Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data

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In the context of an equilibrium asset-pricing model, the dynamics of the instantaneous real interest rate and the instantaneous rate of expected inflation are estimated. Unlike previous models, we allow real interest rates and inflation to be mutually dependent processes. The model is estimated as a state-space system that includes observations on various maturity Treasury bills and NBER-ASA survey forecasts of inflation. Over the period 1968-1988, we find evidence that instantaneous real interest rates and expected inflation are significantly negatively correlated. Real interest rates also display greater volatility and weaker mean reversion than expected inflation.

Analysis of the relationship between interest rates and inflation can be traced back to the work of Irving Fisher (1896). Yet, economists continue to devote a significant amount of theoretical and empirical effort to this subject. Cox, Ingersoll, and Ross (1985b) and

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Richard (1978) made important contributions by developing equilibrium, "arbitrage-free" bond pricing models that explicitly recognize the effect of inflation on the term structure of interest rates. These theoretical models provided a framework for a number of empirical studies, such as Oldfield and Rogalski (1987). Stambaugh (1988), Gibbons and Ramaswamy (1986), Heston (1988), and Pearson and Sun (1988).

A basis for this research is the presumption that a suitable model of the term structure of interest rates requires multiple sources of uncertainty, perhaps reflected by different dynamics for real interest rates and inflation. Recent empirical evidence supports this view. Stambaugh (1988) considers a latent variable model of the term structure, where logs of bond prices are linear functions of multiple (unobserved) state variables. Using a generalized method of moments (GMM) test of conditional expected Treasury-bill returns, he rejects the hypothesis that interest rate movements can be described by a single-state variable. However, he finds that a two- or three-state variable model, where these state variables are assumed to be mutually independent, is sufficient to characterize Treasury-bill returns. Although his analysis does not attempt to identify these variables, he suggests that variables that determine the real rate of interest and the expected inflation rate would be obvious candidates.

In the context of a model of real interest rates and inflation given by Cox, Ingersoll, and Ross (1985b), hereafter referred to as CIR, Gibbons and Ramaswamy (1986) present an innovative technique for estimating the parameters of the process followed by the real interest rate. Their GMM estimation considers nonlinear restrictions that the model places on unconditional means and covariances of real Treasury-bill returns. Their technique does not require -making specific assumptions regarding the process followed by inflation, other than the assumption that it be mutually independent of the process followed by the real interest rate. In a similar manner, Heston (1988) gives GMM estimates that result from imposing both linear and nonlinear CIR model restrictions on conditional mean bond returns. The bivariate CIR model is also the basis of a maximum likelihood technique presented by Pearson and Sun (1988). However, each of these empirical analyses retains the assumption that inflation and real interest rates follow independent processes.[1]

This article is in the same spirit as the empirical studies cited above in that it seeks to estimate the dynamics of real interest rates, as well

[1] This Independence assumption is necessary for obtaining a closed-form solution to each of the two-factor models in CIR (1985b).

as inflation, in the context of an equilibrium model of bond prices.[2] However, it departs from the previous literature by weakening the assumption of state-variable independence. The processes followed by the real interest rate and inflation are allowed to be jointly dependent.[3] Moreover, an estimation technique is presented that allows us to identify and estimate the nature of this dependence in terms of both expected and instantaneous (unexpected) movements in real interest rates and inflation.

Why might it be important to allow real interest rates and inflation to be dependent? Economic theory suggests that most exogenous variables will have simultaneous effects on both real interest rates and inflation, and thus induce correlation between them. For example, technological change, commonly modeled as an exogenous variable, can directly influence investment opportunities, real rates of return (interest), and output growth. Except when monetary policy is completely accommodating to this change in output, there will also be a change in the level of inflation via output's effect on money demand.[4] Alternatively, a government's monetary policy may itself be modeled as an exogenous variable, although a more complete model might treat it as endogenous. While there are certain restrictions that can be placed on monetary models, such that they can display "monetary (super-)neutrality," most models would predict that anticipated and/or unanticipated changes in money growth will have an influence on the real economy. In summary, there is at most a very small set of macroeconomic models that would predict independence between real interest rates and inflation.

[2] It should be noted that other multivariate equilibrium bond-pricing models, such as Brennan and Schwartz (1982), choose not to model processes for real interest rates and inflation explicitly, but instead specify state-variable processes for yields on bonds of different maturities. In addition, alternative "regression-based" approaches have been used to investigate the relationship between interest rates and inflation [e.g., Fama (1975, 1988) and Mishkin (1987)]. While these studies impose few model-specific restrictions on the processes followed by real interest rates and inflation, they are unable to identify the specific dynamics of these processes, such as unexpected (instantaneous) versus expected movements, or magnitudes of mean reversion in real rates and inflation. This limits their usefulness in obtaining bond-volatility parameters applicable to pricing and hedging interest-sensitive contingent claims, such as options and futures. Instead, these articles have been useful in examining more general questions, such as whether forward rates predict (or possess information on) future inflation or real interest rates.

[3] Recent work by Boudoukh (1990) also considers the pricing of bonds when real measures are correlated with inflation. He considers a discrete-time model of an endowment economy, so that consumption equals output each period, and assumes that log inflation and log consumption growth follow a vector autoregressive process, with the variance of inflation following an ARCH process. The model's parameters are estimated using data on aggregate consumption and inflation. Using these estimates, bond prices are approximated (closed-form solutions are not possible) using a Gaussian quadrature method. The statistics generated from these bond prices are compared to those of actual bond prices. A number of results are consistent with those of this article, including the observation that correlation between inflation and consumption growth plays an important role in determining the properties of the term structure.

[4] Fama (1981) makes a similar argument.

The model presented in this article fits within the equilibrium framework of CIR (1985a) in that our assumptions regarding state-variable processes as well as the "prices" of risk associated with these state variables are consistent with a general equilibrium model. However, our model does not test any of the specific examples given in CIR (1985b). Instead, it is similar to those of Vasicek (1977), Langetieg (1980), and Marsh (1980). As will be discussed below, it possesses both advantages and disadvantages compared to the CIR (1985b) model. An attractive feature of the model is that it can be transformed into a state-space system that allows mutual dependence between the unobserved state variables. This model can then be estimated by maximum likelihood, using a Kalman filter to simplify computation of the likelihood function.

A key aspect of our estimation technique is the use of survey forecasts of Inflation as well as Treasury-bill prices of various maturities. Combining these two types of data allows us to identify the (latent) state variables as the real interest rate and expected inflation, rather than identifying these variables by assuming them to be independent, as was done in other studies. Using survey inflation forecasts offers potential advantages relative to the usual practice of inferring investors' expectations of inflation from actual inflation. If survey data is a more direct measure of the expectations of bond-market investors, it would not be as prone to small sample problems associated with other types of rational expectations estimates. Hence, more powerful estimates of bond-price dynamics could result.

The article's econometric results indicate that the instantaneous (short) rate of real interest is significantly more volatile than the instantaneous level of expected inflation, during both the 1970s and 1980s. This is strikingly at odds with a number of earlier studies that found changes In nominal interest rates as primarily reflecting fluctuations in expected inflation rather than fluctuations in real rates (e.g., Fama, 1975; Mishkin, 1981; Fama and Gibbons, 1982). Unexpected changes in real interest rates and inflation are also found to be significantly negatively correlated. Tests of the hypothesis that real interest rates and inflation follow independent processes are conclusively rejected. In addition, real interest rates display weaker mean reversion than inflation.

The plan of the article is as follows. In Section 1, the empirical work is motivated by showing its consistency with a simple intertemporal capital asset-pricing model. In Section 2, I derive the equilibrium term structure of interest rates that results from this model. In Section 3, the technique used to estimate this model is discussed, while, in Section 4, the data chosen for the empirical work is described. An analysis of the empirical results follows in Section 5.

1. An Intertemporal Model

Consider an economy similar to that of Merton (1971, 1973a) and CIR (1985a, 1985b). The following three assumptions are made.

Assumption 1. *Infinitely lived individuals maximize utility of the form*

$$E_t \int_t^{\infty} e^{-\rho s} \ln(C(s))\, ds, \tag{1}$$

where C(t) is the time t consumption of a representative individual.[5]

Assumption 2. *There exists a single capital-consumption good. A single technology is available that transforms capital, K(t), into output according to the process*[6]

$$dK/K = \alpha_k(t)\,dt + \sigma_k\, dz_k, \tag{2}$$

where the expected rate of return on capital, $\alpha_k(t)$, also varies stochastically, following the process

$$d\alpha_k = (a_1 + b_{11}\alpha_k(t) + b_{12}\pi(t))\,dt + \sigma_\alpha\, dz_\alpha, \tag{3}$$

and where $\pi(t)$ is the expected rate of inflation, which follows the process

$$\begin{aligned} d\pi &= (a_2 + b_{21}\alpha_k(t) + b_{22}\pi(t))\,dt + \sigma_\pi\, dz_\pi, \\ dz_\alpha\, dz_\pi &= \rho_{\alpha\pi}\, dt. \end{aligned} \tag{4}$$

Assumption 3. *The monetary policy of the government is assumed to result in a process for the nominal price level, P(t), of the form*

$$dP/P = \pi(t)\,dt + \sigma_p\, dz_p, \qquad dz_k\, dz_p = \rho_{kp}\, dt, \tag{5}$$

with the expected rate of inflation, $\pi(t)$, following the process previously specified in Equation (4).[7]

Similar to CIR (1985b), we assume an exogenous nominal price level process without explicitly modeling the nature of the money

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[5] Duffie and Epstein (1989) show that this specification of utility an be generalized to a recursive utility formulation, where the intertemporal elasticity of substitution Is unity but the coefficient of relative risk aversion can be different from unity. The resulting form of the process for the equilibrium real interest rate is the same as that derived below.

[6] This assumption could be weakened to allow for multiple technologies, without changing the form of the process followed by the equilibrium real interest rate [e.g., see CIR (1985a].

