Design and Marketing of Financial Products

Dilip Madan
Badih Soubra
University of Maryland

Marketing costs are introduced into the security design environment outlined in Allen and Gale (1988). It is shown that splitting the firm's cash flow between products enhances their investor appeal and reduces marketing costs. We also explain bow the extremal product design in Allen and Gale is thereby avoided and how in simple cases, debt, equity, or warrants can be optimal. Furthermore, we illustrate in general terms bow the optimal solution employs portfolios of option-type products, and we give an example of two optimal products that share profits in seven of eigbt states.

The most common forms of financial contracts are debt and equity. However, Allen and Gale (1988) in their path-breaking article show that these products are not necessarily optimal when market incompleteness follows from short-sale restrictions. In fact, their optimal products have an extreme payout pattern that never splits the firm's cash flow between securities in any state.

The basis for market incompleteness here extends

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to include marketing considerations suggested by Ross (1989a, 1989b). Marketing costs are incurred in the process of finding and approaching prospective investors and are also reflected in unsold inventory. Costs of marketing are reduced primarily by designing products with a wider investor appeal. Their inclusion in the model successfully avoids the extremal design structure noted above. Unlike the-issuing costs of Allen and Gale that depended only on the number of securities issued, these marketing costs also depend on both the security design structure and the issue price. In addition, we suppose that the financial products are created and marketed by an intermediary as in Merton (1989).

The primary contribution of this article is to bridge the gap between the abstract ideas in Allen and Gale (1988) and the need to address more specific questions about financial product design without altering the basic motivation for the design of securities by an intermediary. This motivation Is the variation in individual state prices; and a simple and parsimonious description of this diversity, helpful to researchers studying optimal designs, is introduced. Each agent in the economy is modeled as having a distribution of state prices completely characterized by individual dependent means and variances. The mean and variance of these individual means and variances are the four parameters that characterize the distribution of state prices.

Our economic model consists of a large number of consumers who invest in financial products that are created by an intermediary to meet perceived investor demands. These products are backed by the intermediary's holdings of the securities issued by firms. We consider the partial equilibrium problem of the Intermediary who wishes to design and sell financial products with a view to maximizing revenues net of marketing costs.

It is shown that optimal products typically display profit sharing in the higher profit states. In simple cases; this may involve the use of debt, equity, or warrant-type securities. More generally, in constructing optimal products, the cash flow can first be split into specialized products of the type defined by Allen and Gale (1988), which amount to options on the firm's value conditioned on sets of states. The optimal product is then a portfolio of these specialized products. An example with eight states and a continuum of investors is presented, displaying two optimal products that are a proper portfolio of specialized products.

Apart from exploiting differences in individual state prices for product design, other important design considerations include managerial remuneration to mitigate agency problems [for an extensive survey see Hart and Holmstrom (1987)], signaling quality to avoid adverse selection [Nachman and Noe (1990)], the allocation of control rights to increase efficiency [Aghion and Bolton (1988)], and corporate control considerationsto alleviate managerial entrenchment [Grossman and Hart (1988); Harris and Raviv (1988, 1989)]. For a survey of these issues the reader is referred to Allen (1990). Our results abstract from such considerations.

The remainder of the article is organized as follows. The economic setting is described in Section 1. Examples of optimal products in the context of two marketing strategies are presented in Section 2. In Section 3, general results on the properties of optimal products are provided. Methods for product design are presented in Section 4, and Section 5 is the conclusion.

1. The Economic Setting

The economy has a large number of consumers, who also invest in financial products. There are also many firms with claims to their cash flows regularly traded in active secondary security markets. In addition to these investors and firms, there are also a large number of financial intermediaries contributing to market efficiency through the creation of financial products catered to specialized investor demands. The liabilities of the intermediaries are covered in the secondary markets for the firm's securities, with transactions and marketing costs prohibiting investors from directly creating these products.

In this article, a single risk-neutral intermediary's problem of creating and marketing financial products is modeled. A typical financial product is defined by its claims on state-contingent cash flows $y(s)$ in state $s \in S$, with the products being backed by the state-contingent. cash flows $x(s)$. The cash flows $x(s)$ could be those of a single firm, or more generally the liquidation value of a portfolio. The financial products are sold at time 0, and the cash flows are realized at time 1. For most applications in this article, there will be a finite number of states and M products. The intermediary's problem consists of defining limited liability or nonnegative payoffs, $y_i(s)$, for product i in state s, which add up to the cash flow $x(s)$. For extremal products, the payoffs $y_i(s)$ are either $x(s)$ or 0, while interior products involve sharing of the cash flow in at least one state.

Investors are indexed by $\alpha \in A$, where the set A is large enough to permit continuity of the distribution of individual investor valuations. The utility function, $U(c_0, c_1(s), \alpha)$, for investor α, is monotonically increasing, strictly concave, and is defined over current and state-contingent consumption $(c_0, c_1(s))$. Define the marginal rate of substitution, $\omega(s, \alpha)$, between state-contingent consumption $c_1(s)$ and current consumption c_0, by

$$\omega(s, \alpha) = \frac{\partial U(c_0, c_1(s), \alpha)/\partial c_1(s)}{\partial U(c_0, c_1(s), \alpha)/\partial c_0}. \tag{1}$$

Individual α 's valuation, $q(\alpha)$, of the state-contingent payoff $y(s)$ is then

$$q(\alpha) = \sum_{s \in S} \omega(s, \alpha) y(s). \tag{2}$$

Typically, in incomplete markets equilibria, these valuations differ across investors. If unlimited purchases can be made by all investors in the economy (with costless information and no transaction costs), but short sales are not permitted, then the Allen and Gale result obtains: only the investors that value the financial product the most hold it, and the market price equals this maximum personal valuation.[1] We suppose that the marketing costs, involved in identifying and contacting investors with the highest personal valuation for a product, are exorbitant. Furthermore, personalized search or transactions costs could motivate the possible eventual existence of personalized values above market prices if a general equilibrium interpretation was sought. The intermediary is viewed as marketing financial products in the light of information obtained on the distribution of the personalized values for its financial products.

