An Intemporal Model of Asset Prices in a Markov Economy with a Limiting Stationary Distribution

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A testable single-beta model of asset prices is presented. If state variables have a long-run stationary joint density function, then the rate return on a very long-term default-free discount bond will be peefectly correlated with the representative investor's marginal utility of consumption. Thus, the covariance of an asset's return with the return on such a bond will be an appropriate measure of the asset's riskiness. The model can be, therefore, applied or tested even though the market portfolio or aggregate consumption may not be observable. It also is shown that the expected rate of return on a very long-term bond is equal to its variance. This proposition can be tested to determine whether state variables follow stationary processes.

A single-beta multiperiod equilibrium model of asset prices in an economy with stochastic investment and consumption opportunities is developed. The contribution of this article is that the model can be tested

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without observing the market portfolio or aggregate consumption.[1] Merton (1973), Rubinstein (1976), Breeden (1979), and Cox, Ingersoll, and Ross (1985a) have shown that a general measure of an asset's riskiness is the covariance of its return with the representative investor's marginal utility of wealth or consumption. In this article, I show that under a certain set of assumptions the intertemporal behavior of a "very long-term" default-free discount (VLD) bond (to be made precise) captures the behavior of the investor's marginal utility. Breeden (1983) also presents a similar idea in a single-good economy.[2]

The article's major results rest upon the assumption that the movement of exogenousvariables that describe the state of the economy can be represented by a system of Markov processes having a limiting stationary joint density function.[3] The assumption that state variables are Markov also appears in the multiperiod asset pricing models mentioned at the beginning of this article [see also Breeden and Litzenberger (1978), Stapleton and Subrahmanyam (1978), Prescott and Mehra (1980), and Constantinides (1980)]. However, since I assume that state variables have limiting stationary density functions, the present model becomes a *special* case of these models.

The major contribution of this paper is the development of an asset pricing model with the following characteristics.

(a) A single beta, measured in terms of the covariability of an asset's return with the return on a VLD bond, is the appropriate measure of risk.

(b) The pricing model accepts nominal values as inputs and requires no estimate of inflation.

(c) The return on a VLD bond can be used to estimate prices of Arrow-Debreu securities.

There are two potential problems in applying or testing this model. First, the state variables may not have limiting stationary distributions and bond prices will not be, therefore, useful in estimating the riskiness of securities. Second, rates of return on VLD bonds may not be observable. This should not be a serious problem if our major assumption holds and if there are pure-discount bonds with fairly long maturities (e.g., 25 to 30 years). As shown in Section 3.4, depending on

[1] Problems with using data on only a subset of all assets to test the Sharpe-Lintner model are discussed in Roll (1977). Some empirical tests of the consumption CAPM are presented in Grossman, Melino, and Shiller (1987), and Breeden, Gibbons, and Litzenberger (1989).

[2] There are some overlaps between the two articles, but as discussed in the text, Breeden did not fully explore the implications of his idea. In particular, its major result holds under conditions that are considerably more general than stated in Breeden. I wish to thank the referee for bringing this article to my attention.

[3] This assumption will be discussed fully in Section 3. This property of Markov processes is discussed in Orey (1971) and Futia (1982) for discrete-time processes; and it is discussed In Itô and Nisio (1964). Karlin and Taylor (1981). and Gardiner (1985) for continuous-time processes.

the rate at which the joint transition density function of the state variables converges to a stationary distribution, the potential bias may not be significant. Furthermore, Proposition 5 of the same section demonstrates that when this article's assumptions are satisfied, the excess return on a VLD bond is equal to its variance. This result can be tested to determine whether the article's assumptions are violated.

A few other studies have considered the effects of the limiting properties of Markov processes on equilibrium asset prices. Merton (1975) develops an asymptotic theory of growth under uncertainty and presents the steady-state distribution of the economy. Merton also derives a closed-form solution for the short-run behavior of the riskless rate of interest. Lucas (1978) investigates some short-run and long-run properties of asset prices in an exchange economy using a discrete-time Markov process framework. Recently, Duffie et al. (1988) have proved Lucas's results using a less restrictive set of assumptions (e.g., they allow for heterogeneous investors).

Whether state variables have a limiting stationary density function is an empirical question. Several recent studies offer some *indirect* evidence concerning this property of state variables. Shiller and Perron (1985), Poterba and Summers (1988)) and Fama and French (1988) present empirical results suggesting that common stock returns may follow mean-reverting processes. On the other hand, Lo and MacKinlay (1988) present results that reject both the random-walk and the mean-reverting hypothesis. Finally, several papers [e.g., Nelson and Plosser (1982), Harvey (1985), Campbell and Mankiw (1987), Clark (1987), Perron (1988), and Cochrane (1988)] have tested for the existence of unit roots in the time series of certain macroeconomic variables (e.g., GNP, CPI, etc.), The existence of unit roots indicates that these variables follow nonstationary processes.

The plan of the article is as follows. In Section 1, an overview of the major result of the article is presented and an informal discussion of its derivation is offered. The article's assumptions are presented in Section 2. The model is discussed in Section 3, and some concluding comments are offered in Section 4.