[7] The formulation In Equation (5) allows the government's monetary policy to be modeled as one in which real capital shocks to the price level may or may not be accommodated. $\rho_{kp} = -1$ could be regarded as a nonaccommodating monetary policy, while, if $\rho_{kp} = 0$, capital shocks are completely

supply and demand that would give rise to it.8 However, unlike CIR, the specifications in Equations (2)-(5) allow for the possibility of interaction between the real return on capital and the rate of inflation. As mentioned in the introduction, a number of theoretical models justify this general specification. Monetary models predict that technological change, which affects the efficiency (return) of capital, influences the economy's rate of inflation.[9] Equation (4) captures this effect by allowing movements in expected inflation to depend on the current expected return on capital (through a nonzero b_{21}) and innovations in the expected return on capital (through a nonzero $\rho_{\alpha\pi}$).

In addition to the return on capital affecting inflation, there may well be a causal relationship working in the opposite direction. Fischer and Modigliani (1978) outline a number of ways that inflation can affect the real economy. Many of these influences are a result of nonindexed private contracts or the taxation of nominal asset returns. In particular, Feldstein (1980) demonstrates a link between higher expected inflation and lower asset valuation. Equation (30) allows for this effect by making the expected real return on capital depend on the level of, and innovations in, expected inflation.[10]

The solution to the consumption and portfolio choice problem described by Assumptions l-3 is similar to others found in the literature and is outlined in Appendix A. Individuals' optimal consumption behavior implies that aggregate consumption is proportional to aggregate real capital:

$$C(t) = \zeta K(t). \tag{6}$$

Let f(t) be the (instantaneous) nominal yield on a currently maturing, default-free discount bond, and let r(t) be the (instantaneous) real yield on a currently maturing, default-free indexed bond. In equilibrium, the nominal interest rate equals

$$i(t) = \alpha_k(t) - \sigma_k^2 + \pi(t) - \sigma_p^2 - \rho_{kp}\sigma_k\sigma_p \tag{7}$$

and the equilibrium real interest rate is

$$r(t) = \alpha_k(t) - \sigma_k^2. \tag{8}$$

[8] Foresi (1990) extends CIR (1985a) by assuming an explicit process for the central bank's supply of money and by allowing agents' utility to depend on real money balances. The price level process can be derived endogenously rather than specified as exogenous. He finds explicit solutions for the instantaneous nominal interest rate, but closed-form solutions for finite maturity bonds can only be derived for very simple cases.

[9] For example, see Siegel's (1983) extension of the Sidrauski (1967) model.

[10] We do not wish to Imply that Equations (3) and (4) can be regarded as "structural" equations in the sense of Lucas (1976). Rather, they should be considered as reduced forms. The economy is not modeled at the level of detail needed to evaluate government tax or monetary policy changes. In fact, we find evidence of certain parameter shifts over the period 1979-1982, when a change In monetary policy was announced.

Thus, the instantaneous riskless real interest rate has dynamics similar to the dynamics of the expected rate of return on risky capital:

$$dr = d\alpha_k = (a_1 + b_{11}\sigma_k^2 + b_{11}r(t) + b_{12}\pi(t))dt + \sigma_\alpha\, dz_\alpha. \quad (9)$$

The state of the economy is determined by the level of real wealth, K(t), along with the bivariate system, consisting of the real interest rare and expected inflation. Letting $s(t)' = (r(t)\ \pi(t))$ and $\sigma_\alpha\, dz_\alpha = \sigma_r\, dz_r$, Equations (9) and (4) can be rewritten in the form

$$ds(t) = (A + Bs(t))dt + \sigma dZ, \quad (10)$$

where

$$A' = (a_1 + b_{11}\sigma_k^2 \quad a_2 + b_{21}\sigma_k^2),$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

and

$$(\sigma dZ)' = (\sigma_r dz_r \quad \sigma_\pi dz_\pi),$$

with

$$\sigma dZ(\sigma dZ)' = \Sigma dt,$$

where

$$\Sigma = \begin{pmatrix} \sigma_r^2 & \sigma_{\pi r} \\ \sigma_{r\pi} & \sigma_\pi^2 \end{pmatrix}.$$

The bivariate process for $r(t)$ and $\pi(t)$ will be jointly stationary if the real parts of the eigenvalues of B are negative. Note that this bivariate Ornstein-Uhlenbeck process will not restrict either $r(t)$ or $\pi(t)$ from becoming temporarily negative. In one sense, this is an advantage relative to the "square root" process models in CIR (1985b), which restrict $r(t)$ and $\pi(t)$ to be positive, since in the absence of a costless storage technology, $r(t)$ could be negative, while expected deflation implies $\pi(t)$ could be negative.[11] Nevertheless, we would ideally like to impose the restriction that the nominal interest rate be nonnegative [i.e., $i(t) = r(t) + \pi(t) - \sigma_p^2 - \rho_{kp}\sigma_k\sigma_p \geq 0$]. Unfortunately, this leads to significant difficulties in deriving solutions for equilibrium bond prices, so that this restriction will not be imposed

[11] Our estimates of $r(t)$ are consistent with the general view that real interest rates were negative during a large part of the 1970s (Figure 7). It is also likely that $\pi(t)$ was negative during certain historical episodes, such as in Great Britain during the early 1920s, when the British attempted a return to the gold standard at pre-war parity.

in the analysis that follows.[12] However, we can anticipate that the restriction that f(t) be positive, together with the characteristic that $r(t)$ and $\pi(t)$ may individually become negative, will lead to estimates of b_{12}, b_{21}, and/or $\sigma_{r\pi}$ that reflect negative correlation between $r(t)$ and $\pi(t)$.

2. Derivation of the Term Structure

We can now derive the term structure of interest rates for (zero net supply, default-free) nominal bonds. Following the work of Richard (1978) and Langetieg (1980), let $N(\tau, s(t))$ be the nominal price at date t of a discount bond that pays \$1 with certainty at date $t + \tau$. Since the bond price depends on the vector of state variables, $s(t)$, Ito's lemma allows us to write its dynamics as

$$\begin{aligned} dN(\tau, s(t))/N(\tau, s(t)) &= \mu_n(s(t))\,dt \\ &\quad + (N_r\sigma_r/N)\,dz_r + (N_\pi\sigma_\pi/N)\,dz_\pi, \end{aligned} \tag{11}$$

where

$$\mu_n = (1/N)[N_r\mu_r + N_\pi\mu_\pi + \tfrac{1}{2}N_{rr}\sigma_r^2 + \tfrac{1}{2}N_{\pi\pi}\sigma_\pi^2 + N_{r\pi}\sigma_{r\pi} - N_\tau], \tag{12}$$

where μ_r and μ_π are the instantaneous expected changes in $r(t)$ and $\pi(t)$, respectively, given in Equation (10).

A currently maturing bond is instantaneously riskless in nominal terms, and thus $N_r = N_\pi = N_{rr} = N_{\pi\pi} = N_{r\pi} = 0$ as the bond price approaches its maturity value of $N(\tau = 0, s(t)) =$ \$1. Defining the yield on this bond to be the nominal interest rate, $i(t, s(t))$, we have

$$= i(t, s(t))\,dt. \tag{13}$$

If bon& are priced such that no arbitrage opportunities exist in equilibrium, then a standard hedging argument can be employed to show that the expected rate of return on a τ maturity bond must be of the form

$$\mu_n = i(t) + \chi_r N_r\sigma_r/N + \chi_\pi N_\pi\sigma_\pi/N, \tag{14}$$

where χ_r and χ_π are the "factor risk premia" or "market prices of risk" of a unit of standard deviation from the real interest rate and inflation, respectively. CIR (1985a) shows that, in equilibrium, the factor risk premia equal

$$\chi_s = \left(\frac{-U_{CC}(C^*)}{U_C(C^*)}\right)\operatorname{cov}(C^*, s), \tag{15}$$

[12] As with the Vasicek (1977) and Langetieg (1980) models, omitting this restriction still leads to sensible bond prices when the nominal interest rate is currently positive.

where $U(C^*)$ is the utility function evaluated at individuals' optimal level of consumption, and $\mathrm{cov}(C^*, s)$ is the covariance between changes in optimal consumption and changes in the state variable, s. Given that utility is logarithmic, $C^* = \zeta K(t)$, and using Equation (10), we have $\chi_r = \sigma_k \sigma_r \rho_{kr}$ and $\chi_\pi = \sigma_k \sigma_\pi \rho_{k\pi}$.

Equating the right-hand sides of Equations (12) and (14) gives us the standard partial differential equation that the equilibrium bond price, $N(\tau, s(t))$, must satisfy. Richard (1978, p. 49, Equation 57) shows that the solution to this equation can be written in the form

$$N(\tau, s(t)) = E_t \exp\left[-\int_t^{t+\tau}\left(i(v) + \frac{1}{2}\psi'\Sigma^{-1}\psi\right)dv - \int_t^{t+\tau}\psi'\Sigma^{-1}\sigma\, dZ(v)\right], \tag{16}$$

where $\psi' = (\sigma_r \chi_r \quad \sigma_\pi \chi_\pi)$, and where $i(t) = r(t) + \pi(t) - \sigma_p^2 - \rho_{kp}\sigma_k\sigma_p$.

Some insight may be gained by comparing this formula to that of a forecast of inflation over the life of the bond (i.e., the period from t to $t + \tau$). From the stochastic process for inflation, Equation (5), we obtain

$$E_t[P(t+\tau)/P(t)] = E_t \exp\left[\int_t^{t+\tau}\left(\pi(v) - \frac{1}{2}\sigma_p^2\right)dv + \int_t^{t+\tau}\sigma_p\, dz_p(v)\right]. \tag{17}$$

Note the similarity in 'Equations (16) and (17), especially the first terms under the integral signs on the right-hand side of each equation. One component of the bond price is the expectation of an exponential function of the integral of the nominal interest rate, of which one component is the instantaneous mean rate of inflation, $\pi(t)$. Similarly, the expected rate of inflation over the bond's life [Equation (17)] has this same component.

