

Pricing Stock Options under Expected Increasing and Decreasing Price Cases

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Several studies have shown that the distributions of logarithmic returns of many securities exhibit persistent skewness. The S&P 500 often follows patterns of persistent change that are characterized by skewness. In option pricing, skewness in a binomial process impacts the values of the up and down parameters and changes their asymptotic properties such that for a large number of subperiods, they depend on the variance, skewness, and mean. We illustrate how the Johnson, Pawlukiewicz, and Mehta skewness-adjusted model can be used to calibrate a binomial tree for increasing and decreasing stock price cases where the end-of-the period distribution is characterized by skewness. We then show spot and futures option price differences between the skewness-adjusted binomial model and the traditional binomial option pricing model from simulations in which there is an expectation that the underlying security prices will be increasing or decreasing similar to the stock index trends that have been observed historically.

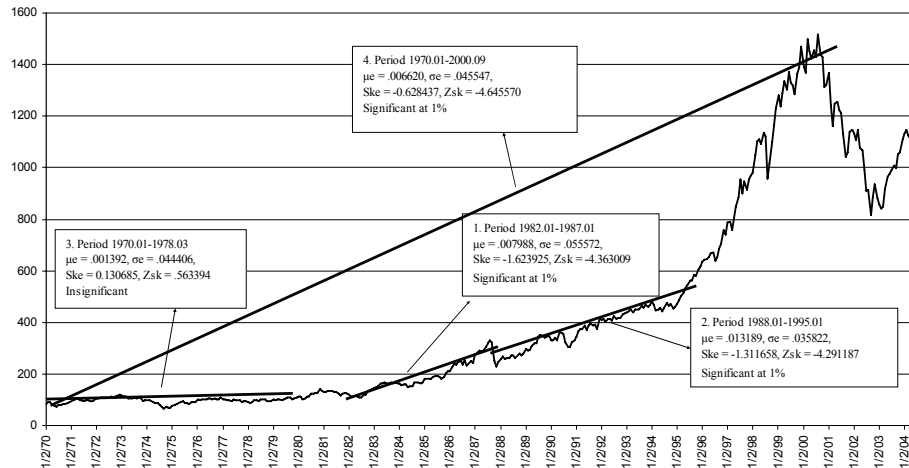
Introduction

Several empirical studies have provided evidence that the distributions of stock and stock index returns exhibit persistent skewness (for example, Kon, 1984; Aggarwal and Rao, 1990; and Turner and Weigel, 1992). We conduct a statistical analysis of the S&P 500 using the D'Agostino, Belanger, and D'Agostino tests of normality (1990) and find that over multiple periods of time the index often has followed patterns of persistent increases that are characterized by skewness. For example, the intermediate periods from 1982 to 1987 and 1988 to 1995 saw a trend of increasing stock prices characterized by positive average logarithmic returns and negative skewness for the S&P 500 (Figure 1). Similarly, normality tests applied to

the longer period from 1970 to 2000 also show a period characterized by a positive average logarithmic return and negative skewness for the S&P 500. In contrast, for the period from 1970 to 1978, the D'Agostino, Belanger, and D'Agostino tests show a trend of stable prices characterized by a low average logarithmic return, but with skewness that is insignificant.

Figure 1—Historical S&P 500 Rates

Boxes show the monthly mean, μ_e , standard deviation, σ_e , and skewness, Sk_e , of the S&P 500's logarithmic returns, $\ln(S_t/S_{t-1})$. Also shown is the Z-value, Z_{sk} , for testing the null hypothesis that skewness is zero. The test statistic is that suggested by D'Agostino, Belanger, and D'Agostino (1990)



Empirical studies by Black (1975), MacBeth and Merville (1979), and Emanuel and MacBeth (1982) have reported pricing biases associated with the Black-Scholes (B/S) option pricing model (OPM) (1973). These biases generally are thought to be the result of the model's assumption that the option's underlying security's logarithmic return is normally distributed. Studies by Stein and Stein (1991), Wiggins (1987), and Heston (1993) have demonstrated that when skewness exists, the Black-Scholes model consistently misprices options. To address the pricing bias resulting from the assumption of normality, Jarrow and Rudd (1982) and Corrado and Tie Su (1996) have extended the Black-Scholes model to account for cases in which there is skewness in the underlying security's return distribution. Similarly, Câmara and Chung (2006) and Johnson, Pawlukiewicz, and Mehta (JPM) (1997) have extended the Cox, Ross, and Rubinstein (CRR) (1979) and Rendleman and Bartter (RB) (1979) binomial option pricing model to include skewness. In addressing skewness, Câmara and Chung show how the up (u) and down (d) parameters in the binomial process are obtained from skewness and kurtosis of the distribution implied by market prices. Johnson, Pawlukiewicz, and Mehta, in turn, show that in

the binomial modeling of security price patterns, the existence of skewness impacts not only the values of the up and down parameters, but also the probabilities of the underlying security increasing or decreasing each period. Specifically, a binomial process that converges to an end-of-the-period distribution of logarithmic returns that is normal will have equal probabilities of the stock increasing or decreasing each period, while one that converges to a distribution that is skewed will not.

In their paper, Johnson, Pawlukiewicz, and Mehta also show that the presence of skewness affects the relative contribution of the mean to the values of u and d . In the case of a positive mean, the mean becomes more important in determining the value of u , the greater the negative skewness. In contrast, in the case of a negative mean, the mean becomes more important to the value of d , the greater the positive skewness. The presence of skewness also changes the asymptotic properties of the u and d parameters in the Johnson, Pawlukiewicz, and Mehta skewness model. Specifically, for a large number of subperiods, n , the Cox, Ross, and Rubinstein/Rendleman and Bartter model depends only on the volatility of the underlying asset. Skewness, though, changes the order of magnitude as n becomes large, making u and d dependent on all three moments—variance, skewness, and mean.¹

Objective

The noted observations on stock price trends suggest that in cases where stock prices are expected to increase or decrease, pricing biases may result when using the Cox, Ross, and Rubinstein/Rendleman and Bartter binomial option pricing model or the Black-Scholes option pricing model where the underlying security's logarithmic return is based on the assumption of normality. The purpose of this paper is three-fold:

1. To show the applicability of the Johnson, Pawlukiewicz, and Mehta skewness-adjusted binomial option-pricing model to the binomial pricing of stock options when increasing or decreasing stock price cases are expected;
2. To illustrate the pricing differences that exist between the Johnson, Pawlukiewicz, and Mehta skewness model and the Cox, Ross, and Rubinstein/Rendleman and Bartter and Black-Scholes option pricing model models, and
3. To extend the Johnson, Pawlukiewicz, and Mehta skewness model from the pricing of spot options to the pricing of futures options.

We begin by examining the characteristics of skewed binomial distributions that have different expected values. Next, we define Johnson, Pawlukiewicz, and Mehta's

¹ The skewness-extended Black-Scholes models have the feature that the option price depends only on the underlying security's second and third moments and not on the mean. Because the binomial model converges to the Black-Scholes model, one could infer that the observed pricing biases associated with the Black-Scholes model may be due not only to the omission of skewness, but also the mean.

estimating formulas for the binomial probabilities and the up and down parameters that incorporate skewness. Using the skewness-adjusted model, we illustrate how the valuation of a stock's call and put options are impacted by binomial processes that are calibrated to end-of-the period distributions that reflect increasing and decreasing stock price cases. We then show with a numerical simulation the differences in option prices found with the Cox, Ross, and Rubinstein/Rendleman and Bartter binomial model and the skewness-adjusted binomial model given different stock price scenarios. After comparing the skewness model with the Cox, Ross, and Rubinstein/Rendleman and Bartter model, we then show how the Johnson, Pawlukiewicz, and Mehta model can be extended to the pricing of futures options. We conclude the paper with a note on how the skewness-adjusted binomial option pricing model can be estimated using skewed volatility smiles.

Skewness-Adjusted Binomial Model

Cox, Ross, and Rubinstein derive the u and d parameters for the binomial option pricing model by setting the equations for the expected value and variance of the logarithmic return of the underlying stock equal to their empirical values. The resulting equations are solved simultaneously for u and d under the assumption that the probability of the stock increasing in one period (q) is 0.5. When skewness exists, though, the probability of an increase in the stock price in one period must be treated as an unknown. In this case, the expressions for the mean, variance, and skewness of the return distribution are set equal to their empirical values and the resulting system of equations is solved simultaneously for the three unknown parameters: u , d , and q . In both approaches, the methodology for estimating u and d is based on specifying the characteristics of a distribution of logarithmic returns that follows a binomial process.

