Disappearing Evidence of Chaos in Security Returns: A Simulation

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Recent evidence suggests stock market returns may be characterized by deterministic nonlinear processes—chaos, as it commonly is described. If true, these assertions present a strong challenge to the theory of efficient markets and question modern financial research based on the standard statistical assumptions of independence and normality in the return-generating process. Until now, researchers have not distinguished nonlinear behavior observed in the return series of single securities and those of security portfolios. This is unfortunate, for the two behave differently. A portfolio of chaotic securities loses its chaotic character as additional securities are added. Results obtained through simulation analysis suggest that researchers seeking evidence of chaos in the financial markets should look to the behavior of individual securities or even lines of business rather than to aggregate portfolios or market indices.

INTRODUCTION

Recent empirical evidence of nonlinear dependencies in financial time series suggests security prices may be more predictable than formerly thought (Hinich and Patterson, 1985; Scheinkman and LeBaron, 1989; Peters, 1991a, 1991b; Hsieh, 1991). Seemingly random price and return sequences may arise from deterministic nonlinear dynamical systems. Nonlinear models of security price behavior historically have been avoided due to their complexity and because they generally lack well-defined solutions. A new field of inquiry, colloquially known as *chaos theory*, has arisen that attempts to describe the characteristics of seemingly random systems, including the behavior of security price series. This paper discusses the effect of aggregating known chaotic series of simulated security prices and assesses the resulting security portfolio characteristics. Sugihara, Grenfell, and May (1990) and McNevin and Neftci (1992) have examined similar problems in the context of ecological and business cycle time series. Simulation analysis is used in this paper to develop predictions of portfolio behavior that are subsequently verified using actual stock market data.

BACKGROUND

The roots of modern chaos research generally are traced to meteorologist Edward Lorenz's attempts to predict future weather patterns using a system of nonlinear equations in the early 1960s (Lorenz, 1963). Lorenz is credited with discovery of the *butterfly effect*,

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the notion that small changes in initial conditions can engender large variations in future events.[1] Later models allowed him to identify regions of predictable, deterministic, periodic behavior and areas of nondeterministic, aperiodic behavior. Researchers in disciplines as diverse as biology, physics, astronomy and economics have found similar patterns in numerous natural phenomena. Their collective work comprises the field of research known as *chaos theory.*[2]

STOCHASTICITY, NONLINEARITY, AND CHAOS

A key characteristic of chaotic series is their seemingly random behavior. This arises in part from researchers' greater familiarity with linear systems. A given series of data points may be deterministic or stochastic, i.e., arising from the application of known mathematical equations or generated from a probability function. Deterministic nonlinear systems frequently are called chaotic. Datasets generated by nonlinear processes resemble (at least superficially) those generated by linear stochastic processes, and many software random number generators rely on chaotic equations for their "random" numbers. Chaos further is characterized as either low or high complexity. Low complexity chaos involves relatively few parameters and is fairly easy to detect. If one were able to identify the important parameters in a low dimensional chaotic system, one could hope for better predictive ability. Highly complex chaotic processes can be impossible to detect using limited datasets.

CORRELATION DIMENSION

Nonlinearity is a necessary, but not sufficient, condition for chaotic behavior. Randomly generated nonlinear data will fill a given n-dimensional space in a uniform manner, whereas true chaotic datasets leave "holes." Grassberger and Procaccia (1983) develop a metric for measuring the amount of space filled by a particular dataset, the correlation dimension (α_M). Grassberger and Procaccia's algorithm tests the relatedness of sequential subsets of the original series using the correlation integral (C_M). First, an embedding dimension (M) is chosen reflecting the length of each subseries, called M histories. The degree of similarity between subsequent M histories is defined as ε. Dividing the correlation integral by ε and taking the natural logarithm produces the series' correlation dimension. The lower the correlation dimension is, the greater is the probability of chaos. Grassberger and Procaccia report correlation dimensions for known chaotic systems ranging from 1.00 to 7.50. (The correlation dimension for a truly random process equals infinity.)

