

On Computing Complete Distributions for American and European Standard and Exotic Options on Stocks Paying Discrete Dividends with Applications to Stochastic Dominance Analysis

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This paper proposes using a finite Markov chain as an intuitive, accurate, and versatile method for the analysis of ordinary and exotic European and American calls and puts on dividend and non-dividend paying stocks. The approach presented in this paper allows the valuation of options under a wide variety of conditions. Previous work in the literature by Duan and Simonato (2001) and others developed Markov chain methods for standard calls and puts on non-dividend paying stocks. This paper extends their work with a finite Markov chain method that applies to both standard and exotic calls and puts on stocks paying discrete dividends. Because the approach approximates the complete probability distribution for the option, analyses of stochastic dominance are practical and are illustrated by examples in this paper.

Introduction

This paper shows that finite Markov chains can be applied as an intuitive and versatile method for the analysis of ordinary and exotic European and American calls and puts on dividend and non-dividend paying stocks. The approach allows the valuation of options under a wide variety of conditions. Because finite Markov

*The authors wish to thank an anonymous reviewer for several helpful, thoughtful comments.

chains can be applied to approximate the entire distribution, stochastic dominance analysis becomes practical. This paper extends previous work on the finite Markov chain approach to ordinary and exotic options on stocks paying periodic discrete dividends.

Many different option pricing methods have been developed in the literature, but most have important limitations. Closed form solutions, such as the seminal model by Black and Scholes (1973), price European options on non-dividend paying stocks accurately, but are relatively unintuitive and encounter problems in pricing American options on securities with discrete dividends. Lattice approaches such as the binomial and trinomial trees of Cox, Ross, and Rubinstein (1979), Figlewski and Gao (1999), Rendleman and Bartter (1979), and Boyle (1988) are more intuitive, but do not provide paths for moving to numerous prices in initial time periods as occurs naturally. Instead, these methods simulate the multi-state outcomes observed in markets by artificially subdividing time into small increments. Monte Carlo simulation is intuitively appealing and is the only tractable method for certain option types. A rich and growing literature shows simulation of various sorts is the method of choice for models with multiple driving factors such as stochastic volatility and jump components or if there are several correlated underlying securities. (See, for example, Andersen and Broadie, 2004.) Unlike lattice and Markov chain approaches, Monte Carlo simulation does not suffer the curse of dimensionality and, therefore, seems the appropriate choice in such multifactor, multiasset models. Its use in simpler, one dimensional models, however, could disguise important conceptual issues and is less tractable in cases requiring backward induction.

The Markov chain approach is more intuitive because it uses the same nodes in every time period and allows the user clearly to see the probability that an underlying asset price moves from any one state to another in a given interval of time. These features make the option pricing process transparent. Furthermore, Duan, Dudley, Gauthier, and Simonato (1999) and Duan and Simonato (2001) show that the Markov chain approach has excellent accuracy and convergence properties that compare favorably to those of large-scale binomial trees and Monte Carlo simulations when the methods are applied to standard European and American calls and puts on non-dividend paying stocks. The Markov chain approach is effective in part because the time step length and number of discrete asset prices (or nodes) are set independently. Hodges, Theis, and Haensly (2004) provide numerical examples showing that a relatively large number of nodes in every period (analogous to the approach in finite difference methods) facilitates accurate estimates not only of option prices, but also the risk dimensions commonly known as *Greek letters* (or, simply, *the Greeks*) even in cases where early exercise plays a major role.

The Markov approach has greater flexibility than most other methods. It can accommodate longer maturities, pre-defined monitoring dates, and discrete as well as other non-Gaussian probability distributions and thus may have greater applicability

in practice. The approach naturally lends itself to approximating the entire distribution of an option's return and allows the assessment of risk by means of stochastic dominance analysis. The major computational burden is relatively straightforward matrix multiplication. This paper illustrates the flexibility of the Markov chain approach by presenting an analysis of standard and exotic options by means of stochastic dominance. Duan, Dudley, Gauthier, and Simonato (1999) and Duan and Simonato (2001) develop a Markov chain approach for valuation of standard European and American call and put options on non-dividend paying stocks. This paper extends their work by developing and applying finite Markov chain methods to standard and exotic European and American options on stocks with discrete periodic dividends.

Finite Markov Chain Models for the Underlying Stock Price

This section describes the general features of Markov chain modeling of stock prices and briefly outlines the construction of the state space and transition probabilities. Stock prices frequently are treated as continuous-time, continuous-state stochastic processes denoted as $\{S(t): t \geq 0\}$. To approximate the process by a discrete-time, discrete-state Markov chain, $\{S(0), S(1), S(2), \dots\}$, requires that the range $(0, \infty)$ of stock prices be partitioned into a finite number of non-overlapping intervals to form the discrete states. The approximation also assumes that trades occur at discrete time intervals.

Insofar as stock price is a random variable, the immediate goal is to compute the distribution of $S(n)$ in any given time period n . Knowledge of the probabilistic rules governing the evolution of the stochastic process are a prerequisite to computing the distribution of $S(n)$. In both the continuous and discrete forms of Markov chains, the stochastic processes describing stock price movements must satisfy the Markov property. The discrete case requires for every n and all states $i_1, i_2, \dots, i_n$ that

$$\text{Prob}\{S(n) \in i_n | S(n-1) \in i_{n-1}, S(n-2) \in i_{n-2}, \dots, S(0) \in i_0\} = \text{Prob}\{S(n) \in i_n | S(n-1) \in i_{n-1}\}.$$

That is, the probability of the process entering any given state depends only upon what occurred in the immediately preceding period. Therefore, a full description of the evolution of the process requires knowledge only of the conditional probabilities of going from one state to another in a single step. For a Markov chain with k distinct states there are only k^2 transition probabilities.