The results of Langetieg (1980) allow us to solve for the equilibrium price of a bond of any maturity.[13] Denote the log of the bond price as $n(\tau, s(t)) = \ln N(\tau, s(t))$. Using Equations (16) and (10), the formula for $n(\tau, s)$ can be shown to take the form

$$\begin{aligned} n(\tau, s) &= K_1(\tau) - [1\ 1]\Gamma(\tau)s(t) \\ &\equiv K_1(\tau) + \alpha_1(\tau)r(t) + \alpha_2(\tau)\pi(t), \end{aligned} \tag{18}$$

[13] Specifically, see Equations (30), (32), and (33) of Langetieg (1980, pp. 86-89).

where $K_1(\tau)$ is a constant, given the bond's maturity, τ, and $\Gamma(\tau)$ is a 2×2 matrix given by $\Gamma(\tau) \equiv B^{-1}(\exp(B\tau) - I)$. Hence, $\alpha_1(\tau)$ and $\alpha_2(\tau)$ are defined as

$$\alpha_1(\tau) \equiv -(\Gamma_{11}(\tau) + \Gamma_{21}(\tau)),$$
$$\alpha_2(\tau) \equiv -(\Gamma_{12}(\tau) + \Gamma_{22}(\tau)),$$

where $\Gamma_{ij}(\tau)$ is the (i, j)th element of $\Gamma(\tau)$.

Similarly, using Equations (17) and (10), we can solve for the optimal forecast of the price level, given the current price level, $P(t)$, real interest rate, $r(t)$, and instantaneous rate of inflation, $\pi(t)$. Denoting the time t optimal forecast of the price level at $t + T$ as $E_t[P(t + T)]$, we find it satisfies

$$\begin{aligned} \ln(E_t[P(t + T)/P(t))] &= K_2(T) + [0 \ 1]\Gamma(T)s(t) \\ &\equiv K_2(T) + \alpha_3(T)r(t) + \alpha_4(T)\pi(t), \end{aligned} \quad (19)$$

where $K_2(T)$ is a constant, given the forecast horizon, T and $\alpha_3(T)$ and $\alpha_4(T)$ are defined as

$$\alpha_3(T) \equiv \Gamma_{21}(T), \qquad \alpha_4(T) \equiv \Gamma_{22}(T).$$

An interesting feature of the log bond prices and log inflation forecasts, given by Equations (18) and (19)) respectively, is that they are both linear in the *state* variables." Also, the coefficients of these state variables are functions of only the elements of the matrix B from the state-variable process, as well as either τ or T. Since the bivariate state-variable process, given in Equation (10), is conditionally homoskedastic, log bond prices will also be conditionally homoskedastic. This is a possibly unattractive characteristic of the model as compared to the models of CIR (1985b), which allow for a particular type of conditional heteroskedasticity. The results of previous research challenge the assumption of homoskedasticity. Fama (1984), using a regression approach, finds time variation in the term premia of longer maturity Treasury bills. Oldfield and Rogalski (1987) find variance instability in log Treasury-bill price changes over the period 1959 to 1981. While the heteroskedasticity found in these two articles is based on conditioning sets that are different from that of the present article, so that the specification of Equation (18) is not conclusively rejected, we do not want to take a firm stand on the complete accuracy of the specification given in Equation (18). In Section 5, we will examine evidence of heteroskedasticity during the 1968-1988 sample period, and will attempt to gauge its influence by looking at subperiod samples of the data.

[14] This linearity feature is also shared by the two Independent-factor models of CIR (1985b). See Stambaugh (1988) on this point.

However, in spite of its disadvantage of homoskedasticity, Equation (18) has the attractive feature of being a closed-form solution for bond prices, which also allows parameterization of the correlation between real interest rates and expected inflation. In addition, this specification for bond prices is consistent with many stochastic interest-rate option pricing models, such as Merton's (1973b) general stochastic interest-rate option pricing model, Grabbe's (1983) model of options on foreign exchange, and Jamshidian's (1989) model of options on bonds. Hence, the parameter estimates that we obtain from this article can be conveniently used in empirical applications of the models.[15]

3. Estimation of the Model

In this section, we outline a method of estimating the parameters of the previous model. This bond-pricing model was formulated in continuous time, so that analytic solutions for bond prices could be obtained. However, since the data used to estimate the model are available only at discrete observation intervals, it will be useful to examine this model when viewed at discrete intervals. We show that this model can be transformed into a discrete-time state-space system.[16]

The process for the state variables, $s(t)$, given by Equation (10) above, can be rewritten in a discrete-time form as

$$s(t+\delta) = \gamma(\delta) + \Phi(\delta)s(t) + v_t(\delta), \tag{20}$$

where

$$\Phi(\delta) = e^{B\delta} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

is the fundamental solution matrix of the state-variable process, and where

$$\gamma(\delta) = \int_0^{\delta} \Phi(\omega)A \, d\omega,$$

and $v_t(\delta)$ is a normally distributed random variable with zero mean

[15] Equations (11) and (18) imply that the variance of a τ-period bond equals $\alpha_1^2(\tau)\sigma_r^2 + \alpha_2^2(\tau)\sigma_\pi^2 + 2\sigma_{r\pi}\alpha_1(\tau)\alpha_2(\tau)$. Most option-pricing models require integration of this expression with respect to τ. It is straightforward to show that this integration leads to a closed-form expression. Note that this expression does not Involve any unobserved variables, such as $r(t)$ or $\pi(r)$, which makes it valuable in empirical applications.

[16] A state-space representation is also used by Hansen and Singleton (1987) to test asset-pricing models by a generalized method of moments (GMM) technique. Their procedure allows for greater generality In consumers' utility but relies on the use of consumption data.

and covariance matrix

$$Q = \int_0^\delta \Phi(\omega)\Sigma\Phi'(\omega)\,d\omega.$$

Equation (20) is a "state transition equation" in a discrete-time state-space model, where the state variables, the instantaneous real interest rate and expected inflation, are unobservable to the econometrician. A corresponding set of "measurement equations" can be constructed from data on bond prices and inflation forecasts.

As is discussed in more detail below, we compute hypothetical τ maturity bond prices by linearly interpolating the yields derived from the average of observed bid and ask quotes of bonds whose maturities immediately surround τ. This constructed bond price is assumed to measure a "true" τ maturity bond price with error. Specifically, the log of this constructed bond price, $n^*(\tau, s)$, takes the form

$$n^*(\tau, s(t)) = n(\tau, s(t)) + e_1(t), \tag{21}$$

where $e_1(t)$ is independently and identically distributed over time.[17]

A second measurement equation for the survey data on inflation forecasts assumes that survey forecasts are noisy measures of the "true" market forecast of inflation. Denoting a survey inflation forecast as $E_t^*[P(t+T)/P(t)]$, we make the assumption

$$\ln E_t^*[P(t+T)/P(t)] = \ln E_t[P(t+T)/P(t)] + e_2(t), \tag{22}$$

where e,(t) Is also assumed to be independently and identically distributed over time with possibly a constant nonzero mean. A nonzero mean would allow survey forecasts to be biased in a simple way. The assumption that the additive errors, e_1 and e_2, are each identically distributed implies that measurement error has a standard deviation that is proportional to either the level of bond prices or the level of expected inflation, since Equations **(21)** and (22) are in log form. In the following discussion, we will, make the additional assumption that e_1 and e_2 are normally distributed. However, our estimation technique will still produce best-linear unbiased estimates of the parameters in the absence of this assumption.

Now, suppose that at date *t* we can observe (with error) the prices of discount *bonds* at M maturities, $\tau_1, \ldots, \tau_M$, and survey forecasts of inflation for *N* periods into the future, $T_1, \ldots, T_N$. Then, we can write

[17] Given that our empirical work uses bid-ask prices of bonds sampled at monthly intervals, this Independence assumption is justified. It might not be justified if the sampling frequency were dally or hourly. As in the case of other empirical tats of equilibrium bond-pricing models [e.g., Brown and Dybvig (1986) or Stambaugh (1988)], It is necessary to assume some form of measurement noise in order to prevent rejection of the model with probability 1. One Indication of the model's plausibility will be the magnitude of these measurement errors.

these observations in matrix form as

$$y(t) = K + \alpha s(t) + e_t, \tag{23}$$

where

$$y(t)' = [n^*(\tau_1, s) \cdots n^*(\tau_M, s) \ln(E_t^*[P(t + T_1)/P(t)]) \cdots \ln(E_t^*[P(t + T_N)/P(t)])],$$

K is an $M + N$ vector of constants, and a is an $(M + N) \times 2$ matrix whose first M rows are of the form $(\alpha_1(\tau_i)\ \alpha_2(\tau_i))$, $i = 1, \ldots, M$, and whose last N rows are of the form $(\alpha_3(T_j)\ \alpha_4(T_j))$, $j = 1, \ldots, N$. The $M + N$ vector of error terms is distributed as $N(0, R)$, where the covariance matrix R has diagonal elements $(\sigma^2_{\tau 1} \ldots \sigma^2_{\tau M}\, \sigma^2_{\tau 1} \ldots \sigma^2_{TN})$.

Equations (20) and (23) now comprise a state-space model. However, another slight simplification can be obtained by defining the state variables not as $s(t)' = (r(t)\ \pi(t))$, but as the deviations of the real interest rate and expected inflation from their unconditional (long-run) means. We can define a new vector of state variables, $x(t)' = (x_1(t)\ x_2(t))$ with zero unconditional mean:

$$x_1(t) = r(t) - \frac{\gamma_1(1 - \phi_{22}) + \gamma_2\phi_{12}}{(1 - \phi_{11})(1 - \phi_{22}) - \phi_{12}\phi_{21}},$$

$$x_2(t) = \pi(t) - \frac{\gamma_2(1 - \phi_{11}) + \gamma_1\phi_{21}}{(1 - \phi_{11})(1 - \phi_{22}) - \phi_{12}\phi_{21}}.$$

With this state-variable transformation, the state-space model becomes

$$x_t = \phi x_{t-1} + v_t, \tag{24a}$$

$$y_t = \alpha x_t + \beta + e_t, \tag{24b}$$

with $\binom{v_t}{e_t}$ distributed normal with covariance matrix $\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$, and where β is an $(M + N)$-dimensional vector of constants and, without loss of generality, δ in the state-transition equation [(20)] has been set to 1.