The Grassberger and Procaccia algorithm has been criticized on at least three fronts. First, it is biased downward with respect to small sample sizes, increasing the probability of finding chaos where none exists (Ramsey and Yuan, 1989; Ramsey, Sayers, and Rothman, 1990). Second, infinite correlation dimensions are impossible to verify with finite data sets. The larger the embedding dimension, the more data are required to estab-

[1] This phenomenon takes its name from the suggestion that an event as apparently inconsequential as the flapping of a butterfly's wings in Japan can cause a hurricane in Brazil a month later.

[2] James Gleick's (1987) best-selling book, *Chaos*, provides an excellent introduction to this field of research.

lish the presence or absence of chaos. Third, it is unable to distinguish true chaos from some nonlinear stochastic models (Scheinkman and LeBaron, 1989).

BDS STATISTICS

The BDS test statistic attempts to overcome shortcomings of the correlation dimension. Developed by Brock, Dechert, and Scheinkman (1987), it tests for nonlinearity based on the null hypothesis that a data series is independent and identically distributed (IID). The BDS statistic is distributed asymptotically normal and avoids the biases of the correlation dimension. It is robust in detecting the departure from independent and identically distributed returns in non-IID data generated from a number of models (Hsieh, 1991).

Unfortunately, rejecting the hypothesis of independent and identically distributed returns is not enough to claim a data series is chaotic. Hsieh (1991) discusses three alternative explanations: linear dependence; nonstationarity; and nonlinear stochastic processes. Linear dependence is generally not applicable, as data series are usually prefiltered to remove linear correlations. Changing economic conditions make it difficult to argue for parameter stationarity in long data series. The final explanation, in conjunction with conditional heteroskedasticity, is used by Hsieh (1991) as an explanation for the behavior of returns on the S&P 500 and the CRSP equally-weighted and value-weighted indexes.

LYAPUNOV EXPONENTS

A less-powerful method for detecting the presence of chaos in financial time series involves Lyapunov exponents. Chaotic series display sensitive dependence to initial conditions. Lyapunov exponents are used to measure this sensitivity. For every dimension in phase space there is a corresponding Lyapunov exponent. It is impossible to compute all possible Lyapunov exponents for a system without knowledge of the underlying equations of motion. Brock (1986) and others have shown that if the largest Lyapunov exponent is positive, however, the system is chaotic.

CHAOS IN THE FINANCIAL MARKETS

A substantial literature exists on various applications of chaos theory to the study of financial economics (cf. Savit, 1988; Baumol and Benhabib, 1989; Brock, Hsieh, and LeBaron, 1991; Peters, 1991b; Hsieh, 1991). Empirical investigations of economic and financial time series have yielded mixed results, as shown in Table 1.

Four of the five studies involving stock or bond time series use datasets composed of aggregate market indices or portfolios, e.g., the S&P 500 and the CRSP equally-weighted and value-weighted indexes. One should be careful not to generalize from these results to the behavior of single securities. Portfolios constructed from series exhibiting known chaotic behavior have different characteristics than their underlying components.

METHODS

Simulated return series are generated using a set of five nonlinear equations reported by May (1976) as demonstrating chaotic behavior under certain conditions. These equations, or *chaos generators*, are presented in Table 2 with their respective threshold values where chaotic behavior begins.

Table 1—Empirical Evidence of Chaos in Economic and Financial Time Series

Author and Year	Data Series	Method	Results
Hinich and Patterson (JBES 1985)	Daily returns for 15 individual stocks	Bispectrum test	Returns are generated by nonlinear, non-Gaussian process
Scheinkman and LeBaron (JB 1989)	Weekly returns on CRSP value-weighted index	BDS test, shuffle test	Evidence of nonlinearity
Hsieh (JB 1989)	Daily changes in foreign exchange rates	BDS test	Strong nonlinear dependence
Frank, Gencay, & Stengos (EER 1988)	GNPs of Japan, Italy, Germany and UK	Correlation dimension, Lyapunov exponent	No evidence of chaos for any country, although Japan shows nonlinearity and is empirically distinct from others
Brock and Sayers (JME 1988)	Employment, industrial production, unemployment, and pig iron production	Correlation dimension, Lyapunov exponent, Brock's residual test	Weak evidence of chaos, but series are nonlinear
Frank and Stengos (JME 1988)	Canadian national income data	Correlation dimension, Brock's residual test	Little evidence of chaos
Peters (FAJ 1989)	Monthly returns on SP 500 and 30 year T-bonds	Hurst exponent	Biased random walk
Peters (FAJ 1991)	Monthly SP 500 returns detrended for inflation	Correlation dimension, Lyapunov exponent	Underlying low dimension chaotic attractor
Hsieh (JF 1991)	CRSP equally weighted and value-weighted index; weekly, daily, and 15 minute SP 500 returns	BDS test	Not independent and identically distributed; not chaotic; explained by conditional heteroskedasticity