The one-step transition probabilities are the elements of the one-step transition matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & & p_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix},$$

where:

$$p_{ij} = \text{Prob}\{S(n) \in j | S(n-1) \in i\}.$$

To compute the distribution of $S(n)$, $n = 1, 2, \dots, N$, requires knowledge of the distribution of the stock price in the initial period. Let the row vector, $\pi_0 = (\pi_1(0), \pi_2(0), \dots, \pi_k(0))$, be the starting vector whose elements are nonnegative and sum to unity. This starting vector is the distribution of $S(0)$. In the case where the initial stock price is deterministically known, π_0 has a one in the appropriate coordinate and zeros elsewhere. The row vector $\pi_1 = \pi_0 P$ is the corresponding distribution for $S(1)$, because

$$\pi_j(1) = \text{Prob}\{S(1) \in j\} = \sum_{i=1}^k \pi_i(0) p_{ij}.$$

Summing amounts to adding the probabilities of all possible ways of being in state j in time period 1. The distribution of the process in two steps is $\pi_2 = \pi_1 P = \pi_0 P^2$. In general the distribution of $S(n)$ is $\pi_n = \pi_0 P^n$.

Because the state space and transition probabilities play such prominent roles in Markov chains, an outline of their construction is instructive. The state space for stock prices will refer to intervals on $(0, \infty)$, usually of unequal length, with values large enough to accommodate the distribution of asset prices at maturity date T . Suppose k discrete interval states exist and define the values $b_0, b_1, \dots, b_k$ as lower and upper bounds of each of k states. Further assume the distribution of returns, not necessarily normal, has mean μ and standard deviation σ and the initial stock price is $S(0)$. Let $0 < \beta_0 < \beta_1 < \dots < \beta_k < 1$ be a monotonically-increasing, equally-spaced sequence of cumulative probabilities from a normal distribution, Φ , having mean μT and standard deviation $\sigma T^{1/2}$. In the numerical examples, we let $\beta_0 = (1-m)/2$ and $\beta_k = (1+m)/2$, where $m = \text{Prob}\{Z < 2 + \log[\log(k)]\}$, and Z is a standard normal random variable. As Duan and Simonato (2001) state, this is a possible choice for m in order that the discrete Markov process converges to the continuous process for increasing values of k . The bounds then can be defined by the sequence

$$b_i = S(0)e^{\Phi^{-1}(\beta_i)}, \quad i = 0, 1, \dots, k,$$

where:

$$\beta_i = (1-m)/2 + im/k.$$

The statement that the stock price is in state i will mean that the price is between b_{i-1} and b_i . When a single value for the price is needed, however, the midpoint, s_i , of the interval will be used. In what follows, the column vector of midpoints $\mathbf{s} = (s_1, s_2, \dots, s_k)$ serves as a list of state space outcomes for stock prices. The one-step transition probabilities are computed as

$$p_{ij} = \text{Prob}\{b_{j-1} \leq S(t+h) < b_j | S(t) = s_i\}, \quad (1)$$

where h is the discrete time interval. Strictly speaking, this probability should be conditioned on the event $b_{i-1} \leq S(t) < b_i$, requiring a numerical approximation of the appropriate double integral. Use of various Gaussian quadrature approximations failed to result in any appreciable increase in accuracy and took an order of magnitude longer to compute. Even in those cases where the transitional matrix need be computed but once and time is not a consideration, we find the simpler conditional assumption preferable.

Many scenarios depend on the assumption that stock price returns are normally distributed. In that case, the stock price itself would be lognormally distributed with parameters easily calculated from the mean and variance of returns and the time to expiration. In those scenarios, it is sensible to define interval states as above. If the returns are assumed to follow a distribution other than normal, however, then it is likely that even the form of the terminal distribution of stock prices is unknown. Beside the fact that the lognormal distribution covers the entire range of possible outcomes, using the mean and variance of non-normal distributions to develop the state space as if the returns were normal can be justified for two reasons. First, distributions with the same mean and variance usually do not generate wildly different outcomes. Second, much of the interest in using non-normal distributions is to compare results to the normal. Using the same state spaces, e.g., those described above, facilitates the comparison.

Distribution of Standard Options

Stock Prices and European Calls and Puts

Constructing the Markov chain approximation to the distribution of stock prices and European options is straightforward once the state space and one-step transition probabilities are defined. Let N be the number of discrete time intervals of length h until the expiration date T , i.e., $Nh = T$, and π_0 the starting vector with a one in the coordinate containing the initial stock price, $S(0)$, and zeros elsewhere. Approximate the probability density of stock prices at the expiration date by $\pi_N = \pi_0 P^N$ with outcomes s . If r is the continuously compounded risk-free interest rate and risk neutrality is assumed, then $S(0) = \pi_N s e^{-rT}$. That is, the discounted expected stock price equals the initial stock price.

Because European options can be exercised only at maturity, the vector of probabilities, π_N , can be used in an obvious way to construct the probability densities of both call and put options. Let K be the strike price of the option. Define the payoff column vector f_N , where:

$$f_i(N) = \max(s_i - K, 0) \quad (2a)$$

if a call option and

$$f_i(N) = \max(K - s_i, 0) \quad (2b)$$

if a put option. Approximate the probability density of a European option at expiration by π_N with outcomes f_N , where equation (2a) applies if a call option and (2b) if a put. Given this distribution, the expected present value of the option, or its price, is $\pi_N f_N e^{-rT}$.

Stock Prices and American Calls and Puts

Finding the distributions for stock prices themselves or for European options permits a straightforward application of the n -step transition matrix. American options, in which early exercising is a possibility, present a bigger challenge. Closed form models can be adapted and work well for one expected dividend, e.g., see Whaley (1982). Lattice methods easily can be adapted for continuous dividend yields. The middle ground—multiple discrete dollar dividends—present difficulties for closed form and lattice methods, and the solutions presented in the literature generally are complicated and unintuitive. The Markov chain method, on the other hand, easily and intuitively accommodates multiple discrete dividend payments and applies the same general approach for one or many expected dividends over the life of the option.