The analysis of this article is partial equilibrium. We view the intermediary as entering an economy that is otherwise in equilibrium, and address the design and marketing question. Secondary markets in these products are precluded as they are specially designed by the intermediary and investors attempting a resale would be in direct competition with the intermediary. The intermediary designs the product, prices it at q, and then markets the product earning proceeds P that are net of marketing costs MC, so that

$$P = q - \mathrm{MC}. \tag{3}$$

Marketing costs have two components, unsold inventory and transactions costs. In general, we would expect marketing costs to be higher the smaller the proportion of the population that is willing to purchase the product, and our formulation reflects this.

Consider first the modeling of unsold inventory. We interpret $q(\alpha)$, defined by Equation (2), as the reservation price of an investor of type α for the product with payoffs $y(s)$. If the product is offered to the investor at a price less than $q(\alpha)$, the investor demands a nonnegative quantity d_α; otherwise, he demands zero. When the intermediary approaches an individual, it does not know either $q(\alpha)$ or

[1] Alternatively, if securities are restricted to debt and equity, then the issuance of debt an be advantageous [Senbet and Taggart (1984)].

d_a. We assume that the quantities $q(a)$ and d_a are uncorrelated in the population. This would occur, for example, if the slopes of individual demand curves are uncorrelated with the level of personalized transaction costs. It follows from these assumptions that the value of unsold inventory is

$$\text{Unsold inventory} = qF(q), \tag{4}$$

where we suppose without loss of generality that the aggregate product demand is unity and $F(q)$ is the proportion of the population with $q(a)$ below q.

In the first example, the investors are. approached once (single-offer case) and marketing costs include unsold inventory plus a transactions cost of θ per dollar of revenue. Since the revenue is $q(1 - F(q))$, we have that

$$\text{MC} = qF(q) + \theta q[1 - F(q)]. \tag{5}$$

Note that MC is increasing in $P(q)$ as θ is less than unity, and therefore marketing costs rise with the proportion of the population disinterested in the product.

In the second example, investors are repeatedly approached (repeated-offer case) until the total inventory is sold. While there is in this scenario no unsold inventory, the transactions costs associated with these repeated approaches are much higher. Following Ross (1989a), it is shown below that marketing costs are given by

$$\text{MC} = \theta q/[1 - F(q)]. \tag{6}$$

Note that marketing costs are again directly related to the proportion $F(q)$.

There is a monopoly element in the single-offer case, whereby one models the demand curve through unsold. inventory. The repeated-offer case is closer to the spirit of incorporating search costs explicitly, and we view the single-offer case as a more tractable simplification of this more general model.

Both the examples studied here reflect price-dependent marketing costs endogenously modeled as functions of the proportion of prospective buyers. Since costs of searching for investors. are part of marketing costs, low-priced products would, be expected to have low marketing costs reflected in both (5) and (6) above. The dependence on price is important as the intermediary has to make a simultaneous decision on product design and price, which jointly determine expected marketing costs. This contrasts with the assumptions in Allen and Gale, where issuing costs, though in general dependent on both the financial structure and product design, are independent of product prices.

2. Examples of Optimal Products

2.1 The single-offer case

In thesingle-offer case, the intermediary offers its products to investors at a preset price q_i for the *ith* product. The offers are made to a select group of investors chosen from a particular distribution network. The particular individuals are picked randomly from, this collection; The problem faced by the intermediary is to both design products and set their prices to maximize its proceeds.

For a given product structure, if the prices are. set too low, then there is a high probability of the offers being taken up, but at a low level of proceeds. Conversely, if the prices are set too high, the proceeds may be low as a result of insufficient sales. As noted earlier, we view this as a special case of a marketing cost. These trade-offs imply the existence of an optimal price for each product structure.

The pricing problem' for a given product with an arbitrary state-contingent payoff involves the choice of q to maximize proceeds obtained on substituting (5) into (3):

$$P = (1 - \theta)q[1 - F(q)]. \tag{7}$$

For notational convenience, let $G(q)$ be 1 - $F(q)$. For a set of M products, therefore, the pricing problem amounts to maximizing, over choices q_i, the objective:

$$P = \sum_{i=1}^{M} (1 - \theta) q_i G_i(q_i). \tag{8}$$

The possibility that optimal products may not be of the extremal Allen and Gale (AG) type is illustrated by the examples below. In fact, in these examples, the payoff pattern of these products resembles debt and equity. Of course, the precise form of the optimal products depends upon the upon parameter values. For the examples of this section, θ is taken to be zero.

Example 1. Suppose that there are two equiprobable states at time period 1, s_0 and s_1, with $x(s)$ given by the vector (1, 2). Let there be two types of investors, a_0 and a_1, in the proportions 1 - λ and λ. The marginal rates of substitution for each state, in terms of time-zero consumption, for the two investors are given by

$$W = \begin{array}{c c} & \begin{array}{cc} s_0 & s_1 \end{array} \\ \begin{array}{c} \alpha_0 \\ \alpha_1 \end{array} & \begin{bmatrix} 1 & 0 \\ 0.25 & 0.75 \end{bmatrix} \end{array}. \tag{9}$$

If only one-product is issued, the personalized valuations are 0.5

and $\frac{7}{8}$ by α_0 and α_1, respectively. The intermediary has the choice of pricing the product at 0.5 or $\frac{7}{8}$. The proceeds are 0.5 or $\frac{7}{8}\lambda$; and, clearly, the single-product pricing strategy depends on the value of λ. For λ less than $\frac{4}{7}$, the single product proceeds are 0.5.

For the AG extremal products, the payoffs are (1, 0) for the first product and (0, 2) for the second in the states (s_0, s_1), respectively. For λ less than ¾, the best prices for the AG products are 0.5 and 0.75, respectively. The proceeds of 0.5 + 0.25λ then dominate the single-product case.

Consider now the products, with payoffs of (1, 1) for the first product (debt) and (0, 1) for the second product (equity). The optimal prices are $\frac{1}{2}$ and $\frac{3}{8}$, with proceeds of $\frac{1}{2} + \frac{3}{8}\lambda$. Notice that these products, "debt and equity," dominate the AG products. The improved value resulting from debt and equity is due to the fact that these products appeal to a larger proportion of the population and thereby reduce marketing costs.

Example 1 above demonstrated the possible optimality of nonextremal or interior products for a discrete set of investors. The resulting distributions F_i were therefore step functions leading to a local insensitivity of F_i with respect to price changes q_i. For the case of a continuum of investors; the optimality of interior products is demonstrated in Example 2 below.