Table 2–Equations Used to Generate Chaotic Security Returns

No.	Equation	Tuning Parameter	Threshold Values: Point of Instability	Beginning of Chaotic Region
1	$X_{t+1} = a\,X_t(1 - X_t)$	a	a = 3.000	a > 3.6786
2	$X_{t+1} = X_t \exp[r\,(1 - X_t)]$	r	r = 2.000	r > 2.8332
3	$X_{t+1} = a\,X_t$ if $X_t < 0.50$; $X_{t+1} = a\,(1 - X_t)$ if $X_t > 0.50$	a	a = 1.000	a > 1.4142
4	$X_{t+1} = \lambda X_t$ if $X_t < 1$; $X_{t+1} = \lambda X_t^{(1-b)}$ if $X_t > 1$	b	b = 2.000	b > 2.6180
5	$X_{t+1} = \lambda X_t\,[1 + X_t]^{-B}$	λ, B	varies	varies

Adapted from May (1976)

Equation 1, the logistic equation, exemplifies the behavior of a chaos generator. For this equation, $X_{t+1} = a\,X_t(1 - X_t)$. For $a < 3.0$ and $0 < X_t < 1.0$, the series is stable and converges to a fixed value of X_{t+1} (Panel A). At $a = 3.0$ the first bifurcation occurs, and X_{t+1} may take one of two values (Panel B). At approximately $a = 3.4$ another bifurcation occurs, and values of X_{t+1} oscillate between four points (Panel C). Chaos begins at $a = 3.6786$, and values of X_{t+1} fail to demonstrate any regular periodicity (Panel D). The behavior of this system is illustrated in Figure 1.[3]

The behavior demonstrated by the logistic equation is typical of nonlinear equations and accounts for their extreme sensitivity to initial conditions. Because of this, even a slight difference in the initial condition (X_0) yields an entirely different series of future values of X. The divergence between two series initiated with slightly different starting conditions increases with time.

PHASE DIAGRAMS

Financial data commonly are expressed in the form of a time series, i.e. periodic observations are plotted against time. Viewed in this fashion, financial time series often appear to have been derived in random fashion—the efficient markets hypothesis would have it so. Another method for expressing time series data involves the notion of phase space. A phase diagram plots momentum. The x-axis no longer is defined as time; rather, it is the value of a particular variable at a given point in time (X_t). The y-axis plots the value of the same variable lagged some finite number of periods (X_{t-n}). Figure 2 shows normal random variates in both time and phase space (at $n = 1$). Note the cobweb appearance of the phase diagram and the apparent unrelatedness of each observation to its predecessor. Panel C provides a frequency distribution indicating the variables are normally distributed.

An interesting result arises when observations generated from an equation with known chaotic properties are plotted in phase space and time. Figure 3 presents phase

[3] An interesting characteristic of the logistic equation is that when a equals 3.8284 the series again oscillates between three points.

