Intertemporal Arbitrage Pricing Theory

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It is shown that the arbitrage pricing theory holds in each infinitesimal period of a continuous trading model under the assumption that dividend payoffs are functionals of factor and idiosyncratic uncertainty. This generalizes the one-period model's result that the arbitrage pricing theory holds under the assumption that price changes in a given period satisfy a factor structure. since instantaneous returns in a multiperiod model are endogenously determines the theory is derived under assumptions that may be viewed as restricting more primitive characteristics of the economy than the assumptions made for the one-period model.

The one-period arbitrage pricing theory (ART), first studied by Ross (1976), is one of the central models of the theory of capital markets and has attracted much attention in the literature [e.g., Huberman (1983), Chamberlain and Rothschild (1983), Ingersoll (1984), and Reisman (1988)]. This model states that if the payoffs of a sequence of assets satisfy a factor structure, then the expected return of each asset in the sequence is approximately a linear combination of the covariances of the factors with the return of the asset.

I consider a continuous trading economy in which

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each asset i is a claim to a liquidating dividend payoff $X_i(T)$. Returns over the entire period need not satisfy a factor structure in the usual sense. However, it is assumed that terminal payoff can be split into two parts. One of these is dynamically spanned by K factor portfolios with value processes $V_1, \ldots, V_K$, and the other is conditionally uncorrelated across assets at each time, that is,

$$X_i(T) = F_i + N_i,$$

where F_i is the payoff of a continuously adjusted portfolio of the factor portfolios. We refer to the term $N_i = X_i(T) - F_i$ as a "residual," and suppose that $cov_t(N_i, N_j) = 0$, for each $i \neq j$ and all t. Because F_i need not be linear in the factor portfolios, this is a more general hypothesis than directly assuming a factor structure, and it does not imply that expected returns over the entire period satisfy the usual APT relation. Here the systematic risks are dynamically spanned by the factors, rather than being the payoffs of static portfolios of the factors. Of course, with continuous trading, the dynamic span of the factor portfolios is much richer than the span of static portfolios. Further, instead of assuming an exact dynamic factor structure for the payoffs [$\text{cov}_t(N_i, N_j) = 0$], an approximate factor structure along the lines of Chamberlain and Rothschild (1983) is assumed. In other words, it is assumed that the residuals N_i are "dynamically weakly correlated." This means that there is only weak correlation in the local changes in the expected idiosyncratic risks $E_t[N_i]$.

An example of an economy with an exact dynamic factor structure is the following. For each t, the factors $V_1, \ldots, V_K$ are Itô processes, $cov_t(dV_K, dV_j) = 0$, for every $k \neq j$, and $\text{var}_t(dV_k) = dt$, for each k. The risk-free rate is deterministic. Each payoff $X_i(T)$ satisfies some integrability conditions and is a functional of the sample paths of $V_1, \ldots, V_K$ and a Wiener process B_i, where the dB_i are uncorrelated across assets and uncorrelated with dV_k for each k.

I demonstrate that, subject to some regularity conditions, the absence of arbitrage and the existence of a dynamic factor structure imply that instantaneous returns satisfy an APT relation. The APT result is that, for most assets and at most times, the local risk premium of the asset is approximately a linear function of its betas with respect to the factors. This approximation result is in the same spirit of the one-period model.

If we consider the expected idiosyncratic risks under the risk-neutral probability, say $E_t^*[N_i]$, and we adopt the weak-correlation hypothesis for the local changes in these expectations, then the APT would be relatively straightforward to derive. The reason is that these local changes are the residuals from projecting the local rates of return on $dV_1, \ldots, dV_K$. Thus, the weak correlation hypothesis would be equivalent to assuming that local changes in prices have a factor structure at each instant. Applying the one-period APT logic at each instant would yield the dynamic APT. However, this is an assumption about the pricing of the risks N_t rather than an assumption about the risks themselves. This assumption is not implied by the dynamic weak correlation hypothesis used in this article, as will be demonstrated by example.

To summarize, I show that the APT holds in each infinitesimal period of a continuous trading model under the assumption that dividend payoffs are functionals of factor and idiosyncratic uncertainty. This generalizes the one-period model's result that the APT holds under the assumption that price changes in a given period satisfy a factor structure. Since instantaneous returns in a multiperiod model are endogenously determined, the theory in this article is derived under assumptions that may be viewed as restricting more primitive characteristics of the economy than the assumptions made for the one-period model.

The result is presented in Section 1, with the proof being given in the Appendix. Technical remarks regarding the interpretation of the assumptions and the derivation of the results and their implications are contained in Section 2.

1. The Intertemporal APT

We consider an economy with continuous trading over a time interval $[0, T]$, in which the sources of uncertainty are a countable number of independent Wiener processes.[1] We assume there is a risk-free asset with instantaneous rate of return of 0 and a liquidating value of 1. There is a sequence of other assets, with prices $X_i(t)$. Each asset is a claim to its liquidating value[2] $X_i(T)$. We make a no-arbitrage assumption so that there exists a risk-neutral probability Q. This means that Q assigns zero probability to the same events as does the underlying measure P, and that $X_i(t) = E_t^*[X_i(T)]$ for each i, where E^* denotes the expectation under Q.

