Risk Aversion and the Intertemporal Behavior of Asset Prices

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In this article, we characterize ecoonomies in which both cash flows and forward prices follow random walks. We show in the case of geometric random walks that the preferences of the representative investor are of the constant proportional risk-aversion type. We also show the conditions under which spot prices follow random walks and under which the equivalent martingale measure is non-state-dependent.

The proposition that asset prices follow a random walk, or the closely related proposition that rates of return on assets are serially independent, has been asserted frequently in both theoretical and empirical research. In this article, we look at the utility-theoretic conditions under which these assertions can be justified.

Samuelson (1965) shows that "properly anticipated (forward) prices fluctuate randomly," regardless of the distribution of the underlying cash payments on the asset. This result, however, relies on the assumption that investors are risk neutral or, if they are risk averse, that the appropriate adjustment factor for risk

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is nonstochastic. Under risk neutrality, the forward price of the asset follows the path of the conditional expectation of the underlying cash payoff. Thus, for example, if the asset yields a cash flow that evolves as a geometric (arithmetic) random walk, then the forward price of the asset will also follow a geometric (arithmetic) random walk. In a subsequent article, Samuelson (1973) shows that "properly discounted" spot prices of assets fluctuate randomly. However, his conclusion again relies on an assumption that the relevant discount rate is nonstochastic.[1]

For the case of risk aversion, investigation of the conditions under which forward prices fluctuate randomly requires an assumption about the stochastic process generating cash payoffs. It is natural to restrict attention to the case in which a given cash flow to be paid at time s, X_s, evolves as a random walk and to consider the conditions under which its market price also follows a random walk. This is the purpose of the present article.

LeRoy (1973) was the first to discuss the validity of the Samuelson results under risk aversion for particular investor utility functions. LeRoy assumes that earnings follow a linear autoregressive process, and shows that the rate of return on an asset is not serially independent for the case of constant absolute risk aversion (CARA). When the cash payoff on an asset and aggregate consumption follow geometric random walks, Ohlson (1977) and Rubinstein (1976) show that constant proportional risk aversion (CPRA) is a sufficient condition for the rate of return on the asset to be serially independent.[2] In the literature to date, the emphasis has been on *sufficient* conditions for stock prices to evolve randomly over time. In this article, we are concerned with the *necessary* conditions for stock prices to follow a random walk.[3]

The structure of the article is as follows. In Section 1, we consider a no-arbitrage economy and derive necessary and sufficient conditions for the forward price of a cash flow to follow a random walk. We show that, in the case where the cash flow evolves as an arithmetic random walk, the pricing function is exponential. In Section 2, we consider

[1] Stone (1975) points out that for the case of stochastic discount rates, additional restrictions have to be imposed on their behavior over time for the Samuelson results to hold.

[2] Note that Rubinstein (1976, theorem 2), states that if the rate of growth of the firm's dividend stream and aggregate consumption follow random walks, then the rate of return on the stock follows a random walk. (See Rubinstein, 1976, p. 410, corollary.) However, Franke (1984) correctly restates Rubinstein's result. Franke credits Rubinstein with showing that constant proportional risk aversion is a sufficient condition for "serial independence of the market excess return" (see Franke, 1984, p. 440).

[3] Franke (1984) derives necessary and sufficient conditions for a related characteristic of asset prices: the serial independence of the rate of return on the market portfolio of all risk assets. In a recent article, LeRoy and Raymon (1989) look at a related question. They investigate the necessary and sufficient conditions for the neutrality of money. A sufficient condition is that the money stock follows a random walk.

an economy in which the representative investor is risk averse and maximizes the expected value of a time-additive utility function, and aggregate consumption follows a random walk. We first show that in this economy the forward price of the cash flow will follow an arithmetic random walk if and only if the representative investor has CARA utility. We then establish analogous results for the case in which aggregate consumption and the cash flow follow geometric random walks. In Section 2, we also establish conditions for spot prices to follow random walks. Since the spot price of an asset depends upon the risk-free rate as well as the forward price, we derive conditions under which the risk-free rate is nonstochastic. We show, in the case where aggregate consumption and the cash flow follow geometric random walks, that the random walk in forward prices translates into a random walk in spot prices if and only if the coefficient of proportional risk aversion is a constant.