Binomial Process

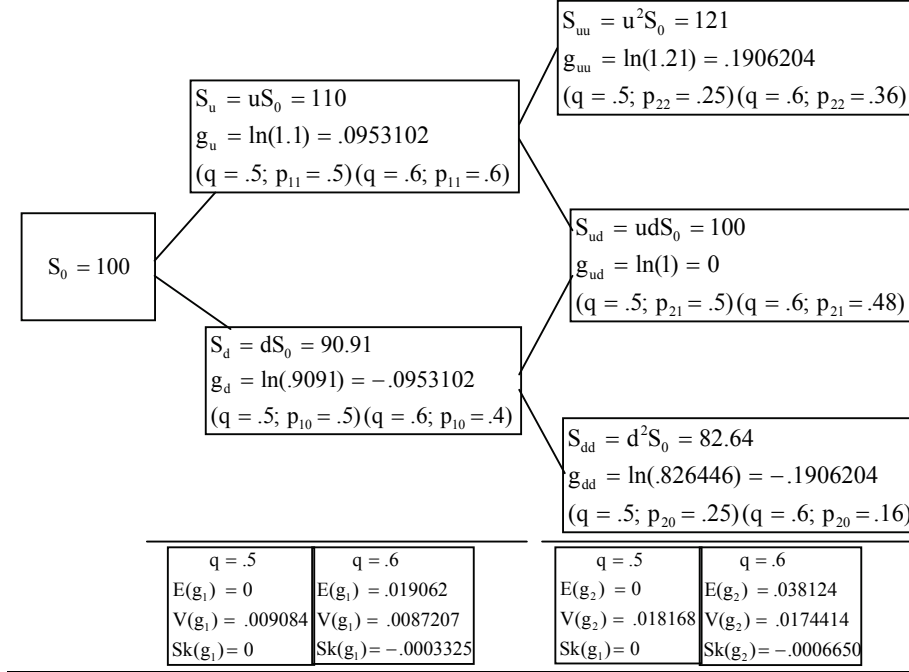
A binomial distribution of a stock price, S , and its corresponding logarithmic return, $g_n = \ln(S_n/S_0)$, is shown in Exhibit 1. The possible returns presented in the exhibit reflect an initial price of $S_0 = 100$, an upward parameter of $u = 1.1$, and a downward parameter of $d = 1/u = 0.9091$. The distribution's expected logarithmic return, variance, and skewness, Sk , can be defined in terms of u , d , the number of periods, n , and the probability of the rate increasing in one period, q :

$$E(g_n) = \sum_{j=0}^n p_{nj} g_{nj} \quad (1)$$

$$E(g_n) = n[q \ln u + (1 - q) \ln d] = nE(g_1)$$

Exhibit 1—Binomial Process of Logarithmic Returns

$u = 1.1$ and $d = 1/1.1 = 0.9091$



$$V(g_n) = E[g_n - E(g_n)]^2 = \sum_{j=0}^n p_{nj} [g_{nj} - E(g_n)]^2 \quad (2)$$

$$V(g_n) = n q (1 - q) [\ln(u/d)]^2 = nV(g_1)$$

$$Sk(g_n) = E[g_n - E(g_n)]^3 = \sum_{j=0}^n p_{nj} [g_{nj} - E(g_n)]^3 \quad (3)$$

$$Sk(g_n) = n[q(1 - q)^3 - q^3(1 - q)] [\ln(u/d)]^3 = nSk(g_1)$$

where:

g_1 = Logarithmic return for one period;

j = Number of increases in n periods;

p_{nj} = Probability of j increases in n periods, where:

$$p_{nj} = \frac{n!}{(n-j)!j!} q^j (1-q)^{n-j}.$$

In the example in Exhibit 1, the expected value, variance, and skewness are shown for each period's distribution ($n = 1$ and 2) at the bottom of the exhibit under two cases: $q = 0.5$ and $q = 0.6$. In examining each distribution's moments, several well-known characteristics of the binomial process should be noted. First, the expected value, variance, and skewness of the logarithmic return are equal to their moment values for one period times the number of periods defining the total period. Second, the skewness for each period's distribution is zero when there is an equal probability of the rate increasing or decreasing in one period, and skewed when there is not (as illustrated in the case when $q = 0.6$). Thus, a sufficient condition for symmetry is that there is an equal probability of the stock increasing or decreasing in each period. Finally, the expected value is equal to zero for the case in which $q = 0.5$.

The last property is the result of assuming not only that there is an equal probability of an increase or decrease each period, but also that u and d are inversely proportional or, equivalently, that the proportional increase in each period ($\ln u$) is equal in absolute value to the proportional decrease ($\ln d$). If the distribution of the logarithmic return at the end of n periods had, for example, a positive expected value and zero skewness, then the underlying binomial process would have been characterized by the proportional increase in each period exceeding in absolute value the proportional decrease, with the probability of the increase in each period being 0.5 . If the distribution also had negative skewness, then the probability of the increase in one period would have exceeded 0.5 . On the other hand, if the distribution of the logarithmic return had a negative expected value and zero skewness, then the underlying binomial process would have been characterized by the proportional decrease in each period exceeding in absolute value the proportional increase and with $q = 0.5$. If the distribution also had a positive skewness, then q would have been less than 0.5 :

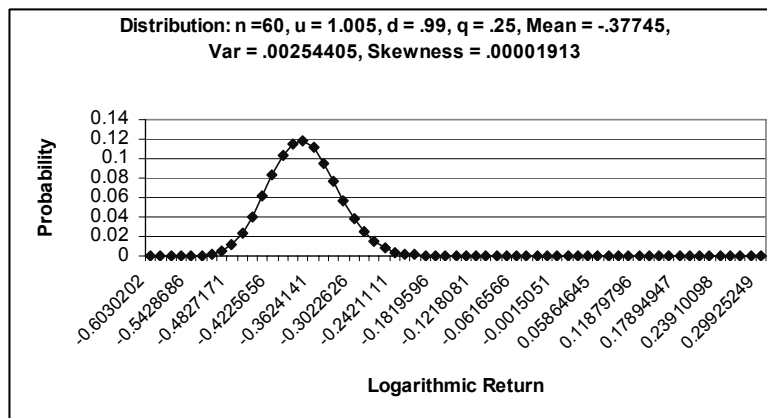
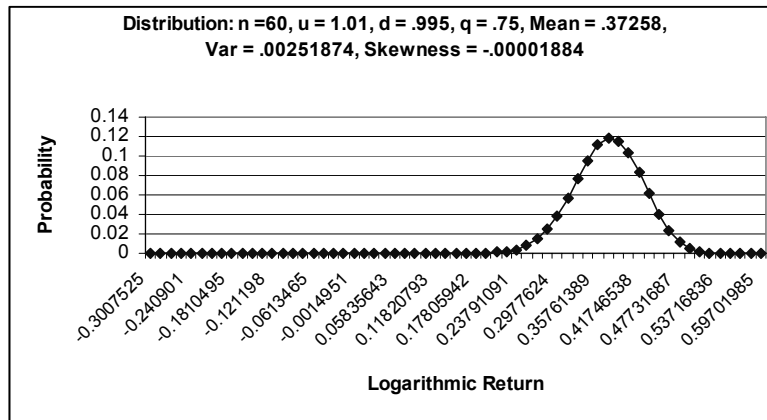
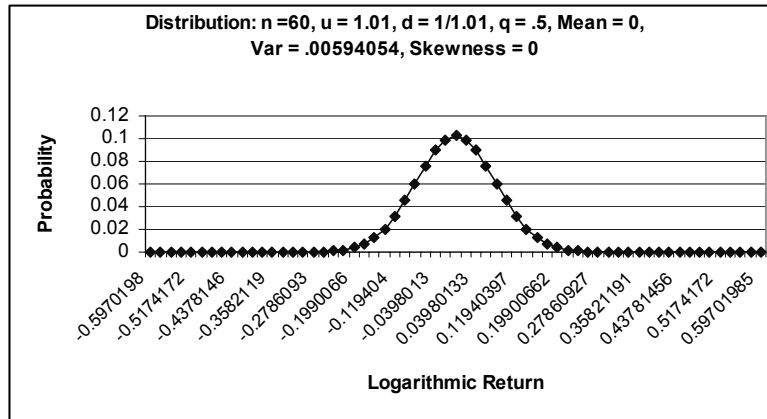
$$E(g_n) > 0 \text{ and } Sk(g_n) = 0 \Rightarrow |\ln u| > |\ln d| \text{ and } q = 0.5$$

$$E(g_n) > 0 \text{ and } Sk(g_n) < 0 \Rightarrow |\ln u| > |\ln d| \text{ and } q > 0.5$$

$$E(g_n) < 0 \text{ and } Sk(g_n) = 0 \Rightarrow |\ln d| > |\ln u| \text{ and } q = 0.5$$

$$E(g_n) < 0 \text{ and } Sk(g_n) > 0 \Rightarrow |\ln d| > |\ln u| \text{ and } q < 0.5$$

Exhibit 2 shows three end-of-the period distributions resulting from a binomial process in which the number of periods to expiration is $n = 60$. The probability distribution in the top exhibit is generated from a binomial process in which $u = 1.01$, $d = 1/1.01$ ($|\ln u| = |\ln d|$) and $q = 0.5$. The distribution approaches a normal distribution with $E(g_n) = 0$, $V(g_n) = 0.00594054$, and $Sk(g_n) = 0$. The middle distribution

Exhibit 2—Constant, Increasing, and Decreasing Stock Distributions

reflects an increasing stock price case in which $u = 1.01$, $d = 0.995$ ($|\ln u| > |\ln d|$), and $q = 0.75$. The distribution is negatively skewed with a positive mean: $E(g_n) = 0.37258$, $V(g_n) = 0.00251874$, and $Sk(g_n) = -0.00001884$. Finally, the bottom distribution reflects a decreasing stock price case in which $u = 1.005$, $d = 0.99$ ($|\ln u| < |\ln d|$), and $q = 0.25$. This distribution is positively skewed with a negative mean: $E(g_n) = -0.37745$, $V(g_n) = 0.00254405$, and $Sk(g_n) = 0.00001913$.

