega = \ln K - \ln K^*$. The slope of this function is zero at the extremities when the frontier is placed optimally. Numerical values of the parameters are $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$, $\gamma = -1$ or 0.9.

ditions (5a) and (5b) imply that p reaches the values s and $1/s$ at the two extremities. At the instant when shipping is activated, the good located in the country of abundance (from where shipping originates) is s times less valuable than the good located in the country where it is scarce. Moreover, because the boundaries have been chosen optimally, the slope $p'(\omega)/p$ at the boundary points is zero, by virtue of the super-contact optimality conditions (6a)-(6d). As a result, the function $\ln p(\omega)$ tapers off, as one moves from the central point and approaches the boundaries.[22]

It is not hard to see that, in the absence of shipping, the relationship between $\ln p$ and ω would be a straight line [$\ln p = (\gamma - 1)\omega$]. When goods are transferable, p is closer to parity than it would be under autarky. That is, when K is abundant relative to K^*, the price of K is below the costless-trade price of 1 and above the autarkic price:[23]

$$\text{when } K > K^*, \qquad 1 > p \equiv \frac{V_1}{V_2} > \left[\frac{K}{K^*}\right]^{\gamma-1} = e^{(\gamma-1)\omega}. \qquad (14)$$

[22] If the boundaries had not been chosen optimally, the function would still taper off, but to a nonzero slope on the edges.

[23] One could easily show that this property of the price is equivalent to our previous result (11) concerning consumption behavior in relation to the stocks of goods. This provides an alternative rationale for the result.

This "bias within the band" [24] in the price is due to the asset character of the two stocks of goods and to the resulting dependence of today's relative price on tomorrow's price. The bias is present, even at times at which no shipping effectively takes place, because of the anticipation that shipping will eventually prevent the price from escaping from the interval *[s, 1/s]*.

4.2 The process for the LOP deviation

We have noted in Section 3 that the relative capital stock ω exhibits a centrifugal tendency. Because a monotone function links 1n p to ω, the centrifugal behavior of ω induces a similar behavior for 1n p. At any point where no shipping occurs, the conditional probability of an outward movement for 1n p, over some sufficiently short but finite period of time, is greater than that of an inward movement.[25]

We have also noted (Section 4.1) that the 1n p (ω) function, relating the LOP deviation to the quantity imbalance, tapers off as one reaches the boundaries (see Figure 3), reflecting the anticipation that shipping will prevent the price from escaping from the interval *[s, 1/s]*. Because of the shape of the function, the probability mass for 1n p is displaced toward the boundaries. Hence, the 1n p process is "near" its boundaries even more frequently than the ω process. The stationary (unconditional) probability density function (not shown) is U-shaped, as was that for the physical imbalance ω, but is more accentuated, with even more probability mass concentrated on the edges.

Figure 4 shows a randomly drawn sample path[26] of the 1n p process over a period of 2400 days. It illustrates that the process is indeed most of the time near one of the boundaries, which means that LOP deviations-and the associated trade imbalance? typically last "a long time." These are periods of "quiescence," such as those that have been observed in the foreign-exchange markets. They are interrupted by rare periods of "turbulence," when the international economy, under the influence of a succession of random shocks in the right direction, crosses over to the other boundary. At such times, the

[24] In Figure 3, the 1n p (ω) function nowhere coincides with the straight line marked "autarky." The two lines are not even tangent to each other at the central point. In some ways, this bias is analogous to the "honeymoon effect" identified by Krugman (1987.1991) in his study of exchange-rate target zones. Central banks, in his model, intervene in the foreign-exchange market only when the exchange rate reaches the boundaries of a band. Nonetheless, the exchange rate within the band is affected by this intervention, at a time when no intervention effectively takes place.