In Section 3, we relate our results to the equivalent martingale measure of Harrison and Kreps (1979), and show, for the cases of arithmetic and geometric random walks in cash flows, that the conditional equivalent martingale measure is not state dependent only if investors have CARA or CPRA utility, respectively. In Section 4, we conclude and discuss some of the implications of our results.

1. A Characterization of Economies in Which Cash Flows and Forward Prices Follow Random Walks

We consider a pure-exchange economy in which the distribution of future cash flows is given. The cash flow of a particular firm at time s is denoted by X_s. The expectation of X_s at time t, conditional on the information available, is given by

$$\underset{t}{E}(X_s), \qquad t < s.$$

We assume that X_s follows an arithmetic random walk:

$$X_s = X_t + y_{t+1} + y_{t+2} + \cdots + y_s,$$

where the successive changes $y_{t+1}, y_{t+2}, \ldots$ are identically and independently distributed. This implies that X_s can be written

$$X_s = X_t + y_{t,s}, \tag{1}$$

where $y_{t,s}$ is distributed independently of the state of the world at time t. We do not restrict the expectation of $y_{t,s}$ to be zero; hence, X_s is a random walk with drift.

In the absence of arbitrage, the forward price at time t of the cash flow X_s can be written as

$$F_t(X_s) = \underset{t}{E}(X_s\phi_s), \qquad \underset{t}{E}(\phi_s) = 1, \tag{2}$$

where $F_t(X_s)$ is the forward price at time t of the cash flow, X_s for delivery at time s, and $\phi_s > 0$ is the pricing function for cash flows at time s.[4] In turn, the spot price or value at time t of the cash flow X_s can be written as

$$V_t(X_s) = \underset{t}{E}(X_s\phi_s)B_{t,s}, \tag{3}$$

where $B_{t,s}$ is the price at time t of a pure discount bond that pays one dollar at time s. Note that, by definition, the forward and spot prices, at time s, equal the realization of the cash flow X_s:

$$F_s(X_s) = V_s(X_s) = X_s. \tag{4}$$

We now establish a result that characterizes economies in which both the cash flow and the forward price follow arithmetic random walks.

Lemma 1. *If X_s follows an arithmetic random walk, a necessary and sufficient condition for the forward price to also follow an arithmetic random walk is that $Cov_t(X_s, \phi_s)$ is nonstochastic.*

Proof. Necessity: If the forward price of X_s follows an arithmetic random walk,

$$F_s(X_s) = F_t(X_s) + \epsilon_{t,s}, \tag{5}$$

where $\epsilon_{t,s}$ is distributed independently of $F_t(X_s)$. Hence, from (2) and (4), it follows that

$$X_s = \underset{t}{E}(X_s\phi_s) + \epsilon_{t,s} = \underset{t}{E}(X_s) + \underset{t}{\mathrm{Cov}}(X_s, \phi_s) + \epsilon_{t,s},$$

since $E_t(\phi_s) = 1$. Taking expectations at time t, we find

$$\underset{t}{\mathrm{Cov}}(X_s, \phi_s) = \underset{t}{E}(-\epsilon_{t,s}),$$

which is nonstochastic, since by the random walk assumption $\epsilon_{t,s}$ is independent of the state of the world at time t.

Sufficiency: If $\mathrm{Cov}_t(X_s, \phi_s)$ is nonstochastic, Equation (5) can be rearranged to yield

$$\epsilon_{t,s} = X_s - \underset{t}{E}(X_s) - \underset{t}{\mathrm{Cov}}(X_s, \phi_s),$$

[4] See, for example, Harrison and Kreps (1979) for a discussion.

using Equations (2) and (4). Substituting from (1), after taking expectations, yields

$$\epsilon_{t,s} = y_{t,s} - \underset{t}{E}(y_{t,s}) - \underset{t}{\mathrm{Cov}}(X_s, \phi_s),$$

which, by the definition of a random walk in X_s and the nonstochastic nature of $\mathrm{Cov}_t(X_s, \phi_s)$ is independent of the state of the world at time t. It follows, therefore, that $F(X_s)$ is an arithmetic random walk.

