Mean Reversion and Consumption Smoothing

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Most models of the evolution of wealth and consumption assume that wealth volatility and risk premium are constant. But in fact, volatility declines, and risk premium seems to decline, as wealth rises. A model that allows mean reversion in the sense that the risk premium declines as wealth rises can help explain both the "consumption smoothing puzzle" and the "equity premium puzzle." In an example of such a model that gives us an analytic solution, direct and derived risk aversion are both constant, but differ.

Hall (1978, 1988), Hansen and Singleton (1983), Deaton (1987), and others find consumption surprisingly smooth relative to income and prices. This "consumption smoothing puzzle" is closely related to the "equity premium puzzle" described by Mehra and Prescott (1985) and Prescott (1986). In a conventional model with additive separable utility, Mehra and Prescott are unable to reconcile such facts as smooth consumption, a high risk premium in the stock market, and a low real interest rate.

Can we explain consumption smoothing in a conventional model? Or must we turn to models with "imperfections" such as liquidity constraints [Zeldes (1989)], habit formation [Constantinides (1988), Sundaresan (1989)], and "rule-of-thumb" consumption [Campbell and Mankiw (1989)]? Must we rely on mea-

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surement problems [Flavin (1988)] to explain the relation between consumption and income?

Imperfections are very important in explaining the path of consumption and the relation between consumption and wealth. But I think some features of a model that has no imperfections help explain why consumption is smoother than wealth.

We can explain consumption smoothing in a conventional model [as in Merton (1969, 1971, 1973) and Brock (1982)] if we allow wealth variability and the market's risk premium to vary with wealth and time. In fact, the changes we need to explain consumption smoothing are precisely the changes we observe: Wealth variability falls as wealth rises [Black (1976), Schwert (1988, 1989)], and stock prices show mean reversion [Fama and French (1988), Poterba and Summers (1988)]. Mean reversion means that expected return falls as wealth rises.

We can measure changes in volatility more easily than we can measure changes in expected return. Merton (1980) and Summers (1986) point out that it is hard to estimate the process by which expected returns change. Also, the amount of mean reversion in the market may change over time. Black (1988) suggests that changes in mean reversion may help explain the 1987 market crash.

In fact, Kim, Nelson, and Startz (1989) point out that the Fama and French estimates of mean reversion do not differ significantly from zero for the postwar period beginning after World War II. Thus we can take a fall in volatility when wealth rises as quite certain, but we musts take a fall in expected return when wealth rises as much less certain. So we never know whether mean reversion is present; and if it is present, we do not know how large it is.

Intuition suggests that the markets will generally show some mean reversion. If the amount of risk that people must bear falls, the market must pay them less for bearing it. So if volatility falls as wealth rises, the market's risk premium may fall as wealth rises too. However, intuition may be faulty. Abel (1988) gives a general equilibrium model where both risk and risk premium vary with wealth. But in his model, risk and risk premium vary together only when the coefficient of relative risk aversion is less than one. They move in opposite directions when the coefficient is greater than one.

Another intuitive argument is that changes in the "discount rate" may cause changes in wealth. This can give mean reversion, too. For example, suppose all investments are fixed rather than fluid and that people become less risk averse. As a result, the risk premium will fall and the market's level will rise.

Cecchetti, Lam, and Mark (1988) and Kandel and Stambaugh (1988) also discuss the tie between mean reversion and consumption smoothing. Kandel and Stambaugh observe that the price of risk can fail to measure relative risk aversion when the consumption growth rate shows serial correlation. The model below has a related feature: Relative risk aversion in the direct and derived utility functions may differ. So in thinking about risk aversion, we must carefully separate these two measures.

In fact, in the model below, saying that consumption is smoother than wealth is almost the same as saying that derived relative risk aversion taken from utility of wealth is smaller than direct relative risk aversion taken from utility of consumption. Naik and Ronn (1988) make a similar point when they discuss the importance of the differencing interval in estimating relative risk aversion. The difference between these two measures of risk aversion is sufficient to explain the consumption-smoothing puzzle and most of the equity premium puzzle.