[25] Here is a precise mathematical statement of what I mean by 1n p being a "centrifugal process." Consider a real number x: $\ln s < x < 0$. Consider a current instant t. Then there exists $T(x, t) > t$ such that, for any t': $t < t' \leq T$, it is true that

$$\mathrm{Prob}[\ln p_{t'} \leq x \mid \ln p_t = x] > \mathrm{Prob}[\ln p_{t'} \geq x \mid \ln p_t = x].$$

Remark: T is less than infinity.

[26] This simulation is borrowed from Shrikhande (1991).

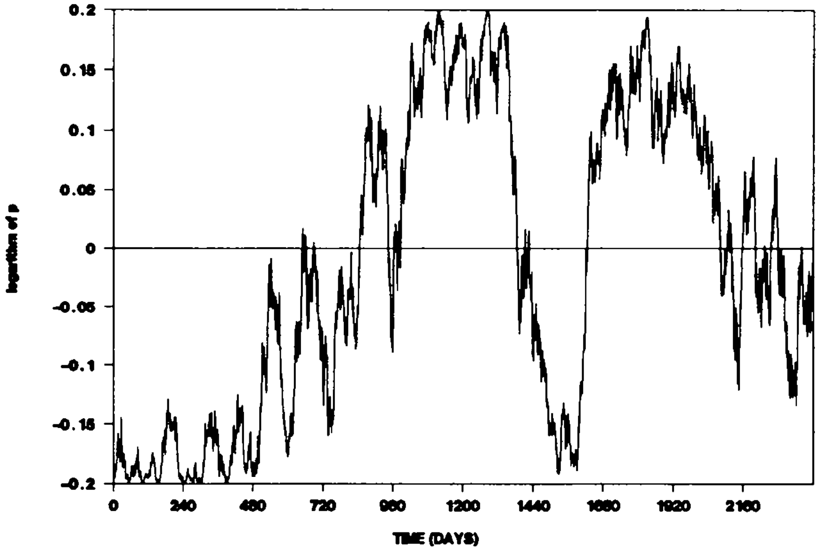

Figure 4
Sample path of the real exchange rate
A simulation of the level of the logarithm of the real exchange rate $\ln p$ over 2400 days. Numerical values of parameters are $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$, $\gamma = -1$. If ρ, σ, and α are interpreted as rates per year, the time scale is in days and the total simulation represents approximately 10 years of observations.

volatility is observed to be larger than when the economy is near the boundaries (see Frenkel and Levich, 1975).

Figure 5 shows the conditional expected change and the conditional standard deviation (in fact, the signed diffusion coefficient) of the ln p process as functions of the current level of ln p. The same numerical values are used as before, along with two values of risk aversion, $1 - \gamma = 2$ and $1 - \gamma = 0.1$.

As suggested by the simulation, the LOP deviation, in sharp contrast to ω, is strongly *heteroskedastic*. The conditional standard deviation is larger at the central point than near the boundaries and becomes zero (with an infinite slope) at the boundaries. The shape of the diffusion function causes the central point to act as a repellent and accounts for the U shape of the stationary density.

Surprisingly, the instantaneous drift of the process (see Figure 5) formally exhibits *mean* reversion. The drift is negative when the deviation from parity is positive and positive when the deviation is negative. The process of deviations from the LOP is certainly not a martingale, even though capital market efficiency (or rational expectations) has been assumed. As we know, two factors are responsible for the behavior of ln p: (i) the behavior of ω is centrifugal; and (ii) the price function ln p (ω) tapers off and exhibits a curvature near the