Skewness-Adjusted u , d , and q Parameters

When skewness exists, the u , d , and q values defining a binomial process can be found by setting the equations for the population moments (equations (1), (2), and (3)) equal to their respective empirical values, then solving the resulting equation system simultaneously for u , d , and q . That is:

$$n[q \ln u + (1 - q) \ln d] = \mu_e \quad (4a)$$

$$nq(1 - q)[\ln(u/d)]^2 = V_e \quad (4b)$$

$$n[q(1 - q)^3 - q^3(1 - q)][\ln(u/d)]^3 = \delta_e, \quad (4c)$$

where:

μ_e , V_e , δ_e = The empirical values of the mean, variance, and skewness of the logarithmic return for a period equal in length to n periods.

The values of $\ln u$, $\ln d$, and q that satisfy this system of equations are:

$$\ln u = \frac{\mu_e}{n} + \left[\frac{V_e(1 - q)}{nq} \right]^{1/2} \quad (5)$$

$$\ln d = \frac{\mu_e}{n} - \left[\frac{V_e q}{n(1 - q)} \right]^{1/2} \quad (6)$$

$$q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4V_e^3}{n\delta_e^2} + 1 \right]^{-1/2}, \quad - \text{ if } \delta_e > 0, \text{ and } + \text{ if } \delta_e < 0. \quad (7)$$

Several technical points should be noted in examining the equations for the skewness-adjusted parameters. First, note that in equation (4c) the direction of the skewness of the empirical distribution has an impact on the size of q . Because n and $\ln(u/d)$ are strictly positive, if δ_e is positive (negative), then q is less (greater) than 0.5. If skewness is zero, then $q = 0.5$, and the equations simplify to the Cox, Ross, and Rubinstein formulas for $\ln u$ and $\ln d$:

$$\ln u = \frac{\mu_e}{n} + \left[\frac{V_e}{n} \right]^{1/2} \quad (8a)$$

$$\ln d = \frac{\mu_e}{n} - \left[\frac{V_e}{n} \right]^{1/2} \quad (8b)$$

Second, the presence of skewness affects the relative contribution of the mean to the values of u and d . In the case of a positive mean, the mean becomes more important in determining the value of u , the greater the negative skewness (or equivalently the more q exceeds 0.5). By contrast, in the case of a negative mean, the mean becomes more important in determining the value of d , the greater the positive skewness. For example, if $u = 1.1$ and $d = 0.95$ and there was no skewness ($q = 0.5$), then $E(g_1)$ would equal $.0220189$ ($= [(0.5) \ln(1.1) + (0.5) \ln(0.95)]$) and $V(g_1)$ would equal $.005373145$ ($= 0.5[\ln(1.1) - .0220189]^2 + 0.5[\ln(0.95) - .0220189]^2$). If these values were the estimated ones, then $\ln u$ would be equal to 0.09531 ($u = 1.1$) and the mean would contribute 23 percent to the value of u ($0.0220189/0.09531$) and the variance would contribute 77 percent ($(.005373145)^{1/2}/0.09531$). If skewness were present such that $q = 0.6$, then $E(g_1) = 0.0365$, $V(g_1) = 0.00514$, and $Sk(g_1) = -0.0001503$. If these values were the estimated ones, then the $\ln u$ would again be 0.09531 , but q would be 0.6 and the mean's contribution to u would be 38.3 percent ($0.0365/0.09531$). Finally, if skewness were such that $q = 0.8$, then $E(g_1) = 0.057186$, $V(g_1) = 0.0058138$, and $Sk(g_1) = -0.0006649$. Again, if these were the estimated moment values, then the $\ln u$ would still be 0.09513 , but q would equal 0.8 and the mean's contribution to the value of u would be 60 percent ($0.057186/0.09531$).

Finally, as Johnson, Pawlukiewicz, and Mehta (1997) show, the skewness-adjusted u and d parameters do not have the same asymptotic properties as the Cox, Ross, and Rubinstein/Rendleman and Bartter parameters. In the Cox, Ross, and Rubinstein/Rendleman and Bartter model, the μ_e/n term in equation (8) goes to zero faster than the $[V_e/n]^{1/2}$ term does. Thus, for large n , the Cox, Ross, and Rubinstein/Rendleman and Bartter model depends only on the volatility of the underlying asset:

$$u = e^{\left[\frac{V_e}{n} \right]^{1/2}}$$

$$d = e^{-\left[\frac{V_e}{n} \right]^{1/2}}$$

In the skewness model, equation (5) for $\ln u$ includes a $(1-q)/q$ term, and equation (6) for $\ln d$ includes a $q/(1-q)$ term, both of which change the order of magnitude as n gets large. Specifically, for the case of negative skewness, the $(1-q)/q$ term can be rewritten as:

$$\frac{1-q}{q} = \frac{4(V_e)^3}{2n(\delta_e)^2 + 4(V_e)^3 + 2\sqrt{n(\delta_e)^2 + 4(V_e)^3} \sqrt{(\delta_e)^2}} \quad (9)$$

The expression $(1-q)/q$ in equation (9), in turn, has the same order of magnitude as $1/n$. This can be seen by observing that the term $[(1-q)/q]/[1/n]$ approaches the constant $(V_e)^3/(\delta_e)^2$ as n gets large. That is:

$$\frac{(1-q)/q}{1/n} = \frac{4(V_e)^3}{2(\delta_e)^2 + [4(V_e)^3]/n + 2\sqrt{(\delta_e)^2 + [4(V_e)^3]/n} \sqrt{(\delta_e)^2}} \quad (10)$$

and in the limit:

$$\frac{(1-q)/q}{1/n} \rightarrow \frac{(V_e)^3}{(\delta_e)^2}, \quad \text{as } n \rightarrow \infty \quad (11)$$

With $(1-q)/q$ having the same order of magnitude as $1/n$, the term $[(1-q)/q][(V_e)/n]^{1/2}$ in equation (5) approaches a constant multiplied by $[(V_e)]^{1/2}/n$, as n becomes large. As a result, for the case of large n , the second term in equation (5) approximates a constant divided by n , which is in the same form as the first term, μ_e/n . Consequently, both terms in equation (5) for $\ln u$ contribute equally, even when n is large. Thus, as n becomes large, $\ln u$ depends not only on the variance and skewness, but also on the mean. By contrast, equation (6) for $\ln d$ is defined in terms of $q/(1-q)$. In this case, as n gets large, $\ln d$ approaches the constant δ_e/V_e and only the μ_e/n term approaches zero. Thus, for the case of negative skewness with a large n , $\ln d$ depends on the variance and skewness, but not on the mean, while $\ln u$ depends on all three moments. Just the opposite asymptotic relationships occur when skewness is positive ($\delta_e > 0$). In this case, the mean, variance, and skewness determine $\ln d$ (even when n is large), while just the variance and skewness determine $\ln u$.

To illustrate the relative importance of the mean for the case of large n , suppose the estimated annualized mean, variance, and skewness of a stock that has an option expiring in one year are

$$\mu_e^A = 0.0733$$

$$V_e^A = 0.0103164$$

$$\delta_e^A = -0.000302485$$

If we subdivide the year into 10 subperiods ($n = 10$), then the mean term in equation (5) contributes 26.20 percent to the value of $\ln u$:

$$[\mu_e / n] / [\ln u] = [0.0733/10] / .02798 = 0.2620$$

and the second term (defined in terms of the variance and skewness) contributes 73.80 percent to the value of $\ln u$; if we subdivide the year into 100 periods ($n = 100$), the mean term contributes 18.76 percent. For larger n , the contribution of the mean is approximately the same: when $n = 500$, the relative contribution of the mean is 17.58 percent, and when $n = 10$ million, the contribution is 17.248 percent.² By contrast, if skewness is zero (the Cox, Ross, and Rubinstein/Rendleman and Bartter model), the relative contribution of the mean term is 18.59 percent when $n = 10$, 6.73 percent when $n = 100$, 3.13 percent when $n = 500$, .72 percent when $n = 10,000$, and .022 percent when $n = 10$ million.

Option Valuation

If a stock is expected to increase in the future, then the value of an option on that stock should be valued by a binomial model that reflects a positive expected logarithmic return and possibly negative skewness. In contrast, if a stock is expected to decrease in the future, then the value of an option on the stock should be valued by a binomial model that reflects a negative expected logarithmic return and possibly positive skewness.