1. An Overview of the Model

For the moment, assume that investors are identical. Then, it is well known that [e.g., see Rubinstein (1976) and Breeden (1979)]

$$\mu_i(t) - r(t) = -\mathrm{Cov}[dU_c(t)/U_c(t),\ dS_i(t)/S_i(t)], \tag{1}$$

where $\mu_i(t)$ is the expected *nominal* rate of return on asset *i*, $r(t)$ is the current *nominal* riskless rate of interest, $U_c(t)$ is the investor's marginal utility of *nominal* consumption expenditure, and $S_t(t)$ is the current price of security *i.* I will show that under the assumptions of this article the *nominal* rate of return on a VLD bond will be perfectly correlated with $-dU_c(t)/U_c(t)$. To see this, let $B(t, T)$ be the price of a default-free pure-discount bond that will pay one dollar at time T. We can write [e.g., see Cox, Ingersoll, and Ross (1985a)]

$$B(t, T) = \mathrm{E}[U_c(T) \mid Y(t)]/U_c(t), \tag{2}$$

where $\mathrm{E}[\cdot \mid \cdot]$ is the conditional expectation operator and $Y(t)$ is a vector of state variables representing all available and relevant information at time t.

Consider the behavior of the same bond over $(t, t + dt)$:

$$B(t + dt, T) = \mathrm{E}[U_c(T) \mid Y(t + dt)]/U_c(t + dt).$$

As t increases to $t + dt$, new information is received by the market, represented by $[Y(t + dt) - Y(t)]$; this will generate two types of effects. First, it induces the investor to change his investment-consumption policies, leading to a change in the current marginal utility of consumption expenditure from $U_c(t)$ to $U_c(t + dt)$. Second, the new information causes the investor to change the conditional expectation of the future value of his marginal utility of consumption from $\mathrm{E}[U_c(T) \mid Y(t)]$ to $\mathrm{E}[U_c(T) \mid Y(t + dt)]$. The underlying assumption of this article is that $Y(t)$ has a limiting stationary density function, which implies that $\mathrm{E}[U_c(T) \mid Y(t)]$ will *not* depend on $Y(t)$ when T is large enough, and, thus, the second effect is eliminated;[4] that is, the numerator of the expression on the right-hand side of Equation (2) remains unchanged as new information is received.[5] We, therefore, have

$$dR(t) = \lim_{T \to \infty} \frac{B(t + dt, T)}{B(t, T)} = \frac{U_c(t)}{U_c(t + dt)}.$$

In a continuous-time framework, this indicates that the *nominal* rate of return on a VLD bond is perfectly *negatively* correlated with the rate of change in the investor's marginal utility of *nominal* consumption expenditure. By combining this with Equation (1), this pricing model is obtained:

$$\mu_i(t) - r(t) = \mathrm{Cov}[dR(t), dS_i(t)/S_i(t)]. \tag{3}$$

[4]Note that all expectations are implicitly conditioned upon current wealth. In an exchange economy, the behavior of aggregate wealth is exogenously determined for it will be a function of state variables (e.g.. see the closed-form solution of Section 3.2). In a production economy, aggregate wealth will be one of the state variables, and the restrictions imposed on the production technology will ensure that it displays the required stationarity. See also the discussion following Assumption 4.

[5]Darrell Duffie has pointed out to me that if there is a very long-term asset that provides a contingent cash flow upon maturity, its rate of return can also be used to measure riskiness of securities. For example, one may consider a very long-term pure-discount bond that is not default-free.

Breeden (1983) argues that when aggregate real consumption follows a mean-reverting process, prices of options on long-term default-free *purchasing power* bonds will be perfectly positively correlated with changes in real aggregate consumption expenditure. He does not, however, extend the results to a multigood economy, and he foresees two potential problems in practical applications of the results.

First, Breeden discusses how options on *purchasing power* bonds can permit an efficient allocation of risk. Breeden argues that "options traded are likely to be based on nominal prices, rather than upon real prices as the theory suggests," (p. 6) and if variations in nominal bond prices are mostly caused by variability in inflation, then "allocational benefits discussed are not likely to be so significant for options on nominal bonds" (p. 6).

Second, Breeden states that under certain assumptions[6] bond prices may not be stochastic, although aggregate consumption and wealth are. In this case, "options on bond prices cannot span consumption possibilities, and thus cannot promote efficient allocation of risk" (p. 6). As Breeden argues, the first problem is more serious (the second one is discussed in Section 3.3). In this article, however, I show that such a problem does not exist.

Breeden and Litzenberger (1978) show that if two states are equally likely to occur, prices of Arrow-Debreu securities for the two states will be equal if and only if aggregate consumption is the same in the two states. Using the results of Rubinstein (1974) or Constantinides (1982), we may restate this proposition as follows: prices of Arrow-Debreu securities for the above states will be equal if and only if the marginal utility of the composite investor is the same in the two states. Thus, following Breeden and Litzenberger, if the price of an asset spans the states of the composite investor's marginal utility of consumption, then its price will span all *relevant* states. This result holds in a single-good as well as a multigood economy. It must be noted that unlike Breeden (1983), I am notconcerned with the allocational role of long-term bonds; rather my objective is to show that in a complete market, bond prices can be used to determine the behavior of the composite investor's marginal utility.

2. Assumptions and Definitions

I present the structure of my economy, which is similar to that of Merton (1973), Breeden (1979), Cox, Ingersoll, and Ross (1985a), and Huang (1987). Thus, I will keep my discussion brief.