Example 2. There are two equiprobable states, s_0 and s_1, with $x(s)$ again given by the vector (1,2); Let A, the set of investors, be the interval [0, 1], with the proportion of investors of type below a being given by the cumulative distribution function α^2, for $\alpha \in [0, 1]$. The marginal rates of substitution are now given by real-valued functions

$$\omega_0(\alpha) = 1 - \alpha/2 \quad \text{and} \quad \omega_1(\alpha) = \alpha.$$

The objective is to choose y_0, and y_1, the payoffs to the first product in the states s_0 and s_1, respectively, to maximize proceeds. The payoffs to the second product are, therefore, $1 - y_0$ and $2 - y_1$ in states s_0 and s_1, respectively. The individual valuations of the two products for Individual α are

$$q_1(\alpha) = \tfrac{1}{2}[(1 - \alpha/2)\, y_0 + \alpha y_1]$$

and

$$q_2(\alpha) = \tfrac{1}{2}[(1 - \alpha/2)(1 - y_0) + \alpha(2 - y_1)].$$

The functions $F_i(q_i, y_0, y_1)$ in this case are

$$F_i(q_i, y_0, y_1) = \int_0^1 I_{\{x>0\}}(q_i - q_i(\alpha))2\alpha \, d\alpha,$$

Where $I_{\{b>0\}}$ is the indicator function of the $\{b \mid b > 0\}$. Hence, $I_{\{b>0\}}(b)$ is unity for $b > 0$ and zero otherwise. The integration (10) was performed numerically using Mathematica.[2] The optimization of the proceeds was done by the FindMinimum function of Mathematica, employing one-dimensional searches to verily the optimum.

The maximum proceeds for the intermediary, when only one product uct $(y_0 = 1; y_1 = 2)$ is issued, is 0.6724 with the product priced at 0.80. The extremal AG product pair is given by $(y_0 = 1; y_1 = 0)$ for which the maximum proceeds are 0.6358 and the products are sold for 0.2525 and 0.5813. If debt and equity are issued with debt payoffs of $(y_0 = 1; y_1 = 1)$, then the optimum proceeds are 0.7201 and debt is sold for 0.5532, while the equity is sold for 0.3017. Finally, the globally optimal products are $(y_0 = 1.0000; y_1 = 0.9500)$ with proceeds of 0.7254 and prices of 0.5417 and 0.2977, respectively.

2.2 The repeated-offer case

In the repeated-offer case, the intermediary sets a price on a designed product and begins the search process by approaching a client with an offer to sell the product at the set price. If the client purchases the product, the search is stopped; otherwise the search continues with other clients being approached until the product is sold. If the price is set too high, the search will continue for a longer period of time with commensurately higher marketing costs.

We now formally model the proceeds from the repeated-offer strategy. The revenue raised from the issue is q. The marketing cost of this issue depends. on the number of approaches to clients to complete the sale. In modeling this cost, we follow Ross (1989a) and take the expected cost to be proportional to the expected number of approaches. However, instead of supposing that the cost is constant per client, implying a large total cost if many clients are contacted, we model the cost as a constant commission charge of θq for shares successfully marketed in a single approach and a commission of $2\theta q$ for those successfully sold in two approaches, and so on. The net revenue or proceeds from the issue are then

$$P = q - \sum_{k=1}^{\infty} k\theta q(1 - F(q))F(q)^{k-1},$$

where $(1 - F(q))F(q)^{k-1}$ is the proportion sold on the kth approach. Summing the infinite series, we get

$$P = q - \theta q/(1 - F(q)). \tag{11}$$

[2] *Mathematica: A System for Doing Mathematics by computer,* by Stephen Wolfram (Addison-Wesley, Reading, Mass., 1988).

The second term in (11) is the commission cost associated with marketing the product. Note that the expected number of approaches for a sale or the expected waiting time is $1/(1 - F(q))$.[3]

In the general case where M products are issued, the proceeds are

$$P = \sum_{i=1}^{M} \left[q_i - \frac{\theta q_i}{1 - F(q_i)} \right]. \tag{12}$$

Example 3. This example illustrates that debt and equity can be optimal in the repeated-offer. case. Again, we have two states with payoffs 1 and 2 in the states s_0 and s_1, respectively. The marginal rates of substitution $\omega(s, \alpha)$, for a continuum of individuals $\alpha \in [0, 1]$, include among them three individuals: those that have a higher relative valuation of a dollar in state s_0, those that value a dollar in state s_1 more, and those whose valuations for the two state-contingent dollars are identical. Let the state prices for states s_0 and s_1 by individual α be given by

$$\omega_0(\alpha) = \tfrac{1}{2} - 4(\alpha - \tfrac{1}{2})^3 \quad \text{and} \quad \omega_1(\alpha) = \tfrac{1}{2} + 4(\alpha - \tfrac{1}{2})^3.$$

Note that by design all individuals agree on the valuation of the debt security with payoffs (1,1), implying that these products are all sold in a single approach.

We construct the distribution of individuals to be centered at $\alpha = \frac{1}{2}$, the individual that is indifferent between the two states. The cumulative distribution function for the population is assumed to be

$$G(\alpha) = [H(\alpha) - H(0)]/[H(1) - H(0)],$$

where $H(\alpha) = [1 + \exp(-\gamma(\alpha - \frac{1}{2}))]^{-1}$.

The values for θ and γ used were 0.01 and 15, respectively. The solution was obtained employing Mathematica. The sale of a single product yields optimal proceeds of 0.7371 at a price of 0.7456. For the case of two AG products, the optimal proceeds are 0.7364, with the product paying (1, 0) being priced at 0.2473, while the product paying (0, 2) is priced at 0.4996. The global optimum design, however, is debt and equity with the payoffs (1, 1) and (0, 1), respectively. The proceeds are 0.7406, debt is priced at 0.5, and equity is priced at 0.2489.

The expected waiting time for the single issue is 1.14. For the two AG products, the waiting times are 1.19 and 1.50 for the (1,0) and (0, 2) products, respectively. On the other hand, for the global optimum products, debt and equity, the expected waiting time is 1 for debt and 1.29 for equity.

[3] Notice that the marketing cost in (11), given by the second term, is identical to (6) above.