Since the sources of uncertainty are Wiener processes, each asset price is an Itô process.[3] We write its differential as

$$dX_i(t) = \alpha_i(t)\, dt + \sigma_i(t)\, dZ_i(t),$$

[1]This allows us to prove that various processes are Itô processes. For technical reasons, we augment the information generated by the Wiener processes by all subsets of zero probability events.

[2]The generalization to nonzero and stochastic risk-free rate is straightforward. Similarly, it is straightforward to allow interim dividends by letting X_i denote the price of the asset when dividends are reinvested.

[3]This is proved in Back (1989), following an argument of Duffie and Huang (1985) and using the martingale representation theorem of Hitsuda and Watanabe (1976).

where Z_i is a Wiener process.[4] A trading strategy in the assets is a sequence $q = (q_1, q_2, \ldots)$ of nonanticipative processes such that

$$\sum_{i=1}^{\infty} E^*\left[\int_0^T \theta_i(t)^2 \sigma_i(t)^2 \, dt\right] < \infty.$$

The associated value process is

$$V(t) = V_0 + \sum_{i=1}^{\infty} \int_0^t \theta_i(s) \, dX_i(s),$$

where V_0 is a constant. We consider K fixed trading strategies $f^1, \ldots, f^K$ with associated value processes $V_1, \ldots, V_K$. For simplicity of notation we assume that $V = (V_1, \ldots, V_K)$ is a Q- Wiener martingale. We wish to relate the asset risk premium $\alpha_i(t)$ to its risk as measured by the covariances of its price change with the price changes of the hedging portfolios:[5]

$$b_{i,k}(t) \, dt = \mathrm{cov}_t(dX_i(t), dV_k(t)). \tag{1}$$

A trading strategy in the factor portfolios is a trading strategy of the form $q = x^1 f^1 + \cdots + x^K f^K$, where the x^k are nonanticipative real-valued processes. We assume that for each asset i, there is a trading strategy in the factor portfolios with payoff F_i such that

$$X_i(T) = F_i + N_i, \tag{2}$$

where the martingales $h_i(t) = E_t[N_i]$ satisfy the condition that, for each k,

$$\mathrm{cov}_t(d\eta_i(t), dV_k(t)) = 0,$$

and the residuals N_i are "dynamically weakly correlated." The meaning of this is as follows. By Hitsuda-Watanabe (1976), the h_i are Itô processes. Let $G_M(t)$ denote the matrix of local covariances $\mathrm{cov}_t(dh_i(t), dh_j(t))$, for $i,j = 1,\ldots, M$, and let $\mu_M(t)$ denote the largest eigenvalue of $G_M(t)$. Set $\mu(t) = \sup_{M \geq 1} \{\mu_M(t)\}$. We assume that $\mu(t)$ is finite for almost all t and almost all states of the world, and moreover that $\int_0^T \mu^2(t) \, dt$ is essentially bounded,[6] that is,

[4]The infinite sum here represents a limit in the space of square-integrable martingales (under the probability measure Q).

[5]Formally, $\mathrm{cov}_t(dX_i, dV_k)$ is defined as $d\langle X_i, V_k\rangle_t$, where the brackets denote the predictable joint variation process [see Elliott (1982)) This process will always exist in this article and will be absolutely continuous since we deal only with Itô processes (because of the assumption that the sources of uncertainty are Wiener processes).

[6]For a random variable x, ess.sup $(x) = \inf\{c \in R: x \leq c$ almost surely$\}$.

$$\text{ess.sup}\left(\int_0^T \mu^2(t)\, dt\right) < \infty. \tag{3}$$

To get some intuition as to the meaning of dynamic weak correlation, notice that if for each time t we have that cov $_t(N_i, N_j) = 0$ for every $i \neq j$, then it can be proved that the local covariance matrix of the residuals, $G_M(t)$, is diagonal. For h_i to satisfy our definition of dynamic weak correlation, some integrability conditions on the diagonal elements of these covariance matrices should be imposed.

The relationship between the local covariance matrix, $G_M(t)$, and the conditional covariance matrix of $h_1(T), \ldots, h_M(T)$ at time t, $\hat{G}_M(t)$, is

$$\hat{G}_M(t) = E_t\left[\int_t^T G_M(s)\, ds\right].$$

This follows since

$$\text{cov}_t(\eta_i(T), \eta_j(T)) = E_t[\eta_i(T), \eta_j(T)] - \eta_i(t)\eta_j(t) = E_t\left[\int_t^T d\langle\eta_i, \eta_j\rangle\right].$$

This implies that if cov $_t(N_i, N_j) = 0$, for $i \neq j$, then $G_M(t)$ is diagonal. In addition, this also implies that for each t the largest eigenvalue of $G_M(t)$ is smaller than $\text{ess.sup}(\int_t^T \mu_M^2(s)\, ds)$. This shows that dynamic weak correlation of the residuals implies a conditional weak correlation of the residuals in the sense of Chamberlain and Rothschild (1983) at every time, though the converse may not be true in general.