[6] For example, if the investor's measure of relative (absolute) risk aversion is constant and aggregate consumption follows geometric (arithmetic) Brownian motion, interest rates and thus bond prices will not be stochastic through time.

Assumption 1. Assume that there exists an $(N \times 1)$ -dimensional vector of state variables, $Y(t)$, that describes the state of the world at time t. The movement of these variables is determined by a system of time-homogeneous stochastic differential equations of the form [e.g., see Arnold (1974)]

$$dY(t) = \theta(Y)\,dt + \delta(Y)\,dZ(t), \tag{4}$$

where $\theta(Y)\colon \mathbf{R}^N \rightarrow \mathbf{R}^N$, $\delta(Y)\colon \mathbf{R}^N \rightarrow \mathbf{R}^{N\times N}$ and is nonsingular, and $Z(t) = \{Z_1(t), \ldots, Z_N(t)\}$ are independent standard Brownian motions. We also assume that the above system has a limiting stationary density function. That is, the following limit exists:

$$\lim_{s\rightarrow\infty} \Pr\{Y(t+s) \in \mathscr{A} \mid Y(t) = x_t\} = \int_{\mathscr{A}} \psi(Y)\,dY. \tag{5}$$

According to this expression, the probability that the state of the economy will be in set $\mathscr{A}$ at time $t + s$ is independent of its state at t when s approaches infinity. This probability density function is represented by y (Y). Further restrictions must be imposed on the parameters of the system if y (Y) is to exist. These restrictions and some specific solutions for y (Y) are presented in Appendix A.

Assumption 2. There are N risky securities. Prices of these assets are endogenously determined and are, therefore, functions of the state variables. If the market price of security i, $S_t(Y, t)$, is a continuous function with continuous partial derivatives, then its movement can be represented by a stochastic differential equation. Though state variables may follow-for example, mean-reverting processes-not all asset prices may follow such processes. Depending on the functional form of $S_t(Y, t)$, the price of an asset may, for example, follow a geometric Brownian motion [see Gihman and Skorohod (1972), p. 31].

Assumption 3. Security markets are perfect and trading takes place continuously. Short selling is allowed with the full use of the proceeds, and investors can borrow and lend at the same *nominal* risk-free rate of interest. There are no redundant assets, and because the number of securities is equal to the number of state variables, security markets are complete and the allocation of risk is Pareto optimal [see Breeden (1984) and Duffie and Huang (1985)].[7]

[7] The assumption that markets are complete is required only to show that a representative investor will exist. In the absence of this investor, equilibrium prices will depend upon the distribution of wealth, and then one has to show that these state variables will also follow stationary processes; for a model of this type, see Dumas (1989).

Assumption 4. There are Q investors, indexed by $q = 1, \ldots, Q$, who have homogeneous beliefs. An investor seeks to maximize an objective function of the form

$$E\left[\int_0^\infty U^q(C^q(\tau), P(\tau), \tau)\, d\tau \mid Y(0)\right],$$

where $C^q(t)$ is his nominalconsumption expenditure and $P(t)$ is a J- dimensional vector of prices of consumption goods; $U(\cdot\ t)$ is the *indirect* utility of consumption and is equal to $\exp\{-rt\}\ V(\cdot)$, where $0 < r < 1$. It is assumed that $V(\cdot)$ is increasing, and strictly concave in C^q and twice differentiable with respect to C^q and P.[8] Each investor is endowed with a vector of consumption goods, denoted by $k^q(Y(t), t) \geq 0$, which is a continuous function of Y and t, and is a bounded function of t. Aggregate endowment is denoted by $k(Y(t), > 0$, and aggregate consumption expenditure by $C(Y(t), t) = k(Y(t), t)^T P(Y(t), t)$.

Though the basic framework of this article is that of an exchange economy, the results also will hold in a production economy. For example, its results will hold in the production economies of Merton (1975) and especially Bourguignon (1974).

Constantinides (1982) and Huang (1987) show that when securities markets are complete, one can construct an economy in which heterogeneous investors are replaced with a central planner, while the equilibrium values of economic variables (e.g., asset prices) remain equal to those of the original economy. Given our assumptions, a representative investor will exist and we use $U(C(\cdot), P(\cdot), t)$ to denote his utility function. [See also Rubinstein (1974) for a discussion of aggregation conditions in *incomplete* markets.]

3. The Model

Here, I first obtain an endogenously determined return-generating process for default-free bonds. To further illustrate the result, I then present some closed-form solutions. Finally, I use these results to study properties of pure-discount bonds with very long maturities.

3.1 Return-generating processes of default-free bonds

The current price of a default-free discount bond that pays one dollar at maturity, T, and the composite investor's marginal utility of con-

[8] The above assumptions regarding the investor's utility function can be relaxed somewhat. For example, it is not necessary for the utility function to be separable in the time preference. It is only required that $\lim_{t\to\infty}[\partial U_c(t)/\partial C + \partial U_c(t)/\partial P]/U_c(t)$ be bounded. Furthermore, preferences may be state dependent.

sumption expenditure satisfies Equation (2). We now use a result reported by Gihman and Skorohod (1972) to express this equation in terms of the *unconditional* expected value of a *conditional* Markov process.