By increasing the number of products, it is possible to raise the total proceeds; however, expected transactions costs would increase because of a rise in expected waiting times. If the products are extremal, and this is not matched by population tastes, the increase in commission costs may more than offset the gains in total proceeds.

3. Properties of Optimal Designs

The examples of the previous section demonstrate the possibility that interior products maybe optimal. In this section, we study the general design problem; isolating conditions under which interior products are optimal. It is shown that interior optimality requires sufficient diversity in individual state prices. Since the general design problem can be quite complex, involving as part of the specification the choice, of the number of products and states,.as well as the marketing strategy, in the interests of parsimony, we shall in the rest of the article restrict attention to the case of two products, an arbitrary number of states, and the single-offer marketing strategy.

The objective for this design problem is to maximize. P from the sale of two products. By Equation. (8),

$$P = q_1(1 - F_1(q_1)) + q_2(1 - F_2(q_2)).$$

In the analysis of the design problem it is useful to identify the states with cash flows x, and define the payoff to the first product in terms of the state-contingent proportion $a(x)$. The payoffs to the two products are therefore $a(x)x$ and $(1 - a(x))x$, respectively, where $0 \leq a(x) \leq 1$.

The functions $F_i(q_i)$, $i = 1,2$, yielding the proportion of the population valuing product i less than q_i, can be defined as follows. Let $\omega(x, \alpha)$ be individual α's marginal rate of substitution for the cash flow state x. With the true probability distribution of profits, given by $H(x)$, let $dQ(x, \alpha) = \omega(x, \alpha)\,dH(x)$; then Q is a personalized change of measure over x for each α that incorporates discounting. Individual valuations for the products are given by the expectations under Q of the payoffs. Define Ψ, the proportion of the population rejecting the single offer, by

$$\Psi(q, a) = \Theta\left(\left\{\alpha \,\middle|\, \int xa(x)\, dQ(x, \alpha) \leq q\right\}\right),$$

where Θ is a measure on the space of individuals.

It follows that

$$F_1(q_1, a) = \Psi(q_1, a) \qquad \text{and} \qquad F_2(q_2, a) = \Psi(q_2, (1 - a)).$$

Note that Ψ is increasing in q, decreasing in a, and homogenous of degree 0 in q and a.

Proposition 1. *Proportional revenue-sharing plans, with $a(x) = \lambda$ a positive constant less than 1, provide no increase in proceeds over issuing a single product whose payoffs are identical to the cash flows x.*

Proof: See the Appendix.

By optimizing out the sale prices q, one may write the design objective solely in terms of the sharing rule $a(x)$. For this purpose, define $V(a)$ as the proceeds from the optimal pricing and sale of a single product with contract shares $a(x)$ and payoffs $a(x)x$. Then

$$V(a) = \max_{q} q(1 - \Psi(q, a))$$

and the two-product design problem can then be written as

$$\max_{a} V(a) + V(1 - a).$$

This problem is analogous in structure to the maximization of the total production of two identical producers sharing an n-dimensional unit input cube as an Edgeworth box of resource endowments. The structural properties of the single-product proceeds function V are critical to the solution.

Proposition 2. *The proceeds $V(a)$ from optimal pricing and sale of a single product with contract shares a is homogeneous of degree 1 in a.*

Proof: See the Appendix.

Hence V is analogous to a production function, with V_a nonnegative and V satisfying constant returns to scale. The maximization of $V(a) + V(1 - a)$ is then like obtaining a point on the production possibility frontier. If V is quasiconcave, then, since both producers have the same V function, the contract curve is the line $a_1 = a_2$ and any solution implies that the vector $a = \lambda$ for a constant λ. By Proposition 1 this solution is dominated by the issue of a single product. Hence, if more than one product is optimal, then V is not quasiconcave.

The theorem below shows that if the optimally priced proceeds function V is quasiconvex in the contract shares a, then AG products are optimal. Sufficient conditions for the quasiconvexity of V are presented in Propositions 3 and 4.

Theorem 1. *For two states and V quasiconvex, the optimal products are AG.*

Proof. See the Appendix.

Proposition 3. *$V(a)$ is quasiconvex in a if Ψ (q, a), the probability of product rejection, is quasiconcave in a.*

Proof: See the Appendix.

Proposition 4. *If the ordering of individuals based on their valuation of a product is independent of the product, then Ψ (q, a) is quasiconcave.*

Proof: See the Appendix.

For preferences restricted in this way, quasiconvexity of V and the associated optimality of AG products follow from Propositions 3 and 4. Since individual product valuations are linear in the coefficients a that define the product, the hypothesis of Proposition 4 is satisfied if the ordering of individuals based on personalized state prices is independent of the state.

These results show that interior optimality, though possible by the examples of Section 2, is not necessarily the rule. For such interior optimality, one has to have state-preference structures with divergent state rankings across individuals and optimally priced proceeds V neither quasiconcave nor quasiconvex in contract shares a. Of course, for functions employing cumulative density functions like $F_i(q_i)$, variations from local quasiconcavity to local quasiconvexity can arise. Observe that the examples of the previous section built in the required diversity in individual state prices. The next section takes up the formulation and investigation of product design problems.

4. The Design Problem

We consider in this section the general n-state, two-product, single-offer problem with payoff proportions given by. the vector $a = (a_1, a_2, ..., a_n)$ in n successive cash-flow states indexed by i. Our first objective is to simplify the formulation of the optimally priced proceeds function $V(a)$, for general preference structures.

For $i = 1, ..., n$, let $v_i(\alpha)$ be the value to individual a of the ith AG product with payoff equal to the cash flow x_i if state i occurs, and zero otherwise. Let v_i be the random variable of these values with probability distribution induced by the measure on the individuals. For the purpose of determining $V(a)$, let $v = (v_1, v_2, ..., v_n)$ be the joint random vector of the individual valuations for these n AG products.

Since individual valuations are linear, being performed at the margin by an integration operation, the random variable q for the individual valuation of the product a is given by

$$q = a^T v,$$

where the superscript T denotes transposition.

