

A Simple and Student-Friendly Approach to the Mathematics of Bond Prices

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In his seminal paper Malkiel (1962) rigorously developed and proved five theorems showing the relationship between changes in yield to maturity and bond price movements. These theorems are important in the eyes of the academicians and practitioners and find a place in virtually all finance textbooks dealing with the pricing of bonds. The proofs of the theorems are based entirely on calculus and seem to be challenging for finance professionals and students. Hence, finance textbooks usually describe these five theorems using figures and numerical examples to illustrate the results instead of providing any logical proof. In this paper we prove the Malkiel (1962) theorems using simple algebra and provide an alternate way of understanding the results.

Introduction

In his seminal paper Malkiel (1962) rigorously examined the relationship between the yield to maturity of bonds and their market prices. He developed and proved five theorems and showed how the relationship between the changes in yield to maturity and bond price movements depend on the coupon, the time to maturity, and the existing yield to maturity. Malkiel's (1962) five bond theorems have become the cornerstone in understanding the relationship between bond prices and their various determinants. There is no book that we are aware of dealing with bond prices and related topics that does not refer to these five theorems.

The original presentation by Malkiel is very elegant and uses calculus to prove these theorems. The proofs to the theorems seem so challenging that almost all of the professional and academic textbooks state the theorems and provide only numerical examples and figures to convey the points. For example, Sharpe (1990, p. 383) presents the five theorems and uses numerical examples to show that the theorems are correct. He asserts "it is important for a bond analyst to understand thoroughly these

properties of bond prices, since they are valuable in forecasting how bond prices will respond to changes in interest rates.” Kolb (1992 pp. 229-237) accepts the difficulty of the proofs and writes that “the five bond principles discussed in this section were proven rigorously by Burton Malkiel in a now famous article. Here the principles are illustrated rather than proven.” Kolb (1992) uses numerical examples and figures to illustrate the five theorems. Francis (1991, pp. 383-385), Hirt and Block, (1999, chapter 12), Charles P. Jones (1998, pp. 292-296), and Bodie, Kane, and Marcus (2004, p. 341) state the theorems and illustrate them using figures. Francis and Ibbotson (2002, pp. 579-580) use calculus to show the inverse relationship between a bond’s price and its yields and further depict the relationships graphically.

The illustration of the theorems through figures and numerical examples simply tests the theorems and conveys the theorems as an accepted truth. Without any logical proof, it is hard to appreciate the results of the theorems. A simple and nonrigorous proof will help finance students appreciate the importance of the five theorems. In this paper, we provide a simple and student-friendly theoretical approach to understanding these theorems. Only a rudimentary knowledge of algebra is required.

Proofs of the Malkiel Theorems

The market price P of a bond is the present value of the future cash flows and is calculated using the following equation:

$$P = \frac{Cpn}{1+r} + \frac{Cpn}{(1+r)^2} + \dots + \frac{Cpn}{(1+r)^n} + \frac{FV}{(1+r)^n} . \quad (1)$$

In the above equation, FV , r , and Cpn are, respectively, the face value, the yield to maturity, and the coupon payments for the bond. If c is the coupon rate, then

$$Cpn = FV \times c . \quad (2)$$

Substituting from equation (2) in equation (1), we have the price of the bond as:

$$P = \frac{FV \times c}{1+r} + \frac{FV \times c}{(1+r)^2} + \dots + \frac{FV \times c}{(1+r)^n} + \frac{FV}{(1+r)^n} . \quad (3)$$

The coupon rate c and the yield to maturity r are both in fractions and can be expressed by the equation

$$c = r + x . \quad (4)$$

where x is a natural number. The coupon rate c of the bond is fixed. Therefore, x will increase (decrease) with a decrease (increase) in yield to maturity r . Substituting equation (4) in equation (3) we get

$$P = FV \times \left[\frac{r+x}{1+r} + \frac{r+x}{(1+r)^2} + \dots + \frac{r+x}{(1+r)^n} \right] + \frac{FV}{(1+r)^n} \quad (5)$$

$$P = FV \times \left[(r+x) \left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^n} \right) \right] + \frac{FV}{(1+r)^n}.$$