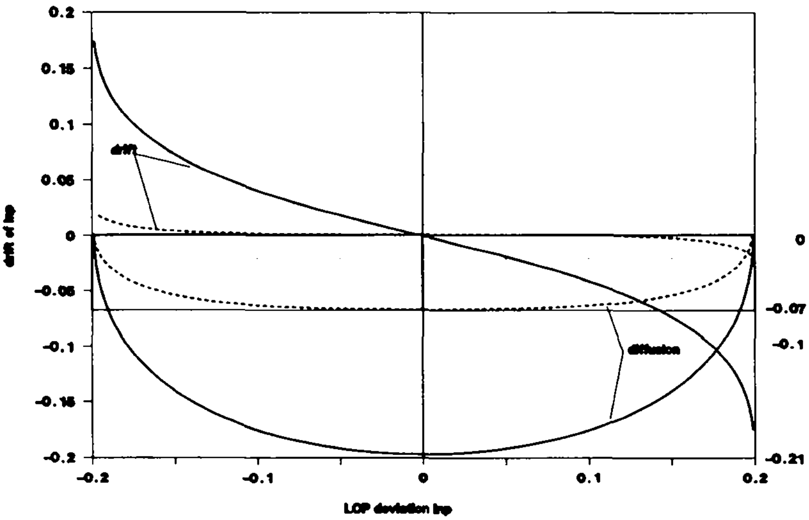

Figure 5
The LOP deviation process
The process for $\ln p$ is more complex than that of the physical imbalance ω because the variance is not constant. The drift and the signed diffusion functions are drawn here for the numerical values of parameters: $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$, $\gamma = -1$ (solid lines) or 0.9 (dotted lines). The drift function exhibits mean reversion but the process is not a reverting one; the volatility (diffusion coefficient) is variable.

edges, in anticipation of shipping. This second effect, via Ito's lemma, more than offsets the first one and causes the price process overall to exhibit instantaneous mean reversion.

However, as the probabilistic reasoning based on ω has shown, the drift of $\ln p$ is not representative of the behavior of the process for $\ln p$. The drift becomes dominant on the edges only, where the diffusion is null and where it is the pure product of the impending reflection. The correct qualitative appraisal of probabilistic behavior within the band is provided by the drift function of ω, not $\ln p$, since the ω process is homoskedastic, whereas the process for $\ln p$ is not. On this count, as noted, $\ln p$ is centrifugal.

Comparing, in Figure 5, the graphs obtained for $1 - \gamma = 2$ and for a low risk aversion, $1 - \gamma = 0.1$, provides some confirmation of the conjecture that the Roll (1979) "ex ante PPP" obtains in the limit. As one approaches risk neutrality, the drift function approaches the zero axis. Observe, however, that the diffusion coefficient is also progressively reduced, so that the price process is characterized by less and less agitation as one approaches risk neutrality.

4.3 Remarks concerning empirical studies

Empirical studies of real exchange rates leave most issues unsettled. With a few exceptions, deviations from PPP have been found to follow a process that could not be distinguished statistically from a martingale [Rogalski and Vinso (1977), Roll (1979), Adler and Lehman (1983), Huizinga (1987)]. These studies have relied, in various ways, on statistical models with a linear mean equation and on procedures, some of which only are robust to heteroskedasticity. Their failure to reject the martingale hypothesis may conceivably be understood if the underlying process is of the type identified here. Linear mean equations are unlikely clearly to identify a process such as the one for 1n p in which heteroskedasticity plays center stage, in which nonlinearities in the mean are intertwined with a definite form of time-varying variance, and in which long-run behavior is very different from short-term behavior, since reversion manifests itself only when the deviation from parity has become wide enough.

Published measurements of sample autocorrelations of exchange rates at various lags, although statistically nonsignificant, are not inconsistent with our conjectured model since short-run autocorrelations are frequently found positive while long-run autocorrelations are found negative. Engel and Hamilton (1990), along with many practitioners, have observed that the dollar goes through "long swings" or lasting phases of upward and downward movements. They have reasoned that these "trends" can be made compatible with market efficiency if one imagines that, at each instant, there is a positive probability of switching from the upward to the downward trend and vice versa. Their "switching-random walks" statistical model has received some validation from the data. Sample paths of the real exchange rate produced by simulation of our model exhibit long swings as well.