Our goal is to estimate the parameters of the instantaneous real interest rate and expected inflation dynamics given by Equation (10), and test the validity of the hypothesized bond-pricing formula [Equation (18)]. Specifically, we wish to estimate the elements of the matrix B and the elements of the covariance matrix Σ. Note that in the state-space system (24), both 4 and a are uniquely determined by the parameters b_{11}, b_{12}, b_{21}, and b_{22} (i.e., the matrix B). In addition, the covariance matrix Q is uniquely determined by the covariance matrix Σ and the matrix B. Therefore, we propose to fit the model (24) subject to the cross-equation restrictions on the parameters of ϕ, α, and Q, and thus obtain estimates of B and Σ.

A general problem in estimating multiple-factor (state-variable) models of asset pricing is that processes followed by the individual unobserved factors cannot be identified. For example, this lack of identification occurs in empirical work based on the Arbitrage Pricing Theory of Ross (1976) and in empirical work on bond pricing, such as Oldfield and Rogalski (1987) and Stambaugh (1988). Empirical articles, such as Gibbons and Ramaswamy (1986) and Heston (1988)) are able to identify the process followed by the real interest rate by assuming that it is independent of the inflation process. In the current article, the problem of identification is solved without resorting to this independence assumption. Instead, identification is achieved by utilizing data on survey inflation forecasts as well as bond prices. The combination of these two types of data is necessary to identify uniquely the individual states $r(t)$ and $\pi(t)$ of the model. If only data on bond prices or only data on inflation forecasts are used, there exist orthogonal transformations of the states and the estimated parameters that fit the data equally well.[18]

Following the work of Engle and Watson (1981, 1983), we now describe a maximum-likelihood technique to estimate the parameters of the system in (24). Define the vector of "innovations," η_t, as

$$\begin{aligned}\eta_t &= y_t - \hat{y}_t \\ &= y_t - \alpha\Phi E_{t-1}[x_{t-1} \mid y_{t-1}, y_{t-2}, \ldots, y_1] - \beta. \end{aligned} \tag{25}$$

Let $\hat{x}_{t-1} \equiv E_{t-1}[x_{t-1} \mid y_{t-1}, y_{t-2}, \ldots, y_1]$ be the optimal forecast of the unobservable state vector given the current and past values of the measurement equation observations. Also, let H_t denote the time t covariance matrix of the innovations vector, η_t. Both $\hat{x}_{t-1}$ and H_t can be computed from a Kalman filter recursion, given starting estimates for $\hat{x}_0$ and the covariance of x_0, denoted G_0, and given values of the parameters Φ, α, β, Q, and R (or equivalently given values for the parameters B, Σ, β, and R). $\hat{x}_t$ and H_t are computed as follows. Define $\hat{G}_t \equiv \Phi G_{t-1}\Phi' + Q$. Then, we have

$$\hat{x}_t = \Phi\hat{x}_{t-1} + \hat{G}_t\alpha' H_t^{-1}(y_t - \alpha\Phi\hat{x}_{t-1} - \beta), \tag{26}$$

$$G_t = \hat{G}_t - \hat{G}_t\alpha' H_t^{-1}\alpha\hat{G}_t, \tag{27}$$

$$H_t = \alpha\hat{G}_t\alpha' + R. \tag{28}$$

Once η_t and H_t are computed using the Kalman filter, we can calculate the log-likelihood of the observations, y_t. Schweppe (1965) has shown that the log-likelihood can be written in terms of the innovations as

$$L = \sum_t L_t = \sum_t -\frac{1}{2}(\log|H_t| + \eta_t' H_t^{-1}\eta_t). \tag{29}$$

[18] The proof is available from the author upon request.

The remaining step is to find, by using a numerical iteration technique, those parameter values B, Σ, β, and R that maximize the above function.

4. Data

A monthly time series of Treasury-bill prices was obtained from the Center for Research in Security Prices (CRSP) tape. Similar to Gibbons and Ramaswamy (1986), we constructed time series of (hypothetical) 30-, 90-, 180-, and 345-day Treasury-bill prices by linearly interpolating the yields derived from the average of bid and ask prices of Treasury bills whose maturities immediately surround the desired maturity. This time series of Treasury-bill prices was created to coincide with the period over which survey inflation forecasts were available, 1968-1988.

Since 1968, the National Bureau of Economic Research (NBER) and the American Statistical Association (ASA) have made quarterly surveys of professional forecasters' predictions of the GNP price deflator. A recent article by Keane and Runkle (1990) provides an in-depth analysis of the characteristics of these forecasts. Unlike other studies that have analyzed survey forecasts, Keane and Runkle find that they are unable to reject the hypothesis that the NBER-ASA price level forecasts are unbiased and rational in the sense that publicly available information cannot be used to improve the forecasts. They attribute their differing results to their more careful consideration of the correlation between individual forecasters' prediction errors, to their use of unrevised rather than revised GNP price-deflator data, and to their more detailed consideration of exactly what information was available to the forecasters. In addition, Keane and Runkle emphasize that, unlike other surveys, the NBER-ASA survey is limited to professional forecasters. Being professionals, these forecasters have an economic incentive to utilize all publicly available information.[19] Since the model of our article is that of a "representative" individual, we chose the median forecast of the NBER-ASA survey respondents as our (noisy) measure of the "market's" forecast of the price level.[20]

[19] The Livingston survey collects the predictions of economists. whereas the University of Michigan survey polls consumers. Keane and Runkle believe that, relative to professional forecasters. these two groups lack the incentive to compile accurate foreasts, which could be one reason why previous research based on these surveys has often rejected the hypothesis that these forecasts rationally utilized all available information.

[20] Recall that the estimation Equation (22) allows for Independent measurement errors. Our empirical technique is also valid even if the median survey forecast possesses a constant bias. The median, rather than the mean, forecast was chosen to reducc the influence of possibly extreme individual forecasts that could result from inaccurate data entry or the existence of a few uninformed forecasters.

The NBER-ASA survey reports forecasts of the level of the GNP price deflator at five future quarterly dates. At the time that forecasts are submitted by the respondents (the end of May, August, November, and February of each year), these quarterly dates are 1, 4, 7, 10, and 13 months into the future. Note that in order to construct inflation forecasts of the form $E_t[P(t+T)/P(t)]$, we would need to know what the forecasters believe is the current level of the GNP deflator, $P(t)$, when they make their forecasts. This poses a problem because the GNP deflator is only reported on a quarterly basis, and then only with a lag. Hence, the GNP deflator, $P(t)$, is not truly observed at the time of these forecast submissions.

This problem can be resolved if we modify the way in which inflation forecasts are constructed, so that the forecasted price change is a price change that begins one month from the time the forecasts are submitted. In other words, if t is measured in months, we can construct price forecast ratios (inflation forecasts) of the form $E_t[P(t+T)/P(t+1)]$, where $T = 4, 7, 10$, or 13. Using Equation (19), we can see that these inflation forecasts take the form

$$\begin{aligned}\ln(E_t[P(t+T)/P(t+1)]) \\ = K_2(T) - K_2(1) + (\alpha_3(T) - \alpha_3(1))r(t) \\ + (\alpha_4(T) - \alpha_4(1))\pi(t).\end{aligned} \tag{30}$$

As with Equation (19), the right-hand side of (30) is linear in the state variables, with the coefficients ‘multiplying these state variables being functions only of T (equal to 4, 7, 10, or 13) and the elements of the matrix B. Therefore, Equation (30) will be a suitable measurement equation that we can use in place of (19), and this is done in the empirical work that follows.

The estimation technique described in the previous section assumed that bond prices and inflation forecasts were sampled at the same frequency. However, since the NBER-ASA survey forecasts are available at quarterly intervals, while bond prices from CRSP are available on a more frequently monthly basis, our estimation technique must be modified in order to take full advantage of all the available data. In Appendix B, we outline how the state-space model can be reformulated and estimated when the observation interval for inflation forecasts differs from that of the bond prices.

Table 1 shows summary statistics of the CRSP Treasury-bill prices and the NBER-ASA median inflation forecasts for the period 1968-1988. Part A of the table shows that the term structure of interest rates was upward sloping, on average, with the average difference between the yields on 30-day and 345-day bills being just over 100 basis points. The average predicted annual inflation rate was approximately the same (about 5.1%), whether the predictions were made over 3-, 6-,

Table 1
Summary statistics of Treasury-bill yields (243 monthly observations) and NBER–ASA median predicted inflation rates (81 quarterly observations), 1968.10–1988.12

A: Sample means of annualized, continuously compounded Treasury-bill yields and median predicted inflation forecasts

Bond maturity	Yield	Prediction interval	Inflation rate
30-day	.0703	3 month	.0514
90-day	.0758	6 month	.0515
180-day	.0788	9 month	.0512
345-day	.0808	12 month	.0512

B: Annualized sample standard deviations of first differences of log bond prices and log median inflation forecasts

Bond maturity	Monthly first differences	Prediction interval	Quarterly first differences
30-day	.00269	3 month	.00261
90-day	.00647	6 month	.00522
180-day	.01256	9 month	.00669
345-day	.02325	12 month	.00838

C: Correlation coefficients of quarterly first differences of log bond prices (BP) and log median inflation forecasts (IF)

	30-day BP	90-day BP	180-day BP	345-day BP	3-month IF	6-month IF	9-month IF
90-day BP	.945						
180-day BP	.869	.972					
345-day BP	.825	.940	.989				
3-month IF	−.124	−.109	−.113	−.095			
6-month IF	−.108	−.118	−.118	−.099	.824		
9-month IF	−.140	−.118	−.105	−.082	.824	.922	
12-month IF	−.178	−.159	−.170	−.158	.791	.874	.918

9, or 12-month intervals (T = 4, 7, 10, or 13, respectively). As one would expect, part B of Table 1 shows that bond volatility (as proxied by the sample standard deviation of monthly first differences of log bond prices) increases with the bond's maturity. The sample standard deviation of the change in log inflation forecasts is also increasing with the prediction interval.[21] Part C shows evidence of high correlation between changes in log bond prices of different maturities and also high correlation between log inflation predictions made over different (partially overlapping) time intervals. However, there is a relatively small and negative correlation between log bond price changes and log-predicted inflation forecast changes.

[21] Being sample standard deviations, a constant mean is used to compute the residual from each first difference. Of course, our theoretical model implies this mean is not constant but changing with the level of the unobserved state variables. Hence, the statistics in Table 1 are meant only to be suggestive.