Using the formula for the present value of an annuity,^a the above equation becomes

$$P = FV \times \left[\frac{r+x}{r} - \frac{r+x}{r \times (1+r)^n} \right] + \frac{FV}{(1+r)^n}$$

$$P = FV \times \left[\frac{r}{r} + \frac{x}{r} - \frac{r}{r(1+r)^n} - \frac{x}{r(1+r)^n} \right] + \frac{FV}{(1+r)^n}$$

$$P = FV + \frac{FV \times x}{r} - \frac{FV}{(1+r)^n} - \frac{FV \times x}{r(1+r)^n} + \frac{FV}{(1+r)^n}.$$

Thus, the resulting expression for the price of bond is:

$$P = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^n} \right]. \quad (6)$$

The term in the brackets is the present value of a \$1 annuity and always will be nonnegative. From equation (6) we derive the following simple and universally known relationships:

- When $x = 0$, the coupon rate is equal to the yield to maturity and the price of the bond is equal to the face value of the bond.
- For $x > 0$, i.e., when the coupon rate is greater than the yield to maturity, the price of the bond is

$$P = FV + \text{a positive number}$$

Hence, the price of the bond is greater than the face value and the bond is a premium bond.

- For $x < 0$, i.e. when the coupon rate is less than the yield to maturity, the price of the bond is

$$P = FV + \text{a negative number}$$

The price of the bond is less than the face value and the bond is a discount bond.

^a The present value of an annuity of n years and an interest rate r is given by

$$\left(\frac{1}{r} - \frac{1}{r(1+r)^n} \right)$$

Using this simple exposition we now offer alternative proof of the Malkiel (1962) theorems.

Theorem 1: Bond prices move inversely to the yield to maturity of bonds.

To prove the theorem, we have to show that for an increase in yield to maturity, the bond price will decrease and vice versa. We can prove theorem 1 easily from equation (6), where the price of bond is given by:

$$P = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^n} \right].$$

When the yield to maturity of bond increases, the present value of \$1 annuity decreases and the value of x as per expression (4) decreases (for a premium bond, the new value of x is lower than the previous value, and for a discount bond, the new value of x is more negative than the previous value), thus reducing the price of the bond. Conversely, when the yield to maturity decreases, the value of a \$1 annuity increases and the value of x also increases (the new value of x is more than the previous value for a premium bond, and the new value of x is less negative for the discount bond), thus increasing the price of bond. Hence, whether the bond is a premium or discount bond, bond prices move inversely to bond yields.

Theorem 2: For a given change in yield to maturity from the nominal yield, the change in bond prices are greater, the longer is the term to maturity.

In his paper Malkiel (1962) defines nominal yield i_0 as

$$i_0 = \frac{Cpn}{FV}.$$

This is the same as the coupon rate. Hence, for the nominal yield (coupon rate) equal to the yield to maturity, the price of the bond is same as its face value, i.e. $P = FV$. We assume two bonds (bond 1 and bond 2) with the same coupon rate and face value but different maturities n_1 and n_2 , where $n_1 > n_2$. For the same change in the yield to maturity from nominal yield, i.e., for the same x , the prices for the two bonds are:

$$P_1 = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^{n_1}} \right] \quad (7)$$

$$P_2 = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^{n_2}} \right]. \quad (8)$$

Obviously, the longer the maturity of annuity, the higher the value of the annuity will be. Because the bracketed term in the above expressions is the present value of a \$1 annuity, we have $(P_1) > (P_2)$.

As the bonds were trading at face value before the change in yield, the absolute value of the change in bond 1 is greater than the absolute value of the change in bond 2, i.e., $|P_1 - FV| > |P_2 - FV|$

We have used absolute values because the changes in premium and discount bonds are in opposite directions. For a premium bond, x is positive and, from equations (7) and (8), the price for a longer maturity bond is greater than the price for a shorter maturity bond, i.e. $P_1 > P_2$. For a discount bond, x is negative; hence, we are subtracting more from FV in equation (7) than we are in equation (8), leading to $P_1 < P_2$.