Proposition 1. *If the system of stochastic differential equations (4) satisfies certain existence and uniqueness conditions, then its solution will be Markov and will satisfy the relationship*

$$\Pr\{\phi(Y(T)) \in \mathscr{B} \mid Y(t) = x_t\} = \Pr\{\phi(Y(x_t, T)) \in \mathscr{B}\}, \qquad (6)$$

where f is any continuous and bounded function, and $Y(x_t, T)$ is the solution of Equation (4) at time $T > t$ subject to the initial condition, $Y(t) = x_t$.

Proof: See Gihman and Skorohod (1972), theorem 1, pp. 67 and 288. ∎

According to Equation (6), the *conditional* probability that $f(\cdot)$ will be in set $\mathscr{B}$ at time T is equal to the *unconditional* probability that it will be in set $\mathscr{B}$, if $f(\cdot)$ is written as a function of the solution to the system of stochastic differential equations.

The following corollary plays an important role in the derivation of the return-generating process of default-free discount bonds.

Corollary 1. *If $f(x_t)$ is r-times differentiable and it satisfies certain growth conditions, then the function*

$$\Psi(x_t) = \mathrm{E}[\phi(Y(T)) \mid Y(t) = x_t] = \mathrm{E}[\phi(Y(x_t, T))] \qquad (7)$$

will be r-times differentiable with respect to x_t and, in fact,

$$\frac{\partial \Psi(x_t)}{\partial x_t} = E\left[\frac{\partial \phi(Y(x_t, T))}{\partial Y(x_t, T)} \frac{\partial Y(x_t, T)}{\partial x_t}\right]. \qquad (8)$$

Proof. See Gihman and Skorohod (1972), corollary 1, p. 62. ∎

Suppose $Y(t) = x_t$, then we can characterize time T aggregate consumption expenditure and consumption goods prices as $C(Y(x_t, T), T)$ and $P(Y(x_t, T), T)$, respectively. Equation (2) can thus be written in its unconditional form as

$$B(t, T) = \mathrm{E}[U_c(C(\cdot, T), P(\cdot, T), T)] \times U_c(C(\cdot, t), P(\cdot, t), t)^{-1}.$$

This expression is an alternative form of Equation (2). The major advantage of using this is that we can analyze the *direct* effects of new information on current as well as future values of the investor's marginal utility of consumption expenditure. We now use this expression to derive an endogenously determined return-generating process for a default-free discount bond.

Proposition 2. *If Assumptions 1-4 are satisfied,*[9] *the nominal rate of return on a default-free discount bond will be*

$$\frac{dB(t,T)}{B(t,T)} = \left[r(t) + \left(\frac{H(t)}{U_c(t)} - \frac{\mathrm{E}[H(T)Y_x(T)]}{\mathrm{E}[U_c(T)]}\right)\delta(t)\delta(t)^{\mathrm{T}}\frac{H(t)^{\mathrm{T}}}{U_c(t)}\right]dt + \left(\frac{H(t)}{U_c(t)} - \frac{\mathrm{E}[H(T)Y_x(T)]}{\mathrm{E}[U_c(T)]}\right)\delta(t)\; dZ(t). \tag{9}$$

For brevity we have used $U_c(t)$ to denote $U_c(C(\cdot, t), P(\cdot, t), t)$. Here, $H(t) = -U_{cc}(t)C_y(t) - U_{cp}(t)P_y(t)$, which represents the sensitivity of $U_c(t)$ to changes in the time t values of state variables; $r(t)$ is the short-term nominal riskless rate of interest; and $Y_x(T)$ denotes the derivative of $Y(x_t, T)$ with respect to its initial condition, x_t. Matrix transpose is denoted by "$^{\mathrm{T}}$."

Proof. See Appendix B. ■

According to Equation (9), the expected *nominal* rate of return on a default-free discount bond is equal to the short-term *nominal* riskless rate plus a premium, which is affected by two factors. First, it is affected by $H(t)$, which is related to the investor's preference. Second, it is affected by $Y_X(t)$, which captures the time-series properties of state variables. For instance, if $\mathrm{E}[Y_x(\tau)] = 1$, the state variables will follow nonstationary processes (i.e., they possess unit roots); whereas if $\mathrm{E}[Y_x(t)] < 1$, they will follow stationary processes.

From Equation (9) we can see that the risk premium and the standard deviation of the bond's return reflect the relative sensitivity of the investor's current marginal utility and its expected future value to unexpected changes in the state variables. The change in the current marginal utility resulting from the arrival of new information is measured by $-H(t)d(t)\,dZ(t)$, while the change in its expected future value is measured by $-E[H(T)Y_x(T)]\,d(t)\,dZ(t)$. Note that $H(t)$ tends

[9] It is not necessary to assume that the state variables have a limiting stationary density function to prove this proposition.

to be greater than zero,[10] and thus the size and the sign of the premium on the bond's return are significantly affected by the stochastic properties of state variables, $Y_x(\tau)$. For instance, if $Y_x(\tau)$ is uniformly negative, the state of the economy displays negative autocorrelation, and the premium will always be positive and rather large.