Lemma 1 establishes that the risk adjustment of the expected cash flow must be nonstochastic for the forward price to follow a random walk. We will now show that for this to be true, for an arbitrary cash flow process X_s, ϕ_s must be an exponential function of X_s.

Theorem 1. *If the cash flow X_s follows an arbitrary, nondegenerate arithmetic random walk, the forward price of the cash flow $F(X_s)$ also follows an arithmetic random walk if and only if the pricing function ϕ_s is an exponential function of X_s.*

Proof. Sufficiency: We write $\phi_s = \phi(X_s)$ and assume $\phi(X_s)$ can be specified as

$$\phi(X_s) = ae^{\alpha X_s}. \tag{6}$$

Since, by definition, $E_t(\phi_s) = 1$, we can write

$$a = [e^{\alpha X_t} \underset{t}{E}(e^{\alpha y_{t,s}})]^{-1}.$$

Substituting in (6) and simplifying using (1), we have

$$\mathrm{Cov}(X_s, \phi_s) = \mathrm{Cov}[y_{t,s}, e^{\alpha y_{t,s}}/\underset{t}{E}(e^{\alpha y_{t,s}})],$$

which is nonstochastic; and, by Lemma 1, the forward price follows a random walk.

Necessity: Suppose the forward price follows a random walk. By Lemma 1, it follows that $\mathrm{Cov}_t(X_s, \phi_s)$ is nonstochastic. Consider two possible outcomes of X_t: $X_t = X_t'$ and $X_t = X_t' + v$. Since $\mathrm{Cov}_t(X_s, \phi_s)$ is nonstochastic, it is independent of the state of the world at time t and hence of X_t. We can, therefore, write

$$\underset{t}{\mathrm{Cov}}[y_{t,s}, \phi(X_t' + y_{t,s})] = \underset{t}{\mathrm{Cov}}[y_{t,s}, \phi(X_t' + v + y_{t,s})]. \tag{7}$$

For (7) to hold for arbitrary, nondegenerate $y_{t,s}$, we require that

$$\phi(X_t' + y_{t,s}) - \phi(X_t' + v + y_{t,s}) + k = 0, \tag{8}$$

where k is nonstochastic. However, since $E_t(\phi_s) = 1$, we can establish, by taking expectations at time t, that $k = 0$ and

$$\phi(X'_t + y_{t,s}) = \phi(X'_t + v + y_{t,s}). \tag{9}$$

It follows from Brennan and Kraus (1976, theorem 7) that

$$\phi(X_s) = ae^{\alpha X_s}. \tag{6}$$

Theorem 1 restricts the nature of the pricing function if the cash flow and its forward price are to follow arithmetic random walks. Before discussing the implications of these results for restrictions on preferences in a representative agent economy, we turn briefly to the case of geometric random walks.

When X_s follows a geometric random walk,

$$X_s = X_t y_{t+1} y_{t+2} \cdots y_s,$$

where the successive y_{t+i} are independently and identically distributed. Hence, we can write

$$X_s = X_t y_{t,s}, \tag{10}$$

where $y_{t,s}$ is distributed independently of the state of the world at time t. As a corollary of Theorem 1 we now have the following.

Corollary 1.1. *If the cash flow X_s follows an arbitrary, nondegenerate geometric random walk, the forward price of the cash flow also follows a geometric random walk if and only if the pricing function ϕ_s is a power function of X_s.*[5]

Proof. By a similar argument to that used in the proof of Lemma 1 it follows in the case of geometric random walks that the forward price follows a random walk if and only of $\text{Cov}_t\,[X_s/E_t(X_s), \phi(X_s)]$ is nonstochastic. Using (10) this implies that

$$\frac{1}{\underset{t}{E}(y_{t,s})} \underset{t}{\text{Cov}}\,[y_{t,s}, V(\ln X_t + \ln y_{t,s})]$$

is nonstochastic, where $V(\cdot)$ is defined by

$$V(\ln X_t + \ln y_{t,s}) = \phi(X_t y_{t,s}). \tag{11}$$

[5] An interesting special case of this theorem is that when X_s is lognormal, the forward price follows a random walk if and only if the pricing function ϕ_s is a power function. The result explains why the Rubinstein (1976) and Brennan (1979) option-pricing framework in discrete time is essentially similar to that of Black and Scholes (1973). The former assume directly that the pricing function is of the constant proportional risk aversion type (power function). The latter assumes, in effect, a special case of a geometric random walk in forward prices.