Based on the above proof, *ceteris paribus*, for a premium bond, the price of a semiannual coupon bond will be higher than for an annual coupon bond. Similarly, for discount bond, the price of a semiannual coupon bond will be less than for an annual coupon bond. Some of the textbooks mention the result for premium bonds because it is intuitive (Ehrhardt and Brigham, 2006, pp. 199-200; Brealey and Myers, 2003, pp. 48-49; Brealey, Myers, and Marcus, 2004, p. 119). The result for discount bonds between annual versus semiannual coupon payment is counterintuitive, however; that is, for a discount bond, the value of the bond will be lower for the semiannually compounded coupons than their annual counterparts.

Theorem 3: The percentage change described in theorem 2 increases at a diminishing rate as the time to maturity increases.

Consider three bonds with same coupon rate and same face value but maturity N , $N+1$, and $N+2$. The above theorem says that for a given change in yield from the nominal yield, the percentage change from N to $N+1$ will be greater than the percentage change from $N+1$ to $N+2$. Using equation (6), the prices for the three bonds (with maturities N , $N+1$, and $N+2$) are

$$P_1 = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^N} \right] \quad (9)$$

$$P_2 = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^{N+1}} \right] \quad (10)$$

$$P_3 = FV + FV \times x \left[\frac{1}{r} - \frac{1}{r(1+r)^{N+2}} \right]. \quad (11)$$

As the bonds were trading at their face value before the change in yield, the percentage change in the price of the three bonds is

$$\frac{P_1 - FV}{FV}, \frac{P_2 - FV}{FV} \text{ and } \frac{P_3 - FV}{FV}, \text{ respectively}$$

for bonds with maturity N , $N+1$, and $N+2$. To show that percentage price changes increase at a diminishing rate as the maturity of a bond increases, we have to show that the difference in $P_2 - P_1$ is greater than the difference $P_3 - P_2$. From equations (9), (10), and (11):

$$P_2 - P_1 = FV \times x \left[\frac{1}{r(1+r)^N} - \frac{1}{r(1+r)^{N+1}} \right] \quad (12)$$

$$P_3 - P_2 = FV \times x \left[\frac{1}{r(1+r)^{N+1}} - \frac{1}{r(1+r)^{N+2}} \right]. \quad (13)$$

Dividing equation (13) by equation (12), we get

$$P_3 - P_2 = \frac{P_2 - P_1}{1+r}.$$

For $r > 0$, the denominator in the above equation is greater than 1 and $P_2 - P_1$ is greater than $P_3 - P_2$, indicating that the percentage price change increases at a diminishing rate as the maturity of a bond increases.

Theorem 4: Price movements resulting from equal absolute (or, what is the same, from equal proportionate) increases and decreases in the yield to maturity are asymmetric; that is, a decrease in yield to maturity raises bond prices more than the same increase in yield to maturity lowers prices.

From equations (4) and (6), for an increase of k in the yield to maturity, the new price of the bond is:

$$P_k = FV + FV \times (x - k) \left[\frac{1}{r+k} - \frac{1}{(r+k)(1+r+k)^n} \right].$$

Subtracting equation (6) from above, we get the change in price as:

$$\Delta P_k = FV \times (x - k) \left[\frac{1}{r+k} - \frac{1}{(r+k)(1+r+k)^n} \right] - FV \times x \left[\frac{1}{r} - \frac{1}{(r)(1+r)^n} \right]. \quad (14)$$

For a similar decrease k in the yield to maturity the changed price of the bond is:

$$P_{-k} = FV + FV \times (x + k) \left[\frac{1}{r - k} - \frac{1}{(r - k)(1 + r - k)^n} \right].$$

Subtracting equation (6) from above, we get the change in price as:

$$\Delta P_{-k} = FV \times (x + k) \left[\frac{1}{r - k} - \frac{1}{(r - k)(1 + r - k)^n} \right] - FV \times x \left[\frac{1}{r} - \frac{1}{(r)(1 + r)^n} \right]. \quad (15)$$