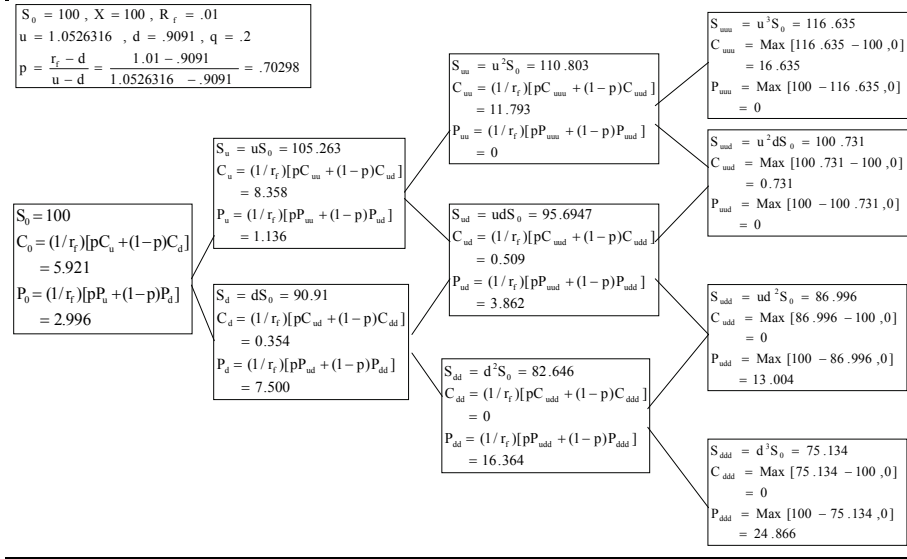
As an example, suppose a stock currently is priced at 100 and there is a market expectation of a bear market over the next year such that the distribution of logarithmic returns for the stock has the following annualized mean, variance, and skewness: $\mu_e^A = -0.19794536$, $V_e^A = 0.010314819$, and $\delta_e^A = 0.00090724$. Given these estimated moment values, the four-month u , d , and q values defining a three-period binomial tree ($n = 3$) that would calibrate a binomial distribution to this empirical distribution would be $u = 1.0526316$, $d = 0.9091$ and $q = 0.2$:

$$u = e^{\frac{\mu_e^A}{n} + \left[\frac{V_e^A(1-q)}{nq} \right]^{1/2}} = e^{\frac{-0.19794536}{3} + \left[\frac{(0.010314819)(1-0.2)}{(3)(0.2)} \right]^{1/2}} = 1.0526316$$

² It should be noted that because all the moments contribute to either $\ln u$ or $\ln d$ as n becomes large, the n value in which the skewness-adjusted model approaches a continuous one depends on the relative values of μ_e , V_e , and δ_e . For the case of $\delta_e < 0$, the term $(1-q)/q$ approaches a constant divided by n in the limit. The critical value, n^* , therefore can be found by solving for the n that makes $(1-q)/q$ (equation (9)) equal to a large proportion (e.g., 0.99) of the limit (equation (11)). Defining the proportion as $1 - \varepsilon$, where ε is equal to the proportion of error (e.g., $\varepsilon = 0.01$), the n^* that is equal to $1 - \varepsilon$ of the limit is

$$n^* = \frac{(V_e)^3(1-\varepsilon)(1+\sqrt{1+\varepsilon})}{(\delta_e)^2\varepsilon}$$

(The same equation also holds for the case of $\delta_e > 0$.) In the illustrative example ($\mu_e = 0.0733$, $V_e = 0.0103164$, and $\delta_e = -0.000302485$), $n^* = 2,382$ for an $\varepsilon = 0.01$.

Exhibit 3—Binomial Call and Put Values under Decreasing Stock Price Case

$$d = e^{\frac{\mu_e^A}{n} - \left[\frac{V_e^A q}{n(1-q)} \right]^{1/2}} = e^{\frac{-0.19794536}{3} + \left[\frac{(0.010314819)(0.2)}{(3)(1-0.2)} \right]^{1/2}} = 0.9091$$

$$q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4(V_e^A)^3}{n(\delta_e^A)^2} + 1 \right]^{-1/2} = \frac{1}{2} - \frac{1}{2} \left[\frac{4(0.010314819)^3}{(3)(0.00090724)^2} + 1 \right]^{-1/2} = 0.2$$

Exhibit 3 shows the binomial valuations of a European call option, C_0 , and put option, P_0 , on the stock, with each option having an exercise price of $X = 100$ and one-year expiration [or, equivalently, an expiration at the end of the three four-month periods ($n = 3$)] and with the four-month risk-free rate assumed to be $R_f = 1$ percent. The binomial option values of $C_0 = 5.921$ and $P_0 = 2.996$ are obtained recursively by determining the options' four intrinsic values in period 3 given the four possible stock prices and then using the single-period binomial option pricing model at each node to price the options equal to the values of their replicating portfolios:

$$C_0^* = \frac{1}{r_f^n} \left[\sum_{j=0}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{(n-j)} \left[\text{Max}((u^j d^{(n-j)} S_0) - X, 0) \right] \right]$$

$$P_0^* = \frac{1}{r_f^n} \left[\sum_{j=0}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{(n-j)} \left[\text{Max}(X - (u^j d^{(n-j)} S_0), 0) \right] \right]$$

where:

r_f = One plus the risk-free rate per period, R_f ;

$$p = (r_f - d)/(u - d).$$

In contrast to a decreasing stock price case, suppose the market expects the stock to increase over the next year, with the distribution of the stock's logarithmic returns having the following estimated annualized moments of $\mu_e^A = 0.19796845$, $V_e^A = 0.010316438$, and $\delta_e^A = -0.00090746$. The four-month u , d , and q values for a three-period binomial tree that would calibrate a binomial distribution to this distribution would be $u = 1.1$, $d = 0.95$, and $q = 0.8$. Exhibit 4 shows the binomial valuations of the stock, call, and put: $C_0 = 6.235$ and $P_0 = 3.294$.

A comparison of the option values for the two cases shows:

1. The value of the call option under the increasing-stock price case is 6.235 compared to only 5.921 under the decreasing-price scenario.
2. The value of the put option under the increasing-price case is 2.996 compared to a put value of 3.294 under the decreasing-price scenario.

Thus, the binomial pricing of a tree calibrated to an increasing-price case results in values of a call option that are greater and a put option that are lower than the values obtained using a tree calibrated to a decreasing-price scenario. Note that given an option's limited loss feature, such price relations may not always be the case. That is, it is possible that a call option could be priced less under an increasing price scenario, where the benefits of the call's limited loss features are less than under a decreasing case. Similarly, it is possible that a put could be valued less under a decreasing price case than an increasing one.

Exhibit 4—Binomial Call and Put Values under Increasing Stock Price Case

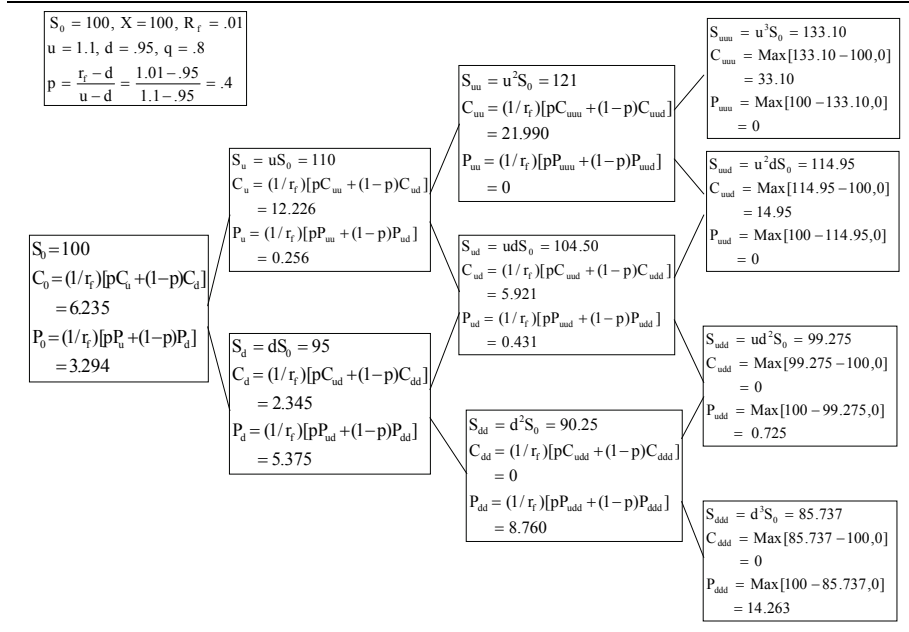


Exhibit 5—Valuation of a Call Option Using a 60-Period Binomial Model

$n = 60$, $R_f^A = 0.06$, $S_0 = 100$, $X = 100$, Expiration = 1 year

Stock-Price Case	Call Price	American Put Price	European Put Price
Constant Price Trend			
$\mu_e^A = 0$			
$V_e^A = .02352883$			
$\delta_e^A = 0$	9.17	4.13	3.09
$u = 1.02$, $d = 1/1.02$, $q = 0.5$			
Increasing Price Trend			
$\mu_e^A = .74036^a$			
$V_e^A = .01002599$			
$\delta_e^A = -.00014965$	7.71	2.60	2.05
$u = 1.02$, $d = 0.99$, $q = 0.75$			
Decreasing Price Trend			
$\mu_e^A = -.71764^b$			
$V_e^A = .00960186$			
$\delta_e^A = .00014026$	7.54	2.50	1.88
$u = 1.01$, $d = 0.980392$, $q = 0.25$			