Two sets of issues must be addressed when constructing the transition matrix in the Markov chain approach: what effect do dividends have on transition probabilities, and what should be the time interval between possible trades? Suppose that the asset pays a dividend of d dollars per time period, and assume that the price falls by d on the ex-dividend date. Suppose that the stock is in state i on a dividend date. Then, instead of going from s_i to interval j , the stock price must now go from $s_i - d$ to interval j . Therefore the one-step transition probabilities in P are

$$p_{ij} = \text{Prob}\{b_{j-1} \leq S(t+h) < b_j \mid S(t) = s_i - d\} \quad (3)$$

rather than the probabilities defined in equation (1). Thus, in the Markov chain approach, discrete dividend payments require a parametric rather than a structural change in the computational procedures.

For American call options, optimal early exercise only occurs immediately before an ex-dividend date, e.g., see Merton (1973). In equation (3), let h be the time between dividend payments. The computational efficiency of the Markov chain approach improves the fewer the trading periods. As the number of dividend payments over the life of the option increases, efficiency declines. When frequency of dividend payments is sufficiently high, then it makes sense to approximate dividend yield as a continuous process and use lattice methods rather than finite Markov chain methods. The focus of the current paper, however, is on the middle ground between one expected dividend and continuous dividend yields.

The outcomes space also must be modified to allow for early exercise when the underlying stock pays dividends. For the American call option, at each period $n-1$ the option holder chooses between exercising the option and receiving $s_i - K$ in state i or

continuing to hold (or sell) an option whose value is the discounted expected future value of the option at period n . For the standard option, this decision is independent of the path taken to arrive at state i and depends only on the stock price s_i and the possible outcomes in the next period.

The backward induction begins with the payoff vector $\mathbf{f}_N$ defined in equation (2a), where we assume either that no dividend is expected at period N or that the call option expires immediately before the stock goes ex-dividend. The discounted present value of the expected payoff at N conditional on the stock price being in state i at period $N-1$ is

$$v_i(N-1) = e^{-rh} \sum_{j=1}^k p_{ij} f_j(N),$$

where r is the continuously compounded risk-free rate. The option holder compares $v_i(N-1)$ to the payoff $s_i - K$. Thus, the call option value at period $N-1$, conditional on state i , is

$$f_i(N-1) = \max \left[s_i - K, e^{-rh} \sum_{j=1}^k p_{ij} f_j(N) \right].$$

The general backward induction that defines $\mathbf{f}_{n-1}$, $n = 1, 2, \dots, N$, is

$$f_i(n-1) = \max \left[s_i - K, e^{-rh} \sum_{j=1}^k p_{ij} f_j(n) \right] \quad (4)$$

or, in vector form,

$$\mathbf{f}_{n-1} = \max(\mathbf{s} - K\mathbf{1}, e^{-rh}\mathbf{P}\mathbf{f}_n), \quad (5)$$

where $\mathbf{1}$ is a k by 1 column vector of ones.

An important practical advantage that the Markov chain approach has over lattice methods when valuing American call options is that improvements in accuracy do not require increasing the number of time steps. If the discrete dividends occur at fixed natural time periods such as quarterly or monthly, then increasing the number of time steps in a lattice method requires adjustments that do not generalize in a simple fashion. The adaptive mesh model developed by Figlewski and Gao (1999) circumvents this difficulty by grafting a high-resolution tree on a low-resolution tree close to maturity. If such high resolution is required, however, the Markov chain approach can be adapted similarly by defining a special transition matrix for the last period in which the number of states at period N is increased as needed.

For American put options on dividend paying stocks, the option value must be modified in a slightly different manner. The vector $\mathbf{f}_N$ of option values at expiration is defined as in equation (2b). But the vector $\mathbf{f}_n$ of option values for $n < N$ must take into account that the put option holder is never worse off, and sometimes is better

off, by not exercising the put immediately before the stock goes ex-dividend and instead waiting until immediately after the stock goes ex-dividend.

Define the value of the put option at period $n-1$, conditional on the process being in state i , but immediately after the stock has gone ex-dividend, as

$$f_i(n-1) = \max \left[K - (s_i - d), e^{-rh} \sum_{j=1}^k p_{ij} f_j(n) \right]. \quad (6)$$

That is, the put option holder either chooses to exercise and receive the difference between the strike price K and the ex-dividend stock price $(s_i - d)$ or not exercise because the value $v_i(n-1)$ of continuing to hold the option (or sell it) is greater. The vector form for the backward induction process for the American put option on a dividend-paying stock is

$$\mathbf{f}_{n-1} = \max(K\mathbf{I} - (\mathbf{s} - d\mathbf{I}), e^{-rh}\mathbf{P}\mathbf{f}_n), \quad (7)$$

for $1 < n \leq N$, where $\mathbf{f}_N$ is defined by equation (2b). The vector $\mathbf{f}_0$ is defined by equation (7) if time 0 is immediately prior to the stock going ex-dividend. Otherwise, define $\mathbf{f}_0$ by setting d equal to zero in equation (7).

Unlike for the American call option, early exercise of an American put may be optimal at points in time other than ex-dividend dates, e.g., see Merton (1973). If the time interval between dividends is too long, then the iterative process may not test the put option for early exercise sufficiently often. The above procedure can be generalized in a straightforward manner to add additional decision points. Label the transition matrix from period $n-1$ to n as $\mathbf{P}_n$. Modify the transition probabilities in equation (3) and the option values in equation (7) by replacing d with d_n and letting $t = (n-1)h$ in equation (3). Then the above approximation procedure can be extended to scenarios where the dividend process d_n , $n = 0, 1, 2, \dots, N-1$, is not constant; in particular, the American put may be evaluated at a sequence of points in time where $d_n = 0$ if the stock does not go ex-dividend at n .