Diebold and Nerlove (1985) and Hsieh (1989) have tested for "nonlinearities" in nominal exchange rates and found that a generalized autoregressive conditional heteroskedasticity model is a fair representation of the data. Meese and Rose (1990) and Diebold and Nason (1991), in studies of nominal exchange rates, have used general-purpose, nonparametric statistical models in an attempt to discriminate between the relative role of nonlinearities in the mean and heteroskedasticity. However, nonparametric procedures have proved insufficiently powerful.

Ours is the first general-equilibrium model that endogenously produces the kind of nonlinearity and heteroskedasticity that have previously been introduced into purely statistical models aiming to fit exchange-rate data.[27]

[27] The present model may also serve to examine empirically the behavior of the relative level of stock indexes across the world.

5. The Real Interest-Rate Differential

As soon as the relative price of two goods, or two varieties of a good, fluctuates over time, the rate of interest measured in units of one of them is not equal to the rate of interest measured in units of the other. Furthermore, if the fluctuations of the relative prices are random, a financial asset that would be riskless when its rate of return is measured in units of one of the goods is not riskless when its rate of return is measured in terms of the other. One must, therefore, be careful to distinguish four quantities that are conceptually quite different.

Quantity 1 (denoted r) is the rate of interest on an asset which is riskless in terms of K (the good located at home). This is the own rate measured in units of K itself.

Quantity 2 is the expected value of the rate of return on this same K-riskless asset but measured in terms of K^* (the good located abroad). Quantities 1 and 2 differ only by the expected value of the change in the relative price (i.e., they differ if and only if the price does not follow a martingale process).

Quantity 3 (denoted r^*) is the own rate of interest on an asset that is riskless in terms of K^*. Quantities 2 and 3 differ by a "risk premium." [28]

Quantity 4 is the expected rate of return on the K^*-riskless asset measured in terms of K. Quantity 4 differs from quantity 3 by the expected price change. Quantity 4 differs from quantity 1 by a risk premium.

In what follows, the spread between quantity 1 (denoted r) and quantity 3 (denoted r^*) is measured and is called "the" real interest-rate differential. But this spread is broken down into a component related to the expected price change and a residual that constitutes a risk premium.

5.1 Derivation of the interest-rate differential

In a market economy, the riskless interest rate is equal to the discount factor of utilities (ρ in our notation) minus the conditionally expected rate of change in the (undiscounted) indirect utility of wealth:[29]

$$r = \rho - \frac{E\,d(V_1)}{V_1\,dt}, \tag{15a}$$

[28] In what follows, the term "risk premium" is used somewhat improperly. It does not mean that the designated quantity would be equal to zero under risk neutrality, only that it would be zero in the absence of risk.

[29] In the following equations, E is the expected value operator conditional on the current values of K and K^*.

$$r^* = \rho - \frac{E\,d(V_2)}{V_2\,dt}. \tag{15b}$$

Equations (15a) and (15b) imply

$$r^* - r = \frac{E\,d(V_1)}{V_1\,dt} - \frac{E\,d(V_2)}{V_2\,dt}. \tag{16}$$

It is also true that, in each country, the rate of interest is related to the expected rate of return a on productive assets:[30]

$$\alpha = r - \frac{1}{dt}\frac{dK}{K}\frac{dV_1}{V_1}, \tag{17a}$$

$$\alpha = r^* - \frac{1}{dt}\frac{dK^*}{K^*}\frac{dV_2}{V_2}. \tag{17b}$$

Equations (17a) and (17b) explain how, under uncertainty, the two real rates of interest can be different in the two countries, even though the technological coefficients are equal.

Based on the definition (12) of p, the following identities are all equally true:

$$\frac{dV_1}{V_1} - \frac{dV_2}{V_2} \equiv \frac{dp}{p} + \frac{dp}{p}\frac{dV_2}{V_2}, \tag{18}$$

or

$$\frac{dV_2}{V_2} - \frac{dV_1}{V_2} \equiv \frac{d(1/p)}{1/p} + \frac{d(1/p)}{1/p}\frac{dV_1}{V_1}, \tag{19}$$

or yet

$$\frac{dV_1}{V_1} - \frac{dV_2}{V_2} \equiv d(\ln p) + d(\ln p)\left[\frac{1}{2}\frac{dV_1}{V_1} + \frac{1}{2}\frac{dV_2}{V_2}\right]. \tag{20}$$

Taking expectations as in (16), Equation (18) provides the relationship between interest rates that is established by people residing in the foreign country, while (19) provides the same relationship from the point of view of the home country.