A sufficient condition on the terminal asset payoffs, $X_i(T)$, under which the decomposition (2) with dynamically uncorrelated residuals is possible is derived in Example B1 of Appendix B. This condition is that, for each i, the payoff $X_i(T)$ is a functional of the sample paths of $V_1, \ldots, V_K$, and B_i, where $\{B_i : i \geq 1\}$ is a sequence of P- Wiener processes independent across i such that cov $_t(dB_i(t), dV_k(t)) = 0$ for each t almost surely, and the $X_i(T)$ satisfy some integrability conditions. The proof is based on the fact that such functionals can be represented as the sum of stochastic integrals with respect to the V_k's and B_i, where the F_i in decomposition (2) is the sum of the integrals with respect to the V_k's, and N_i is the integral with respect to B_i.

It is important to notice that we do not make the assumption that price changes have a factor structure; that is, we do not assume that

$$dX_i(t) = b_{i,0}(t)\, dt + \sum_{k=1}^{K} b_{i,k}(t)\, dV_k(t) + de_i(t),$$

where the $b_{i,k}$ are nonanticipative processes and e_i is a P- local martingale such that the various e_i are locally uncorrelated at each instant. The fact that asset payoffs have the decomposition (2) with locally uncorrelated residuals does not imply that local changes in asset prices have a strict factor structure. This is demonstrated in Example B2 of Appendix B.

The only other assumption needed concerns the state price density process. This process is the martingale $\psi(t) = E_t[dQ/dP]$, where dQ/dP denotes the Radon-Nikodym derivative of Q with respect to P. If an asset with a price process $dX(t) = d\psi(t) + v(t)\,dt$ is traded, then by Girsanov's theorem v will be given by [see Back (1990) and Ross (1989)]

$$\mathrm{cov}_t(d\psi(t),\ d\psi(t)/\psi(t)) = \nu(t)\ dt.$$

We assume that $E[\psi(T)^2] < \infty$), and that the expected cumulative absolute risk premium of the above artificial asset is finite, that is,

$$E\left[\int_0^T |\nu(t)|\ dt\right] < \infty. \tag{4}$$

Theorem 1. *There exist nonanticipative processes (the prices of the factor risks)* $\lambda_1, \ldots, \lambda_K$, *such that*

$$\sum_{i=1}^{\infty}\left(\int_0^T \left|\alpha_i(t) - \sum_{k=1}^{K}\lambda_k(t)\, b_{i,k}(t)\right|\ dt\right)^2 < \infty \qquad \text{a.s.} \tag{5}$$

A sketch of the proof and its relation to the one-period model and a discussion of the necessity of the assumptions are given in Section 2. The proof of Theorem 1 is given in Appendix A.

Theorem 1 implies that, for most assets and at most times, $\alpha_i(t)$, the local risk premium of asset i, is approximately a linear combination of its betas with respect to the hedging portfolios. This approximation result is in the same spirit as the one-period model.

2. Remarks

Technical remarks regarding the relationship between the one-period model and the intertemporal model, a sketch of the derivation of the result, and comments on the implications of the assumptions are contained in this section.

First, we examine the relationship between the results derived here and the one-period model. Using the method of the proof for the one-period model, we will show that, for each i,

$$X_i(t) = F_i(t) + \eta_i(t) + R_i(t),$$

where $F_i(t)$ is the price of the payoff F_i in the decomposition (2) at time t, $h_i(t)$ is the expectation at time t of the residual term N_i in (2), and

$$\sum_{i=1}^{\infty} R_i(t)^2 < \infty. \tag{6}$$

Therefore, $X_i(t)$ is given approximately by

$$X_i(t) \approx F_i(t) + \eta_i(t).$$

$R_i(t)$ is the approximation error at time t, and (6) implies that the approximation errors are small in the sense that the sum of their squares is finite.

The payoff $X_i(T)$ is priced by decomposing it into a sum of two payoffs, F_i and N_i. The first part of the decomposition, F_i, is spanned by the factors, and its price at time t is given by $F_i(t)$, the arbitrage price at time t of the payoff F_i. The second part in the decomposition, N_i, is approximately priced by its conditional expected value at time t, $h_i(t)$. It was proved before that the assumption of weak dynamic correlation implies that the $h_i(T)$ are conditionally weakly correlated at each time in the sense of Chamberlain and Rothschild (1983). Therefore, it follows from the one-period version of the APT studied by Reisman (1988) that the sum of the squared approximation mistakes is finite.

The process R_i has the representation

$$dR_i(t) = \rho_i(t)\, dt + d\epsilon_i(t),$$

where r_i is a nonanticipative process and e_i is a P- local martingale. The proof of Theorem 1 is based on demonstrating that the r_i are small in the appropriate sense.

If the idiosyncratic risks are locally uncorrelated with the state price density, that is, if

$$\mathrm{cov}_t(d\psi(t), d\eta_i(t)) = 0,$$

at each instant, then it follows from Girsanov's theorem that $E_t^*[\eta_i(T)] = h_i(t)$. This implies that R_i is zero, which implies that r_i is zero, and we have exact pricing.[7] It should be noted that in this article we make no assumptions on the state price density, except for an integrability condition.

To sketch the proof of Theorem 1, we consider (without any loss

[7] Sufficient conditions on the primitives of the economy that imply that ψ is V adapted (and therefore we have exact pricing) were studied independently by Chamberlain (1988).

of generality) the case in which $F_i = 0$ for each i. In this case,

$$dX_i(t) = \rho_i(t)\ dt + d\epsilon_i(t) + d\eta_i(t).$$

One can view the decomposition as follows: $d\eta_i(t)$ is attributable to changes in our expectations about the terminal payoff of the asset, and $d\epsilon_i(t)$ is attributable to changes in the risk premium of the terminal payoff due to changes in our expectation about the payoff.