A useful probability structure on v, motivated by the linearity of q in v, is that of multivariate normality, with the implied normality of q. The complete distribution function is then described by the first two moments. Supposing v has means given by the vector μ, with covariance matrix Σ, then the variable q is normally distributed with mean, $a^T\mu$, and variance $a^T\Sigma a$. The probability of product rejection is

$$\Psi(q, a) = N((q - a^T\mu)/(a^T\Sigma a)^{1/2}),$$

where N is the cumulative distribution function of a standard normal variate.

It is useful to define Ψ directly in terms of the standard normal variate z and for this purpose we define the product's expected valuation by $\mu(a)$, and the standard deviation of the individual product valuations by $\sigma(a)$, where

$$\mu(a) = a^T\mu, \qquad \sigma(a) = (a^T\Sigma a)^{1/2}.$$

Let

$$z = (q - \mu(a))/\sigma(a),$$

so that $\Psi(q, a) = N(z)$. The function $V(a)$ can now equivalently be defined in terms of $\mu(a), \sigma(a)$, and z by

$$V(a) = \max_z([\mu(a) + \sigma(a)z](1 - N(z))),$$

or setting the coefficient of Individual price variation,

$$c(a) = \sigma(a)/\mu(a),$$

We may write

$$V(a) = \mu(a)\max[(1 + c(a)z)(1 - N(z))].$$

Define the function $\Phi(c)$, to be determined once for use in all applications, by

$$\Phi(c) = \max[(1 + cz)(1 - N(z))];$$

then we may write that

$$V(a) = \mu(a)\Phi(c(a)).$$

Observe that the optimally priced proceeds function $V(a)$ is proportional to the price $\mu(a)$ with the factor of proportionality $\Phi(c)$ depending on just the coefficient of variation c induced by the contract shares a. One may therefore interpret $\Phi(c)$ as the premium associated with diversity in individual valuations measured by c. Many design problems for two or more products or states may be numerically solved using as input the vector μ and matrix Σ, with the above formulation of $V(a)$, once $\Phi(c)$ has been determined.

4.1 The diversity premium function $\Phi(c)$

The maximization of the function defining $\Phi(c)$ has a first-order condition obtained by equivalently differentiating the log of the objective to yield

$$\frac{c}{1 + cz} - \frac{n(z)}{1 - N(z)} = 0.$$

The positive contribution to the derivative, is falling in z, approaching 0 hyperbolically as z tends to infinity, and approaching infinity as z tends to $-(1/c)$. The negative contribution is in absolute value tending to zero as z tends to $-\infty$ and, as z tends to infinity, by l'Hôpital's rule, it tends to the 45° line. Hence, there is a unique value of z maximizing the objective, given by $z^*(c)$, that may be positive or negative. As c is increased, the negative contribution is unaffected while the positive contribution rises and hence $z^*(c)$ rises with c; the effect on $\Phi(c)$ is ambiguous when $(1 + cz)$ rises and $(1 - N(z))$ falls. In fact, by the envelope theorem, $\Phi_c = z^*(c)(1 - N(z^*(c))$, and so is negative or positive depending on the sign of $z^*(c)$. For Φ_c to be zero, one must have $z^*(c) = 0$, or $c = 2n(0) = 0.7979$.

The diversity premium function $\Phi(c)$ was calculated exactly for a grid of c values using Mathematica to solve the first-order condition for z^*. An approximation to $\Phi(c)$ was then constructed by regressing. the function values thus obtained onto a matrix defined by the constant term and the first seven powers of c evaluated at these grid points. The resulting approximation for $\Phi(c)$ is

$$\Phi(c) = 0.98 - 1.9628c + 3.00674c^2 - 2.2959c^3 + 0.9906c^4 - 0.2409c^5 + 0.03075c^6 - 0.0016c^7,$$

for $0 \leq c \leq 5$.

4.2 Examples of interior optima.

Even for two states and two products, there are examples of interior optima. Consider a mean vector for the AG values of (1, 2) with standard deviations of 0.5 and 2 and a correlation of –.5. In this case, using the function $\Phi(c)$, we observe that the value of issuing one product is 1.58, while that of issuing two AG products is 1.53, with the optimal value being 1.68. The optimal value of a_1 is 0, while that of a_2 is 0.71.

In all the above examples provided, sharing has occurred in only one state. We present now a three-state example with the optimal product-sharing payoffs in two of the three states. The basic idea is to make the second and third states much like each other and related to the first state in individual valuations in a manner analogous to our two-state example above. Suppose then that the means are (1, 2, 3) for the three states and the coefficients of variation are (0.5, 1, 1) or that the standard deviations are (0.5, 2, 3). Furthermore, let the correlation matrix R be

$$R = \begin{bmatrix} 1.0 & -.5 & -.5 \\ -.5 & 1.0 & .9 \\ -.5 & .9 & 1.0 \end{bmatrix}.$$

As may be observed, state valuations for states 2 and 3 are related and they share the same coefficient of variation. These two states also have a negative correlation with the first state of –.5.

The optimal product in this case is interior and involves profit-sharing in both the second and third states with payoff, proportions in these states being .13 and .08, respectively. This optimal product also receives the entire payoff of state 1. Basically, given the negative correlation of state-1 valuations with states 2 and 3, it is desirable to offer the first product some payoff In these states to boost the demand for it. This does not detract too much from the "second and third state" product as this has high valuation variance in any case. The optimal proceeds are 3.16, the single product fetches 2.59, and the highest AG value is 3.04.

4.3 Implementing product designs

In this subsection, methods are presented for obtaining information on the first and second moments of the distribution of individual valuations of AG products. The moments of this distribution of personalized valuations for AG products must be indirectly inferred as the products are not-traded. Once these moments have been obtained, the procedures described earlier in this section may be applied to construct optimal products.

We identify the state space with the time 1 cash flow continuum x and employ a finite set of approximating AG products defined by a partition of the positive real line into intervals given by the cutoffs $\{K_0 = 0, K_1, K_2, \ldots, K_{n-1}, K_n = \infty\}$. Define, for $j = 1, \ldots, n$, the payout of the *jth* AG product as y_j, where y_j equals x provided $K_{j-1} < x < K_j$, and is zero otherwise. Personalized valuations for an arbitrary cash flow $y(x)$, by individual i, are given by

$$V_i(y) = \int y(x) q_i(x) \, dx,$$

where q_i is the personalized state price function. Normalizing the integral of just q_i to the present value of a certain future dollar, which we take to be e^{-r}, let $h_i = q_i e^r$ and observe that h_i integrates to unity and is therefore a pseudopersonalized probability. $V_i(y)$ is then the discounted expected value of y with respect to the personalized density h_i.