3.2 Closed-form solutions

To further illustrate this article's results, we derive closed-form solutions corresponding to Equation (9) for a multigood monetary economy. Suppose we write Equation (4) as

$$dY_i(t) = \nu_i(\eta_i - Y_i(t))\,dt + \delta_i\,dZ_i(t), \qquad i = 1,\dots,N, \tag{4'}$$

where ν_i, η_i, δ_i are positive constants. Under this assumption, it can be shown that $\partial Y_i(x_{it}, T)/\partial x_{it} = \exp\{-\nu_i(T - t)\} < 1$, indicating that the process is stationary. Also, assume that the investor's direct utility function can be written as

$$U(C_1, \dots, C_J, t) = \frac{e^{-\rho t}}{1-\gamma}\left(\prod_{j=1}^{J} C_j^{\beta_j}\right)^{1-\gamma}, \tag{10}$$

where $\beta_j > 0$, $\Sigma_j \beta_j = 1$, and $\gamma > 0$. This utility function is employed because it leads to a well-defined price index [see Grauer and Litzenberger (1979)]. Note that these assumptions are needed only to solve Equation (9) for a simple case; they are *not* needed to obtain the main results of the article.

It is shown in Appendix C that the closed-form solution corresponding to Equation (9) is

$$\frac{dB(t, T)}{B(t, T)} = r(t)\,dt + \sum_{i=1}^{N} \varphi_i^2 \delta_i^2 (1 - e^{-\nu_i \tau})\,dt + \sum_{i=1}^{N} \varphi_i \delta_i (1 - e^{-\nu_i \tau})\,dZ_i(t), \tag{11}$$

where $\tau = T - t$ is the time to maturity of the bond, and φ_i is a known function of the degree of relative risk aversion and other parameters of the model.

Many interesting results can be obtained from Equation (11), but their full discussion will distract from the main idea of this article. It is, however, important to point out the following.

(a) If aggregate consumption and the price index follow geometric Brownian motions (i.e., $\nu_i = 0$, $i = 1,\dots, N$), then the rate of return will be nonstochastic and equal to $r(t)$.

[10] Note. $H(\tau)/U_c(\tau) = \gamma\partial \ln C/\partial Y + \Sigma_j (\gamma\beta_j - m_j)\partial \ln P_j/\partial Y$. Here, γ is the measure of relative risk aversion, and b and m, are, respectively, the average share and the marginal share of consumption good j in the investor's budget.

(b) The risk premium on the bond's return is always positive, and both the premium and the variance of the return are *increasing* functions of time to maturity.

(c) The bond's return will be stochastic as long as nominal or real consumption expenditures is stochastic (see Appendix C).

3.3 Return-generating processes of very long-term default-free bonds

I now examine the properties of returns on VLD bonds when the system of Markov processes that describes the movement of the economy has a limiting stationary density function.

Proposition 3. *If Assumptions 1-4 are satisfied, the nominal rate of return on a VLD bond can be expressed as*

$$dR(t) = \left[r(t) + \frac{H(t)\delta(t)\delta(t)^{\mathsf{T}}H(t)^{\mathsf{T}}}{U_c(t)^2} \right] dt + \frac{H(t)}{U_c(t)}\delta(t)\ dZ(t). \quad (12)$$

Proof. The proof is obtained by allowing $T \to \infty$ in Equation (9). Consider the behavior of E[$H(T)Y_x(T)]/$ E[$U_c(T)]$ as $T \to \infty$. When the system displayed in Equation (4) has a limiting stationary density function, its solution, $Y(x_t, T)$, becomes independent of its initial condition, as $T \to \infty$. Consequently, $Y_x(x_t, T)$ will approach zero, and, given Assumptions 1 and 4, E[$H(T)\ Y_x(T)]/$ E[$U_c(T)$] approaches zero as well when $T \to \infty$. ∎

In Equation (12) the expected rate of return on a VLD bond is equal to the current riskless rate of interest plus a risk premium, which is equal to the variance of the rate of change in the marginal utility of consumption. This equation shows that the rate of return on a VLD bond is perfectly negatively correlated with the rate of change in the investor's marginal utility of nominal consumption expenditure. This relationship holds as long as bond returns are stochastic and whether real aggregate consumption expenditure is constant or not. To see this, suppose I is a wealth invariant price index. Since $U_c(C, I)$ is homogeneous of degree -1, we have $U_c(C, I) = IU_c(C/I)$. This shows that the marginal utility of nominal consumption expenditure will be stochastic even when real consumption expenditure, C/I, is constant.

Breeden (1986) has shown that, under certain assumptions, prices of default-free bonds will be nonrandom through time, making Proposition 3 invalid. The constancy of the *nominal* rate of interest is an empirical issue that can be tested without difficulty. In any event, as long as there is some risk in the economy *and* nominal interest rates are random, returns on VLD bonds can be used to measure that risk.

It is instructive to write Equation (12) for the case of a VLD bond. If the distribution function of the bond return is not degenerate (i.e., $\varphi_i \neq 0$ and $\nu_i > 0$, $i = 1,\dots,N$), then we can let $(T - t) \to \infty$ in Equation (12) to obtain the following expression:

$$dR(t) = r(t)\ dt + \sum_{i=1}^{N} \varphi_i^2 \delta_i^2\ dt + \sum_{i=1}^{N} \varphi_i \delta_i\ dZ_i(t). \tag{13}$$

3.4 A testable model of asset prices

Here, I present the first testable implication of the results discussed in Section 3.3.