For both the American call and put, the option's price at $n = 0$ is the expected value $\boldsymbol{\pi}_0 \mathbf{f}_0$, where $\boldsymbol{\pi}_0$ is the starting vector. The backward induction identifies which states in each time period result in the option's exercise. If the American call option is on a non-dividend paying stock, then early exercise is never optimal. The call option's price defined by the backward induction in equation (5) simplifies to the same formula as the European call:

$$\boldsymbol{\pi}_0 \mathbf{f}_0 = \boldsymbol{\pi}_0 \mathbf{P}^N \mathbf{f}_N e^{-Nrh} = \boldsymbol{\pi}_N \mathbf{f}_N e^{-rT}.$$

On the other hand, early exercise may be optimal for an American put even if the stock pays no dividend, so the backward induction in equation (7), with d set equal to zero, still must be applied.

Once the option values $\mathbf{f}_n$ are determined for each period n , $n = 1, 2, \dots, N$, the remaining task is to construct the probability distribution for Ω , the outcomes space.

The outcomes space is defined to include zero as well as the set of all possible exercise values discounted to the present at the risk-free rate. For American call options, Ω includes zero (the call expires out-of-the-money), $(S(0) - K)$ if $S(0) > K$, and all nonzero elements of the vectors $e^{-rh}\mathbf{f}_N$, $e^{-2rh}\mathbf{f}_N, \dots, e^{-Nrh}\mathbf{f}_N$, where $\mathbf{f}_N$ is defined by equation (2a). For American put options on stocks whose ex-dividend dates are $n = 1, 2, \dots, N-1$, the outcomes space Ω includes zero (the put expires out-of-the-money), $(K - S(0))$ if $S(0) < K$, and all nonzero elements of the vectors $e^{-rh}(\mathbf{f}_N + \mathbf{d1})$, $e^{-2rh}(\mathbf{f}_N + \mathbf{d1}), \dots, e^{-(N-1)rh}(\mathbf{f}_N + \mathbf{d1})$, and $e^{-Nrh}\mathbf{f}_N$, where $\mathbf{f}_N$ is defined by equation (2b).

The construction of the probability distribution for Ω requires an indicator function for option exercise. Define $\eta_i(n)$ to equal one if the option is exercised in state i at period n and zero otherwise. For example, if the option is an American call option on a dividend-paying stock,

$$\eta_i(n) = \begin{cases} 0 & \text{if } s_i - K \leq v_i(n) \\ 1 & \text{if } s_i - K > v_i(n) \end{cases}, \quad (8)$$

where:

$$v_i(n) = \begin{cases} e^{-rh} \sum_{j=1}^k p_{ij} f_j(n+1) & \text{for } n = 0, 1, 2, \dots, N-1 \\ 0 & \text{for } n = N \end{cases}$$

$f_i(N)$ is defined by equation (2a), and $f_i(n)$, $n < N$, is defined by equation (4). Define $H(n)$, $n = 0, 1, 2, \dots, N$, as a k -dimensional diagonal matrix with diagonal elements $\eta_i(n)$, $i = 1, 2, \dots, k$.

The first step of the iterative process assigns probabilities to option values at $n = 0$ for states in which the option is exercised immediately. Partition the probabilities in the row vector π_0 as $\pi_0^c = \pi_0 H_0$ and $\pi_0^{nc} = \pi_0 (I - H_0)$, where I is the identity matrix. Define $\mathbf{e}_0 = H_0 \mathbf{f}_0$. The positive elements of $\mathbf{e}_0$ are assigned the corresponding nonzero values in π_0^c as their probabilities and the remaining probability is reserved for future events via π_0^{nc} . In the next step, define

$$\pi_1^c = \pi_0^{nc} P H_1,$$

$$\pi_1^{nc} = \pi_0^{nc} P - \pi_1^c,$$

$$\mathbf{e}_1 = e^{-rh} H_1 \mathbf{f}_1.$$

The positive elements of $\mathbf{e}_1$ are assigned the corresponding nonzero values in π_1^c as their probabilities and the remaining probability is reserved for future events via π_1^{nc} . The iterative process continues in this manner through the last step, where:

$$\pi_N^e = \pi_{N-1}^{ne} P H_N,$$

$$\pi_N^{ne} = \pi_{N-1}^{ne} P - \pi_N^e,$$

$$\mathbf{e}_N = e^{-rN\Delta t} H_N \mathbf{f}_N.$$

In summary, the probability distribution for Ω is defined by assigning each non-zero element of $\mathbf{e}_n = e^{-rnh} H_n \mathbf{f}_n$ a probability equal to the corresponding element of π_n^e , $n = 0, 1, 2, \dots, N$; the zero value outcome (expiration out-of-the-money) is assigned the probability $\pi_N^{ne} \mathbf{1}$. Let F be the cumulative distribution function. In terms of the above notation,

$$F(x) = 0 \quad \text{if } x < 0, \quad (9a)$$

$$F(x) = \sum_{i=1}^k \pi_i^{ne}(N) + \sum_{n=0}^N \sum_{i=1}^k \pi_i^{ne}(n) \eta_i(n) I\{e_i(n) \leq x\} \quad \text{if } x \geq 0. \quad (9b)$$

Note that by construction,

$$\begin{aligned} \pi_0^e + \pi_1^e + \pi_1^{ne} &= \pi_0^e + \pi_0^{ne} P H_1 + \pi_0^{ne} P (I - H_1) \\ &= \pi_0^e + \pi_0^{ne} P, \end{aligned}$$

and thus

$$\begin{aligned} (\pi_0^e + \pi_1^e + \pi_1^{ne}) \mathbf{1} &= \pi_0^e \mathbf{1} + \pi_0^{ne} P \mathbf{1} \\ &= \mathbf{1}, \end{aligned}$$

because $P \mathbf{1} = \mathbf{1}$ (the rows of P sum to 1) and $\pi_0^e + \pi_0^{ne} = \pi_0$. At step n ,

$$\begin{aligned} \pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_n^e + \pi_n^{ne} &= \pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_{n-1}^{ne} P H_n + \pi_{n-1}^{ne} P (I - H_n) \\ &= \pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_{n-1}^{ne} P \end{aligned}$$

and thus, reusing the fact that $P \mathbf{1} = \mathbf{1}$,

$$\begin{aligned} (\pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_n^e + \pi_n^{ne}) \mathbf{1} &= (\pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_{n-1}^{ne} P) \mathbf{1} \\ &= (\pi_0^e + \pi_1^e + \dots + \pi_{n-1}^e + \pi_{n-1}^{ne}) \mathbf{1} \end{aligned}$$