In order to facilitate the comparison with empirical studies that use logarithms of exchange rates, we choose to develop the third, hybrid, and symmetric expression numbered (20).[31] Substitute it into

[30] The reader is reminded that products of stochastic differentials. such as $dK \times dV_1$, are construed as being nonstochastic. provided that equations that contain them are understood to hold in the quadratic-mean sense. $(dK/K) \times (dV_1/V_1)/dt$ is the instantaneous covariance of the home rate of return with the marginal utility of home capital.

[31] But Figure 6 also displays an asymmetric risk premium (computed from the point of view of the foreign investors).

(16) to obtain the following decomposition of the real-rate differential:

$$r^* - r = \frac{E\,d(\ln p)}{dt} + \frac{1}{dt}(\ln p)'(\sigma\,dz - \sigma\,dz^*)\left[\frac{1}{2}\frac{dV_1}{V_1} + \frac{1}{2}\frac{dV_2}{V_2}\right], \quad (21)$$

where $(\ln p)'$ refers to the derivative $d \ln p / d\,\omega$ of the function $\ln p\,(\omega)$, defined in Equation (13). However, on grounds of symmetry, we have

$$\sigma\,dz\left[\frac{dV_2}{V_2} - \frac{dV_1}{V_1}\right] = \sigma\,dz^*\left[\frac{dV_1}{V_1} - \frac{dV_2}{V_2}\right]$$

so that

$$dz - \sigma\,dz^*)\left[\frac{1}{2}\frac{dV_1}{V_1} + \frac{1}{2}\frac{dV_2}{V_2}\right] = \sigma\,dz\frac{dV_1}{V_1} - \sigma\,dz^*\frac{dV_2}{V_2}.$$

However, the right-hand side of this last equation, by virtue of (17a) and (17b), is $-(r^* - r)dt$. Substituting this result into (21), we get finally

$$r^* - r = \frac{E\,d(\ln p)}{dt} - (\ln p)'(r^* - r), \quad (22)$$

or

$$\frac{E\,d(\ln p)}{dt} = [1 + (\ln p)'](r^* - r). \quad (23)$$

Equation (22), via Equation (13) and the function $I\,(\omega)$ computed in Section 2, gives the real-interest-rate differential as a function of the physical imbalance, ω, while Equation (23) relates the expected exchange-rate change to the interest-rate differential (the forward premium, by covered interest-rate parity).

The second term on the right-hand side of (22) is a simple expression for the risk premium on foreign exchange. It says that the risk premium arises from the adjustment of the relative price of capital in relation to capital imbalance.

As noted, this “risk premium,” by an abuse of language, is all inclusive. It contains terms which would not necessarily vanish under risk neutrality. The theory of international asset pricing [Adler and Dumas (1983)] has identified two kinds of premiums on financial assets. The first kind, the “pure” risk premium, which would be zero under risk neutrality, is traditionally related to the conditional covariance of the asset with a benchmark portfolio return. The second kind is a comingled risky-inflation premium, which arises from the heterogeneity of the investor population and which would vanish under purchasing power parity (if the CAPM is written in terms of real