By a similar argument to that used in the proof of Theorem 1, $V(\cdot)$ must be exponential. Hence,

$$\phi(X_s) = a \exp[\alpha(\ln X_t + \ln y_{t,s})] = aX_s^{\alpha}. \qquad (12)$$

2. Asset Prices in a Representative Investor Economy in Which Aggregate Consumption Follows a Random Walk

In this section, we assume that the preferences of investors in the economy can be replaced by those of a representative investor whose utility function can be written in additive-separable form as

$$U(C_t, \ldots, C_s) = U(C_t) + \cdots + \rho_s U(C_s), \qquad (13)$$

where C_s is aggregate consumption in period s. The first-order conditions from the representative investor's maximization problem imply that the pricing function ϕ_s can be written as (see Rubinstein, 1976)

$$\phi_s = U'(C_s)/\underset{t}{E}[U'(C_s)]$$

and the price of the discount bond $B_{t,s}$ can be written as

$$B_{t,s} = \rho_s \underset{t}{E}[U'(C_s)]/U'(C_t).$$

We also assume that aggregate consumption follows an additive random walk. Hence, we can write $C_s = C_t + z_{t,s}$, where $z_{t,s}$ is independent of C_t.[6]

Theorem 2. *In a representative investor economy, in which cash flows and aggregate consumption follow arbitrary, nondegenerate, arithmetic random walks, asset forward prices follow arithmetic random walks if and only if the representative investor has CARA preferences.*[7]

Proof. Sufficiency: If

$$U'(C_s) = ae^{\alpha C_s},$$

then

$$\underset{t}{\mathrm{Cov}}(X_s, \phi_s) = \underset{t}{\mathrm{Cov}}[y_{t,s}, e^{\alpha z_{t,s}}/\underset{t}{E}(e^{\alpha z_{t,s}})], \qquad (14)$$

[6] Note that in a closed economy, the random walk property for aggregate consumption follows from the assumption of a random walk for individual cash flows.

[7] LeRoy (1973) shows the sufficient condition for the case of a first-order autoregressive process.

which is nonstochastic by the assumption that $z_{t,s}$ and $y_{t,s}$ are independent of the outcomes at time t.

Necessity: Suppose that $\mathrm{Cov}_t(X_s, \phi_s)$ is nonstochastic. Hence, for two values of C_t, $C_t = C'_t$ and $C_t = C'_t + v$, we have

$$\mathop{\mathrm{Cov}}_t[y_{t,s}, \phi(C'_t + z_{t,s})] = \mathop{\mathrm{Cov}}_t[y_{t,s}, \phi(C'_t + v + z_{t,s})].$$

For this to hold for an arbitrary, nondegenerate $y_{t,s}$, we require

$$\phi(C'_t + z_{t,s}) - \phi(C'_t + v + z_{t,s}) + k = 0.$$

From the argument used in the proof of Theorem 1, if $U'(C_s)$ is a function such that

$$\phi(C_s) = U'(C_s)/\mathop{E}_t[U'(C_s)],$$

then

$$U'(C_t) = ae^{\alpha C_s}.$$

Theorem 2 establishes that in a representative investor economy in which aggregate consumption, the cash flows of firms, and the forward prices of the cash flows follow arithmetic random walks, the representative investor has CARA preferences.