By mathematical induction on n , this last expression equals 1. Hence the probabilities assigned to the positive outcomes in $\mathbf{e}_n$, $n = 0, 1, 2, \dots, N$ and to the zero outcome total to one, i.e., we have constructed a probability distribution.

Because most approaches available in the literature focus solely on valuations, we are able to provide accuracy comparisons only with respect to the backward induction process which prices an American dividend paying option by the expected value, $\pi_0 f_0$. Table 1 compares Markov chain approximations to Geske and Shastri's (1985) results derived from binomial and explicit finite difference models. Their binomial models are based on $n = 140$, $n = 160$, and $n = 140$, for maturities 1, 4, and 7 months, respectively; whereas their explicit finite difference models use 200 steps in asset price and 320 steps per month. The finite Markov model has 400 states and as many time periods as ex-dividend dates plus two. The stock price in each instance is \$40 with strike prices of \$35, \$40, and \$45. The standard deviation is 0.3 while the risk free interest rate is 5 percent. The stock pays a \$0.50 dividend at times 0.5, 3.5, and 6.5 months.

Table 1—Comparison of American Call Option Values Assuming Discrete Dividend Payments

Strike Price	Model	Maturity in Months		
		1.0	4.0	7.0
35	Binomial Fixed Cash Dividend	5.10	5.73	6.34
	Explicit Finite Difference	5.10	5.74	6.35
	Finite Markov Model with 400 states	5.1018	5.7426	6.3458
40	Binomial Fixed Cash Dividend	1.27	2.69	3.55
	Explicit Finite Difference	1.27	2.70	3.57
	Finite Markov Model with 400 states	1.2686	2.6961	3.5662
45	Binomial Fixed Cash Dividend	0.12	1.04	1.82
	Explicit Finite Difference	0.12	1.04	1.83
	Finite Markov Model with 400 states	0.1212	1.04039	1.8322

Unlike a continuous dividend case where several time periods would be desired to capture many dividend payments, the finite Markov chain results are based on only as many time periods as there are ex-dividend dates (plus the initial and terminal dates). Geske and Shastri halted execution when the price changed less than one cent and therefore reported prices to two decimal places. The finite Markov chain results round to the explicit finite difference price in every case, and there is no apparent loss of accuracy in its use.

Distribution of Exotic Options

Exotic or path-dependent options have claims contingent on the price path traveled by the underlying asset. This section provides examples of how the Markov chain approach applies to path-dependent options. Although there are many varieties of exotic options, barrier options best exemplify the possibilities of the Markov approach. Barrier options may be classified as either knock-out options, which cease to exist when the price of the underlying asset reaches a specified barrier price, or knock-in options, which come into existence only when the price of the underlying

asset reaches a barrier price. Barrier options also may be classified by the nature of the states of the underlying asset as described in the following subsections.

Down-and-out Options

A down-and-out European option ceases to exist (and thus has a payoff of zero) if, at any time prior to or at maturity T , the underlying price S_t falls to or below a specified level, B , where $B < S(0)$. Otherwise the payoff at expiration is $\max(S_T - K, 0)$ if a call option and $\max(K - S_T, 0)$ if a put. Let s_{q+1} be the smallest midpoint of a stock price state such that $s_{q+1} > B$. Then a transition to any of states $1, 2, \dots, q$ results in the option being knocked out; these states will be labeled as the red states. The other $k-q$ states will be labeled as the green states. The probability distribution of payoffs at time T depends on the probability of reaching one of the green states without passing through any of the red states. Partition the one-step transition matrix P as follows:

$$P = \begin{bmatrix} R_{q,q} & U_{q,k-q} \\ D_{k-q,q} & G_{k-q,k-q} \end{bmatrix} \quad (10a)$$

$$R = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,q} \\ p_{2,1} & p_{2,2} & & p_{2,q} \\ \cdot & \cdot & \cdots & \cdot \\ p_{q,1} & p_{q,2} & \cdots & p_{q,q} \end{bmatrix} \quad (10b)$$

$$U = \begin{bmatrix} p_{1,q+1} & p_{1,q+2} & \cdots & p_{1,k} \\ p_{2,q+1} & p_{2,q+2} & & p_{2,k} \\ \cdot & \cdot & \cdots & \cdot \\ p_{q,q+1} & p_{q,q+2} & \cdots & p_{q,k} \end{bmatrix} \quad (10c)$$

$$D = \begin{bmatrix} p_{q+1,1} & p_{q+1,2} & \cdots & p_{q+1,q} \\ p_{q+2,1} & p_{q+2,2} & & p_{q+2,q} \\ \cdot & \cdot & \cdots & \cdot \\ p_{k,1} & p_{k,2} & \cdots & p_{k,q} \end{bmatrix} \quad (10d)$$

$$G = \begin{bmatrix} p_{q+1,q+1} & p_{q+1,q+2} & \cdots & p_{q+1,k} \\ p_{q+2,q+1} & p_{q+2,q+2} & & p_{q+2,k} \\ \cdot & \cdot & \cdots & \cdot \\ p_{k,q+1} & p_{k,q+2} & \cdots & p_{k,k} \end{bmatrix} \quad (10e)$$

The probabilities of transitions between red states are listed in R ; from a red to a green state, U ; from a green to a red state, D ; and between green states, G . For each

n , partition the row vector $\pi_n = [\pi_{Rn}, \pi_{Gn}]$, where π_{Rn} is a 1 by q row vector corresponding to the red states and π_{Gn} is a 1 by $k-q$ vector corresponding to the green states. Only paths that do not pass through a red state can result in a payoff for a down-and-out option. The vector of payoff probabilities at expiration for the green states is $\pi_{G0}G^N$. This formula is determined by performing the matrix multiplication for $\pi_N = \pi_0P^N$ with the partitioned matrix in equation (10a) and then isolating that part of the formula for π_{GN} that corresponds to paths that never pass through a red state. To the extent that the sum of the probabilities $\pi_{G0}G^N$ is less than one, the remaining probability corresponds to the zero payoff.