5. Estimation Results

This section reports maximum likelihood estimates for the parameters B, Σ, β, and R. First, the model was estimated using the monthly Treasury-bill prices along with the quarterly NBER-ASA inflation forecasts over the period 1968-1988. The elements of the observation vector, y_t, were ordered to equal the log of 30-, 90-, 180-, and 345-day bond prices, and 3-, 6-, 9-, and 12-month inflation forecasts, respectively. In order to simplify the estimation process, some further assumptions were made. First, the covariance matrix R was restricted to be diagonal with elements r_i^2, where the standard deviations of the measurement errors were assumed to be the same for each of the log bond prices ($=r_1$), and the same for each of the log inflation forecasts ($=r_2$). The purpose of this assumption was to reduce the number of parameters to be estimated to a more reasonable level. Second, the vector of constants, β, in the measurement equations (24b) were estimated without imposing potential cross-equation restrictions implied by the theoretical model. It can be shown that the elements of the vector β are functions of the elements A, B, Σ, and x, as well as of σ_p, $\rho_{kp}\sigma_k$, and the parameters that would indicate a (possible) constant bias between survey and "market" forecasts of inflation. While it would certainly be of interest to recover estimates of parameters other than B, Σ, and R, attempts to estimate the system by imposing these additional cross-equation restrictions on β proved to be unsuccessful.[22] The results when all other cross-equation restrictions were imposed is given in the first column of Table 2.

The parameter estimates of the covariance matrix, Σ, indicate significant negative correlation ($\rho_{r\pi} = -0.376$) between the innovations of the instantaneous real rate and expected inflation.[23] In addition, the standard deviation of the instantaneous real rate, σ_r, is more than twice as large as that of expected inflation, σ_π. This result, that the instantaneous real interest rate is more volatile than expected inflation, seems surprising given that a number of earlier studies, such as Fama (1975), Mishkin (1981), and Fama and Gibbons (1982), conclude that changes in inflation play a dominant role in interest-rate movements.

Turning to the estimates of the elements of matrix B (i.e., b_{ij}, $i, j = 1, 2$), it can be shown that these point estimates imply negative eigenvalues, consistent with the state variables following a stationary

[22] Several different numerical algorithms were attempted, but each was unsuccessful in converging to a maximum value for the likelihood function. Apparently, the likelihood function is rather "flat" along the dimension of these additional parameters.

[23] This is consistent with the results of other research on interest rates, such as Summers (1983) and Mishkin (1987).

Table 2
Maximum-likelihood estimates of state-space model parameters using observations on monthly 30-, 90-, 180-, and 345-day Treasury-bill prices and quarterly 3-, 6-, 9-, and 12-month NBER–ASA inflation forecasts

	Sample periods				
Parameters	1968.10–1988.12	1968.10–1978.12	1978.12–1988.12	1979.10–1982.9	1982.10–1988.12
b_{11}	−.14079	−.01987	−.00873	−.45011	.02835
	(.03215)	(.08843)	(.05091)	(.19846)	(.12257)
b_{12}	.07378	.39653	−.03511	−1.5247	.47953
	(.04377)	(.12609)	(.04648)	(.36978)	(.16099)
b_{21}	.10991	−.18498	.09455	.02977	.13379
	(.02548)	(.07754)	(.03521)	(.16592)	(.12146)
b_{22}	−.25347	−.40834	−.32113	−.29206	−.26146
	(.03824)	(.11117)	(.03201)	(.38985)	(.15555)
σ_r	.02665	.01934	.03009	.06610	.01474
	(.00117)	(.00217)	(.00176)	(.01084)	(.00282)
σ_π	.01060	.01128	.01050	.03135	.00827
	(.00083)	(.00136)	(.00132)	(.01016)	(.00310)
$\rho_{r\pi}$	−.37567	−.35362	−.26701	−.42703	−.43453
	(.11800)	(.17683)	(.15701)	(.29255)	(.37567)
β_1	−.00245	−.00546	−.00311	−.00563	−.00287
	(.00200)	(.00169)	(.00511)	(.01845)	(.00214)
β_2	−.00875	−.01720	−.01122	−.02019	−.01022
	(.00589)	(.00477)	(.01535)	(.05546)	(.00643)
β_3	−.01915	−.03583	−.02402	−.04477	−.02146
	(.01162)	(.00923)	(.03049)	(.11080)	(.01335)
β_4	−.03927	−.07013	−.04864	−.09463	−.04282
	(.02180)	(.01670)	(.05809)	(.21158)	(.02652)
β_5	.00969	.01234	.00959	.01413	.00660
	(.00294)	(.00329)	(.00471)	(.01916)	(.00362)
β_6	.01942	.02434	.01961	.02856	.01379
	(.00580)	(.00653)	(.00935)	(.03678)	(.00711)
β_7	.02899	.03601	.02958	.04237	.02137
	(.00856)	(.00969)	(.01386)	(.05291)	(.01045)
β_8	.03861	.04725	.04010	.05722	.02925
	(.01120)	(.01268)	(.01819)	(.06747)	(.01354)
r_1	.00094	.00075	.00099	.00080	.00063
	(.00001)	(.00002)	(.00002)	(.00011)	(.00003)
r_2	.00084	.00093	.00056	.00058	.00052
	(.00003)	(.00006)	(.00004)	(.00014)	(.00006)
Half-life (in years) of an innovation in:					
Real interest rate	5.27	3.86	37.55	2.92	—[1]
Expected inflation	1.85	2.01	1.62	1.94	—[1]
Likelihood ratio test statistic of H_0: $b_{12} = b_{21} = \rho_{r\pi} = 0$					
	19.92[2]	23.90[2]	8.35[3]	52.58[2]	8.97[3]

The b_{ij} are the elements of matrix B of the state-transition equation, $ds = (A + Bs(t))dt + \sigma\, dZ$, where the 2 × 1 vector s is composed of the instantaneous real Interest rate and expected inflation. σ_r^2 and σ_π^2 are diagonal elements, whereas $\rho_{r\pi}\sigma_r\sigma_\pi$ is the off-diagonal element of Σ, where $\Sigma\, dt = \sigma\, dZ(\sigma\, dZ)'$. The β_i's are the intercept terms in the measurement equation $y_t = \beta + \alpha x_t + e_t$, where y_t is the 8 × 1 vector of log bond prices and log inflation forecasts, and x_t is the deviation of $s(t)$ from Its unconditional mean. The error term, e_t, has a diagonal covariance matrix, with r_1 and r_2 equaling the standard deviations for the log bond prices and the log inflation forecasts, respectively. Standard errors are In parentheses.

[1]Point estimates of B Imply a positive eigenvalue, making the system nonstationary.

[2]Rejection at the 1% level, $x^2(3)|1\% = 11.34$.

[3]Rejection at the 5% level, $x^2(3)|5\% = 7.81$.

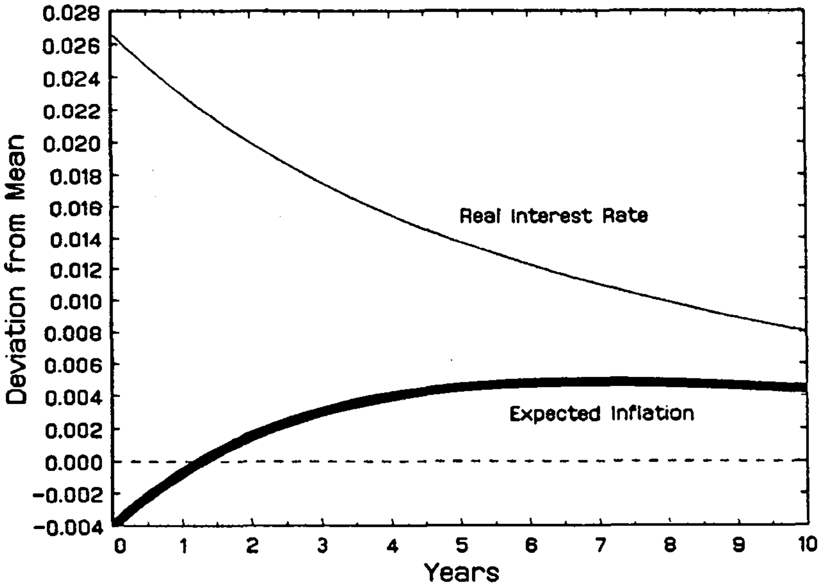

Figure 1
Expected paths of the real interest rate and the rate of inflation following a one standard deviation innovation in the real interest rate, based on model estimates obtained from data over the period 1968.10-1988.12
This Impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$, where x is a vector of the deviations of the real Interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix B are from column 1 of Table 2.

process. These estimates are generally significantly different from zero. The positive off-diagonal elements imply that the expected change in the real rate is positively related to the level of expected inflation; whereas the higher the current real interest rate, the greater is the expected increase in inflation. A likelihood ratio test of the hypothesis of zero overall correlation between real rates and inflation, $b_{12} = b_{21} = \rho_{r\pi} = 0$, results in rejection at a 1 percent significance level.

Based on these parameter estimates, we can describe the dynamics of the instantaneous real interest rate and expected inflation in terms of their response to an innovation [i.e., a realization of dZ in Equation (10)]. Figure 1 shows the expected movement in $r(t)$ and $\pi(t)$ from one standard deviation Innovation in $r(t)$, equaling $\sigma_r = 0.02665$.[24] The real interest rate displays rather weak mean reversion. Its "half-life" (i.e., the time it takes to return one-half way back to its initial value) is 5.27 years. In addition, the correlation between the real rate

[24] Given the nonzero correlation between the innovations in the real rate and expected inflation, a rise of σ_r in the real rate would be expected to produce a change of $\rho_{r\pi}\sigma_\pi$ in the level of expected inflation. For the current set of parameter estimates, this Implies a fall In expected inflation equal to 0.0040.