μ_e^A , V_e^A , and δ_e^A are the annualized mean, variance, and skewness of the stock's logarithmic return. u and d are the upward and downward parameters for a period of length 6 days (360 days/60)

^a $\mu_e = .74036$ would imply that, on average, the price after one year would increase from 100 to 208.87 ($= 100(1 + (.74036/60))^{60}$)

^b $\mu_e = -.71764$ would imply that, on average, the price after one year would decrease from 100 to 48.58 ($= 100(1 - (.71764/60))^{60}$)

Exhibit 5 shows the binomial values for American and European call and put options determined for a more realistic case of $n = 60$. The call and put options each have exercise prices of 100 and expire in one year.³ The tree is subdivided into 60 periods of length six days, with the underlying stock assumed to be currently priced at 100 and paying no dividends and with the annualized risk-free rate assumed to be $R_f^A = 0.06$ (period rate of 0.00097152). The options are valued for the following price scenarios:

- A constant stock price trend in which the distribution at the end of the one-year period is assumed to be characterized by moment values of $\mu_e = 0$, $V_e = 0.02352883$, and $\delta_e = 0$. Using equations (8a), (8b), and (8c), the u and d values (reflecting a six-day period) that calibrate a 60-period binomial tree to this end of the period distribution are $u = 1.02$, $d = 0.980392 = 1/1.02$, and $q = 0.5$.
- An increasing stock price trend in which the distribution at the end of the one-year period is assumed to be characterized by moment values of $\mu_e = 0.74036$, $V_e = 0.01002599$, and $\delta_e = -0.00014965$. The u and d values that calibrate a 60-period binomial tree to this end of the period distribution are $u = 1.02$, $d = 0.99$ and $q = 0.75$ using equations (8a), (8b), and (8c).
- A decreasing stock price trend in which the distribution at the end of the one-year period is assumed to be characterized by moment values of $\mu_e =$

³ With no dividends, there is no early exercise advantage and therefore no difference between American and European call options.

-0.71764, $V_e = 0.00960186$, and $\delta_e = 0.00014026$. The u and d values that calibrate a 60-period binomial tree to this end of the period distribution are $u = 1.01$, $d = 0.980392$, and $q = 0.25$ using equations (8a), (8b), and (8c).

As shown in the exhibit, the binomial valuation for the increasing price trend prices the call at 7.71 compared to a price of 7.54 under the decreasing case. The binomial valuation for the constant stock price trend, in turn, prices the call at 9.17, exceeding both the increasing and decreasing price cases. In this example, the constant trend case has the same proportional increase ($\ln(1.02)$) as the increasing case, but a greater proportional decrease ($|\ln(0.980392)| > |\ln(0.99)|$). As a result, the call option in the constant case provides the same upside potential, but with greater downside protection than in the increasing case. Thus, the higher value reflects the value of the limited loss feature that is greater in the stable case than the increasing case. When compared to the decreasing case, the constant trend case has the same proportional decrease ($\ln(0.980392)$) as the decreasing case, but a greater proportional increase ($|\ln(1.02)| > |\ln(1.01)|$). As a result, the call option in the constant case provides the greater upside potential with the same downside protection as the decreasing case. Thus, the higher call value in the constant case vis-à-vis the decreasing case reflects the greater upside potential in the stable case than the decreasing case.

As for the put options, the binomial valuation for the increasing price trend prices the European and American puts at higher values than the decreasing case. The higher prices for the increasing case reflect the value of greater downside protection the put provides than in the stable case. Moreover, in these cases, the greater value of the protection under the increasing case exceeds the value associated with the greater upside potential the put provides under the decreasing case.

Differences in Option Prices between the Cox, Ross, Rubinstein/Rendleman Bartter Model and the Skewness-Adjusted Model

Exhibit 6 shows the different model values for the above call option, and Exhibit 7 shows the different model values for the American and European puts. The values for the call and put options for different stock prices are generated for both the increasing and decreasing stock price scenarios from the previous example using the discrete 60-period skewness-adjusted model and the Cox, Ross, and Rubinstein/Rendleman and Bartter model, as well as the continuous Black-Scholes model. The American put prices for the skewness-adjusted and the Cox, Ross, and Rubinstein/Rendleman and Bartter models are obtained by constraining the price of the put option at each node to be the maximum of its binomial value or intrinsic value. A comparison of the option values obtained using the skewness-adjusted model with the Cox, Ross, and Rubinstein/Rendleman and Bartter model and Black-

Exhibit 6—Comparison of Skewness-Adjusted Model with Cox, Ross, and Rubinstein/Rendleman and Bartter and Black-Scholes Models – Call Options

60-Period Binomial Model

n = 60, Call Option: X = 100, Expiration = 1 year, Length of Period = 6 days, $R_f^A = .06$

Increasing Stock Price Case: $\mu_e = 0.74036$, $V_e = 0.01002599$, $\delta_e = -0.00014965$

Skewness-Adjusted Model: u = 1.02, d = .99, and q = .75

Cox, Ross, and Rubinstein/Rendleman and Bartter Model: u = 1.01301, d = .98710, and q = .5

Black-Scholes Option Pricing Model: $V_e = 0.01002599$, $R_f = \ln(1.06) = .052689$

Decreasing Stock Price Case: $\mu_e = -0.71764$, $V_e = 0.00960186$, $\delta_e = 0.00014026$

Skewness-Adjusted Model: u = 1.01, d = .98092 and q = .25

Cox, Ross, and Rubinstein/Rendleman and Bartter Model: u = 1.012731, d = .987429, and q = .5

Black-Scholes Option Pricing Model: $V_e = 0.00960186$, $R_f = \ln(1.06) = .052689$

Price Trend	Stock Price	Skewness Model	CRR/RB Model	Difference CRR/RB -Skew	B-S OPM
Increasing	85	0.98	0.68	-0.30	.69
	90	2.32	1.89	-0.43	1.91
	95	4.54	4.09	-0.45	4.11
	100	7.71	7.33	-0.38	7.34
	105	11.63	11.38	-0.25	11.38
	110	16.06	15.94	-0.12	15.95
	115	20.82	20.76	-0.06	20.75
Decreasing	85	0.73	0.64	-0.09	0.64
	90	2.02	1.82	-0.20	1.84
	95	4.30	4.02	-0.28	4.03
	100	7.54	7.26	-0.28	7.27
	105	11.52	11.33	-0.19	11.34
	110	16.03	15.91	-0.12	15.92
	115	20.80	20.74	-0.06	20.74

Note: With no dividends, there is no early exercise advantage and therefore no difference between American and European call options

$$\text{Skewness Model : } u = e^{\frac{\mu_e}{n} + \left[\frac{V_e(1-q)}{nq} \right]^{1/2}} ; d = e^{\frac{\mu_e}{n} - \left[\frac{V_e q}{n(1-q)} \right]^{1/2}} ; q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4V_e^3}{n\delta_e^2} + 1 \right]^{-1/2}$$

$$\text{CRR / RB Model : } u = e^{\left[\frac{V_e}{n} \right]^{1/2}} ; d = e^{-\left[\frac{V_e}{n} \right]^{1/2}} ; q = 0.5$$

Scholes option pricing model illustrates the pricing differences that occur under increasing or decreasing stock price cases. In general, for both price scenarios, the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call and the American and European puts less than the skewness model, with the greatest pricing differences occurring at stock prices of 95 and 100. Specifically, a comparison of the models shows:

Exhibit 7—Comparison of Skewness-Adjusted Model with Cox, Ross, and Rubinstein/Rendleman and Bartter and Black-Scholes Models—American and European Put Options

60-Period Binomial Model

$n = 60$, Put Option: $X = 100$, Expiration = 1 year, Length of Period = 6 days, $R_f^A = 0.06$

Increasing Stock Price Case: $\mu_e = 0.74036$, $V_e = 0.01002599$, $\delta_e = -0.00014965$

Skewness-Adjusted Model: $u = 1.02$, $d = 0.99$, and $q = 0.75$

Cox, Ross, and Rubinstein/Rendleman and Bartter Model: $u = 1.01301$, $d = 0.98710$, and $q = 0.5$

Black-Scholes Option Pricing Model: $V_e = 0.01002599$, $R_f = \ln(1.06) = 0.052689$

Decreasing Stock Price Case: $\mu_e = -0.71764$, $V_e = 0.00960186$, $\delta_e = 0.00014026$

Skewness-Adjusted Model: $u = 1.01$, $d = 0.98092$ and $q = 0.25$

Cox, Ross, and Rubinstein/Rendleman and Bartter Model: $u = 1.012731$, $d = 0.987429$, and $q = 0.5$