We have to show that the absolute value of the price changes due to an equal amount of increase and decrease in the yield to maturity are different and that a decrease in yield to maturity raises bond prices more than the same increase in yield to maturity lowers prices, i.e., $|\Delta P_k| < |\Delta P_{-k}|$. The second term of equation (14) and (15) are same hence we have to show if:

$$FV \times (x - k) \left[\frac{1}{r + k} - \frac{1}{(r + k)(1 + r + k)^n} \right] < FV \times (x + k) \left[\frac{1}{r - k} - \frac{1}{(r - k)(1 + r - k)^n} \right].$$

The present value of an annuity is always positive and is higher for lower yield to maturity and is lower for higher yield to maturity and $(x+k)$ is greater than $(x-k)$. Hence, the right side is greater than the left side, indicating that price movements from equal absolute increase and decrease in yield to maturity are asymmetric and that a decrease in yield to maturity raises the bond price more than the same increase in the yield to maturity lowers the bond price.

Theorem 5: The higher is the coupon carried by the bond, the smaller will be the percentage price fluctuations for a given change in the yield to maturity except for one year securities and consols.

Ceteris paribus, the higher the coupon, the higher is the price of the bond. For two bonds, bond 1 and bond 2, with the coupon rate for bond 1 (c_1) greater than the coupon rate for bond 2 (c_2), using the relationship in equation 4, the prices of the bonds are:

$$P_1 = FV + FV \times (c_1 - r) \left[\frac{1}{r} - \frac{1}{r(1 + r)^n} \right] \quad (16)$$

$$P_2 = FV + FV \times (c_2 - r) \left[\frac{1}{r} - \frac{1}{r(1 + r)^n} \right]. \quad (17)$$

As $c_1 > c_2$, from equations (16) and (17), it is evident that $P_1 > P_2$. For a decrease δ in the yield to maturity, the prices of the two bonds are given by

$$P_{1\delta} = FV + FV \times (c_1 - r + \delta) \left[\frac{1}{r - \delta} - \frac{1}{(r - \delta)(1 + r - \delta)^n} \right] \quad (18)$$

$$P_{2\delta} = FV + FV \times (c_2 - r + \delta) \left[\frac{1}{r - \delta} - \frac{1}{(r - \delta)(1 + r - \delta)^n} \right]. \quad (19)$$

The difference in the prices of the bonds is entirely due to the difference in the second term of the above price equations. The first term FV is the same across all equations. Hence, discarding FV, the percentage price changes for the two bonds can be written as:

Percentage price change for bond 1

$$= \frac{(c_1 - r + \delta) \left[\frac{1}{r - \delta} - \frac{1}{(r - \delta)(1 + r - \delta)^n} \right]}{(c_1 - r) \left[\frac{1}{r} - \frac{1}{r(1 + r)^n} \right]} - 1$$

Percentage price change for bond 2

$$= \frac{(c_2 - r + \delta) \left[\frac{1}{r - \delta} - \frac{1}{(r - \delta)(1 + r - \delta)^n} \right]}{(c_2 - r) \left[\frac{1}{r} - \frac{1}{r(1 + r)^n} \right]} - 1.$$

To prove that higher coupons result in smaller percentage price fluctuations, we have to prove that the percentage price change for bond 2 is greater than the percentage price change for bond 1. Essentially, we have to prove that:

$$\frac{(c_1 - r + \delta)}{(c_1 - r)} < \frac{(c_2 - r + \delta)}{(c_2 - r)}. \quad (20)$$

To prove the above, we use one of the basic properties of natural numbers. If there are two numbers p and q, such that p > q, then the addition of a constant k gives us,

$$\frac{p + k}{p} < \frac{q + k}{q}.$$

The above result is the same as saying that the percentage change in p is less than the percentage change in q, or in the context of equation (20), the higher the coupon rate, the smaller is the percentage price fluctuation due to changes in the yield to maturity of bond.

Conclusion

Malkiel (1962) provides five theorems illustrating the relationship between the yield to maturity of bond and bond prices. He rigorously proved the theorems, but the proofs are based entirely on calculus and are beyond the scope of most professional or academic textbooks on the pricing of bonds. In this paper we prove the theorems using simple algebra. The proofs will help students of finance in understanding the relationships of the movements in yield to maturity and bond prices.

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