Book Review

Prices in Financial Markets. Michael U. Dothan. New York: Oxford University Press, 1990. 360 pp., $29.95 (cloth). ISBN 0-19-505312-5.

This book provides a "unified treatment of selected topics in the theory of financial markets" and an "introduction to the mathematics of this theory." The selection of topics is roughly the same as in Harrison and Kreps (1979) and Harrison and Pliska (1981). Thus, the focus is on arbitrage, state prices, and complete markets. The mathematics consists of the theory of semimartingales and stochastic calculus. The book will be most useful as a supplementary text for Ph.D. courses on multiperiod financial models.

Chapters l-6 cover the finite-state, discrete-time model. Excluding Chapter 5, which I will describe in a moment, the material is a proper subset of that in Chapters 5, 7, and 8 of Huang and Litzenberger (1988) or Chapters 2, 8, and 10 of Ingersoll (1987). The treatment is very precise and includes many numerical examples. This may be helpful in the first Ph.D. course for covering the basic concepts of arbitrage, linear pricing rules, etc. The book would be more useful in this regard, however, if it made a better connection with the existing finance literature.

One irritating characteristic is the use of new names for old concepts. For example, the construct called an "equilibrium price measure" already has one name more than it needs, both "risk-neutral probability' and "equivalent martingale measure" being commonly used. It is never mentioned that these are the same concepts, which creates an unnecessary complication for students, who must learn to read the existing literature. Moreover, the traditional names are superior, because the existence of a risk-neutral probability depends only on the absence of arbitrage or utility maximization and not on equilibrium. Similarly, the term "state prices" is never used. Dothan works instead with state prices times one plus the risk-free rate, which he calls the "likelihood ratio." The most extreme example of avoiding traditional terminology occurs when showing that the return representing the likelihood ratio lies on themean-variance frontier. Dothan's entire discussion is the sentence, "We now prove that the trading strategy ζ has *certain minimal properties*" (emphasis added).

The references for this section of the book are also inadequate. The result that asset risk premia are proportional to their betas with respect to state prices is well-known[1] The same result holds for betas

[1] See, for example, Ingersoll (1987, p. 63). It dates back at least to Beja (1971).

with respect to the projection of state prices on returns[2] Yet, Dothan attributes the single-period version of this model to a 1989 working paper of Dutta and Polemarchakis and states that the multiperiod version is new.

The book would also be more useful as a text if the selection of topics were a bit broader. Even given the deliberately narrow focus of the book, there are surprising gaps in the coverage. For example, one has to consult the end-of-chapter notes to find a mention of the relation between the risk-return formula in terms of state prices and the CAPM and consumption-beta model. Although the Pareto efficiency of complete markets equilibria is discussed at length, there is no mention of the second welfare theorem. Moreover, one will find no discussion of marginal rates of substitution as state prices or of asset pricing in a Markov framework.

An innovative and very useful feature of the discrete-time analysis is Chapter 5. This presents various concepts and results from stochastic calculus (including covariation processes, Id’s lemma, exponential martingales, Girsanov’s theorem, and martingale representation theory) in the context of the discrete model. These concepts are also illustrated with numerical examples. The multiperiod model is then treated in Chapter 6, using the notation and concepts of continuous-time analysis. This should be a very effective pedagogical device.

Chapters 7 and 8 give a good presentation of the Wiener process and the It8 integral. This is rigorous but written at a level that all students should find accessible. There is a nice discussion of Itô's formula, including a sketch of the proof. The usual multiplication rules for stochastic differentials are explained to be a convenient shorthand for calculating covariation processes, which is the appropriate perspective. Chapter 7 also includes a well-written introduction to probability theory, including abstract probability spaces, filtrations, measurability, and convergence concepts.

Chapter 9 concerns the Black-Scholes model. The uniqueness of the risk-neutral probability is demonstrated as in Harrison and Kreps (1979), using Girsanov’s theorem. The Black-&holes formula is calculated by taking expectations under the risk-neutral probability (the binomial model is analyzed in Chapter 6 and the Black-Scholes formula derived there as the limit). Also, the fundamental partial differential equation is derived. There is a slight slip in this calculation when it is claimed that any process (in this case, a contingent claim value V) adapted to another process (the security price P) can be written In feedback form [i.e., $V_t = f(t, P_t)$]. This is not true in general.

[2] See Chamberlain and Rothschild (1983) and Hansen and Richard (1987) for general demonstrations that the return representing the linear pricing rule lies on the mean-variance frontier.

The dynamic completeness of the Black-Scholes model is discussed using an integrability restriction on trading strategies. The conclusion of this discussion is that the market is approximately complete, in the sense that the set of marketed claims equals $L^2(P) \cap L^2(Q)$ [rather than $L^2(P)$, which would be exact completeness], where P is the original probability and Q is the risk-neutral probability. A much better approach is to impose a nonnegative wealth constraint, as in Dybvig and Huang (1988), rather than the restriction on trading strategies used here, which does not have any realistic interpretation. Under the nonnegative wealth constraint, one can show that every contingent claim that has finite cost (i.e., finite expectation under Q) is marketed, which is simpler and more satisfying. This approach is later taken in Chapter 12 for the general continuous-time model. This results in a very confusing discussion in Chapter 12 (p. 295) of the differences between the Black-Scholes model and the general continuous-time model. It is confusing because apparently the best results are obtained for the general model rather than for the special case (the Black-Scholes model). The explanation, of course, is simply that the wrong approach is taken in analyzing the Black-Scholes model.