We suppose that the personalized densities belong to the two-parameter class of lognormal distributions with mean μ_i and variance v_i. Individual *i's* valuation for the *jth* AG product is then given by

$$W_{ij} = E_i e^{-r} x I_{[K_{j-1}, K_j]},$$

where E_i denotes the expectation operator with respect to the density h_i and $I_{[a,b]}$ denotes the indicator function for the the interval $[a, b]$. It is shown in the Appendix that

$$W_{ij} = V_0^i [N(d_{j-1}) - N(d_j)], \tag{13}$$

where V_0^i is $e^{-r}\mu_i$, the present value of the expected value of x with respect to the personalized density h_i, $\sigma_i^2 = \log(1 + v_i)$, and

$$d_j = [\log(V_0^i / K_j)]/\sigma_i + (r/\sigma_i + \sigma_i/2).$$

Three special AG products are first considered with cutoffs given by $K_1 = 1.25$, and $K_2 = 3.0$.[7] We determine the first and second moments of the multivariate distribution of personalized valuations for these products by simulation. We assume that the variation of V_0^i and σ_i across individuals i is adequately described by independent Weibull distributions. The choice of the Weibull was motivated by its robustness in describing nonnegative random variables, and the ease with which the distribution function may be inverted for simulation purposes. The detailed parameter choices for the Weibull are described below. Samples, of size 100 on (V_0, σ) observations are generated. Equation (13) is then applied to determine a sample of 100 observations on personalized valuations for each of these special AG products. Means, variances, and covariances are then computed from this sample of 100 valuations. These were then employed in the optimization procedure used in our earlier examples.

The mean value of V_0 was 4.48 and the standard deviation was 2.43. The corresponding values of the scale and exponent parameters of the Weibull were 1.6 and 5, respectively. The σ distribution Weibull

Table 1
Optimal product displaying sharing in seven of eight states

Asset value interval	AG product mean value	σ for AG values	Optimal proportions	Payoff amounts
0–0.75	0.1460	0.1601	1.0	0.375
0.75–1.0	0.1387	0.1549	.8554	0.7485
1.0–1.25	0.1586	0.1785	.5795	0.6519
1.25–1.5	0.1460	0.1794	.4831	0.6643
1.5–1.75	0.1166	0.1635	.4061	0.6599
1.75–2.0	0.0919	0.1674	.3433	0.6437
2.0–2.5	0.1050	0.2199	.3119	0.7018
2.5–	0.0887	0.2657	.1816	—

Eight states are defined using intervals of asset values. For each state the corresponding AG product receives the entire firm if the value lies in the interval defining the product. Individual valuations for AG products are given by Equation (13) in terms of the individual's assessment of initial value and volatility. The second and third columns present the means and standard deviations of these individual valuations. The fourth column gives the proportion of value due to the first of two optimal products, while the fifth column gives the average payout to this product.

parameters were 0.5 for scale and 1.2 for exponent. The mean value of σ was 0.47 with a standard deviation of 0.3755.

The result in this case was a product that shared payoffs in the third state $[3, \infty]$. One optimal product has payoff proportions of (1, 1, 0.2) and the complementary product has (0, 0, 0.8). The complementary product receives a positive cash flow only when the firm's profits are large. The proceeds from floating one product were 2.53, which was also the value of the best AG alternative. The optimal product gave a return of 2.63.

The optimal product of this example is a simple one that could be implemented by the Issue of a special type of warrant that allows the holder to convert his holding to shares at maturity provided the firm value exceeds an amount equal to K_2; the conversion, however, is done at no cost. For our example, four such warrants must be issued for every share at issue with $K_2 = 3$ and a maturity of one year.

4.4 Optimal products sharing cash flows in multiple states

The optimal products displayed thus far have involved sharing In just one or two states. This enabled one to write optimal products using simple security structures such as debt, equity, or warrants. In general, we do not expect such simple payoff patterns for optimal products. In this section, we illustrate an optimal product that involves a complex system of sharing in seven of eight states.

Keeping r at 10 percent, eight value intervals were used as reported in Table 1. All means, variances, and covariances were computed from averages of 1000 readings. As can be seen from the second column of Table 1, the mean values for the respective AG products were similar and this was by design, constructing the interval cutoffs to

Table 2
Correlation matrix for AG product values

1.0	−.04	−.5	−.58	−.55	−.44	−.39	−.26
−.04	1.0	.18	−.24	−.35	−.33	−.31	−.21
−.5	.18	1.0	.37	−.04	−.18	−.21	−.16
−.58	−.24	.37	1.0	.53	.13	−.02	−.07
−.55	−.35	−.04	.53	1.0	.61	.27	.06
−.44	−.33	−.18	.13	.61	1.0	.52	.20
−.39	−.31	−.21	−.02	.27	.52	1.0	.57
−.26	−.21	−.16	−.07	.60	.20	.57	1.0

This matrix presents the correlations between individual valuations of the eight AG products described by the asset value intervals of Table 1. These correlations, along with the standard deviations of Table 1, define the variance–covariance matrix Σ employed in constructing $V(a)$ as described in Section 4.

have this effect. The third column of Table 1 reports the standard deviation for the respective AG products, while Table 2 provides the AG product correlation matrix. There is positive correlation in the valuations of adjacent AG products and negative correlation for distant ones. These correlations also rise as we go up in the value range.

The value of a single-product issue is 0.5169, while the optimal value is 0.6069. The optimal sharing structure involves sharing in seven of the eight states as reported in column 4, of Table 1. The proportion of value going to one of the two optimal products declines as value increases. This is like debt, but to make a stricter comparison one needs to look at the payoffs involved. Column 5 of Table 1 reports the state-contingent payoffs of the first product. These are obtained by multiplying interval midpoints by the optimal proportion reported in column 4. As can be seen, the product does taper off to a debt-like security; but after it has a hump in the earlier range of payoff 0.75 in interval 2. Figure 1 presents a graph of the optimal payoff function.