Figure 1—The Logistic Equation

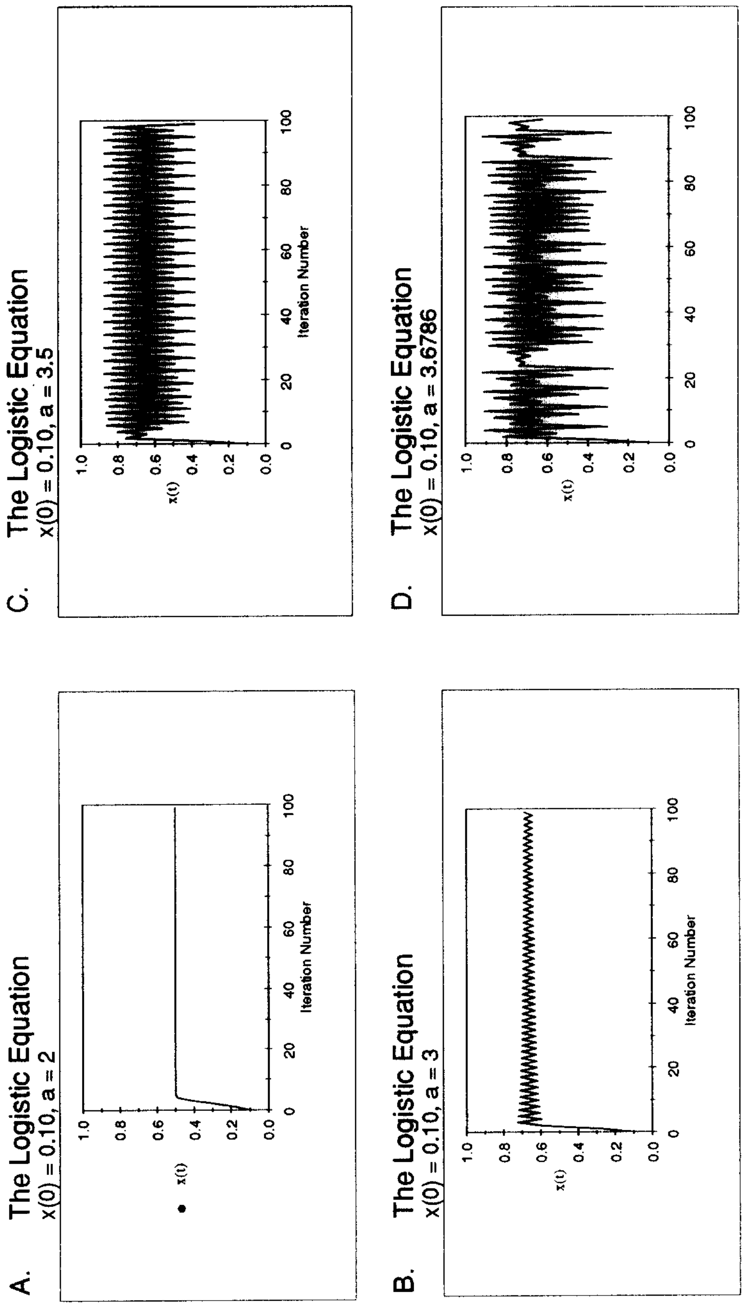
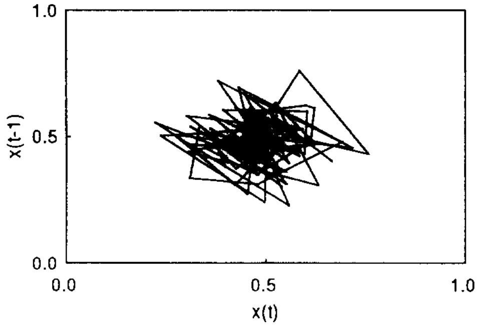
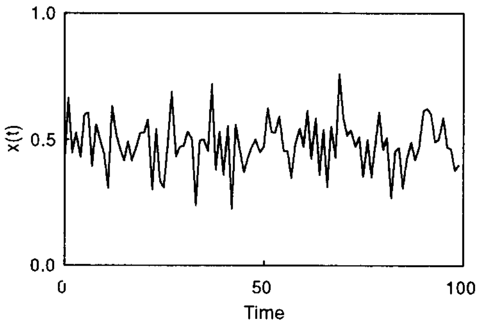