It follows immediately from Corollary 1.1 and Theorem 2 that in the case where a cash flow X_s and aggregate consumption follow geometric random walks the corollary below applies.[8]

Corollary 2.1. *In a representative investor economy, in which a cash flow X_s and aggregate consumption follow arbitrary, nondegenerate, geometric random walks, asset forward prices follow geometric random walks if and only if the representative investor has CPRA preferences.*

It is important to note that the necessity of the CARA utility function follows only when we require the forward price to follow a random walk for *arbitrary* nondegenerate random walks in cash flows. To see this, note that if the cash flow follows a degenerate random walk, that is, if X_s is known at time t, then $F_t(X_s) = X_s$, irrespective of the utility function, and the forward price also follows a degenerate random walk regardless of preferences. Also, a cash flow being independent

[8] In general, it is not possible for all individual cash flows and the aggregate cash flow to follow geometric random walks. This is because the aggregate cash flow is the sum of the individual cash flows. Hence, the assumption in the text that is commonly made elsewhere in the literature can be true only as an approximation.

of aggregate consumption X_s implies $\text{Cov}_t\,[X_s,\ U'(C_s)] = 0$ and $F_t(X_s) = E_t(X_s)$. Hence, in this case also, the forward price follows a random walk irrespective of the utility function.

Two special cases, which are common in the literature, concern asset prices when cash flows are normally distributed [the capital asset pricing model (CAPM)] and when cash flows are generated by a linear factor model [the arbitrage pricing theory (APT)]. If the cash flow of a firm and aggregate consumption are jointly, normally distributed conditional on the state of the world at time t, we have

$$F_t(X_s) = \underset{t}{E}(X_s) + \underset{t}{\text{Cov}}(X_s,\ \phi_s),$$

where

$$\underset{t}{\text{Cov}}(X_s,\ \phi_s) = \{\underset{t}{E}[U''(C_s)]/\underset{t}{E}[U'(C_s)]\}\underset{t}{\text{Cov}}(X_s,\ C_s), \qquad (15)$$

as shown by Rubinstein (1976). This is similar to the standard CAPM derived in the multiperiod case by Stapleton and Subrahmanyam (1978). In this case, Theorem 2 implies that $E_t[U''(C_s)]/E_t[U'(C_s)]$ is nonstochastic if and only if the representative investor has CARA preferences. The result in this case is similar to that derived for the case of additive diffusion processes by Bick (1986).

Now suppose that an m-factor model generates the cash flows in the economy:

$$X_s = \sum_{l=1}^{m} \delta_l L_{s,l} + \epsilon_s, \qquad (16)$$

where the outcomes of the m factors, $L_{s,l}$, explain the cash flow, X_s, with an error ϵ_s. Suppose also that each factor follows an additive random walk. From the extension in Connor (1984) of the Ross (1976) APT, we have

$$\underset{t}{\text{Cov}}(X_s,\ \phi_s) = \sum_{l=1}^{m} \underset{t}{\text{Cov}}\left[L_{s,l},\ U'(C_s)/\underset{t}{E}[U'(C_s)]\right]. \qquad (17)$$

Applying Theorem 2 to each of the factors, the factor risk premia are nonstochastic, if and only if the representative investor has CARA preferences. In this special case, the forward price is given by

$$F_t(X_s) = \underset{t}{\text{Cov}}(X_s,\ \phi_s) + \sum_{l=1}^{m} \delta_l \underset{t}{E}[L_{s,l}]. \qquad (18)$$

Hence, the forward price is itself generated by an m-factor model. This is a basis for the multiperiod APT considered by Garman and Ohlson (1980) and Connor (1984).

2.1. The risk-free rate of interest and spot prices

The spot price of a cash flow can be expressed as a function of the forward price and the risk-free rate of interest for the period from t (the date the spot price is recorded) to s (the date at which the cash flow is paid). Its stochastic behavior depends, therefore, on the behavior of the risk-free rate as well as that of the forward price. In this section, we derive necessary and sufficient conditions for the risk-free rate to be nonstochastic, and for the spot price of the asset to follow a random walk, given that the cash flow follows an arithmetic (or geometric) random walk.