5. Concluding Remarks

The primary objective of this article was to bridge the gap between the abstract ideas in Allen and Gale (1988) and the more specific questions arising in the design of financial products. In addition, noting the importance of a characterization of the diversity of individual state prices for product design, a simple and parsimonious description of this diversity was introduced in terms of four magnitudes, representing the means and variances across individuals of state price means and variances.

In this article, we also introduced marketing costs that are incurred in the process of finding and approaching prospective investors, and that also may be reflected in unsold inventory. It was shown how product designs that share the firm's cash flow significantly enhance their appeal to a wider set of investors and thereby reduce marketing

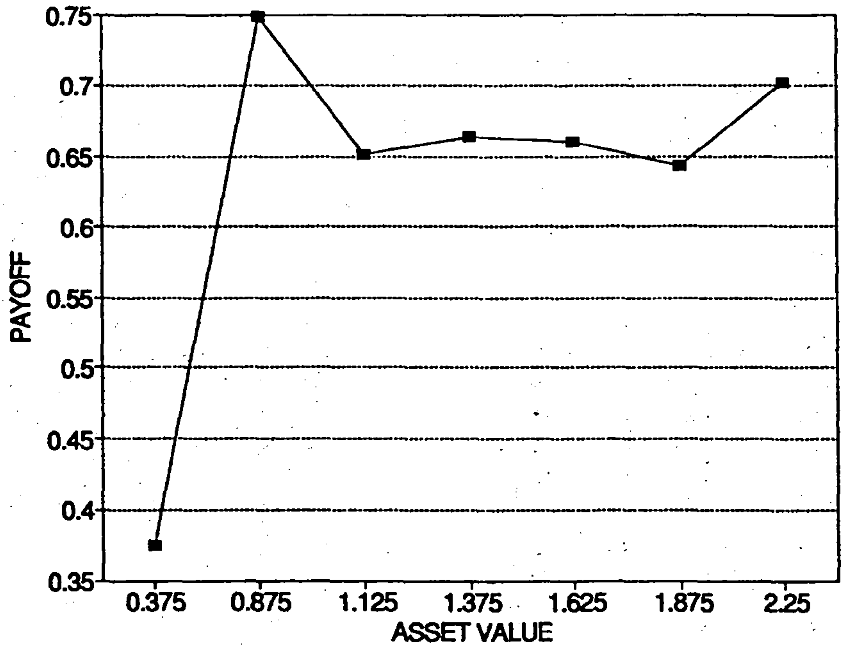

Figure 1
Optimal product payoff as a function of asset values for the eight-state example presented in Table 1

costs. It was shown further how optimal products typically display profit sharing in the higher profit states, and how in simple cases, this could involve just the use of debt, equity, or warrant-type securities; even though, generally one would expect to employ portfolios of option-type products. By way of illustration, an example was presented witheight states and a continuum of investors displaying two optimal products that share profits in seven of eight states.

Appendix

Proof of Proposition 1

Suppose that λx and $(1-\lambda)x$ were issued at prices q_1 and q_2. The net revenue earned would be

$$q_1(1 - F_1(q_1, \lambda\mathbf{1})) + q_2(1 - F_2(q_2, (1-\lambda)\mathbf{1})),$$

where 1 is the vector with all entries unity. In terms of Ψ, we have

$$q_1(1 - \Psi(q_1, \lambda\mathbf{1})) + q_2(1 - \Psi(q_2, (1-\lambda)\mathbf{1})).$$

On the other hand, the firm could be sold at q_1/λ for a net revenue of

$$q_1/\lambda\ (1 - \Psi(q_1/\lambda, 1)),$$

or for a price of $q_2/(1 - \lambda)$ for a net revenue of

$$q_2/(1 - \lambda)(1 - \Psi(q_2/(1 - \lambda), 1)).$$

Since

$$\Psi(q_1, \lambda 1) = \Psi(q_1/\lambda, 1) \stackrel{Def}{=} a$$

and

$$\Psi(q_2, (1 - \lambda)1) = \Psi(q_2/(1 - \lambda), 1) \stackrel{Def}{=} b,$$

sale of the two products yields $q_1(1 - a) + q_2(1 - b))$ while the potential single sales yield $(q_1/\lambda)(1 - a)$ or $(q_2/(1 - \lambda))(1 - b)$. We cannot have $q_1(1 - a) + q_2(1 - b)$ exceeding both the single-sale options, as the two products' net revenues are a linear combination of the two single-sale net revenues.

Proof of Proposition 2

Let $\Phi(q, a) = q(1 - \Psi(q, a))$ and note that

$$\Phi(\lambda q, \lambda a) = \lambda q(1 - \Psi(\lambda q, \lambda a)) = \lambda\,[q(1 - \Psi(q, a)] = \lambda\Phi(q, a);$$

hence, $\Phi(q, a)$ is homogeneous of degree 1 in q, a.

The degree-1 homogeneity of V is a consequence of the homogeneity of Φ. Let

$$V(a_0) = \Phi(q_0, a_0) \geq \Phi(q, a_0)$$

We show that $V(\lambda a_0) = \Phi(\lambda q_0, \lambda a_0) = \lambda V(a_0)$ If there exists q such that $\Phi(\lambda q_0, \lambda a_0) < \Phi(q, \lambda a_0)$, then on division by λ and the degree-l homogeneity of Φ, we have that $(q_0, a_0) < \Phi(q/\lambda, a_0)$, a contradiction of the definition of q_o. Hence, λq_0 maximizes Φ for λa_0, and so $V(a)$ is homogeneous of degree 1 in a.

Proof of Theorem 1

Quasiconvexity of V implies no optima in the interior of the two-dimensional Edgeworth box.

On the face $a_1 = 0$ the set of points $(V(0, a_2), V(1, (1 - a_2)))$ define a curve in two-space that is intersected with a kink by the curve $(V(a_1, 1), V((1 - a_1), 0))$. The kink occurs at the point $(a_1, a_2) =$ (0, 1), where V is typically maximized.