Figure 2—Time-Series and Phase Diagrams for Normal Random Variables

A. Phase Space



B. Time Series



C. Frequency Distribution

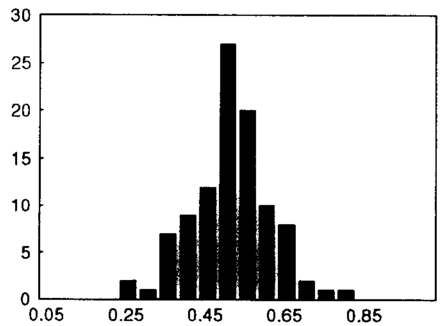
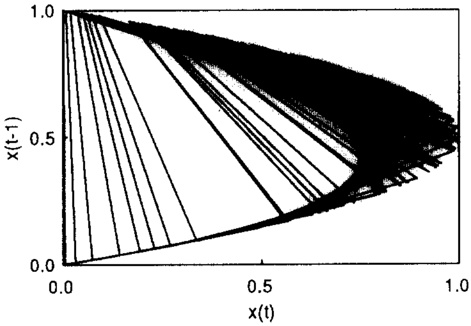
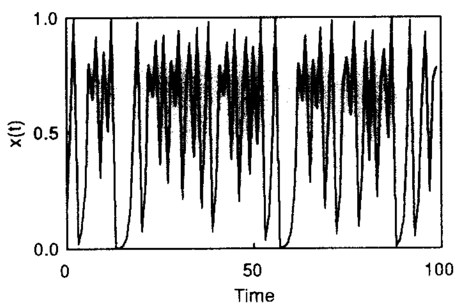


Figure 3—One Firm Portfolio Generated From the Logistic Equation (a = 4)

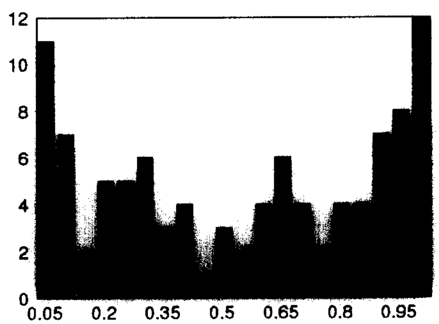
A. Phase Space



B. Time Series



C. Frequency Distribution



space, time series, and frequency distribution results for equation 1 (the logistic equation) at n = 1, a = 4, and X_0 = .00538.

The time series plot of the logistic equation appears superficially similar to that produced from a sequence of random numbers. When plotted in phase space, a discernible pattern appears hinting at the underlying deterministic structure of the generating equation. This is characteristic of chaos—seemingly random information may be generated from a deterministic equation. Each of the chaotic equations described in Table 2 leads to its own particular pattern when plotted in phase space. While these patterns are nonrepetitive, they are confined to a particular region of the phase diagram, the *attractor*.

To investigate the behavior of securities as they are combined into portfolios, return series are created using the chaos generators from Table 2. The tuning values for each equation are set such that each will generate a chaotic series of 5,100 observations. The initial conditions (X_0) for each return series are determined at random, and only the last 100 observations are used in order to eliminate biases from the random number generator. Equally weighted portfolios are constructed by summing the values for a set of return series at time t and dividing by the total number of series in the portfolio. Results from four portfolios (one firm, two firms, 20 firms, and a five firm mixed model) are reported in this paper. The most favorable condition for chaotic behavior in a portfolio should exist when each return series is determined using the same chaotic generator, as was the case for the one, two and 20 firm portfolios. The final five firm portfolio comprises return series generated using all five of the models shown in Table 2. Because nonlinear equations are in general non-additive, portfolios created from a group of chaotic securities are likely to lose their chaotic characteristics and approach the appearance of randomness as the number of securities in the portfolio increases.

GRAPHICAL ANALYSIS

Investigators searching for chaos in financial time series have several methodologies at their disposal. For the most part, these involve statistical tests and inferences regarding the distribution of the simulated portfolio returns. A first pass approach may be to graph the return series in phase space as additional securities are added to the portfolio. Destruction of a well-behaved attractor region would provide evidence for an intuitive belief in disappearing chaos.

STATISTICAL ANALYSIS

Correlation dimensions and BDS test statistics are computed for each portfolio series identified in the previous sections using a program supplied by W. Dechert. Largest Lyapunov exponents are calculated for each of several portfolios using a computer program adapted from Peters (1991b).[4]

[4] Following Peters (1991b, pp. 219-222), the *maximum divergence before a replacement point* (SCALMX) is set to equal 10 percent of the difference between the maximum and minimum values of each series, while the minimum divergence (SCALMN) equals 10 percent of SCALMX.