Proposition 4. *If Assumptions 1-4 are satisfied, risk premiums on asset returns can be expressed according to Equation (3).*

Proof. According to Equation (12), the rate of return on a VLD bond can be expressed as $dR(t) = [dU_c(t)/U_c(t)]^2 - dU_c(t)/U_c(t)$. Note, according to Breeden (1986), we have E $dU_c(t) = -r(t)U_c(t)\ dt$. Thus, we have

$$\mathrm{Cov}[dR(t),\ dS_i(t)/S_i(t)] = -\mathrm{Cov}[dU_c(t)/U_c(t),\ dS_i(t)/S_i(t)]. \tag{14}$$

The proof is completed once Equations (14) and (1) are compared. ■

The central result of this article is that under its set of assumptions the risk premium on any asset's return is equal to its covariance with the rate of return on a VLD bond. One major difficulty in testing this result is that one is not likely to find any VLD bonds. Besides, from an empirical point of view, it is unclear how long the maturity of a bond should be to be considered a very long-term bond. The following proposition provides a partial solution to this problem.

Proposition 5. *The expected excess nominal rate of return on a VLD bond is equal to the variance of its rate of return. That is,*

$$\mathrm{E}[dR(t)] - r(t)\ dt = \mathrm{Var}[dR(t)]. \tag{15}$$

Proof. Follows directly from Proposition 4. ■

This result holds for VLD bonds if state variables have a joint limiting stationary density function. Therefore, Proposition 5 represents another testable implication of the underlying assumptions of this article. Whether Propositions 4 and 5 approximately hold for long-term bonds (i.e., default-free pure-discount bonds with maturities of, say, 20 or 30 years) depends, among other things, on the rate that the joint transition density function of the state variables converges, if at all, to a stationary distribution. In other words, using Equation (9), the covariance between return on risky asset *i* and return on a default-free pure-discount bond of *finite* maturity can be expressed as

$$\begin{aligned}
&\mathrm{Cov}[dB(t,T)/B(t,T),\ dS_i(t)/S_i(t)] \\
&\quad = -\mathrm{Cov}[dU_c(t)/U_c(t),\ dS_i(t)/S_i(t)] \\
&\qquad - \frac{\mathrm{E}[H(T)Y_x(T)]}{\mathrm{E}[U_c(T)]}\mathrm{Cov}[dY(t),\ dS_i(t)/S_i(t)].
\end{aligned}$$

The first term on the right-hand side of this expression is equal to the asset's risk premium. Thus, if one uses the covariance between the return on a risky asset and the return on *fjnite* maturity bond to measure risk, the risk premium will be under- or overestimated by the last term on the right-hand side of the above expression. Therefore, the accuracy of a test of the model using returns on such bonds depends on how significantly $\mathrm{E}[H(T)Yx(T)]/\mathrm{E}[U_c(T)]$ is different from zero; an indication of its significance can be obtained by testing Proposition 5.

Some empirical tests of a relationship similar to the one presented in Equation (15) are discussed in Kazemi and Milonas (1989). They use portfolios of Treasury bonds to create data on the behavior of long-term pure-discount bonds. They conduct nonparametric and conditional tests of Equation (15). Returns on three out of four portfolios created to replicate the behavior of pure-discount bonds did not significantly violate the restrictions imposed by Equation (15) on VLD bond returns.

An empirical test of the asset pricing model presented here will, therefore, consist of two steps. First, data on long-term pure-discount bonds are needed to test Proposition 5 to see whether one of the implications of our assumptions is empirically supported. Second, if the first test is successful, the price dynamics of the same bonds should be employed to verify the results of Proposition 4.

To develop this model, I had to assume that state variables that describe the *nominal* state of the economy have a stationary density function. It is, however, more likely that state variables describing the real state of the economy (e.g., real GNP, real productivity, etc.) may display this property. Then to test the model, one should use only real returns, and as suggested by Breeden (1983), data on purchasing power bonds will be needed to measure risk.

The property of VLD bonds that we just discussed can also be used to calculate prices of Arrow-Debreu securities. The equilibrium prices of these securities are generally written as

$$S(Y(T) \mid Y(t)) = \pi(Y(T) \mid Y(t))\frac{U_c(T)}{U_c(t)}, \qquad (16)$$

where $\pi\ (Y(T) \mid Y(t))$ is the transition density function of the state variables. Define $Z(t, T+s) = B(T, T+s)^{-1} B(t, T+s)$, $s > 0$, as the inverse of the return, over (t, T), of a default-free discount bond maturing at $T + s$, and let $R(t, T)$ denote the inverse of the return on a VLD bond over the same time interval. Given this article's assumptions, we have

$$\begin{aligned} R(t, T) &= \lim_{s\to\infty} Z(t, T+s) \\ &= \lim_{s\to\infty} \frac{U_c(T)}{E_T[U_c(T+s)]} \times \frac{E_t[U_c(T+s)]}{U_c(t)} \\ &= \frac{U_c(T)}{U_c(t)}. \end{aligned}$$

Using this result, Equation (16) can be written as

$$S(Y(T) \mid Y(t)) = \pi(Y(T) \mid Y(t))R(t, T).$$

This characterization of prices of Arrow-Debreu securities can be used to price more complex securities with uncertain income streams.