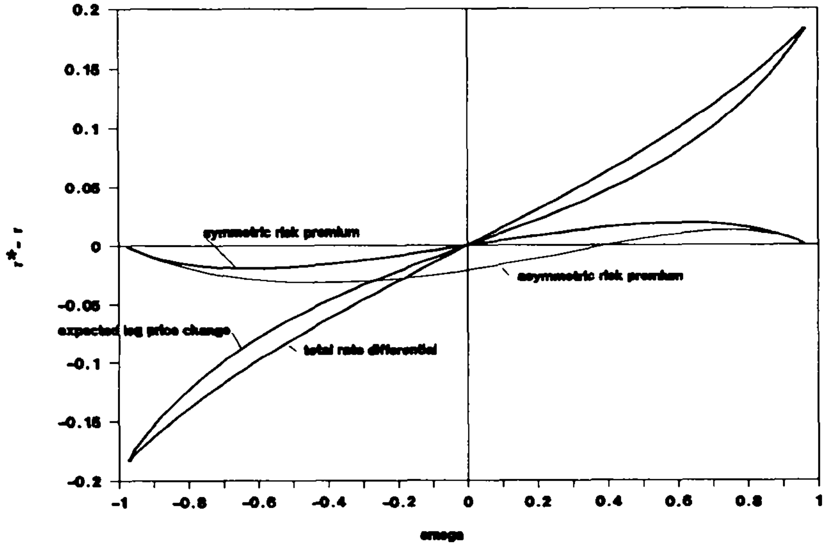

Figure 6
The real rate differential
Decomposition of the real rate differential into two components: The expected change in the price of K relative to K^* and a risk premium, measured symmetrically, as in the text, or asymmetrically from the point of view of a resident of the foreign country [Equation (18)]. Numerical values of the parameters are $\rho = 0.15$, $\sigma = 0.5$, $\alpha = 0.11$, $\gamma = -1$. $\omega = \ln K - \ln K^*$ measures current physical imbalance.

returns). This last is not necessarily zero under risk neutrality but is included in the risk premium of Equation (22).[32]

The interest-rate differential and the risk premium are expressed in (22) as functions of a variable contemporaneous to them, namely, ω. Since the function $\ln p\ (\omega)$ is known and monotonic, they could equally well be expressed as functions of $\ln p$ itself, which is also contemporaneous but is more readily observable than ω. In contrast to CAPMs, which involve expectational variables, Equation (22), like all fully solved-out rational-expectations financial models in the tradition of Lucas (1978), is a directly testable proposition. It can be tested on the basis of contemporaneously observed variables only, thus requiring no extraneous, ad hoc assumptions concerning, for example, the behavior of forecast errors over time.

5.2 Behavior of the interest-rate differential

The rate differential and its components of Equation (22) are plotted as Figure 6 for the same numerical values of the parameters as before.

[32] The risk premium (22) would equal zero under certainty because, in view of the symmetry, K and K^* would grow at the same rate and $\ln p$ would be time invariant. By (23), r would be equal to r^*.

The differential, the expected price change, and the risk premium are zero and change sign at the central point $\omega = 0$. This is the result of the symmetric definition of the price variable $\ln p$ and of the symmetry of the decomposition in (20).

The rate differential behaves very much like the expected price change. They both increase monotonically with the physical imbalance; they both reach a maximum on the edges of the cone. The risk premium notwithstanding, a large real-rate differential remains an indication of strong expected reversion in the LOP deviation. Shipping is triggered when the interest-rate differential reaches its largest possible value; this is also when the deviation from the LOP reaches its largest possible value. Deviations from PPP and deviations from equality of real interest rates are part and parcel of the same phenomenon

The risk premium behaves in an interesting way. Its sign is always such as to increase the absolute value of the interest-rate differential. It is equal to zero on the edges when the $\ln p(\omega)$ function has a zero slope and the price has a zero volatility. It reaches its largest absolute value somewhere between the central point and the edges, so that the risk premium and the expected price change do not covary in the same way in a neighborhood of the central point and in a neighborhood of the extremities. Despite the fairly large amount of risk and risk aversion assumed in our numerical example, the risk premium remains small in comparison to the expected price change and its possible fluctuations over time are also small in comparison with those of the expected price change.