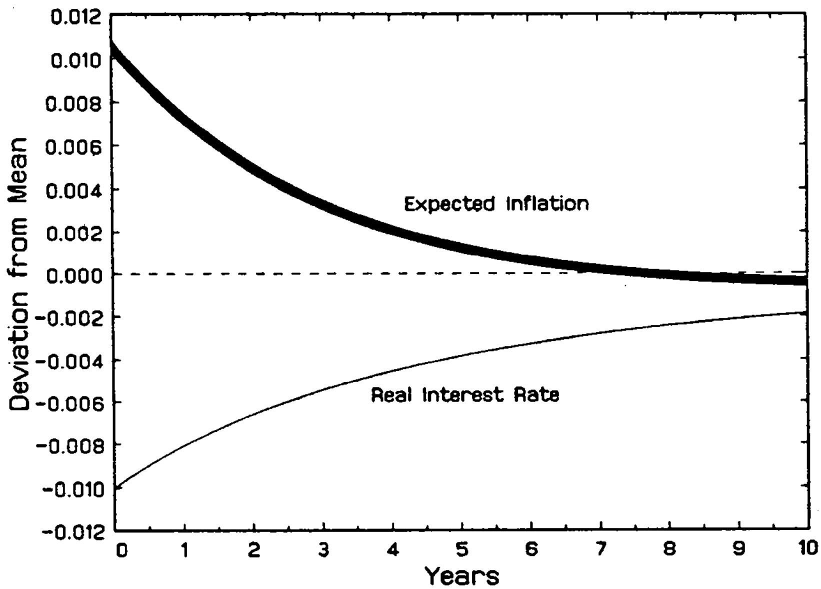

Figure 2
Expected paths of the real interest rate and the rate of inflation following a one standard deviation innovation in the expected inflation rate, based on model estimates using data over the period 1968.10-1988.12
This impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$, where x is a vector of the deviations of the real interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix Bare from column 1 of Table 2.

and inflation indicates that the rate of inflation is expected to "overshoot" its long-run value, first falling below its long-run steady state, then moving above it, before finally falling again.

In a similar manner, Figure 2 shows the expected movement in $r(t)$ and $\pi(t)$ from a one standard deviation movement in r(t). Relative to the real interest rate, the rate of inflation displays stronger mean reversion. Its half-life is 1.85 years.

The estimates of r_1 and r_2, the standard deviations of the "measurement" errors of the log Treasury-bill prices and the log inflation forecasts, respectively, give one indication of how well the model fits the data. Doubling the estimate for r_1 implies that the measurement error on the six-month (180-day) Treasury-bill yield has a standard deviation under 18 basis points. The standard deviation of the difference between the "market" and the NBER-ASA six-month inflation forecast is approximately2 $\times$ 0.00084 = 16.8 basis points. This appears to be a reasonably small level of measurement error.

As an indication of parameter stability, the sample was split in two and the estimation was repeated for each of the two subperiods. The

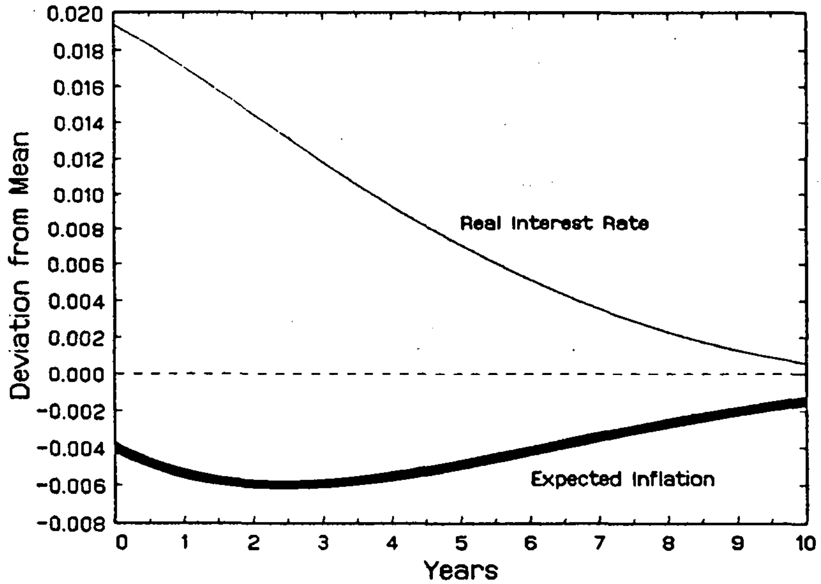

Figure 3
Expected paths of the real interest rate and the rate of inflation following a one standard deviation innovation in the real interest rate, based on model estimates using data over the period 1968.10–1978.12

This impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$ where x is a vector of the deviations of the real interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix B are from column 2 of Table 2.

second column of Table 2 shows the parameter estimates over the 1968-1978 period, while the third column of Table 2 shows estimates for the 1978-1988 period. The estimates of the parameters of the covariance matrix, 2, are fairly similar for the two periods. The estimated standard deviations for expected inflation are very close, 0.0113 and 0.0105. However, the later period shows somewhat greater volatility in the real interest rate, its standard deviation being approximately 50% greater (a, = 0.03 vs. 0.019). The instantaneous correlation between real rates and inflation continues to be negative for the two periods ($\rho_{r\pi} = -0.35, -0.27$).

The major difference between the two subperiods in their implication for the strength of mean reversion is the real rate of interest. As displayed in Figure 3, estimates using only the 1968-1978 data indicate a half-life for the real interest rate of 3.86 years. In contrast, as Figure 5 shows, the real rate's estimated half-life is 37.55 years for the 1978-1988 period. However, there is not a substantial difference in the rate of inflation's mean reversion, as Figures 4 and 6 show. The half-life of inflation is 2.01 years for the 1968-1978 period and 1.62 years for the 1978-1988 period. In addition, in both cases, one can

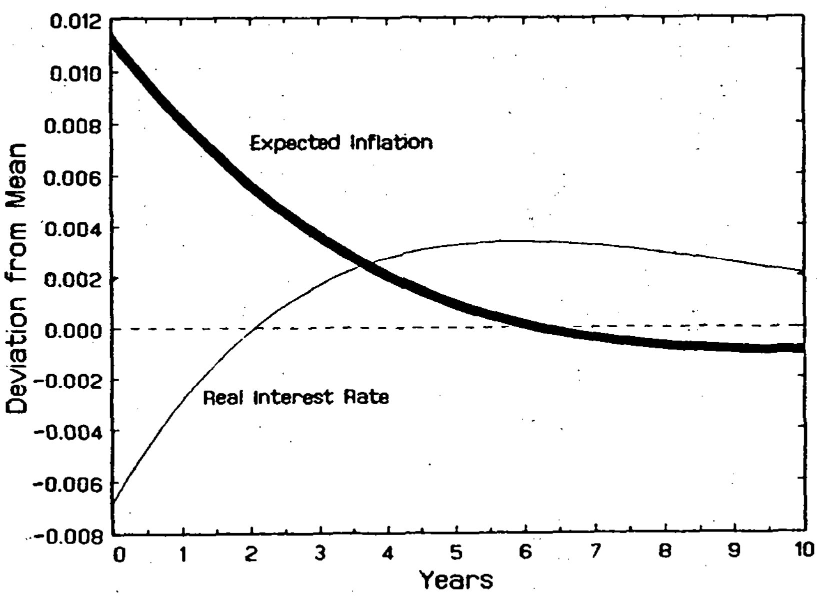

Figure 4
Expected paths of the real interest rate and the rate of inflation following a one standard deviation innovation in the expected inflation rate, based on model estimates using data over the period 1968.10–1978.12
This impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$, where *x* is a vector of the deviations of the real interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix *B* are from column 2 of Table 2.

reject the hypothesis of zero correlation ($b_{12} = b_{21} = \rho_{r\pi} = 0$) between the processes for real rates and inflation, although the rejection is somewhat stronger in the earlier subperiod. All things considered, the qualitative characteristics of the parameter estimates for the two subperiods are similar.

In Figure 7 we graph "smoothed" estimates of the unobserved state variables, *r(t)* and $\pi(t)$, based on the Treasury-bill prices and NBER-ASA forecasts over the full sample period, 1968-1988. A smoothed estimate of a state variable is an optimal estimate based on the full sample of observations, observations that occur both before and after the particular state variable's realization.[25] Consistent with the estimates obtained in Table 2, the real interest rate shows evidence of being more volatile than the expected rate of inflation. In addition, it is noteworthy that our estimate of the instantaneous real interest rate is negative over a substantial part of the 1970s and part of 1980.

[25] In contrast, a filtered estimate of a state variable uses only those observations that precede the realization of the particular state variable. See Anderson and Moore (1979) or Harvey (1981) for a discussion of smoothing.

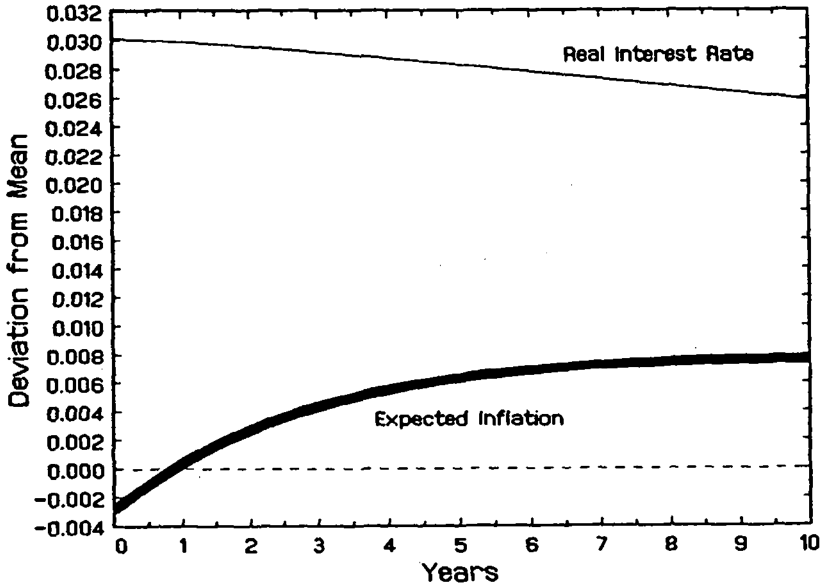

Figure 5
Expected paths of the real interest rate and the rate of inflation following a one standard deviation innovation in the real interest rate, based on model estimates using data over the period 1978.12-1988.12
This impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$, where x is a vector of the deviations of the ml Interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix B are from column 3 of Table 2.