In the representative investor economy where (13) holds, the forward price of the cash flow is

$$F_t(X_s) = \underset{t}{E}\{X_s U'(C_s)/\underset{t}{E}[U'(C_s)]\} \tag{19}$$

and its spot price is

$$V_t(X_s) = \underset{t}{E}\{X_s U'(C_s)/\underset{t}{E}[U'(C_s)]\}B_{t,s}. \tag{20}$$

We will now derive conditions under which the forward and spot prices follow random walks. First we establish the following.

Theorem 3. *(The Risk Free Rate of Interest) If the cash flow and the forward price of the cash flow follow arbitrary, nondegenerate, arithmetic random walks, the risk-free rate of interest is nonstochastic, if and only if the coefficient of absolute risk aversion of the representative investor is a constant.*

Proof. Sufficiency: Under the conditions of the theorem, we know from Theorem 2 that the representative investor has CARA preferences, that is,

$$U'(C_s) = \beta_s e^{\alpha_s C_s}.$$

If $\alpha_s = \alpha_t$, substituting in (20) for the price of a discount bond we have

$$B_{t,s} = \rho_s(\beta_s/\beta_t)\underset{t}{E}[e^{\alpha_s z_{t,s}}],$$

which is nonstochastic since $z_{t,s}$ is independent of the state of the world at time t.

Necessity: If $B_{t,s}$ is nonstochastic, then for any two states $'$ and $''$ at time t with aggregate consumption C'_t and $C''_t = C'_t + v$, where v is arbitrary, then

$$B'_{t,s} = B''_{t,s},$$

or

$$\rho_s \underset{t}{E}[\beta_s e^{\alpha_s(C'_t + z_{t,s})}]/\beta_t e^{\alpha_t C'_t} = \rho_s \underset{t}{E}[\beta_s e^{\alpha_s(C'_t + v + z_{t,s})}]/\beta_t e^{\alpha_t(C'_t + v)}.$$

Since $z_{t,s}$ is independent of C'_t, this implies[9]

$$e^{(\alpha_s - \alpha_t)v} = 1 \qquad \text{and} \qquad \alpha_s = \alpha_t.$$

Corollary 3.1. *If the cash flow, aggregate consumption, and the forward price of the asset follow arbitrary, nondegenerate, geometric random walks, the risk-free rate of interest is nonstochastic if and only if the coefficient of relative risk aversion of the representative investor is a constant.*

Proof. Under these conditions, the representative investor has CPRA preferences. The proof then follows along the lines of the proof of Theorem 3.

We are now in a position to derive the process for spot prices. We can establish the following.

Theorem 4. *(Spot Prices) If the conditions of Theorems 2 and 3 are fulfilled (i.e., if cash flows and aggregate consumption and the forward price follow arbitrary, nondegenerate, arithmetic random walks, and if the coefficient of absolute risk aversion of the representative investor is a constant), then the spot price of the cash flow is a linear function of a random walk. The spot price follows an arithmetic random walk if and only if the risk-free rate of interest is zero.*

Proof. Under the conditions of the theorem we can write the forward price of the asset as

$$F_t(X_s) = \underset{t}{E}(X_s) + k_{t,s},$$

where $k_{t,s} \equiv \text{Cov}_t(X_s, \phi_s)$ is nonstochastic.

Since the risk-free rate is nonstochastic,

$$V_t(X_s) = B_{t,s}\underset{t}{E}(X_s) + B_{t,s}k_{t,s}.$$

Also, since $E_t(X_s)$ follows a random walk, $V_t(X_s)$ is a linear function of a random walk. Note that the value of the cash flow itself follows

[9] Theorem 3 establishes more restrictive conditions for a nonstochastic risk-free rate than those discussed in Hall (1978), who suggests that a random walk in consumption is a sufficient condition.

an arithmetic random walk if and only if $B_{t,s}$ is constant over time, which is the case only if the rate of interest is zero. If $B_{t,s} \neq B_{t-1,s}$, we have

$$V_t(X_s) - V_{t-1}(X_s) = a_t + b_t V_{t-1}(X_s) + \epsilon_t, \tag{21}$$

where

$$a_t = (k_{t,s} - k_{t-1,s}) B_{t,s},$$
$$b_t = B_{t,s}/B_{t-1,s},$$

and ϵ_t is an independent error. If $B_{t,s} = B_{t-1,s}$, then (21) reduces to an arithmetic random walk. However, for $B_{t,s} = B_{t-1,s}$, the rate of interest from $t-1$ to t must be zero (i.e., $B_{t-1,t} = 1$).