5.3 Remarks concerning empirical studies

Empirical tests of uncovered interest rate parity, which have been conducted on foreign exchange markets, have produced evidence of discrepancies between nominal forward premiums and conditional expected values of subsequent spot nominal exchange-rate changes. In Cumby and Obstfeld (1984), among many other studies, realized nominal spot exchange-rate changes were regressed on previous forward premiums (or interest-rate differentials). The regression coefficient was practically always found to be significantly *smaller* than the hypothesized value of 1 (and sometimes to be significantly negative). In order to make sense of Cumby and Obstfeld's results, Fama (1984) and others invoked risk premiums that vary and change sign over time. They reasoned that risk premiums, forward premiums, and conditionally expected exchange-rate changes must covary in a systematic way in order to produce the empirically measured coefficient.

Equation (23) above provides the theoretical analog of Cumby and Obstfeld's regression applied, however, to the real exchange rate.

The slope coefficient is variable and equal to $1 + (1\text{n}\ p)'$. Since $1\text{n}\ p\ (\omega)$ (see Figure 3) is a decreasing function, *the theoretical coeficient is indeed smaller than 1.* Qualitatively at least, the present model is capable of producing a behavior for risk premiums in relation to forward premiums that matches the pattern envisaged by Fama. There always exists a combination of parameter values (shipping cost and risk aversion) that enables our model to match the Cumby-Obstfeld coefficients. In Figure 3, the slope of the $1\text{n}\ p$ function, corresponding to a risk aversion $1 - \gamma = 2$, is, over most of its range, approximatively equal to -0.28. However, larger risk aversions are permissible.

6. Conclusion

Under sluggish quantity adjustment, the deviations from the Law of One Price do not follow a martingale process. In light of the Markov specification of the model, it is not surprising that the process found is an AR(1). It is also not surprising that the deviation remains between two boundary values. The more specific results concern the nonlinearity and heteroskedasticity of this AR(1) process.

The conditional volatility of the deviation is largest when the deviation is zero and smallest (actually zero) at the boundaries when the deviation is at its largest possible value. The conditionally expected change of the deviation is a nonlinear function of the current deviation (see Figure 5). Its sign is formally such as to produce mean reversion (which is strongest when the deviation is largest). Nonetheless, at all times at which no shipment takes place, *the probability of a move of the relative price from parity is larger than that of a move toward parity.* This is true everywhere except on the edges where a transfer of capital takes place. Within the cone, the process for the physical imbalance is centrifugal or divergent, in order that foreigners' consumption claims may be delivered to them. The capital mix spends most of the time near the boundaries where it exhibits a high degree of persistence. Periods during which the deviation from parity is wide and a country is a recurrent exporter or importer tend to last a long time.

This process for the real exchange rate is capable of rationalizing the direction of the deviations from uncovered interest rate parity, which have been found in empirical work. The risk premium, which accounts for these deviations, admits a simple expression equal to the forward premium times the derivative of the real exchange rate with respect to capital imbalance. The expression for the equilibrium forward premium is directly testable since it involves observable, contemporaneous variables only.

The alternative to the real/financial model presented here is to be found in the international macroeconomic literature, which has traditionally emphasized monetary aspects and/or sticky goods prices. But sticky prices and monetary/nominal shocks are not enough to produce a model of LOP deviations in equilibrium. Impediments encountered in transferring physical resources from one place to the other are a needed ingredient that provides room for spatial price discrepancies. It would be desirable to extend our model to incorporate monetary aspects, in order that the impact of monetary shocks, as well as real, productivity shocks, could be endogenized.[33]

Recently," articles by Krugman (1987, 1991) offered another rationale for *nominal* exchange rate reversion, based on central banks' intervention taking the form of "trigger strategies," as when the exchange rate is maintained within a band. The resulting behavior for nominal exchange rates is close to the one we produce here for real ones.

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[33] Stockman and Svensson (1987) and Stulz (1987) describe stochastic optimizing models of inter. national capital Bows, but with money and without deviations from the LOP.

[34] Some time after the first draft of the present article was circulated.

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