A slight modification to the backward induction in equation (5) for the standard version of an American call or equation (7) for the standard version of an American put is required to find the distribution for an American down-and-out option. Entering a red state is the same as a forced exercise with payoff equal to zero. Therefore equation (5) for calls and equation (7) for puts holds for all green states, but now we redefine $f_i(n) = 0$, $n = 0, 1, 2, \dots, N$, if i is a red state (i.e., if $i \leq q$). Construction of the probability distribution for the option proceeds in the same manner as described earlier for standard American options except that the exercise indicator function $\eta_i(n) = 1$ if the process enters a red state (*forced exercise* with a zero payoff) as well as if conventional exercise is justified, e.g., for a call option, if $s_i - K > v_i(n)$. The cumulative distribution function F then may be defined by equation (9) in the same manner as for standard American options.

Up-and-in Options

An up-and-in European option is a regular option that comes into existence only if the barrier is reached, i.e., if at least once prior to or at maturity T , the underlying price S_t rises to or above a specified level, B , where $B > S(0)$. If this occurs, then the payoff at expiration is $\max(S_T - K, 0)$ if a call option and $\max(K - S_T, 0)$ if a put. Otherwise it is zero. Let s_q be the largest midpoint of a stock price state such that $s_q < B$. Define red and green states as in equation (10). Then the vector of probabilities for the red states at expiration, conditional on the path passing only through red states, is $\rho_N = \pi_{R0}R^N$. This formula is determined by performing the matrix multiplication for $\pi_N = \pi_0P^N$ with the partitioned matrix in equation (10a) and then isolating that part of the formula for π_{RN} that corresponds to paths that never pass through a green state. Thus, the vector of probabilities for end states conditional on the path passing through a green state at least once is $\pi_0P^N - [\rho_N, \mathbf{0}_{1,k-q}]$, where $\mathbf{0}_{1,k-q}$ is a 1 by $k-q$ row vector of zeros.

For up-and-in American options, the ability to exercise early depends on whether the process has reached the barrier at least once. Define the green and red states as above for the European up-and-in options. For the American version, the ability to exercise early depends on whether the process previously has visited a green state at least once. The basic computational strategy is to first conduct the

backward induction using equation (5) for calls and (7) for puts in order to identify any permitted and desirable early exercise points along with their payoffs.

The probabilities then are computed with forward induction. An up-and-in option starts in a red state by definition. Define $\pi_0^e = \mathbf{0}$, a 1 by k row vector of zeros, and $\pi_0^{ne} = \pi_0$. (For completeness, define the vector $\mathbf{e}_0$ as before but note that the choice of values is irrelevant because $\pi_0^e = \mathbf{0}$.) The vector of probabilities for the red states at period $n = 1$, conditional on the path having passed only through red states, is $\rho_1 = \pi_{R0}R$, and the vector of probabilities for states conditional on the path having passed through a green state at least once is $\pi_0P - [\rho_1, \mathbf{0}_{1,k-q}]$. Define

$$\pi_1^e = \{\pi_0^{ne}P - [\rho_1, \mathbf{0}_{1,k-q}]\}H_1,$$

$$\pi_1^{ne} = \pi_0^{ne}P - \pi_1^e,$$

$$\mathbf{e}_1 = e^{-rh}H_1\mathbf{f}_1,$$

where the H matrix is defined by equation (8). In the next step, let π_{R1}^{ne} be the first q elements of the row vector π_1^{ne} , and define

$$\pi_2^e = \{\pi_1^{ne}P - [\pi_{R1}^{ne}R, \mathbf{0}_{1,k-q}]\}H_2,$$

$$\pi_2^{ne} = \pi_1^{ne}P - \pi_2^e,$$

$$\mathbf{e}_2 = e^{-2rh}H_2\mathbf{f}_2.$$

The forward iteration continues in this fashion. The cumulative distribution function F then may be defined by equation (9) in the same manner as for standard American options.

Up-and-out Options and Down-and-in Options

An up-and-out European option ceases to exist if, at any time prior to or at maturity T , the underlying price S_t rises to or above a specified level, B , where $B > S(0)$. Otherwise the payoff at expiration is $\max(S_T - K, 0)$ if a call option and $\max(K - S_T, 0)$ if a put. Let s_{q+1} be the smallest midpoint such that $s_{q+1} > B$. Then a transition to any of states $q+1, q+2, \dots, k$ results in the option being knocked out; these states will be labeled as the red states. The other q states will be labeled as the green states. The probability vector for payoffs at time T depends on the probability of reaching a green state at expiration without visiting any of the red states.