4. Concluding Comments

In this article, I considered a *special* case of the intertemporal asset pricing models of Merton (1973)) Rubinstein (1976)) Breeden (1979), and Cox, Ingersoll, and Ross (1985a). I developed an equilibrium model of asset prices in a competitive, multigood, multiasset economy. The model is general enough to allow for stochastic investment and consumption opportunities, and yet is simple enough to be empirically tested.

By assuming that state variables follow Markov processes with limiting stationary density functions, we can show that the rate of return on a very long-term default-free discount bond will be perfectly negatively correlated with the growth rate of the representative investor's marginal utility of consumption. Thus, covariability of an asset's return with return on a very long-term bond would be a sufficient measure of the asset's riskiness. This pricing model can be, therefore, tested and applied without observing the rate of return on the market portfolio or the growth rate of aggregate consumption.[11]

Appendix A

First, consider the limiting stationary density function of a one-dimensional stochastic differential equation. Consider the following case:

$$dY(t) = \theta(Y)\,dt + \delta(Y)\,dZ(t), \qquad Y_1 \leq Y \leq Y_2.$$

The limiting stationary density function of the above equation will be as follows [see Karlin and Taylor (1981), p. 220]:

$$\psi(Y) = C_1 \frac{S(Y)}{K(Y)\delta^2(Y)} + C_2 \frac{1}{K(Y)\delta^2(Y)},$$

where

$$K(Y) = \exp\left\{-\int_{Y_1}^{Y} \frac{2\theta(X)}{\delta^2(X)}\,dX\right\}, \qquad S(Y) = \int_{Y_1}^{Y} K(X)\,dX,$$

and C_1 and C_2 are constants. If the last integral exists, then the limiting stationary density will exist.

Consider the system of stochastic differential equations that appears in Equation (4). The limiting stationary density function of that system, if it exists, will satisfy the partial differential equation [see Gardiner (1985)]

$$\theta_i(Y)\psi(Y) - \frac{1}{2}\sum_j \frac{\partial}{\partial Y_j}[\Omega_{ij}(Y)\psi(Y)] = C,$$

where q_i is the *ith* element of θ, Ω_{ij} is the *ijth* element of $\Omega = \delta\delta^T$, and C is a constant. A *general* solution to the above partial differential equation does not exist unless we set $C = 0$. Under this assumption, the limiting stationary density function, if it exists, will be

$$\psi(Y) = \exp\left\{-\int_{Y_1}^{Y} dX^T\, S(X)\right\},$$

where $S = (S_1, \ldots, S_N)^T$,

$$S_i = \sum_k \Phi_{ik}\left[2\theta_k - \sum_j \frac{\partial \Omega_{kj}}{\partial Y_j}\right],$$

[11] To the extent that investment and consumption opportunities are stochastic, this measure of risk is likely to be nonstationary and, thus, empirical testing of the model will be subject to the same difficulties to which Breeden (1979) is subject [see, e.g., Cornell (1981)].

and Φ_{ik} is the *ikth* element of Ω^{-1}. The necessary and sufficient condition for the existence of a limiting stationary density function is $\partial Si(Y)/\partial Yj = \partial Sj(Y)/\partial Yi$.

Appendix B

Proof of Proposition 2

By applying Itô's lemma to Equation (2), we have

$$\frac{dB(t,T)}{B(t,T)} = \frac{d\mathrm{E}[U_c(T)]}{\mathrm{E}[U_c(T)]} - \frac{dU_c(t)}{U_c(t)} + \left[\frac{dU_c(t)}{U_c(t)}\right]^2 - \frac{d\mathrm{E}[U_c(T)]}{\mathrm{E}[U_c(T)]} \times \frac{dU_c(t)}{U_c(t)}. \tag{B1}$$

First, consider the application of Itô's lemma to E[$U_c(T)$]:

$$d\mathrm{E}[U_c(T)] = \left[\frac{\partial \mathrm{E}[U_c(T)]}{\partial t} + \Phi_t \mathrm{E}[U_c(T)]\right] dt + \left[\frac{\partial \mathrm{E}[U_c(T)]}{\partial x_t}\right] \delta(t)\, dZ(t), \tag{B2}$$

where Φ_t is the infinitesimal operator for the system of Markov processes presented in Equation (4) [Arnold (1974), p. 41]. The two terms inside the first bracket represent Kolmogorov's backward equation and their sum is equal to zero [Arnold (1974), theorem 2.6.3]. We now use Corollary 1 to characterize the remaining term on the right-hand side of Equation (B2):

$$\frac{\partial \mathrm{E}[U_c(T)]}{\partial x_t} = \mathrm{E}\left[U_{cc}(T)\frac{\partial C(T)}{\partial Y(T)}\cdot\frac{\partial Y(T)}{\partial x_t} + U_{cp}(T)\frac{\partial P(T)}{\partial Y(T)}\cdot\frac{\partial Y(T)}{\partial x_t}\right].$$

Thus, we can write

$$d\mathrm{E}[U_c(T)] = E[(U_{cc}(T)C_y(T) + U_{cp}(T)P_y(T))Y_x(T)]\delta(t)\, dZ(t). \tag{B3}$$

Also, the application of Itô's lemma to $U_c(t)$ produces [see Breeden (1986)]

$$dU_c(t) = -r(t)U_c(t)\, dt + U_{cc}(t)C_y(t)\delta(t)\, dZ(t) + U_{cp}(t)P_y(t)\delta(t)\, dZ(t). \tag{B4}$$