Black-Scholes Option Pricing Model: $V_e = 0.00960186$, $R_f = \ln(1.06) = 0.052689$

Price Trend	Stock Price	Skewness Model American	CRR/RB Model American	Difference CRR/RB - Skew	Skewness Model European	CRR/RB Model European	Difference CRR/RB - Skew	B-S OPM European
Increasing	85	15.00	15.00	0	10.32	10.02	-0.30	10.03
	90	10.00	10.00	0	6.66	6.23	-0.43	6.25
	95	5.36	5.15	-0.21	3.88	3.43	-0.45	3.45
	100	2.60	2.27	-0.33	2.05	1.67	-0.38	1.68
	105	1.16	0.91	-0.25	0.97	0.72	-0.25	0.72
	110	0.46	0.33	-0.13	0.40	0.28	-0.12	0.29
Decreasing	115	0.17	0.11	-0.06	0.16	0.09	-0.07	0.09
	85	15.00	15.00	0	10.07	9.98	-0.09	9.98
	90	10.00	10.00	0	6.36	6.17	-0.19	6.18
	95	5.28	5.12	-0.16	3.64	3.36	-0.28	3.37
	100	2.50	2.19	-0.31	1.88	1.60	-0.28	1.61
	105	1.09	0.85	-0.24	0.86	0.67	-0.19	0.68
	110	0.45	0.30	-0.15	0.37	0.25	-0.12	0.26
	115	0.17	0.09	-0.08	0.14	0.08	-0.06	0.08

$$\text{Skewness Model : } u = e^{\frac{\mu_e}{n} + \left[\frac{V_e(1-q)}{nq} \right]^{1/2}} ; d = e^{\frac{\mu_e}{n} - \left[\frac{V_e q}{n(1-q)} \right]^{1/2}} ; q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4V_e^3}{n\delta_e^2} + 1 \right]^{-1/2}$$

$$\text{CRR / RB Model : } u = e^{\left[\frac{V_e}{n} \right]^{1/2}} ; d = e^{-\left[\frac{V_e}{n} \right]^{1/2}} ; q = 0.5$$

- For both the increasing and decreasing price cases, the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call less than the skewness model at all stock prices.
- Under the increasing price case, the skewness model prices the call at 7.71 when the stock is at 100, while the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call 0.38 less at 7.33. The largest price difference for the call under the increasing price case occurs at a stock price of 95

where the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call at 4.09, 0.45 less than the skewness model's price of 4.54.

- Under the decreasing price case, the skewness model prices the call at 7.54 when the stock is at 100, while the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call 0.28 less at 7.26. The largest difference for the call under the decreasing price case occurs at 100 and 95 (skewness = 4.30, Cox, Ross, and Rubinstein/Rendleman and Bartter = 4.02) where the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the call 0.28 less than the skewness model at both prices.
- For both price cases, the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the American put at its intrinsic value when the option is in the money at stock prices of 85 and 90. At the other stock prices, the Cox, Ross, and Rubinstein/Rendleman and Bartter prices the put less than the skewness model, with the greatest price difference (0.33) occurring when the put is at the money.
- For the European put, the largest price difference (0.45) occurs at 95 for the increasing price case (Cox, Ross, and Rubinstein/Rendleman and Bartter = 3.43, skewness = 3.88) and at 100 and 95 (0.28) for the decreasing case.
- As expected, there is no significant difference in the prices for the European call and put options obtained using the Black-Scholes option pricing model and the 60-period Cox, Ross, and Rubinstein/Rendleman and Bartter binomial.

It should be noted that the pricing differences between the Cox, Ross, and Rubinstein/Rendleman and Bartter model relative to the skewness model from these numerical simulations are consistent with the aforementioned empirical studies of Stein and Stein, Wiggins, and Heston who demonstrate that when skewness exists, the Black-Scholes model consistently misprices options.

The Skewness Impact On The Pricing Of Futures Options

As noted, skewness changes the asymptotic properties of the u and d equations, elevating the relative importance of the mean in valuing options. The skewness impact on enhancing the importance of the mean holds for both spot options and futures options. In the case of futures options, however, the skewness impact not only elevates the importance of the mean, but also the risk-free rate and, in the case of index and currency futures options, the asset yield (dividend yield for stock index futures or foreign risk-free yield on currency futures).

To see the impact of skewness on futures options, consider an option on a futures contract on a stock index (or currency) paying a continuous compounded dividend yield (or continuous foreign risk-free rate). The carrying-cost futures price is

$$f_0 = S_0 e^{(R^A - \psi)n_f(t/n)} \quad (12)$$

where:

- S_0 = Current spot price ;
 ψ = Annual dividend yield or foreign risk-free rate;
 R^A = Annual risk-free rate;
 t/n = Length of binomial steps as a proportion of a year;
 n_f = Number of discrete binomial periods to the futures expiration.

In this case, the equation for the risk-neutral probability, p , includes the annual dividend or foreign risk-free yield. Defined in terms of annualized moments, p is

$$p = \frac{e^{(R^A - \psi)(t/n)} - d}{u - d}$$

The binomial model for options on financial futures is defined in terms of the up and down parameters for the underlying spot prices (u and d). It is more common, though, to define the up and down parameters in terms of the futures prices, u^f and d^f . Specifically, if the carrying-cost model holds, then u^f and d^f are given as

$$u^f = \frac{f_u}{f_0} = \frac{uS_0e^{(R^A - \psi)(n_f - 1)(t/n)}}{S_0e^{(R^A - \psi)n_f(t/n)}} = \frac{u}{e^{(R^A - \psi)(t/n)}} \quad (13)$$

$$u = u^f e^{(R^A - \psi)(t/n)}$$

$$d^f = \frac{f_d}{f_0} = \frac{dS_0e^{(R^A - \psi)(n_f - 1)(t/n)}}{S_0e^{(R^A - \psi)n_f(t/n)}} = \frac{d}{e^{(R^A - \psi)(t/n)}} \quad (14)$$

$$d = d^f e^{(R^A - \psi)(t/n)}$$

The skewness-adjusted formulas for estimating the up and down parameters as defined in terms of futures price (u^f and d^f) are:

$$u^f = e^{\sigma_f^A \sqrt{((1-q)/q)(t/n)} + \mu_f^A(t/n)} \quad (15)$$

$$d^f = e^{-\sigma_f^A \sqrt{(q/(1-q))(t/n)} + \mu_f^A(t/n)} \quad (16)$$

where:

σ_f^A and μ_f^A = Annualized standard deviation and mean on the futures price's logarithmic return.

If we assume the skewness for the futures and the skewness for the spot prices logarithmic returns are equal ($\delta_f = \delta_s$), the relationships between the volatility and mean on the futures price's logarithmic return and the spot's volatility and mean are

$$\sigma_f^A = \sigma_s^A \quad (17)$$

$$\mu_f^A = \mu_s^A - (R^A - \psi) \quad (18)$$

where:

σ_s^A and μ_s^A = Annualized standard deviation and mean on the spot price's logarithmic return.

The algebraic derivations of these relations are presented in the appendix. Substituting (17) and (18) into equations (15 and (16), u^f and d^f alternatively can be expressed in terms of the spot mean and variability, the risk-free rate, and the asset yield:

$$u^f = e^{\sigma_f^A \sqrt{((1-q)/q)(t/n)} + \mu_f^A (t/n)} = e^{\sigma_s^A \sqrt{((1-q)/q)(t/n)} + (\mu_s^A - (R^A - \psi))(t/n)} \quad (19)$$

$$d^f = e^{-\sigma_f^A \sqrt{(q/(1-q))(t/n)} + \mu_f^A (t/n)} = e^{-\sigma_s^A \sqrt{(q/(1-q))(t/n)} + (\mu_s^A - (R^A - \psi))(t/n)} \quad (20)$$

Substituting equations (13) and (14) into the equation for p , we obtain the equations for risk-neutral probability and futures call and put option prices defined in terms of the futures up and down parameters:

$$p = \frac{e^{(R^A - \psi)(t/n)} - d}{u - d}$$

$$p^f = \frac{e^{(R^A - \psi)(t/n)} - d^f e^{(R^A - \psi)(t/n)}}{u^f e^{(R^A - \psi)(t/n)} - d^f e^{(R^A - \psi)(t/n)}}$$

$$p^f = \frac{1 - d^f}{u^f - d^f}$$

$$C_0^* = \frac{1}{e^{R^A(t/n)}} [p^f C_u + (1 - p^f) C_d]$$

$$P_0^* = \frac{1}{e^{R^A(t/n)}} [p^f P_u + (1 - p^f) P_d]$$

Note that because u^f and d^f include the risk-free rate and the asset yield (ψ), p^f is determined by both parameters, at least for discrete cases. For a multiple-period model that is defined by a large number of subperiods, the impact of the risk-free rate and dividend yield on u^f and d^f and on the option price depends on the presence of skewness. If there is no skewness, then $q = 0.5$ and u^f and d^f simplify to:

$$u^f = e^{\sigma_s^A \sqrt{t/n} + (\mu_s^A - (R^A - \psi))(t/n)} \quad (21)$$

$$d^f = e^{-\sigma_s^A \sqrt{t/n} + (\mu_s^A - (R^A - \psi))(t/n)} \quad (22)$$

In this case, as the number of subperiods increase, the second term in the exponent of equations (21) and (22)—a term that includes not only the mean, but also the risk-

free rate and asset yield—goes to zero faster than the square root terms. As a result, for a large number of subperiods, the equations for u^f and d^f simplify to

$$\begin{aligned} u^f &= e^{\sigma_s^A \sqrt{t/n}} = u \\ d^f &= e^{-\sigma_s^A \sqrt{t/n}} = d \end{aligned}$$

On the other hand, if there is skewness, then $(1-q)/q$ and $q/(1-q)$ in the exponents of equations (15) and (16) cause the square root terms and the second terms in the exponent to go to zero at the same rate. As a result, for large n , skewness makes the risk-free rate and asset-yield important in pricing futures options.