Consider the slope of the Pareto frontier defined by varying a along the face $a_1 = 0$, followed by moving along the face $a_2 = 1$. Along the face $a_1 = 0$, this slope is

$$-V_2(1, 1 - a_2)/V_2(0, a_2),$$

and on the face $a_2 = 1$, this slope is

$$-V_1(1 - a_1, 0)/V_1(a_1, 1).$$

Consider now what happens to the modulus of the first slope as we increase a_2 from 0 toward 1. The partial of the modulus of the first slope with respect to a_2 is

$$-V_{22}(1, 1 - a_2)/V_2(0, a_2) - (V_2(1, 1 - a_2)/[V_2(0, a_2)]^2)V_{22}(0, a_2)$$

and the partial of the modulus of the second slope with respect to a_1 is

$$-V_{11}(1 - a_1, 0)/V_1(a_1, 0) - (V_1(1 - a_1, 0)/[V_1(a_1, 1)]^2)V_{11}(a_1, 1).$$

We show below that V_{11} and V_{22} are both positive, and so both these terms are negative. Thus, the modulus of the first slope falls as we increase a_2, while the second slope rises as We lower a_1 from 1 toward 0. One has just to graph this slope behavior to observe that the only maxima for $V(a)$ can occur at (0, 0) or at (0, 1). Hence, the optimal products are AG in this case.

The positivity of V_{22} and likewise V_{11} is a consequence of the quasi-convexity and linear homogeneity of V. For such V, the unit isoquant is defined by

$$a_2 = g(a_1) \quad \text{iff} \quad V(a_1, a_2) = 1$$

and satisfies $g' < 0$ and $g'' < 0$. Furthermore, V is implicitly defined by the equation

$$\frac{a_2}{V(a_1, a_2)} = g\left(\frac{a_1}{V(a_1, a_2)}\right).$$

One can now compute V_{22} in terms of g. Differentiating once with respect to a_2, we get that

$$\frac{1}{V} - \frac{a_2}{V^2}V_2 = -g'\frac{a_1}{V^2}V_2,$$

or that

$$V_2 = \frac{V}{a_2 - g'a_1},$$

and this is positive. Differentiating once again implies

$$V_{22} = \frac{V_2}{(a_2 - g'a_1)} - \frac{V}{(a_2 - g'a_1)^2}\left[1 + \frac{a_1^2 g''}{V^2}V_2\right].$$

Substituting for V_2 from above, into the first term of the above expression for V_{22}, it follows that

$$V_{22} = \frac{-a_1^2 g'' V_2}{(a_2 - g' a_1)^2 V'},$$

which is positive. A similar argument shows that V_{11} is positive.

Proof of Proposition 3

Let q_λ be defined by

$$V(\lambda a + (1 - \lambda)b) = q_\lambda(1 - \Psi(q_\lambda, \lambda a + (1 - \lambda)b)).$$

It follows from the definition of V that

$$V(a) \geq q_\lambda(1 - \Psi(q_\lambda, a))$$

and

$$V(b) \geq q_\lambda(1 - \Psi(q_\lambda, b)).$$

The quasiconcavity of $\Psi(q, a)$ implies

$$\Psi(\lambda a + (1 - \lambda)b) \geq \min(\Psi(q, a), \Psi(q, b))$$

and hence that

$$1 - \Psi(q, \lambda a + (1 - \lambda)b) \leq \max(1 - \Psi(q, a), 1 - \Psi(q, b))$$

and therefore that

$$V(\lambda a + (1 - \lambda)b) \leq \max(V(a), V(b)).$$

Hence, V is quasiconvex in a.

Proof of Proposition 4.

On this hypothesis one may organ& individual indexes so that individual product valuations $q(\alpha, a)$ are increasing monotone in a for all products a. It follows that there is then a critical α, say α^*, with $\Psi(q, a)$ being the measure of the interval $[0, \alpha^*]$. Since $\{\alpha \mid q(\alpha, a) \leq q\} = [0, \alpha^*]$ and $\{\alpha \mid q(\alpha, b) \leq q\} = [0, \beta^*]$, for some α^*, β^*, it follows that

$$\{\alpha \mid q(\alpha, a) \leq q\} \cap \{\alpha \mid q(\alpha, b) \leq q\}$$

is one of these two sets, depending on which of the two magnitudes α^*, β^* is smaller. For any point in this intersection, we have both $q(\alpha, a)$ and $q(\alpha, b)$ bounded above by q. Now since agent α's personalized valuations are linear on products, it follows that $q(\alpha, \lambda a + (1 - \lambda)b)$ is also bounded by q. Hence; the intersection is contained in $\{\alpha \mid q(\alpha, \lambda a + 1 - \lambda)b) \leq q\}$, and so

$$\Psi(q, \lambda a + (1 - \lambda)b) \geq \min[\Psi(q, a), \Psi(q, b)],$$

or Ψ is quasiconcave in a.

Derivation of Equation (13)

Individual *i's* valuation for the *jth* AG product is given by

$$W_{ij} = E_i e^{-r} x I_{[K_{j-1},K_j]} = E_i e^{-r}[x I_{[K_{j-1},\infty]} - x I_{[K_j,\infty]}].$$

We now evaluate $E_i e^{-r} x I_{[K_j,\infty]}$. For this purpose, note that by the log-normality of x it follows that

$$x = \mu_i e^{\sigma_i z - \sigma_i^2/2},$$

where z is normal 0.1 and $\sigma_i^2 = \log(1 + \nu_i)$.

Now letting $V_0^i = \mu_i e^{-r}$,

$$E_i e^{-r} x I_{[K_j,\infty]} = e^{-r} K_j E_i I_{[K_j,\infty]} + E_i e^{-r}(V_0^i e^{r+\sigma_i z - \sigma_i^2/2} - K_j)^+, \quad \text{(A1)}$$

because $x I_{[K_j,\infty]} = [K_j + (x - K_j)^+] I_{[K_j,\infty]}$.

The second expectation on the right-hand side of (A1) is precisely the Black-Scholes value for an option with a strike price of K_j and an initial value of V_0^i with variance rate σ_i^2, maturity 1, and interest rate r. Hence, (A1) may be rewritten as

$$E_i e^{-r} x I_{[K_j,\infty]} = e^{-r} K_j E_i I_{[K_j,\infty]} + V_0^i N(d_j) - e^{-r} K_j N(d_j - \sigma_i).$$

It is well-known from options theory that $N(d_j - \sigma_i)$ is precisely $E_i I_{[K_j,\infty]}$ and, therefore,

$$E_i e^{-r} x I_{[K_j,\infty]} = V_0^i N(d_j).$$

Equation (13) now follows on subtraction.

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