This indicates that use of a bond-pricing model that does not restrict the real rate to be positive can be an important consideration.

As mentioned in Section 2, the joint process for real interest rates and inflation assumed in Equation (10) leads to a simple and convenient way of parameterizing instantaneous and longer-term dependence between these two variables. The potential disadvantage of this process is that it implies conditionally homoskedastic changes in log bond prices and log inflation forecasts. By looking at the one-month-ahead model prediction errors [i.e., the η_t's given by Equation (25)], we can obtain a rough indication of the validity of this assumption. Under the model's assumptions, the vector of one-month prediction errors should have zero mean and a steady-state variance that is constant. In other words, given an infinitely long past history of observed log bond prices and survey inflation forecasts, the next period's realization of log bond prices and inflation forecasts should be homoskedastic. However, since we have only a finite history of observables, we should not expect strict homoskedasticity, but a prediction error variance that asymptotes to the steady-state level.

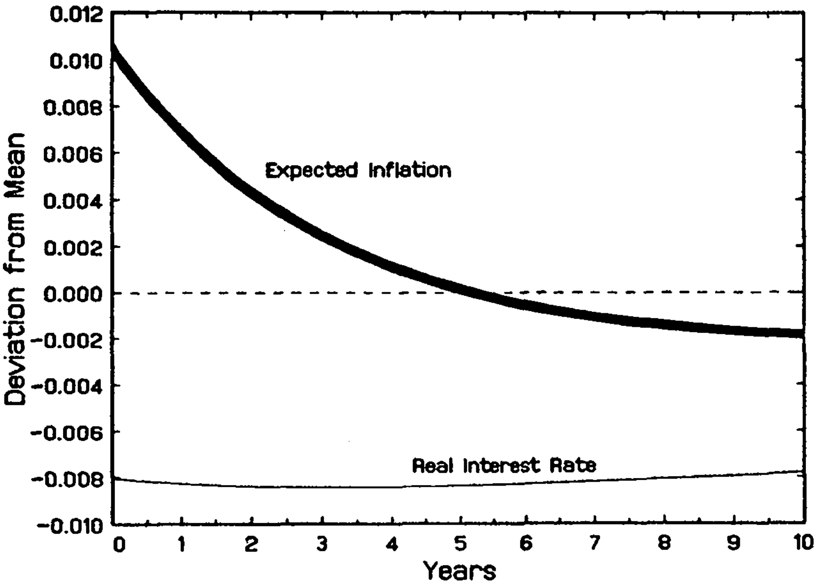

Figure 6
Expected paths of the real interest rate and the rate of inflation following one standard deviation innovation in the expected rate of inflation, based on model estimates using data over the period 1978.12-1988.12
This impulse-response function equals $E_0[x(t)] = x(0)e^{Bt}$, where x is a vector of the deviations of the real interest rate and the expected rate of inflation from their unconditional means. Point estimates for the elements of matrix B are From column 3 of Table 2.

One-month-ahead prediction errors for the log of the 345-day Treasury bill and the log of the one-year-survey inflation forecast are plotted in Figures 8 and 9. There are clear indications of heteroskedasticity in Figure 8. The nine largest (in terms of absolute value) one-month-ahead forecast errors for the log of the 345-day Treasury-bill price occur during 1979.10-1982.9, the period during which the Federal Reserve announced a change in its monetary-policy operating procedures.[26] In comparison, the largest one-month-ahead prediction errors for the log of the one-year inflation forecast do not seem to be concentrated in any one subperiod.

Given the apparent increase in the volatility of Treasury-bill forecast errors during 1979.10-1982.9, this period might be considered a different monetary-policy regime relative to the periods before and after this interval. Research by Hamilton (1988) supports this view, he uses an Expectations Hypothesis model of the term structure in which the process followed by the short-term interest rate (i.e., the interest rate

[26] From October 1979 to October 1981, the stated policy of the Federal Reserve was one of targeting nonborrowed bank reserves, rather than targeting the federal funds rate.

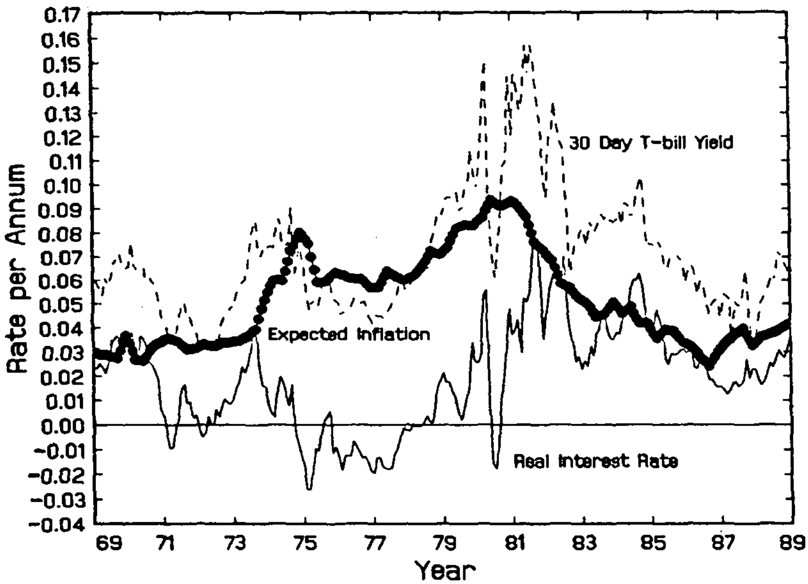

Figure 7
Smoothed estimates of the paths of the instantaneous real interest rate and the instantaneous expected rate of inflation over the period 1969-1989
These smoothed estimates were obtained using observations on monthly 30-, 90-, 180-, and 345-day Treasury-bill prices and quarterly 3-, 6-, 9-, and 12-month NBER-ASA median predicted inflation rates over the period 1968.10-1988.12. Smoothed estimates ate maximum-likelihood estimates of the state-space model's unobserved state variables based on the full sample of observations.

regime) can switch from a low- to a high-volatility process. Hamilton's estimation results indicate a high probability of a regime switch precisely during the October 1979-October 1982 period.[27]

While the theoretical framework of the present article cannot account for a positive probability of a parametric shift of the joint real interest-rate-inflation process, it may be informative to investigate the type of change in parameter estimates over the three-year period 1979.10-1982.91.[28] Although based on only 36 monthly observations, column four of Table 2 shows estimates over this period.

Not surprisingly, the estimates for the standard deviations of the instantaneous real interest rate and expected inflation are significantly higher than during any other sample period; however, the relative magnitudes are approximately the same, with $\sigma r = 0.066$ being somewhat more than twice as high as $\sigma_{\pi} = 0.031$. Interestingly, the instan-

[27] Hamilton finds that heteroskedasticity characterized by regime shifts is far more important in explaining interest rate data than heteroskedasticity of a slowly changing nature, such as embodied In ARCH models.

[28] Our implicit assumption in estimating the model over a particular subperiod is that individuals believe during this subperiod that there is a zero probability of another regime shift ever occurring.

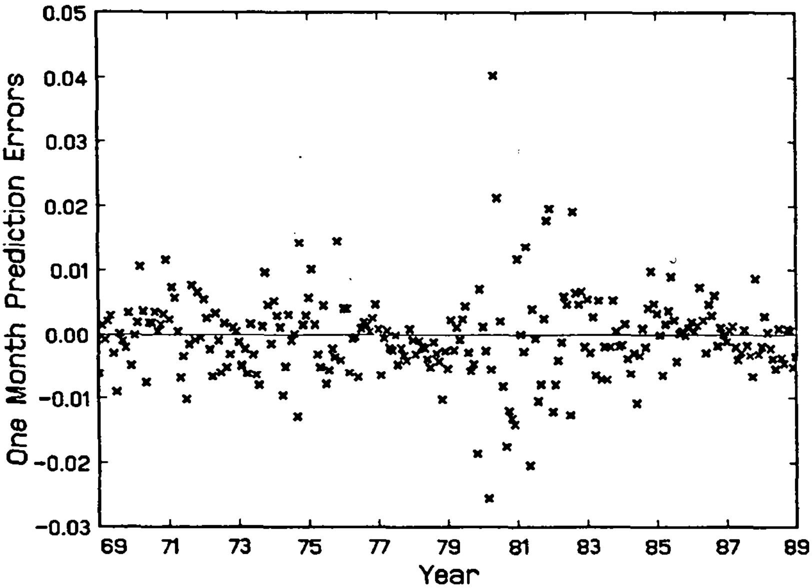

Figure 8
Errors from one-month-ahead Kalman filter predictions of the log of the 345-day Treasury-bill price
These predictions are based on prior observations of monthly 30-, 90-, 180-, and 345-day Treasury-bill prices and quarterly 3-, 6-, 9-, and 12-month NBER-ASA median predicted inflation rates. Defining the log of these observations at month t as the vector y_t, the residual vector $\eta_t = y_t - E_{t-1}[y_t]$ is computed. The residuals that are plotted correspond to those of the log of the 345-day Treasury bill.

taneous correlation between real rates and expected inflation, $\rho_{r\pi}$ = -0.427, is relatively close to the estimated correlation during the earlier 1968.10-1978.12 period. In addition, the implied half-lives for the real interest rate and expected inflation are close to those estimated for the earlier period. The real interest rate continues to display weaker mean reversion than expected inflation. Finally, consistent with the earlier period, the hypothesis of independence between real interest rates and expected inflation is rejected at the 1 percent significance level. In summary, with the exception of higher estimated instantaneous standard deviations, the qualitative behavior of real interest rates and expected inflation is surprisingly similar to the earlier 1968.10-1978.12 period.