Equation (9) is obtained by substituting Equations (B3) and (B4) into (B1). In Equation (B3),

$$Y_x(T) = \frac{\partial}{\partial x_t}\left(x_t + \int_t^T \theta(Y(x_t,\tau,T))\, d\tau\right.$$

$$+ \int_t^T \delta(Y(x_t, \tau, T))\ dZ\ (\tau)\Big)$$

$$= 1 + \int_t^T \theta_Y(\tau) Y_x(\tau)\ d\tau + \int_t^T \delta_Y(\tau) Y_x(\tau)\ dZ(\tau). \qquad \blacksquare$$

Appendix C

To obtain the closed-form solution, we make the further assumption that aggregate endowment of good j is of the form

$$k_j(t) = k_j(0) \exp\left\{\sum_{i=1}^{N} \alpha_i^j Y_i(t)\right\}, \qquad j = 1,\ldots,J,$$

where $kj\ (0) > 0$ and $\alpha_i^j \geq 0$ are constants. Note that $kj(t)$ is lognormally distributed. Money is introduced in this economy by following Rubinstein (1981). At the beginning of each time interval, firms that are endowed with consumption goods sell their outputs to a central auctioneer at equilibrium prices and receive money in exchange. These firms use the cash receipts to make dividend payments to their shareholders, and finally the shareholders must use their cash holdings to purchase consumption goods from the auctioneer. This scenario leads to the unit-velocity quantity equation

$$C(t) = \sum_{j=1}^{J} k_j(t) P_j(t) = M(t),$$

where $C(t)$ is aggregate consumption expenditure and $M(t)$ is the money supply, which is exogenously determined according to the following rule:

$$M(t) = M(0) \exp\left\{\sum_{i=1}^{N} \alpha_i^M Y_i(t)\right\}.$$

Given the investor's preference [Equation (lo)], the equilibrium price of good j will be

$$P_j = \frac{\beta_j M(0)}{k_j(0)} \exp\left\{\sum_{i=1}^{N} (\alpha_i^M - \alpha_i^j) Y_i\right\}.$$

If $I(t)$ denotes the wealth invariant price index, then In $I(f)$ = $\Sigma_j\ b_j$ ln $Pj(t)$, and its growth rate will be $\Sigma_i\ a_i\ Y_i(t)$, where $a_{i.} = \alpha_i^M - \Sigma_j\ \alpha_i^j \beta_j$. Given the above assumptions, the representative investor's marginal utility of consumption when the equilibrium conditions are satisfied will be

$$U_c(Y(t), t) = Ke^{-\rho t} \exp\left\{-\sum_{i=1}^{N} \varphi_i Y_i(t)\right\}, \tag{C1}$$

where $K = M(0)^{-1} (\Pi_{j=1}^{J} k_j(0)^{\beta_j})^{1-\gamma}$, $\varphi_i = \gamma\alpha_i^M - (\gamma - 1)\alpha_i$, and γ denotes the measure of relative risk aversion. If $\alpha_i = 0$, for $i = 1,\ldots, N$, inflation will be nonstochastic, while if $\alpha_i^M = 0$, for $i = 1,\ldots, N$, *nominal* aggregate consumption will be nonstochastic. On the other hand, if $\alpha_i^j = 0$, for $i = 1,\ldots, N$ and $j = 1,\ldots, J$, *real* aggregate consumption will not be random.

Recently, Dybvig, Ingersoll, and Ross (1987) have shown that under surprisingly simple conditions the long-term rate defined as $F(t) = \lim_{T \to \infty} (-(T-t)^{-1} \ln B(t, T))$ must follow a nondecreasing process. That is, for $s > t$, we have $F(s) \geq F(t)$ with probability 1. Their results also point to the conditions under which the long-term rate will be constant through time [constant long-term rate also appears in Cox, Ingersoll, and Ross (1985b)]. The present model is quite consistent with a constant long-term rate. To show this, we can use Equations (2) and (C1) to express the *yield to maturity* of a default-free discount bond as

$$F(t, T) = \rho + \tau^{-1} \sum_{i=1}^{N} \varphi_i \nu_i (\eta_i - Y(t))(1 - e^{-\nu_i \tau})$$

$$- (4\tau)^{-1} \sum_{i=1}^{N} \varphi_i^2 \delta_i^2 \nu_i^{-1} (1 - e^{-2\nu_i \tau}). \tag{C2}$$

Clearly, as τ increases, $F(t, T)$ approaches r, which is constant.

Though the very long-term rate is constant, the *rate of return* on the very long-term bond may be random. To see this, we can present a bond's rate of return in terms of changes in its yield to maturity:

$$\frac{dB(t, T)}{B(t, T)} = F(t, T)\, dt - (T - t)\, dF(t, T) + \frac{1}{2}(T - t)^2\, dF(t, T)^2.$$

If the long-term rate is constant, then clearly $\lim_{T \to \infty} dF(t, T)$ cannot be random. On the other hand, if the rate of return on the VLD bond is random, then $\lim_{T \to \infty} (T - t)\, dF(t, T)$ must be random. It is clear that these two statements are not inconsistent. For example, they apply to the above closed-form solution.

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