Partition the one-step transition matrix P as follows:

$$P = \begin{bmatrix} G_{q,q} & U_{q,k-q} \\ D_{k-q,q} & R_{k-q,k-q} \end{bmatrix}$$

$$G = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,q} \\ p_{2,1} & p_{2,2} & & p_{2,q} \\ \cdot & \cdot & \cdots & \cdot \\ p_{q,1} & p_{q,2} & \cdots & p_{q,q} \end{bmatrix}$$

$$R = \begin{bmatrix} p_{q+1,q+1} & p_{q+1,q+2} & \cdots & p_{q+1,k} \\ p_{q+2,q+1} & p_{q+2,q+2} & & p_{q+2,k} \\ \cdot & \cdot & \cdots & \cdot \\ p_{k,q+1} & p_{k,q+2} & \cdots & p_{k,k} \end{bmatrix}$$

where U and D are defined as in equation (10) except that now the probabilities in U are for transitions from a green to a red state and the probabilities in D are for transitions from a red to a green state, while R holds the probabilities of transitions between red states and G between green states. For each n , partition the row vector $\pi_n = [\pi_{Gn}, \pi_{Rn}]$, where π_{Gn} is a 1 by q row vector corresponding to the green states and π_{Rn} is a 1 by $k-q$ vector corresponding to the red states. Only paths that do not pass through a red state can result in a payoff for an up-and-out option. Analogous to the down-and-out options, the vector of probabilities for payoffs in the green states is $\pi_{G0}G^N$. To the extent that the sum of the probabilities $\pi_{G0}G^N$ is less than one, the remaining probability corresponds to the zero payoff.

A down-and-in European option is a regular option that comes into existence only if the barrier is reached, i.e., if at least once prior to or at maturity T , the underlying price S_t falls to or below a specified level, B , where $B < S(0)$. If this occurs, then the payoff at expiration is $\max(S_T - K, 0)$ if a call option and $\max(K - S_T, 0)$ if a put. Otherwise it is zero. Let s_q be the largest midpoint such that $s_q < B$. Define the green and red states as above for the up-and-out option. Analogous to the up-and-in option, the probability vector for the red states at N , conditional on the path passing only through red states, is $\rho_N = \pi_{R0}R^N$. Thus, the probability vector for end states conditional on the path passing through a green state at least once is $\pi_0P^N - [\mathbf{0}_{1,q}, \rho_N]$, where $\mathbf{0}_{1,q}$ is a 1 by q row vector of zeros.

Computations for up-and-out and down-and-in American options can be handled in a manner analogous to the procedures described earlier for down-and-out and up-and-in American options. The exercise indicator matrices, $H(n)$, $n = 0, 1, 2, \dots, N$, for the up-and-in and down-and-in options are defined in the same manner as standard American options, while $H(n)$ for the down-and-out and up-and-out options are generalized so that $\eta_i(n)$ equals one for either forced exercise with payoff of zero

(when s_i is a barrier state, i.e., a red state) or true exercise (e.g., if $s_i - K > v_i(n)$ for a call option).

Stochastic Dominance

This section begins with a short review of the well-known stochastic dominance theorems, e.g., see chapter 11 in Elton and Gruber (2003). Let

$U_1 = \{\text{utility functions } u: \text{investor prefers more to less, i.e., } du/dx > 0\},$

$U_2 = \{u \in U_1: \text{investor is risk-averse, i.e., } d^2u/dx^2 < 0\},$

$U_3 = \{u \in U_2: \text{investor exhibits decreasing absolute risk aversion, i.e., } d^3u/dx^3 > 0\}.$

Let $F_{1i}(x)$ be the cumulative distribution function for random variable X_i , and define

$$F_{2i}(x) = \int_{-\infty}^x F_{1i}(t) dt$$

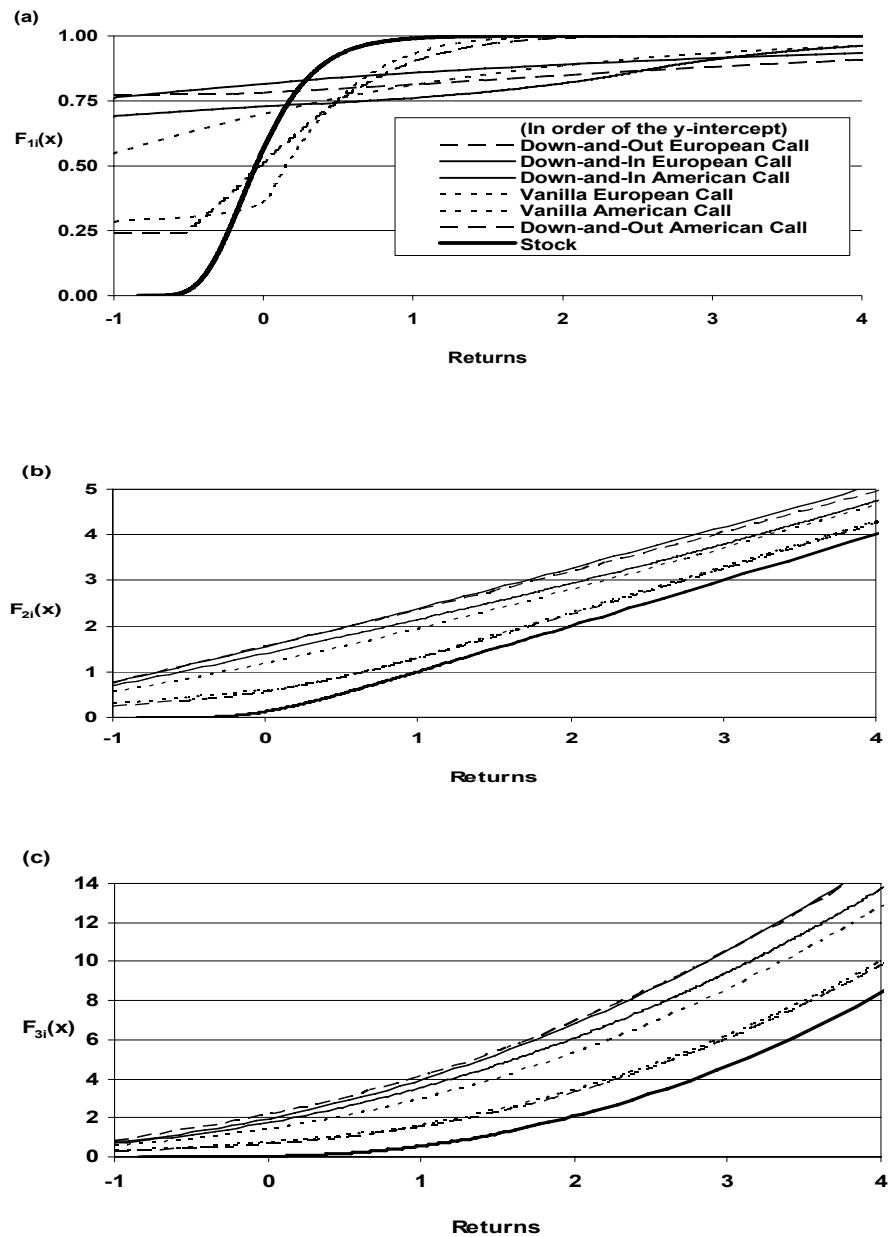
$$F_{3i}(x) = \int_{-\infty}^x F_{2i}(t) dt.$$