It is worth noting that the binomial option pricing model for futures options converges to the seminal Black futures option model (1976) as n becomes large. The Black futures model, in turn, differs from the Black-Scholes option pricing model and the continuous dividend-adjusted Black-Scholes model for spot securities by the exclusion of the risk-free rate and dividend yield in the equations for d_1 and d_2 . This difference, though, only holds given the assumption of normality. If skewness exists, then, as shown, the risk-free rate and asset-yield become important in pricing the futures option.

Exhibit 8 shows different model values for American and European call options on a futures contract on the illustrative stock (described in Exhibit 6), and Exhibit 9 shows the different model values for the American and European put options on the futures contract. The underlying futures contract is assumed to have an expiration of one year and the option is assumed to expire in six months ($t = 0.5$). The stock on the underlying futures contract is assumed to pay an annualized dividend yield of $\psi = 4$ percent and the price on the futures contract is assumed to be equal to its carrying-cost value. The values for the futures call and put options for different spot and futures prices are generated for both the increasing and decreasing stock price scenarios from the previous example using the discrete 60-period skewness-adjusted model and the Cox, Ross, and Rubinstein/Rendleman and Bartter model. Comparing the option values obtained using the skewness-adjusted model with the Cox, Ross, and Rubinstein/Rendleman and Bartter model illustrates the differences in pricing futures contracts that occurs under increasing or decreasing stock price cases. Similar to the spot option cases, the Cox, Ross, and Rubinstein/Rendleman and Bartter model prices the American and European options less than the skewness model for both price scenarios, with the greatest price difference occurring at futures prices closer to the exercise price. Also, as expected, there is no difference in the prices for the European call and put options obtained using the Black futures option model and the 60-period Cox, Ross, and Rubinstein/Rendleman and Bartter binomial.

Exhibit 8—Comparison of Skewness-Adjusted Futures Option Model with Cox, Ross, and Rubinstein/Rendleman and Bartter, American and European Futures Call Options

60-Period Binomial Model, $n = 60$

Futures Call Option: $X = 100$, Option Expiration = 0.5 years, Length of Binomial Period = 3 days,

$R_f^A = 0.06$, Annual dividend yield = $\psi = 0.04$

Futures Contract: Expiration = 1 year

Increasing Security Price Case: $\mu_e = 0.74036$, $V_e = 0.01002599$, $\delta_e = -0.00014965$

Decreasing Security Price Case: $\mu_e = -0.71764$, $V_e = 0.00960186$, $\delta_e = 0.00014026$

Price Trend	Spot Price	Futures Price	Skewness Model	CRR/RB Model	Difference CRR/RB - Skew	Skewness Model	CRR/RB Model	Black Futures Model	Difference CRR/RB - Skew
Increasing	85	86.717	0.19	0.05	-0.14	0.19	0.05	0.05	-0.14
	90	91.818	0.78	0.36	-0.42	0.78	0.36	0.37	-0.42
	95	96.919	2.15	1.47	-0.68	2.15	1.46	1.47	-0.69
	100	102.020	4.65	3.90	-0.75	4.61	3.87	3.87	-0.74
	105	107.121	8.18	7.63	-0.55	8.10	7.53	7.53	-0.57
	110	112.222	12.51	12.26	-0.25	12.32	12.02	12.02	-0.30
Decreasing	115	117.323	17.34	17.32	-0.02	16.96	16.84	16.84	-0.12
	85	86.717	0.19	0.04	-0.15	0.19	0.04	0.04	-0.15
	90	91.818	0.76	0.34	-0.42	0.76	0.34	0.34	-0.42
	95	96.919	2.12	1.42	-0.70	2.12	1.42	1.41	-0.70
	100	102.020	4.58	3.84	-0.74	4.54	3.81	3.81	-0.73
	105	107.121	8.11	7.60	-0.51	8.02	7.50	7.50	-0.52
	110	112.222	12.46	12.25	-0.21	12.26	12.00	12.00	-0.26
	115	117.323	17.33	17.22	-0.11	16.94	16.84	16.83	-0.10

$$f_0 = S_0 e^{(R^A - \psi)n_f(t/n)}$$

$$\text{Skewness Model : } u^f = e^{\sigma_S^A \sqrt{(q/(1-q))(t/n)} + (\mu_S^A - (R^A - \psi))(t/n)};$$

$$d^f = e^{-\sigma_S^A \sqrt{(q/(1-q))(t/n)} + (\mu_S^A - (R^A - \psi))(t/n)}; q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4(tV_e^A)^3}{n(t\delta_e^A)^2} + 1 \right]^{-1/2}; p = \frac{1 - d^f}{u^f - d^f}$$

$$\text{CRR / RB Model : } u^f = u = e^{\sigma_S^A \sqrt{t/n}}; d^f = d = e^{-\sigma_S^A \sqrt{t/n}}; q = 0.5; p = \frac{1 - d^f}{u^f - d^f}$$

$$\text{Black Futures Option Model : } C_0^* = [f_0 N(d_1) - X N(d_2)] e^{-R^A t}$$

$$d_1 = \frac{\ln(f_0 / X) + (\sigma_S^A / 2)t}{\sigma_S^A \sqrt{t}}, d_2 = d_1 - \sigma_S^A \sqrt{t}$$

A Note on Estimating Skewed Distribution with Volatility Smiles

To value spot and futures options using the skewness-adjusted binomial option pricing model requires estimating not only the variance, but also the mean and skewness. A common approach among option traders in valuing options using the Black-Scholes option pricing model or Cox, Ross, and Rubinstein/Rendleman and Bartter binomial option pricing model is to select the volatility based on the option's volatil-

ity smile and its volatility term structure. A volatility smile is a plot of the implied volatilities given different exercise prices. The volatility term structure, in turn, refers to the relation between an option's implied volatility and its time to expiration.⁴ Volatility smiles, in turn, can be used to generate an implied probability distribution for the underlying stock's price or logarithmic return.⁵ When a smile is relatively flat, then the implied distribution of the logarithmic returns approaches a normal distribution. In this case, the smile is consistent with the Black-Scholes model's standard assumption of normality. When the smile is negatively (positively) sloped, however, the implied probability distribution of logarithmic returns tends to be skewed to the left (right). Such a smile is referred to as a volatility skew. Moreover, one approach to estimating the mean, variance, and skewness would be to generate an implied probability distribution for the underlying stock's price or logarithmic return from its skewed volatility smile.

Conclusion

Several empirical studies have provided evidence that the distributions of logarithmic returns of a number of securities exhibit persistent skewness. Our own analysis shows that over intermediate periods of time, the S&P 500 often has followed patterns of persistent increases or persistent decreases that are characterized by skewness. In the binomial pricing of stock options, the existence of skewness in a binomial process impacts the values of the up and down parameters. The presence of skewness also affects the relative contribution of the mean to the values of u and d , and it changes the asymptotic properties of the u and d parameters such that for a large number of subperiods u and d depend on all three moments—variance, skewness, and mean. In this paper, we have illustrated how Johnson, Pawlukiewicz, and Mehta's skewness-adjusted model can be used to calibrate a binomial tree to an increasing stock price case in which the end-of-the period distribution is characterized by positive average logarithmic returns and negative skewness and to a decreasing price case that is characterized by a distribution with negative average logarithmic returns and positive skewness. This skewness-adjusted binomial model, in turn, may be better able to price spot and futures options than the Cox, Ross, and Rubinstein/Rendleman and Bartter approach when there is an expectation that the underlying security price will be increasing or decreasing similar to the stock index trends that have been observed historically.

⁴ See Rubinstein (1985, 1994) and Jackwerth and Rubinstein (1996).

⁵ See D.T. Breeden and R.H. Litzenberger (1978).