It is shown in Example B2 of Appendix B that the fact that the processes η_i are strictly locally uncorrelated does not imply that the processes ϵ_i are also strictly locally uncorrelated.

We decompose each ρ_i as

$$\rho_i = -\rho_{i,1} + \rho_{i,2},$$

so that $d\eta_i - \rho_{i,1}\,dt$ and $d\epsilon_i + \rho_{i,2}\,dt$ are local Q- martingales. Thus, $\rho_{i,1}$ and $\rho_{i,2}$ are risk premia due to each part of the martingale part of the price change. Using the dynamic weak correlation of the η_i, it is proved that the $\rho_{i,1}$ are small in some appropriate sense. It is proved that $d\epsilon_i + \rho_{i,2}\,dt$ is given by the Q- conditional expectation of $\int_0^T \rho_{i,1}(s)\,ds$. Using the fact that the $\rho_{i,1}$ are small, it is demonstrated that the local variances of the ϵ_i are small. Using that fact, it is proved[8] that the $\rho_{i,2}$ are also small.

We prove that the $\rho_{i,1}$ are small in the sense that

$$\sum_{i=1}^{\infty} E^*\left[\left(\int_0^T |\rho_{i,1}|\ dt\right)^2\right] < \infty. \tag{7}$$

[8] The fact that $d\epsilon_i + \rho_{i,2}\,dt$ is given by the Q- conditional expectation of $\int_0^T \rho_{i,1}(s)\,ds$ implies that

$$\int_0^T \rho_{i,1}\ dt = \int_0^T d\epsilon_i + \rho_{i,2}\ dt.$$

Therefore,

$$E^*\left[\left(\int_0^T \rho_{i,1}\ dt\right)^2\right] = E^*[\langle\epsilon_i\rangle_T].$$

It is proved that the $\rho_{i,1}$ are small in the sense that

$$\sum_{i=1}^{\infty} E^*\left[\left(\int_0^T |\rho_{i,1}|\ dt\right)^2\right] < \infty.$$

Therefore, the ϵ_i are small in the sense that

$$\sum_{i=1}^{\infty} E^*[\langle\epsilon_i\rangle_T] < \infty.$$

The proof that the $\rho_{i,2}$ are small follows from the Kunita and Watanabe inequality. since from Girsanov's theorem we have that $\rho_{i,2}\,dt = \psi^{-1}(\epsilon_i, y)$.

We clarify the ideas behind the proof of (7) by deriving a simpler bound of the form

$$\sum_{i=1}^{\infty} \rho_{i,1}^2 < \infty, \tag{8}$$

by using the method of the proof for the one-period model in Reisman (1988) or in Reisman (1986). The derivation of (7) is a technical modification of the derivation of (8).

The reason we derived the stronger bound in (7) is not because we find it more attractive than the weaker bound in (8), but because we need the stronger bound later in the proof in order to demonstrate that the $r_{i,2}$ are small (see note 8).

The proof of (8) goes as follows. From Girsanov's theorem we get that $E_t^*[d\eta_t] = \rho_{i,1}\, dt$. Therefore, for any $\alpha_1, \ldots, \alpha_N$, such that $\sum_{i=1}^N a_i^2 = 1$, we have that

$$\sum_{i=1}^N a_i \rho_{i,1}\, dt = E_t^*\left[\sum_{i=1}^N a_i\, d\eta_i\right] \le C\left(E_t\left[\left(\sum_{i=1}^N a_i\, d\eta_i\right)^2\right]\right)^{1/2},$$

where $C = y(t)^{-1} (E_t[y(T)^2])^{1/2}$ and the last inequality follows from Holder's inequality since for each random variable x, $E_t^*[x] = y(t)^{-1} E_t[y(T)x]$. This implies that

$$\sum_{i=1}^N a_i \rho_{i,1}\, dt \le C\left(\sum_{i=1}^N \sum_{j=1}^N a_i a_j \operatorname{cov}_t(d\eta_i, d\eta_j)\right)^{1/2} \le C\mu,$$

where the last inequality follows from the assumption about dynamic weak correlation of the h_i. By taking $a_i = \rho_{i,1}/(\sum_{i=1}^N \rho_{i,1}^2)^{1/2}$, we obtain that $\sum_{i=1}^\infty \rho_{i,1}^2 < \infty$. This concludes the sketch of the proof of Theorem 1.

It should be noted that while the derivation of (7) requires the use of the global regularity assumptions on μ in (3) and on v in (4), the derivation of (8) requires much weaker assumptions. This is the place in the proof where the global regularity assumptions on μ and v are employed.