Distribution i dominates distribution j in the sense of first-order stochastic dominance, i.e., i is preferred to j for all utility functions in U_1 , if and only if $F_{1i}(x) \leq F_{1j}(x)$ for all x and strict inequality holds for at least one value of x . Distribution i dominates distribution j in the sense of second-order stochastic dominance, i.e., i is preferred to j for all utility functions in U_2 , if and only if $F_{2i}(x) \leq F_{2j}(x)$ for all x and strict inequality holds for at least one value of x . Distribution i dominates distribution j in the sense of third-order stochastic dominance, i.e., i is preferred to j for all utility functions in U_3 , if and only if $F_{3i}(x) \leq F_{3j}(x)$ for all x and strict inequality holds for at least one value of x and the mean of distribution i is greater than the mean of j .

Figures 1a through 1c plot the functions for European and American call options of differing types. The underlying stock is assumed to have a current price of \$80, a volatility of 20 percent, and to pay a quarterly, fixed dividend of \$2. The options are assumed to have a maturity of 2.5 years, strike price of \$70, and, when appropriate, a barrier of \$75. The risk-free interest is 5 percent per annum. Figure 1 summarizes tests for first, second, and third order stochastic dominance among standard, knock-out, and knock-in calls. The knock-out is a down-and-out barrier option and the

Figure 1—Stochastic Dominance Analysis

(a) First Degree Stochastic Dominance or Cumulative Distributions Comparing Returns on all Assets. (b) Second Degree Stochastic Dominance. (c) Third Degree Stochastic Dominance. The legends for panels (b) and (c) are the same as for panel (a)



knock-in is a down-and-in barrier option. For the European options, the expected values are such that $\text{standard} = \text{down-and-out} + \text{down-and-in}$. (In the numerical examples below, this equality was true at 6 decimal places.)

Figure 1a shows the cumulative distributions of returns for the stock itself and for all European and American standard down-and-out and down-and-in call options. It therefore tests for first degree stochastic dominance among these instruments. No asset dominates any other asset with respect to utility functions U_1 , because all distributions intersect each other pair-wise. In these examples second degree stochastic dominance appears to determine the preference ordering. For the European call options, the standard option dominates the down-and-in and down-and-out options, due mainly to the fact that the probability of a minus one return is more than 25 percent greater for the exotic options than for the standard European call. Figures 1b and 1c indicate that although the down-and-in option has only a slight advantage for low returns it is enough to keep it from being stochastically dominated at second and third order dominance by the European down-and-out option.

The preference ordering differs greatly for American call options. The down-and-in American call is second order dominated by both the standard and the down-and-out American calls, and the down-and-out call stochastically dominates the standard at second and third order dominance. Key to understanding these differences is knowing why the down-and-in distribution changes so little while the down-and-out distribution changes so much when switching from European to American options. With the American down-and-in option, there is a classic Catch-22. If the barrier hasn't been reached, then even though the possibility of never hitting the barrier lowers the expected value of holding the option, the investor cannot act on this information because the barrier hasn't been reached. If the barrier has not been reached, then it is irrelevant for present decisions. For the down-and-out option, the investor is free to exercise early as long as the barrier has not been reached. The possibility of the option being extinguished lowers the expected value of holding the option and encourages early exercises. In the present example, early exercising drops the probability of a minus one return for the down-and-out option so much that it is below even that of the standard American call.

Conclusions

Markowitz portfolio selection is limited to assets with normally distributed returns and investors with quadratic utility functions. Stochastic dominance analysis places fewer restrictions on return distributions and utility functions and thus is more suitable for preference ranking of securities such as options. The finite Markov chain method described in this paper can be applied to determine the entire distribution for the option and thus the necessary inputs for a stochastic dominance analysis. In the literature cited and in the examples presented in this paper, the numerical results of the finite Markov chain method are at least as good as all other approximation tech-

niques for the valuation of options where no closed form solution exists. Also, with our innovation in the construction of the transition matrix permitting valuation of American options on discrete dividend paying stocks, researchers need not approximate results by assuming continuous dividend payments or avoid including such options in their sample altogether. This paper uses the lognormal distribution extensively to facilitate comparison with alternative approaches and models in the literature. Other pricing distributions could have been used to form the transitional probability matrix. Deriving other pricing distributions is beyond the scope of this paper. We focus instead on the Markov chain approach as a calculation technique for option pricing models given the distribution assumptions of asset returns. In principle, once one has obtained a pricing distribution—whether lognormal or non-Gaussian cases such as stable Pareto, Cauchy, or normal inverse Gaussian—one can compute the corresponding transitional probability matrix. Nor does the analysis depend upon the stochastic process being stationary. Different transitional matrices could have been used in every time period to reflect changes in risk-free interest, volatility, length of time period, or size of discrete dividend, e.g., the GARCH model for volatility developed by Duan and Simonato (2001).

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