To see what effect the 1979.10-1982.9 subperiod had on the estimates for the second half of the entire sample period, the model was reestimated using only the six years of data from 1982.10-1988.12. This is reported in column 5 of Table 2. While the estimated standard deviation for the instantaneous real interest rate was still higher than that for expected inflation, both of their magnitudes were lower

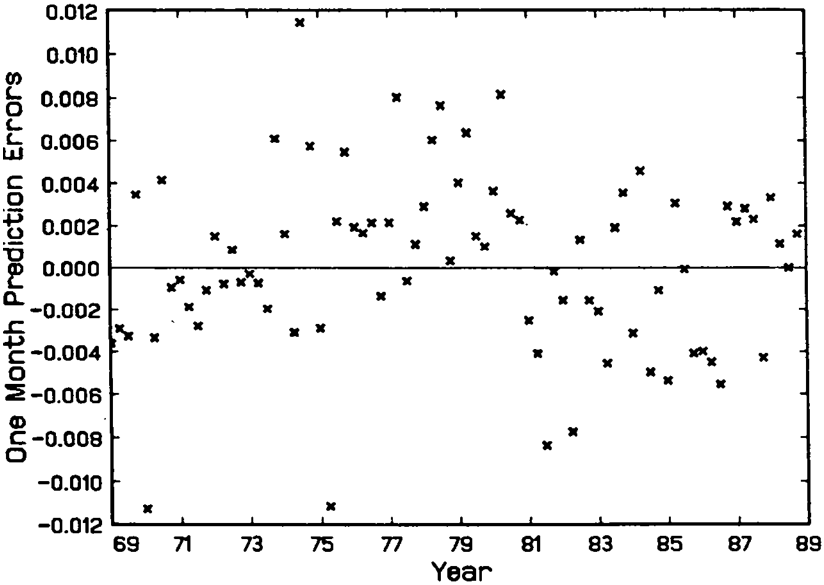

Figure 9
Errors from one-month-ahead Kalman filter prediction of the log of the one-year NBER-ASA median inflation forecast
These predictions are based on prior observations of monthly 30-, 90-, 180-, and 345-day Treasury-bill prices and quarterly 3-, 6-, 9-, and 12-month NBER-ASA median predicted inflation rates. Defining the log of these observations at month t as the vector y_t, the residual vector $n_t = y_t - E_{t-1}[y_t]$ is computed. The residuals that are plotted correspond to those of the 12-month NBER-ASA median inflation forecast.

than during any earlier subperiod ($\sigma_r = 0.015$, $\sigma_\pi = 0.008$). However, the estimated instantaneous correlation was quite similar to estimates obtained for the earlier periods. Also similar to other periods, a likelihood ratio test of the hypothesis $b_{12} = b_{21} = \rho_{r\pi} = 0$ leads to its rejection.

Perhaps because of the relatively short 1982.10-1988.12 sample period, the point estimates of the elements of the matrix B imply a nonstationary process for real interest rates and inflation because one eigenvalue turns out to be positive. Hence, we cannot compute half-lives for $r(t)$ or $\pi(t)$ from these estimates. However, note that with the exception of the element b_{21}, these point estimates do not appear to be exceedingly different from those obtained during the earlier 1968.10-1978.12 period. Thus, the evidence of potential nonstationarity during this subperiod may be weak.

6. Conclusion

An equilibrium bond-pricing model has been formulated and estimated that explicitly accounts for the difference between real and nominal variables. An important feature of the model is that it allows for interdependence between real interest rates and expected inflation, which can be justified on both theoretical and empirical grounds. An estimation technique has been presented that uses NBER-ASA survey inflation forecasts, as well as Treasury-bill prices, to show how these mutually dependent processes can be identified.

The empirical results confirm that real interest rates and inflation follow jointly dependent processes. The instantaneous real interest rate and the instantaneous rate of expected inflation are found to be significantly negatively correlated. In the context of our model, a positive technological shock that increases the expected rate of return on capital, and hence the equilibrium real interest rate, is coincident with a partial fall in expected inflation. This empirical finding could be consistent with a government policy in which money growth rates are not fully accommodating to changes in real returns.

The paper's estimates also indicate that the real interest rate is more volatile than the rate of expected inflation. That expected inflation rates are found to be "sticky' compared to real returns is not an uncommon characteristic of nominal asset prices. For example, fluctuations in real exchange rates, rather than changes in relative inflation rates, dominate nominal foreign exchange rate movements. Mussa (1986) finds this to be true especially during the recent floating exchange rate regime, which covers most of this article's sample period.

Regarding the longer-run dynamics of real interest rates and inflation, we find that the real interest rate displays slower mean reversion than the instantaneous rate of inflation. Again, in the context of the model, where the equilibrium real interest rate possesses the same dynamics as the expected rate of return on (risky) capital, the model would predict weak mean reversion in the expected rate of return on stocks. This prediction is, in fact, validated by Fama and French's (1988) recent empirical analysis of stock returns.

Appendix A

The solution to the consumers' consumption and portfolio choice problem, given by Equations (6) and (8) in Section 1, is outlined here.

Let w_b, w_r, and $w_k (= 1 - w_b - w_r)$ be the proportions of a representative individual's portfolio invested in nominal bonds, real (indexed) bonds, and real capital, respectively. Denoting his total wealth as W, and $J(W, \alpha_k, \pi)$ as the indirect utility function, Merton (1971) shows that the optimality equation is

$$0 = \max_{C, w_b, w_r} \ln(C) + J_W\{W[(1 - w_b - w_r)\alpha_k$$
$$+ (i - \pi + \sigma_p^2)w_b + rw_r] - C\}$$
$$+ \tfrac{1}{2}J_{WW}W^2[(1 - w_b - w_r)^2\sigma_k^2 + w_b^2\sigma_p^2$$
$$- 2\rho_{kp}(1 - w_b - w_r)w_b\sigma_k\sigma_p] - \zeta J$$
$$+ J_{\alpha_k}(a_1 + b_{11}\alpha_k + b_{12}\pi) + J_\pi(a_2 + b_{21}\alpha_k + b_{22}\pi)$$
$$+ \tfrac{1}{2}J_{\alpha_k\alpha_k}\sigma_\alpha^2 + \tfrac{1}{2}J_{\pi\pi}\sigma_\pi^2 + J_{\alpha_k\pi}\rho_{\alpha_k\pi}\sigma_\alpha\sigma_\pi$$
$$+ J_{W\alpha_k}W(\rho_{\alpha_k k}(1 - w_b - w_r)\sigma_k\sigma_\alpha - \rho_{\alpha_k p}w_b\sigma_\alpha\sigma_p)$$
$$+ J_{W\pi}W(\rho_{\pi k}(1 - w_b - w_r)\sigma_k\sigma_\pi - \rho_{\pi p}w_b\sigma_\pi\sigma_p). \tag{A1}$$

Differentiating with respect to C, w_b, and w_r, one obtains

$$C = 1/J_W, \tag{A2}$$

$$J_WW(-\alpha_k + i - \pi + \sigma_p^2) + J_{WW}W^2(-(1 - w_b - w_r)\sigma_k^2 + w_b\sigma_p^2$$
$$- \rho_{kp}(1 - 2w_b - w_r)\sigma_k\sigma_p) - J_{W\alpha_k}W(\rho_{\alpha_k k}\sigma_k\sigma_\alpha + \rho_{\alpha_k p}\sigma_\alpha\sigma_p)$$
$$- J_{W\pi}W(\rho_{\pi k}\sigma_k\sigma_\pi + \rho_{\pi p}\sigma_\pi\sigma_p) = 0, \tag{A3}$$

$$J_WW(-\alpha_k + r) + J_{WW}W^2(-(1 - w_b - w_r)\sigma_k^2 + \rho_{kp}w_b\sigma_k\sigma_p)$$
$$- J_{W\alpha_k}W\rho_{\alpha_k k}\sigma_k\sigma_\alpha - J_{W\pi}W\rho_{\pi k}\sigma_k\sigma_\pi = 0 \tag{A4}$$

Using Equations (A3) and (A4), we can solve for the optimal portfolio proportions, w_b^* and w_r^*. Substituting these proportions back into Equation (Al), we then have a partial differential equation for $J(W, \alpha_k, \pi)$. Merton (1971) shows that the solution to this differential equation is of the form

$$J = (1/\zeta)\ln W + H(\alpha_k, \pi). \tag{A5}$$

Since nominal and real bonds are assumed to be in zero net supply, we have $W(t) = K(t)$. Using conditions (A2) and (A5), we arrive at Equation (6) in the text. Using condition (A5) along with conditions (A3) and (A4), setting $w_b = w_r = 0$, we arrive at Equations (7) and *(8)* in the *text.*

Appendix B

This appendix gives the state-space representation used to estimate the bond-pricing model when bond prices, y_{1t}, are observed at more frequent intervals than are survey inflation forecasts, y_{2t}.

The measurement equation (24b) can be rewritten as

$$y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} x_t + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}. \tag{B1}$$

Now define a new vector of observations, Y_t, where

$$Y_t = \begin{cases} \begin{bmatrix} y_{1t} \\ y_{2t-1} \end{bmatrix}, & \text{when } y_{2t-1} \text{ is observed at time } t-1, \\ \begin{bmatrix} y_{1t} \\ 0 \end{bmatrix}, & \text{when } y_{2t-1} \text{ is not observed at time } t-1. \end{cases} \tag{B2}$$

In addition, define the vector w_t such that

$$w_t = \begin{cases} y_{2t-1} - \beta_2, & \textbf{when } y_{2t-1} \textbf{ is observed at } t-1, \\ 0, & \textbf{otherwise.} \end{cases} \tag{B3}$$

A new vector of state variables, X_t, can be defined where

$$X_t = \begin{bmatrix} x_t \\ w_t \end{bmatrix} = \begin{bmatrix} \Phi & & 0 \\ \alpha_{21} & \alpha_{22} & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} v_t \\ e_{2t-1} \end{bmatrix} \tag{B4}$$

is the form of the new state transition equation.

The measurement equation corresponding to this state equation is

$$Y_t = a_t X_t + b_t + \begin{bmatrix} e_{1t} \\ 0 \end{bmatrix}, \tag{B5}$$

where

$$a_t = \begin{cases} \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & & I \end{bmatrix}, & \text{when } y_{2t-1} \text{ is observed at } t-1, \\ \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & & 0 \end{bmatrix}, & \text{otherwise;} \end{cases}$$

$$b_t = \begin{cases} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, & \text{when } y_{2t-1} \text{ is observed at } t-1, \\ \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

Calculation of the likelihood function when a_t, and b_t are deterministic functions of time is similar to the method given in Section 3 of the text. See Harvey (1981, p. 110) for details.

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