Finally, for the geometric random walk case we have the following.

Corollary 4.1. *If Corollary 3.1 holds (i.e., if the cash flow, aggregate consumption, and the forward price follow arbitrary, nondegenerate, geometric random walks, and if the coefficient of relative risk aversion of the representative investor is a constant), then the spot price of the asset follows a geometric random walk.*

Proof. Under the conditions of the corollary, the forward price $F_t(X_s)$ follows a geometric random walk and the risk-free rate is nonstochastic. Hence, the spot price of the asset

$$V_t(X_s) = B_{t,s} F_t(X_s)$$

is a constant times a geometric random walk and hence itself follows a geometric random walk.

The results in this section are not unexpected. As pointed out originally by Pratt (1964), the exponential utility function implies a lack of wealth effects so that the "price of risk" is unaffected by wealth and in particular is not dependent on previous outcomes of the firm's cash flows and of aggregate consumption. It is this property of the exponential function that ensures that a random walk in the cash flow is translated into a random walk in the forward price. Similarly, the power-utility function implies that demand for risk assets is proportional to wealth, which implies that the price of risk as a proportion of the asset price is constant. This property allows the geometric random walk in cash flows to be translated into a similar process for the forward price and the spot price of the asset.

3. Asset Prices and the "Equivalent Martingale Measure"

We have been concerned in this paper with the conditions under which a random walk in a cash flow is translated into a random walk in the price of the cash flow. In this section, we show that these conditions are equivalent to the existence of an equivalent martingale measure, which is not state dependent.[10] Theorem 2 implies that in a representative investor economy, this condition corresponds to a restriction on the preferences of investors.

We ask the following question: Under what conditions will the conditional, equivalent martingale measure be nonstochastic? We have to be somewhat careful, however, in posing this question. The Harrison and Kreps (1979) results were derived under quite general conditions, whereas the analysis here has assumed that cash flows and aggregate consumption follow random walks. In this section, we will retain our random walk assumption. Assuming a no-arbitrage economy and random walks in cash flows, under what conditions is the mapping from the actual conditional probability measure $\{P\}$ to the conditional equivalent martingale measure $\{P^*\}$ nonstochastic (i.e., non-state-dependent)?

Theorem 5. *If the cash flow of the firm follows an arithmetic random walk, then in a no-arbitrage economy the mapping from the conditional probability measure $P = \{p\}$ to the conditional equivalent martingale measure $P^* = \{p^*\}$ is independent of the state of the world at time t only if the forward price of the cash flow follows an arithmetic random walk.*

Proof. From Harrison and Kreps (1979), in a no-arbitrage economy the forward price can be written

$$F_t(X_s) = \underset{t}{E^*}(X_s), \tag{22}$$

where $E^*_t(\cdot)$ is an expectation taken with respect to an equivalent martingale measure $P^* = \{p^*\}$. If for two states $i = 1, 2$ at time t, $X_t = X_{t,i}$, it follows from the random walk assumption that $X_s = X_{t,i} + y_{t,s}$, where $y_{t,s}$ is independent of the state i at time t. From (22) we have, in state i at time t,

$$\begin{aligned} F_{t,i}(X_s) &= \sum_{i=1}^{n} p^*_i(X_{t,i} + y_{t,s}) \\ &= \underset{t,i}{E}(X_s) + \sum_{i=1}^{n} (p^*_i - p_i)y_{t,s}, \end{aligned}$$

where $E_{t,i}(\cdot)$ refers to an expectation at time t in state i.

[10] The term "equivalent martingale measure" was introduced by Harrison and Kreps (1979).