Exhibit 9—Comparison of Skewness-Adjusted Futures Option Model with Cox, Ross, and Rubinstein/Rendleman and Bartter, American and European Put Options

60-Period Binomial Model, $n = 60$

Futures Put Option: $X = 100$, Option Expiration = .5 years, Length of Period = 3 days, $R_f^A = .06$, Annual dividend yield = $\psi = 0.04$

Futures Contract: Expiration = 1 year

Increasing Security Price Case: $\mu_e = 0.74036$, $V_e = 0.01002599$, $\delta_e = -0.00014965$

Decreasing Security Price Case: $\mu_e = -0.71764$, $V_e = 0.00960186$, $\delta_e = 0.00014026$

		Futures	Skewness	CRR/RB	Difference	Skewness	CRR/RB	Black	Difference
Price Trend	Spot Price	Price	Model	Model	CRR/RB -	Model	Model	Futures	CRR/RB -
			American	American	Skew	European	European	Model	Skew
Increasing	85	86.717	13.34	13.28	-0.06	13.05	12.94	12.94	-0.11
	90	91.818	8.83	8.30	-0.53	8.72	8.30	8.30	-0.42
	95	96.919	5.18	4.49	-0.69	5.14	4.45	4.45	-0.69
	100	102.02	2.67	1.91	-0.76	2.65	1.91	1.91	-0.74
	105	107.121	1.19	0.62	-0.57	1.19	0.62	0.62	-0.57
	110	112.222	0.46	0.16	-0.30	0.46	0.16	0.16	-0.30
	115	117.323	0.15	0.03	-0.12	0.15	0.03	0.03	-0.12
Decreasing	85	86.717	13.33	13.28	-0.05	13.08	12.93	12.93	-0.15
	90	91.818	8.80	8.40	-0.40	8.70	8.28	8.28	-0.42
	95	96.919	5.15	4.44	-0.71	5.11	4.40	4.40	-0.71
	100	102.02	2.60	1.86	-0.74	2.58	1.85	1.85	-0.73
	105	107.121	1.11	0.58	-0.53	1.11	0.58	0.58	-0.53
	110	112.222	0.40	0.14	-0.26	0.40	0.14	0.14	-0.26
	115	117.323	0.12	0.02	-0.10	0.12	0.02	0.02	-0.10

$$\text{Future Price : } f_0 = S_0 e^{(R^A - \psi)n_F(t/n)}$$

$$\text{Skewness Model : } u^f = e^{\sigma_S^A \sqrt{(q/(1-q))(t/n)} + (\mu_S^A - (R^A - \psi))(t/n)};$$

$$d^f = e^{-\sigma_S^A \sqrt{(q/(1-q))(t/n)} + (\mu_S^A - (R^A - \psi))(t/n)}; q = \frac{1}{2} \pm \frac{1}{2} \left[\frac{4(tV_e^A)^3}{n(t\delta_e^A)^2} + 1 \right]^{-1/2}; p = \frac{1 - d^f}{u^f - d^f}$$

$$\text{CRR / RB Model : } u^f = u = e^{\sigma_S^A \sqrt{t/n}}; d^f = d = e^{-\sigma_S^A \sqrt{t/n}}; q = 0.5; p = \frac{1 - d^f}{u^f - d^f}$$

$$\text{Black Futures Option Model: } P_0^* = [X(1 - N(d_2)) - f_0(1 - N(d_1))]e^{-R^A t}$$

$$d_1 = \frac{\ln(f_0/X) + (\sigma_S^A/2)t}{\sigma_S^A \sqrt{t}}, d_2 = d_1 - \sigma_S^A \sqrt{t}$$

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Appendix—Algebraic Derivation of u^f and d^f

Assuming spot and future skewness are equal ($\delta_f = \delta_s$), the skewness-adjusted up and down parameters are

$$u^f = e^{\sigma_f^A \sqrt{(t/n)((1-q)/q)} + \mu_f^A(t/n)} \quad \text{and} \quad d^f = e^{-\sigma_f^A \sqrt{(t/n)(q/(1-q))} + \mu_f^A(t/n)}$$

$$u = e^{\sigma_s^A \sqrt{(t/n)((1-q)/q)} + \mu_s^A(t/n)} u^f \quad \text{and} \quad d = e^{-\sigma_s^A \sqrt{(t/n)(q/(1-q))} + \mu_s^A(t/n)}$$

Substituting these equations into equations (13) and (14) and solving each equations in terms of σ_f^A we obtain:

$$u^f = \frac{u}{e^{(R^A - \psi)(t/n)}}$$

$$e^{\sigma_f^A \sqrt{(t/n)((1-q)/q)} + \mu_f^A(t/n)} = \frac{e^{\sigma_s^A \sqrt{(t/n)((1-q)/q)} + \mu_s^A(t/n)}}{e^{(R^A - \psi)(t/n)}}$$

$$\ln\left(e^{\sigma_f^A \sqrt{(t/n)((1-q)/q)} + \mu_f^A(t/n)}\right) = \ln\left(\frac{e^{\sigma_s^A \sqrt{(t/n)((1-q)/q)} + \mu_s^A(t/n)}}{e^{(R^A - \psi)(t/n)}}\right)$$

$$\sigma_f^A \sqrt{(t/n)((1-q)/q)} + \mu_f^A(t/n) = \sigma_s^A \sqrt{(t/n)((1-q)/q)} + \mu_s^A(t/n) - (R^A - \psi)(t/n) \quad (1)$$

$$\sigma_f^A = \frac{\sigma_s^A \sqrt{(t/n)((1-q)/q)} + \mu_s^A(t/n) - (R^A - \psi)(t/n) - \mu_f^A(t/n)}{\sqrt{(t/n)((1-q)/q)}}$$

$$d^f = \frac{d}{e^{(R^A - \psi)(t/n)}}$$

$$e^{-\sigma_f^A \sqrt{(t/n)(q/(1-q))} + \mu_f^A(t/n)} = \frac{e^{-\sigma_s^A \sqrt{(t/n)(q/(1-q))} + \mu_s^A(t/n)}}{e^{(R^A - \psi)(t/n)}}$$

$$\ln\left(e^{-\sigma_f^A \sqrt{(t/n)(q/(1-q))} + \mu_f^A(t/n)}\right) = \ln\left(\frac{e^{-\sigma_s^A \sqrt{(t/n)(q/(1-q))} + \mu_s^A(t/n)}}{e^{(R^A - \psi)(t/n)}}\right)$$

$$-\sigma_f^A \sqrt{(t/n)(q/(1-q))} + \mu_f^A(t/n) = -\sigma_s^A \sqrt{(t/n)(q/(1-q))} + \mu_s^A(t/n) - (R^A - \psi)(t/n) \quad (2)$$

$$\sigma_f^A = \frac{\sigma_s^A \sqrt{(t/n)(q/(1-q))} - \mu_s^A(t/n) + (R^A - \psi)(t/n) + \mu_f^A(t/n)}{\sqrt{(t/n)(q/(1-q))}}$$

Solving equations (1) and (2) simultaneously for σ_s^A and μ_s^A :
equation (1):

$$\begin{aligned}\sigma_f^A \sqrt{(t/n)((1-q)/q)} \\ = \sigma_S^A \sqrt{(t/n)((1-q)/q)} + \mu_S^A(t/n) - (R^A - \psi)(t/n) - \mu_f^A(t/n)\end{aligned}\quad (3)$$

equation (2) expressed in terms of $\mu_f^A(t/n)$:

$$\begin{aligned}\mu_f^A(t/n) = -\sigma_S^A \sqrt{(t/n)(q/(1-q))} \\ + \mu_S^A(t/n) - (R^A - \psi)(t/n) + \sigma_f^A \sqrt{(t/n)(q/(1-q))}\end{aligned}\quad (4)$$

Substituting (4) into (3) and solving for σ_f^A :

$$\begin{aligned}\sigma_f^A \sqrt{(t/n)((1-q)/q)} \\ = \sigma_S^A \sqrt{(t/n)((1-q)/q)} + \mu_S^A(t/n) \\ - (R^A - \psi)(t/n) + \sigma_S^A \sqrt{(t/n)(q/(1-q))} \\ - \mu_S^A(t/n) + (R^A - \psi)(t/n) - \sigma_f^A \sqrt{(t/n)(q/(1-q))}\end{aligned}\quad (5)$$

$$\begin{aligned}\sigma_f^A \left[\sqrt{(t/n)((1-q)/q)} + \sqrt{(t/n)(q/(1-q))} \right] \\ = \sigma_S^A \left[\sqrt{(t/n)((1-q)/q)} + \sqrt{(t/n)(q/(1-q))} \right]\end{aligned}$$

$$\sigma_f^A = \sigma_S^A$$

Substituting σ_S^A for σ_f^A in equation (4) and solving in terms of μ_f^A :

$$\begin{aligned}\mu_f^A(t/n) = -\sigma_S^A \sqrt{(t/n)(q/(1-q))} \\ + \mu_S^A(t/n) - (R^A - \psi)(t/n) + \sigma_S^A \sqrt{(t/n)(q/(1-q))}\end{aligned}\quad (6)$$

$$\mu_f^A = -[\mu_S^A - (R^A - \psi)](t/n)$$

Thus, the relation between spot and futures variability and mean is:

$$\begin{aligned}\sigma_f^A &= \sigma_S^A \\ \mu_f^A &= -[\mu_S^A - (R^A - \psi)](t/n)\end{aligned}$$

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