Appendix A

Notation

By Girsanov's theorem, there exists a P- Wiener martingale $\mathbf{W} = (W_1, \ldots, W_K)$ and a nonanticipative process $\lambda = (\lambda_1, \ldots, l_K)$, such that

$$d\mathbf{V}(t) = d\mathbf{W}(t) + \lambda(t)\, dt, \tag{A1}$$

where

$$\lambda_k(t)\, dt = -\psi(t)^{-1}\, d\langle \psi, W_k\rangle_t. \tag{A2}$$

Since y is a positive Itô process, it has the representation $dy = fy\, dZ$, where f is a nonanticipative process and Z is a P- Wiener martingale. We will denote $d\Phi = f\, dZ$. We have that

$$\phi^2(t)\, dt = \psi(t)^{-2}\, d\langle\psi\rangle_t = d\langle\Phi\rangle_t = \psi(t)^{-1}\nu(t)\, dt. \tag{A3}$$

(A3) and (4) imply that

$$E^*\left[\int_0^T \phi^2(s)\, ds\right] = E\left[\int_0^T \psi(s)\phi^2(s)\, ds\right]$$
$$= E\left[\int_0^T |\nu(s)|\, ds\right] < \infty. \tag{A4}$$

For a nonanticipative $q = (q_1, \ldots, q_N)$ and an Itô process $X = (X_1, \ldots, X_N)$, we will denote

$$\theta\cdot\mathbf{X}(t) = \sum_{n=1}^{N} \int_0^t \theta_n\, dX_n. \tag{A5}$$

All equalities and inequalities between random variables hold a.s. without being explicit.

Proof of Theorem 1

Equation (3) implies that for each i, $\langle h_i\rangle_T$ is essentially bounded. This implies that $h_i(T) \in L^2(Q)$. Define

$$E_t^*[\eta_i(T)] = \eta_i'(t), \tag{A6}$$

where η_i' is a Q- martingale. Since $X_i(T) = F_i + h_i(T)$ and since X_i is a Q- martingale, we obtain

$$X_i(t) = F_i(t) + \eta_i'(t), \tag{A7}$$

where $E_t^*[F_i] = F_i(t)$ is the value of the payoff F_i at time t. η_i' is also an Itô process and has the representation

$$\eta_i'(t) = \eta_i'(0) + \epsilon_i'(t) + \int_0^t \rho_i(s)\, ds, \tag{A8}$$

where ϵ_i' is a P- local martingale and an Itô process such that $\epsilon_i'(0) = 0$ and r_i is a nonanticipative process.

Since F_i is spanned by $V_1, \ldots, V_K$, it is sufficient to prove the theorem in the case where $F_i = 0$. Therefore, in order to prove the theorem, it is sufficient to show that

$$\sum_{i=1}^{\infty}\left(\int_0^T |\rho_i(s) - \boldsymbol{\lambda}(s)\mathbf{b}_i'(s)|\, ds\right)^2 < \infty, \tag{A9}$$

where $\mathbf{b}'_i = (b'_{i,1}, \ldots, b'_{i,K})$, and, for each k,

$$d\langle W_k, \epsilon'_i \rangle_t = b'_{i,k}(t)\ dt. \tag{A10}$$

To prove (A9), it is enough to show that

$$\sum_{i=1}^{\infty} \left(\int_0^T |\boldsymbol{\lambda}(s)\mathbf{b}'_i(s)|\ ds \right)^2 < \infty, \tag{A11}$$

and that

$$\sum_{i=1}^{\infty} \left(\int_0^T |\rho_i(s)|\ ds \right)^2 < \infty. \tag{A12}$$

Proof of (A11). From the Cauchy-Schwarz inequality, we obtain

$$\left(\int_0^T |(\mathbf{b}'_i(s))\boldsymbol{\lambda}(s)|\ ds \right)^2 \leq \left(\int_0^T |\mathbf{b}'_i(s)|^2\ ds \right) \left(\int_0^T |\boldsymbol{\lambda}(s)|^2\ ds \right). \tag{A13}$$

From the definition of $\mathbf{b}'_i$, we have that $\int_0^T |\mathbf{b}'_i(s)|^2\ ds \leq \langle \epsilon'_i \rangle_T$.

Denote

$$\epsilon_i(t) = \epsilon'_i(t) - \eta_i(t). \tag{A14}$$

Since for each k, $\langle W_k, \eta_i \rangle_t = 0$, then we have also that

$$\int_0^T |\mathbf{b}'_i(s)|^2\ ds \leq \langle \epsilon_i \rangle_T. \tag{A15}$$

This and (A13) imply that

$$\sum_{i=1}^{N} \left(\int_0^T |\mathbf{b}'_i(s)\boldsymbol{\lambda}(s)|\ ds \right)^2 \leq \left(\int_0^T |\boldsymbol{\lambda}(s)|^2\ ds \right) \left(\sum_{i=1}^{N} \langle \epsilon_i \rangle_T \right). \tag{A16}$$

Equation (A1) implies that for each k, $\lambda_k(t)\ dt = -\ d\langle F, W_k \rangle_t$, which implies that

$$\int_0^T \lambda_k^2(s)\ ds \leq \int_0^T \phi^2(s)\ ds < \infty, \tag{A17}$$

where $\int_0^T \phi^2(s)\ ds < \infty$ follows from (A4). The proof of (A11) follows by substituting (A17) in (A16) and observing that Lemma 1 below implies that $\sum_{i=1}^{\infty} \langle \epsilon_i \rangle_T < \infty$. ∎

The proof of (A12) also follows from Lemma 1 below.