1. An Equilibrium Model

We will assume identical individuals who live forever and have additive separable utility functions with constant relative risk aversion. We will assume that the interest rate, the volatility of wealth, and the risk premium depend on wealth and time.

We will assume that investors can borrow or lend. Borrowing and lending are only between investors, so the equilibrium condition for the model is that investors do no borrowing or lending, since all of them are identical.

We do not model technology directly. If investments are fluid, we will reach equilibrium through shifts in the investment mix. If investments have fixed state-dependent payoffs, we will reach equilibrium through shifts in asset prices. The first-order conditions should hold for any combination of fluid and fixed investments.

With identical investors, wealth is just a share of all the assets in the economy. So we can speak of investors as choosing a mix of assets with borrowing or lending.

All variables measuring quantities will be *per capita.* We will use the following symbols:

$u(c)$	instantaneous direct utility function
δ	relative risk aversion in the direct utility function
ρ	utility discount rate
$v(w, t)$	derived utility function
γ	relative risk aversion in the derived utility function
k	ratio of derived to direct relative risk aversion
$r(w, t)$	interest rate
$a(w, t)$	risk premium on assets

$s(w, t)$	wealth volatility
$\sigma(w, t)$	consumption volatility
$x(w, t)$	assets the investor holds
$c(w, t)$	the investor's consumption
b	constant in the derived utility function
g	constant in the consumption function
w	wealth
t	time

The square of a volatility is a variance rate, so the volatilities s and σ both have the dimension "inverse square root of time."

The instantaneous direct utility function is

$$u(c) = \frac{c^{1-\delta}}{1-\delta} \tag{1}$$

Each investor sees himself as choosing x and c. What he does not invest in assets he lends. Thus the equation governing changes in his wealth is

$$dw = (rw + ax - c)\,dt + sx\,dz \tag{2}$$

The maximization equation defines an investor's derived utility function:

$$v(w, t) = \max_{c,\,x}\left[\frac{E(v + dv)}{1 + \rho\,dt} + u(c)\,dt\right] \tag{3}$$

Expanding dv, substituting from Equation (2), taking expected values, and using subscripts to refer to derivatives, we have

$$\begin{aligned} v(w, t) = \max_{c,\,x}\{v + [(rw + ax - c)v_w \\ + v_t + \tfrac{1}{2}s^2x^2v_{ww} - \rho v + u(c)]\,\mathrm{dt}\} \end{aligned} \tag{4}$$

The first-order conditions are Equations (5) and (6):

$$0 = -v_w + u_c(c) \tag{5}$$

$$0 = av_w + s^2xv_{ww} \tag{6}$$

Combining Equations (1) and (5), we have

$$c = v_w^{-1/\delta} \tag{7}$$

In equilibrium, each investor holds all of his wealth in assets, and net lending will be zero. Thus the equilibrium condition is

$$x(w, t) = w \tag{8}$$

Substituting in Equation (4) from Equations (1), (6), (7), and (8), we have

$$0 = (r + \tfrac{1}{2}a)wv_w - \frac{\delta v_w^{(\delta-1)/\delta}}{\delta - 1} + v_t - \rho v \tag{9}$$

Given $r(w, t)$ and $a(w, t)$, Equation (9) constrains $v(w, t)$. To define $v(w, t)$ fully, we need to add boundary conditions. Rather than starting with boundary conditions, we will look for an analytic solution. We will try a solution of the form

$$v(w, t) = \frac{be^{\rho t}w^{1-\gamma}}{1 - \gamma} \tag{10}$$

From Equations (7) and (10), we see that with our trial solution consumption is exponential in time and depends on a power of wealth, where the power is the ratio of derived risk aversion to direct risk aversion:

$$c(w, t) = b^{-1/\delta}e^{-\rho t/\delta}w^k \tag{11}$$

From Equations (6) and (10), we see that with our trial solution a will always be proportional to s^2:

$$a(w, t) = \gamma s(w, t)^2 \tag{12}$$

The ratio of consumption volatility to wealth volatility depends on the sensitivity of consumption to wealth. From Equation (11), we have

$$\frac{\sigma}{s} = \frac{wc_w}{c} = k \tag{13}$$

This means that with our trial solution the ratio of consumption volatility to wealth volatility is exactly equal to the ratio of derived risk aversion to direct risk aversion.