Hence, we can write

$$F_t(X_s) = \underset{t}{E}(X_s) + \underset{t}{\mathrm{Cov}}(X_s, \phi_s),$$

where

$$\underset{t}{\mathrm{Cov}}(X_s, \phi_s) = \sum_{i=1}^{n} (p_i^* - p_i) y_{t,s}$$

is state independent. Hence from Lemma 1, the forward price follows a random walk.

Remark. *We should note that Theorem 5 is not an "if and only if" theorem. It is possible for P^* to be state dependent and for $Cov_t(X_s, \phi_s)$ to be nonstochastic. However it is easy to check that, if $y_{t,s}$ follows a two-state process, a common assumption in option-pricing theory, then the theorem can be stated as an "if and only if" theorem. Also, note that the theorem is not restricted to nondegenerate random walks.*

Corollary 5.1. *In a no-arbitrage, representative investor economy where cash flows and aggregate consumption follow arbitrary, nondegenerate arithmetic random walks, the conditional, equivalent martingale measure P^* is non-state-dependent only if the representative investor has CARA preferences.*

Proof. This follows directly from Theorem 5 and Theorem 2.

We will now simply state the following corollaries for the case of geometric random walks.

Corollary 5.2. *In a no-arbitrage economy, where the cash flow of firm j follows a geometric random walk, the mappping from the conditional probability measure $P = \{p\}$ to the conditional equivalent martingale measure $P^* = \{p^*\}$ is independent of the state of the world at time t only if the forward price of the cash flow follows a geometric random walk.*

Corollary 5.3. *In a no-arbitrage, representative investor economy where the cash flow of the firm and aggregate consumption follow arbitrary, nondegenerate, geometric random walks, the conditional equivalent martingale measure P^* is non-state-dependent only if the representative investor has CPRA preferences.*

4. Conclusions and Implications

The assumption that prices follow a random walk, or that the risk premium on assets is nonstochastic, is common in the literature on valuation theory. Often, the stochastic process for asset prices is assumed to be exogenous, as in Merton's (1973) intertemporal asset-pricing model and the arbitrage pricing model of Ross (1976).[11] The same assumptions about the exogenous process generating prices is usually made in the literature on option pricing. Black and Scholes (1973) and most of the subsequent work on options assumes that the firm's value follows a geometric random walk in continuous time. Similarly, much of the empirical work in financial economics assumes a similar process for asset rates of return. We have shown in this article that the random property of asset prices holds, under risk aversion, only under quite restrictive conditions.

In the theory of asset valuation, the random walk property of asset prices is of crucial importance. As Fama (1970) originally showed, and Constantinides (1980) and Franke (1984) have emphasized, this is one of the key conditions under which the multiperiod valuation problem can be split into a series of period-by-period problems. In order to derive simple valuation relationships (e.g., for use in capital budgeting), the complex general structure of time–state contingent prices has to be restated in terms of end-of-period prices. Thus, we have shown the conditions under which single-period models, such as the capital asset pricing model, can be valid in a multiperiod setting.

The results in this article have some implications for recent work on the volatility of stock prices. Leroy and Porter (1981) and Shiller (1981), among others, claim to have shown the "excessive" volatility of prices. Their work implicitly assumes that the market risk premium is nonstochastic. We have shown the conditions under which varying risk premia will not contribute to stock-price volatility.

Finally, we have shown in Section 3 that restrictions on the utility function of the representative investor are required for the "equivalent martingale measure" of Harrison and Kreps (1979) to be non-state-dependent.

Throughout this article, we have assumed that the cash flows of the firms in the economy follow random walks. We have not considered risk-averse economies in which the cash flows of firms do not follow random walks. It is, of course, possible that cash flows follow some

[11] Only rarely do intertemporal models of value start with an assumption about the process generating cash flows. Examples of this alternative approach include Brennan (1973) and Cox, Ingersoll, and Ross (1985) in a continuous time setting, and Rubinstein (1976) and Stapleton and Subrahmanyam (1978) in a discrete time setting. Garman and Ohlson (1980) and Connor (1984) provide multiperiod APT models based on exogenous cash flow processes.

alternative process, and that forward prices follow a random walk. However, this article has not investigated that possibility, although we suspect that similar necessary conditions on utility functions would be required.

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