Chapters 10 and 11 present the theory of semimartingales and the general stochastic calculus. The presentation is in the form of a succession of definitions and theorems, with little discussion and few proofs. The coverage is very thorough. The only other Finance text that discusses stochastic calculus for semimartingales, as far as I am aware, is Duffie (1988). However, Duffie covers the same material in less than 10 pages. Dothan's presentation is at least as terse as Duffie's but includes many more results and details.

There are a few mistakes in Chapter 11. The statement in Theorem 11.6 that the stochastic integral is a continuous process obviously requires an assumption that the integrator is continuous. Corollary 11.2 states that the Riemann sums, with the integrand evaluated at the left endpoint, converge to the stochastic integral. This requires an assumption that the integrand is left-continuous. Theorem 11.25 states that the stable subspace generated by a vector of local martingales is the set of sums of stochastic integrals. This is in general false, unless the local martingales are orthogonal or a more general definition is used for the stochastic integral.[3] This type of result is relevant for characterizing dynamically complete markets. The same mistake occurs in Harrison and Pliska (1981) and has been noted by Müller (1989). Müller suggests assuming the local martingales (price processes in financial models) are orthogonal, but this means that there

[3] This definition enlarges the class of admissible integrands by not requiring that the Individual stochastic integrals in the vector exist but only that their sum exist. It is due to Jacod (1980). See also Dellacherie and Meyer (1982, p. 385).

are no systematic risks. The appropriate solution is to use the more general definition of the stochastic integral. This approach is taken in Duffie (1988, p. 144), although he does not give details.

Chapters 10 and 11 allow the time horizon to be infinite. It would have been appropriate to include a warning that equivalent martingale measures typically do not exist in infinite-horizon models.

Chapter 12 analyzes the securities-market model with semimartingale price processes. This follows Harrison and Pliska (1981) (with a few more details), except for a result that risk premia are locally proportional to betas measured with respect to state prices[4] and a characterization of state prices in a dynamically complete market (which is so formidable I have not attempted to review it). The hypotheses of the various theorems in this chapter are not explained well.[5] At many places in the chapter, there is a confusion about the need to establish uniform integrability of martingales. Any martingale on a finite interval $[0, T]$ is uniformly integrable.[6] Also, the error noted above concerning market completeness propagates throughout this chapter. This is only a minor problem, since the results can be made correct by generalizing the definition of the stochastic integral. A separate confusion regarding market completeness appears in Corollary 12.3 and Theorem 12.13. If these results were true, then every martingale M would satisfy $E[\sup_{0 \leq t \leq T} |M_t|] < \infty$, which is known to be false.[7]

Abstract preference relations are used in Chapters 9 and 12.[8] In Chapter 9, Dothan states, "To make sense, the Black-Scholes model requires a proof that these prices are indeed equilibrium prices for some population of traders with continuous, increasing, and convex preferences." There can be different opinions about whether a proof of this type is needed at all. However, one clearly does not need to show that the preferences are continuous in the sense of Chapter 9.[9] It is assumed in Chapter 12 that traders have continuous preferences on the consumption set $\Re \times \mathscr{L}^1(\Omega, \mathbf{F}_T, Q)$. This amounts to saying that if the cost of the absolute difference of two contingent claims is

[4] This result was independently derived in Back (1991).

[5] For example, the hypothesis that $1/p_n \in \mathscr{D}$, which appears several times, simply means that each path of the price process p_n is bounded below away from zero. Also, the hypothesis that $(p_n/p_l, 1/z)^Q$ exists, which appears in the fundamental Theorem 12.16. is equivalent to p_n/p_l being a special semimartingale under the original measure P, which an be interpreted as: the local risk premium of security n is finite.

[6] Similarly, the term "finite variation on compacts" appears, even though this is equivalent to plain "finite variation" In this model.

[7] See p. 247 or Durrett (1984, p. 149).

[8] Utility functions and marginal rites of substitution do not appear in the book.

[9] Examples of sensible preferences that are not continuous In that sense are CARA utility functions and any utility functions, such as CRRA, defined only for nonnegative consumption.

small, then traders are nearly indifferent between them. This is a strange and restrictive assumption. However, it does not have any effect on the results of the chapter, since it is never used. In fact, there is no reason in Chapter 12 to even mention preference orderings.

In summary, Chapters 1–4 and 6, and particularly the numerical examples therein, will be a useful supplement to existing texts for teaching the basic concepts of information structures, budget sets, arbitrage, etc. I strongly recommend the introduction to the It6 integral (Chapters 5, 7, and 8) and will make it required reading in our Ph.D. course introducing continuous-time models. Chapters 10 and 11 provide a concise reference for the general stochastic calculus. I would advise interested students to use this as a supplement to a text that provides proofs.[10] I am less happy about the continuous-time finance chapters (9 and 12). Chapter 9 is dominated by Chapter 22 of Duffie (1988), and Chapter 12 is unduly complicated and somewhat confused.

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[10] A good choice In this regard would be Protter (1990).

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