Substituting from Equation (10) into Equation (9), and defining g as $\delta b^{-1/\delta}/(\delta - 1)$, we see that $r + \frac{1}{2}a$ is exponential in time and depends on a power of wealth. Taking the special case where r is constant, we have

$$r + \tfrac{1}{2}a(w, t) = ge^{-\rho t/\delta}w^{k-1} \tag{14}$$

Most authors who allow expected return to vary assume that the interest rate varies while the risk premium is constant. I feel it is more natural to do the reverse. Since the interest rate is easier to measure than the risk premium, investors will respond more to interest rate changes than to risk premium changes. In turn, their responses will dampen the interest rate changes more than the risk premium changes.

We probably want the left-hand side of Equation (14) to be positive, which means that g must be positive. This in turn means

$$\delta > 1 \tag{15}$$

From Equation (14), we find that when $k < 1$ and $\delta > 1$, the derivative of a with respect to w is negative, so the risk premium falls as wealth rises:

$$a_w = 2(k - 1)ge^{-\rho t/\delta}w^{k-2} \tag{16}$$

From Equations (12), (13), and (16), we see that when $k < 1$ and $\delta > 1$:

1. Risk premium falls as wealth rises.
2. Risk falls as wealth rises.
3. Consumption varies less than wealth.

Abel (1988) has a similar result. In his model (1) and (2) hold true only when $\delta < 1$.

Note that the key feature in the models defined above is that risk premium and risk are not constant. Instead they depend on wealth and time. That is why we split the concept of risk aversion into two different concepts that we can choose independently: direct risk aversion and derived risk aversion.

In these models, observations that tell us about risk aversion may bear on either kind of risk aversion. The market's risk premium may tell us about derived risk aversion, while estimates of consumer willingness to shift consumption forward or backward in time may tell us about direct risk aversion.

For example, we may use an equation like (12) to estimate γ. Given that, we can use an equation like (13) to estimate δ.

Since wealth consists of equity, debt, real estate, and human capital, wealth volatility and the risk premium on assets will be substantially lower than equity volatility and risk premium [Brainard and Summers (1987)]. I estimate equity volatility at about 0.18, and Mehra and Prescott (1985) estimate the equity risk premium at 0.06. For a very rough estimate of risk and risk premium for total wealth, divide these numbers by 3 to give $s = 0.06$ and $a = 0.02$.

Mehra and Prescott use estimates of $r = 0.01$ and $\sigma = 0.02$ for the real interest rate and consumption volatility. Thus, my estimate for wealth volatility is 3 times their estimate for consumption volatility. In our simple example, the risk premium and wealth volatility vary with wealth, so we can take these estimates as current values or expected average values. With these inputs to the model, we find

$$\delta = 16.7$$
$$\gamma = 5.6$$

The estimate for δ may appear to be "unreasonably" high. Keep in mind, however, that studies that have estimated risk aversion have not separated it into two kinds (δ and γ). If they did separate it, their conclusions might change sharply.

Remember also that my assumptions are very restrictive. I have excluded real world "imperfections" such as liquidity constraints, habit formation, and "rule-of-thumb" consumption. And in more realistic models, consumption changes and wealth changes will be imperfectly correlated, and both risk and risk premium will depend less on wealth in the long run than in the short run.

In our simple example, the difference between γ and δ allows the risk premium to vary with wealth, which makes consumption smoother than wealth. This helps explain the "consumption smoothing puzzle." As wealth rises, the risk premium falls, so people do not increase consumption by as much as their increase in wealth.

Similarly, the difference between γ and δ helps explain the "equity premium puzzle." It allows the equity premium to be 0.06 while the real interest rate is 0.01 and consumption volatility is only 0.02. (I have not tried to explain the autocorrelation in the consumption series.) Adding "imperfections" and other real-world features will likely give full explanations for both these puzzles.

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