Lemma 1. *Suppose that the assumptions of Theorem 1 hold. Then*

$$\sum_{i=1}^{\infty}\left\{\langle\epsilon_i\rangle_T+\left(\int_0^T|\rho_i(s)|\;ds\right)^2\right\}<\infty. \tag{A18}$$

Proof For each i, define

$$\rho_{1i}(t)\;dt=d\langle\Phi,\eta_i\rangle_t. \tag{A19}$$

Proposition A2 implies that $\int_0^T\rho_{1i}(s)\;ds$ is Q- square integrable. Define

$$Y_i(t)=E_t^*\left[\int_0^T\rho_{1i}(s)\;ds\right],\qquad\forall t\in[0,T], \tag{A20}$$

and

$$\rho_{2i}(t)\;dt=d\langle\Phi,Y_i\rangle_t. \tag{A21}$$

The proof is based on the representation of e_i and r_i given in the proposition below.

Proposition A1. *As defined in (A14), e_i has the representation*

$$\epsilon_i(t)=Y_i(t)-\int_0^t\rho_{2i}(s)\;ds-Y_i(0),\qquad\forall t\in[0,T], \tag{A22}$$

and r_i as defined in (A8) has the representation

$$\rho_i=-\rho_{1i}+\rho_{2i}. \tag{A23}$$

Proof. Since h_i is an Itô process and a P- martingale, it has the representation

$$\eta_i(T)=\int_0^T d_i(s)\;dB_i, \tag{A24}$$

where d_i is a nonanticipative process. From Girsanov's theorem (Elliot, theorem 13.14), we have that

$$B_i^*(t)=B_i(t)-\langle\Phi,B_i\rangle_t,\qquad\forall t\in[0,T], \tag{A25}$$

is a Q- Weiner martingale. Since

$$\int_0^T d_i(s)\;d\langle\Phi,B_i\rangle_s=\langle\Phi,\eta_i\rangle_T=\int_0^T\rho_{1i}(s)\;ds, \tag{A26}$$

then

$$\eta_i(t)=\int_0^t d_i(s)\;dB_i^*+\int_0^t\rho_{1i}(s)\;ds. \tag{A27}$$

$\{\int_0^t d_i(s)\, dB_i^*\}$ is a Q- martingale because it follows from (3) that $\int_0^T d_i^2(s)\, ds$ is essentially bounded. Therefore,

$$\begin{aligned}\eta_i(t) &= E_t^*[\eta_i(T)]\\ &= E_t^*\left[\int_0^T d_i(s)\, dB_i^*\right] + E_t^*\left[\int_0^T \rho_{1i}(s)\, ds\right]\\ &= \int_0^t d_i(s)\, dB_i^* + Y_i(t)\\ &= \eta_i(t) - \int_0^t \rho_{1i}(s)\, ds + Y_i(t).\end{aligned} \tag{A28}$$

Therefore,

$$\begin{aligned}\eta_i'(t) - \eta_i(t) &= Y_i(t) - \int_0^t \rho_{1i}(s)\, ds\\ &= \left(Y_i(t) - \int_0^t \rho_{2i}(s)\, ds\right)\\ &\quad + \left(-\int_0^t \rho_{1i}(s)\, ds + \int_0^t \rho_{2i}(s)\, ds\right),\end{aligned} \tag{A29}$$

where, from Girsanov's theorem, $\{Y_i(t) - \int_0^t \rho_{2i}(s)\, ds\}$ is a P- local martingale. By the definition of e_i in (A14) and (A8),

$$\eta_i'(t) - \eta_i(t) = \eta_i'(0) + \epsilon_i(t) + \int_0^t \rho_i(s)\, ds. \tag{A30}$$

Equation (A29) together with (A30) implies that

$$\begin{aligned}&\eta_i'(0) + \epsilon_i(t) + \int_0^t \rho_i(s)\, ds\\ &\quad = Y_i(0) + \left(Y_i(t) - \int_0^t \rho_{2i}(s)\, ds - Y_i(0)\right)\\ &\quad + \left(-\int_0^t (\rho_{1i}(s) + \rho_{2i}(s))\, ds\right),\end{aligned} \tag{A31}$$

and the proof follows. ■

Bounds on **ρ_{1i}, ρ_{2i}, and ϵ_i**

The next step in the proof is to find bounds for the processes r_{1i}, r_{2i}, and e_i. This is done in Propositions A2-A5.

First, we find a bound on r_{1i}. This bound is the key step in the proof.

Proposition A2.

$$\sum_{i=1}^{\infty} E^*\left[\left(\int_0^T |\rho_{1i}(s)|\ ds\right)^2\right] < \infty. \tag{A32}$$

Proof. Let $N \geq 1$ and let $\mathbf{a} = (a_1, \ldots, a_N)$ be given by

$$a_i(t) = \operatorname{sign}(\rho_{1i}(t))\left(\int_0^t |\rho_{1i}(s)|\ ds\right), \qquad \forall t \in [0, T]. \tag{A33}$$

This implies that

$$\begin{aligned}
&\frac{1}{4}\left(\sum_{i=1}^{N}\left(\int_0^T |\rho_{1i}(s)|\ ds\right)^2\right)^2 \\
&\quad = \left(\int_0^T \sum_{i=1}^{N} a_i(s)\rho_{1i}(s)\ ds\right)^2 = \left(\int_0^T \sum_{i=1}^{N} a_i(s)\ d\langle\Phi, \eta_i\rangle_s\right)^2 \\
&\quad = \left(\int_0^T d\,\langle\Phi, \sum_{i=1}^{N} a_i\cdot\eta_i\rangle_s\right)^2 \\
&\quad \leq \left(\int_0^T d\langle\Phi\rangle_s\right)\left(\int_0^T d\,\langle\sum_{i=1}^{N} a_i\cdot\eta_i\rangle_s\right) \\
&\quad = \left(\int_0^T \phi^2(s)\ ds\right)\left(\int_0^T \mathbf{a}'(s)\,G_N(s)\mathbf{a}(s)\ ds\right) \\
&\quad \leq \left(\int_0^T \phi^2(s)\ ds\right)\left(\int_0^T \left(\sum_{i=1}^{N} a_i^2(s)\right)\mu_N^2(s)\ ds\right) \\
&\quad \leq \left(\int_0^T \phi^2(s)\ ds\right)\left(\int_0^T \mu^2(s)\ ds\right)\left(\sum_{i=1}^{N}\left(\int_0^T |\rho_{1i}(s)|\ ds\right)^2\right).
\end{aligned} \tag{A34}$$

The first equality follows from (A33), since, by integration by parts,

$$\begin{aligned}
&\left(\int_0^T |\rho_{1i}(s)|\ ds\right)^2 \\
&\qquad = 2\int_0^T \left(\int_0^t |\rho_{1i}(s)|\ ds\right) d\left(\int_0^t |\rho_{1i}(\xi)|\ d\xi\right)
\end{aligned}$$

$$= 2 \int_0^T \left(\int_0^t |\rho_{1i}(s)| \; ds \right) |\rho_{1i}(t)| \; dt, \tag{A35}$$

the second equality follows from the definition of r_{1i}, the third equality follows from Elliott (theorem 11.7c)) the first inequality follows from the Kunita and Watanabe inequality (Elliott, theorem 10.11), the next equality follows from the definition of $G_N(t)$, and, since $f^2(t)\ dt$ = dáFñ $_t$, the next inequality follows from the definition of $\mu_N(t)$. The last inequality follows from the fact that, for each $t \in [0, T]$,

$$a_i^2(t) \leq \left(\int_0^T |\rho_{1i}(s)| \; ds \right)^2. \tag{A36}$$

From (A34) it follows that

$$\sum_{i=1}^N \left(\int_0^T |\rho_{1i}(s)| \; ds \right)^2 \leq 4 \left(\int_0^T \phi^2(s) \; ds \right) \left(\int_0^T \mu^2(s) \; ds \right). \tag{A37}$$

The proof of the proposition follows by taking E^* of the above and using the fact that (4) implies that $\int_0^T \phi^2(s)\ ds$ is Q- integrable [see (A4)] and the fact that (3) implies that $\int_0^T \mu^2(s)\ ds$ is essentially bounded. ■

Proposition A3.

$$\sum_{i=1}^\infty E^*[Y_i^2(T)] < \infty \tag{A38}$$

and

$$\sum_{i=1}^\infty \int_0^T d\langle Y_i \rangle_s < \infty. \tag{A39}$$

Proof. By definition $Y_i(T) = \int_0^T \rho_{1i}(s)\ ds$. This implies that

$$E^*[Y_i^2(T)] \leq E^* \left[\left(\int_0^T \rho_{1i}(s) \; ds \right)^2 \right]. \tag{A40}$$

The proof of (A38) follows from Proposition A2. The proof of (A39) follows from (A38) since

$$E^* \left[\int_0^T d\langle Y_i \rangle_s \right] + Y_i^2(0) = E^*[Y_i^2(T)]. \tag{A41}$$

Next, we find a bound for the process r_{2i}.

Proposition A4.

$$\sum_{i=1}^{\infty}\left(\int_0^T |\rho_{2i}(s)|\ ds\right)^2 < \infty. \tag{A42}$$

Proof. From (A21),

$$\int_0^T |\rho_{2i}(s)|\ ds = \int_0^T |d\langle\Phi, Y_i\rangle_s|. \tag{A43}$$

This implies that

$$\sum_{i=1}^{N}\left(\int_0^T |\rho_{2i}(s)|\ ds\right)^2 \le \left(\int_0^T d\langle\Phi\rangle_s\right)\left(\sum_{i=1}^{N}\int_0^T d\langle Y_i\rangle_s\right)$$

$$\le \left(\int_0^T \phi^2(s)\ ds\right)\left(\sum_{i=1}^{N}\int_0^T d\langle Y_i\rangle_s\right). \tag{A44}$$

The first inequality follows from the Kunita and Watanabe inequality (Elliott, theorem 10.11), and the last inequality follows since $f(t)^2\ dt = d\langle\Phi\rangle_t$. The proposition follows from Proposition A3, since

$$\sum_{i=1}^{\infty}\int_0^T d\langle Y_i\rangle_s < \infty. \qquad \blacksquare$$

Next, we find a bound for the process ε_i.

Proposition A5.

$$\sum_{i=1}^{\infty}\langle\epsilon_i\rangle_T < \infty. \tag{A45}$$

Proof. From (A22) and the fact that $\varepsilon_i(0) = 0$, we have that (Elliott, theorem 13.24)

$$\int_0^T d\langle Y_i\rangle_s = \langle\epsilon_i\rangle_T. \tag{A46}$$

The proof follows from Proposition A3. $\blacksquare$

Appendix B

Example B1

This example states conditions on payoffs under which the local correlation matrix of the residuals is diagonal. Let λ be a nonanticipative process such that $d\mathbf{V} - \lambda\, dt$ is a P- local martingale. Define

$$\psi'(t) = \exp\left\{\int_0^t \lambda(s)\, d\mathbf{V}(s) - \frac{1}{2}\int_0^t |\lambda(s)|^2\, ds\right\} \tag{B1}$$

and assume that $E[\psi'(T)] = 1$. Let E' be the expectation operator with respect to the measure $dQ' = \psi'(T)\, dP$. Let $\{B_i;\, i \geq 1\}$ be a sequence of independent P- Wiener processes, and assume that, for each t, $\mathrm{cov}_t(dV_k, dB_i) = 0$ a.s., for each i and k. Assume that for each i, $X_i(T)$ is a functional of the sample paths of $V_1, \ldots, V_K$ and B_i and is Q'-square integrable. Then $X_i(T)$ has the representation (2), where the residuals have a diagonal local correlation matrix. The proof of the above is as follows.

By Girsanov's theorem, applied to the process $(\lambda, 0)$, the vector process $(\mathbf{V}, B_i)$ is a Q'- Wiener martingale. Since $E'_t[X_i(T)]$ is a Q'-martingale, then by the Kunita and Watanabe theorem (Elliott: theorem 12.33), we have that

$$X_i(T) = E'[X_i(T)] + \int_0^T \mathbf{c}_i(s)\, d\mathbf{V}(s) + \int_0^T d_i(s)\, dB_i(s), \tag{B2}$$

where c_i and d_i are nonanticipative processes. The decomposition (2) is obtained by defining

$$F_i = E'[X_i(T)] + \int_0^T \mathbf{c}_i(s)\, d\mathbf{V}(s) \tag{B3}$$

and

$$N_i = \int_0^T d_i(s)\, dB_i(s). \tag{B4}$$

To obtain dynamic weak correlation of the residuals, boundedness conditions on the d_i should be imposed.

Notice that the decomposition (2) was constructed using terminal liquidation payoff and factor prices. No additional information about the structure of the economy or the equilibrium price was needed.

Example B2

The example indicates that a strict dynamic factor structure, such as in the previous example, may not imply that local price changes have a strict factor structure. Assume that, for each i,

$$X_i(T) = B_i(T), \tag{B5}$$

where the B_i are mutually independent P- Wiener processes. In this case, $F_i = 0$ and the residual payoff is $B_i(T)$ (i.e., there exists a dynamic strict factor structure with no factors). We will demonstrate that the local price changes do not have a strict factor structure with no factors. Let λ_i be a nonanticipative process, such that $dB_i^*(t) = dB_i(t) - \lambda_i(t)\, dt$ is a Q- Wiener process. We get that

$$X_i(t) = B_i^*(t) + E_t^*\left[\int_0^T \lambda_i(s)\, ds\right]. \tag{B6}$$

It is not difficult to construct an example in which Q is such that all λ_i are nonanticipative processes adapted to the filtration generated by some fixed Wiener process Z. In this case, each X_i is locally correlated with Z, and therefore the price changes are not strictly locally uncorrelated.

References

Back, K., 1989, "Factorization of Returns in Multiperiod Securities Markets," working paper.

Back, K., 1990. "Asset Pricing for General Processes," forthcoming in *Journal of Mathematical Economics.*

Chamberlain, G., 1988, "Asset Pricing in Multiperiod Securities Markets," *Econometrica,* 56, 1283-1300.

Chamberlain, G., and M. Rothschild, 1983, "Arbitrage Factor Structure and Mean Variance Analysis in Large Asset Markets," *Econometica,* 51, 1281-1304.

Duffie, D., and C. Huang, 1985, "Implementation of Arrow-Debreu Equilibria by Continuous Trading of Few Securities," *Econometrica,* 53, 1337-1365.

Elliott, R., 1982, *Stochastic Calculus and Applications,* Springer, New York

Hitsuda, M., and H. Watanabe, 1976. "On Stochastic Integrals with Respect to an Infinite Number of Brownian Motions and Its Applications," in K. Itô (ed.), *Stochastic Differential Equations: Proceedings of the International Symposium on Stochastic Differential Equations,* Wiley, New York. pp. 57-74.

Huberman, G., 1983, "A Simple Approach to Arbitrage Pricing Theory," *Journal of Economic Theory,* 28, 183-191.

Ingersoll, J., 1984, "Some Results on the Theory of Arbitrage Asset Pricing," *Journal of Finance,* 39, 1021-1039.

Reisman, H., 1986, "Valuation of Claims with Noisy Payoffs," working paper

Reisman, H., 1988, "A General Approach to the APT," *Econometrica,* 56, 473-476.

Ross, S., 1976, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory,* 13, 341-360.

Ross, S., 1989. "Information and Volatility The No Arbitrage Martingale Approach to Timing and Resolution of Irrelevancy," *Journal of Finance,* 44, 1-17.