APPLICATION TO MARKET DATA

The above analyses also are applied to empirical data. Daily return series for the Wilshire 5000, NYSE Index, S&P 500, and Dow-Jones Industrials equity indices are obtained from the *Lotus OneSource* financial database for the period January 1, 1991 to May 21, 1991 (100 days). Correlation dimensions, BDS test statistics, and largest Lyapunov exponents are computed for each index using methods described earlier.

RESULTS

TESTS USING A SINGLE CHAOS GENERATOR

For the sake of brevity, only portfolios created using the logistic equation are reported here.[5] Figures 4 and 5 present phase diagrams, time series plots, and frequency diagrams for portfolios comprising two and 20 firm return series, respectively. These represent numerous trials involving different sets of simulated firms, each of whose initial conditions (X_0) are chosen randomly.

As the number of securities in a portfolio increases, the phase diagram grows less well-defined, more cobweb-like, and becomes difficult to distinguish from the purely random process shown in Table 1. Also, the distribution of observations more nearly approximates a normal, bell-shaped curve.

TESTS USING MULTIPLE CHAOS GENERATORS

A five-series mixed portfolio is constructed using returns generated from each of the five equations in Table 2. Figure 6 shows the combined result. The five-series mixed portfolio is indistinguishable from the random process illustrated in Figure 2.

STATISTICAL ANALYSIS OF SIMULATED FIRMS

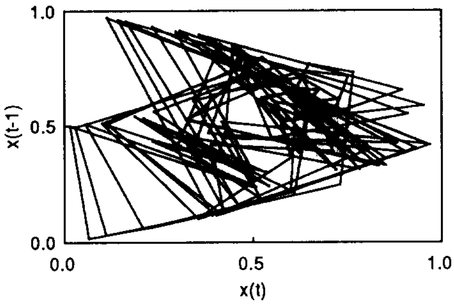
Correlation dimensions (α_M) and BDS statistics and Lyapunov exponents are computed for portfolios created using the logistic equation (one, two, ten, and 20 firms) and a fifth portfolio composed of five purely random series. These results are presented in Table 3. Based on their work with Monte Carlo simulations, Brock, Hsieh, and LeBaron (1991) recommend ε values between one half and three halves the standard deviation of the sample data and M values of between 2 and 5 for small data sets. The ε values in Table 3 are set equal to the standard deviation of the individual data sets.

The correlation dimensions for all regimes of embedding dimension are low and generally decline further as portfolio size increases. As Hsieh (1991) has noted, however, the Grassberger and Procaccia methodology requires a substantial number of data points and may be biased for small samples, such as those employed in our study. The BDS findings are much more supportive. The simulated one firm portfolio, comprising known chaotic returns, strongly rejects the hypothesis of independent and identically distributed returns. Adding even one additional firm destroys the portfolio's chaotic character, with the exception of embedding dimension $M = 10$, which retains its chaotic behavior somewhat longer. The largest Lyapunov exponent declines from 11.62 to 5.04, indicating a breakdown of deterministic chaotic structure as the number of firms in the portfolio is

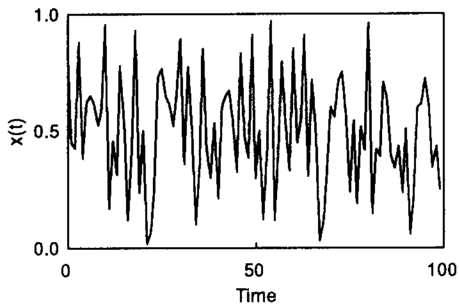
[5] These results illustrate outcomes achieved using any of the other four equations, which are available from the authors upon request.

Figure 4—Two Firm Portfolio Generated From the Logistic Equation (a = 4)

A. Phase Space



B. Time Series



C. Frequency Distribution

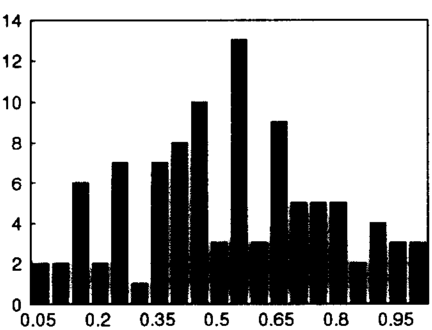


Figure 5—20 Firm Portfolio Generated From the Logistic Equation (a = 4)

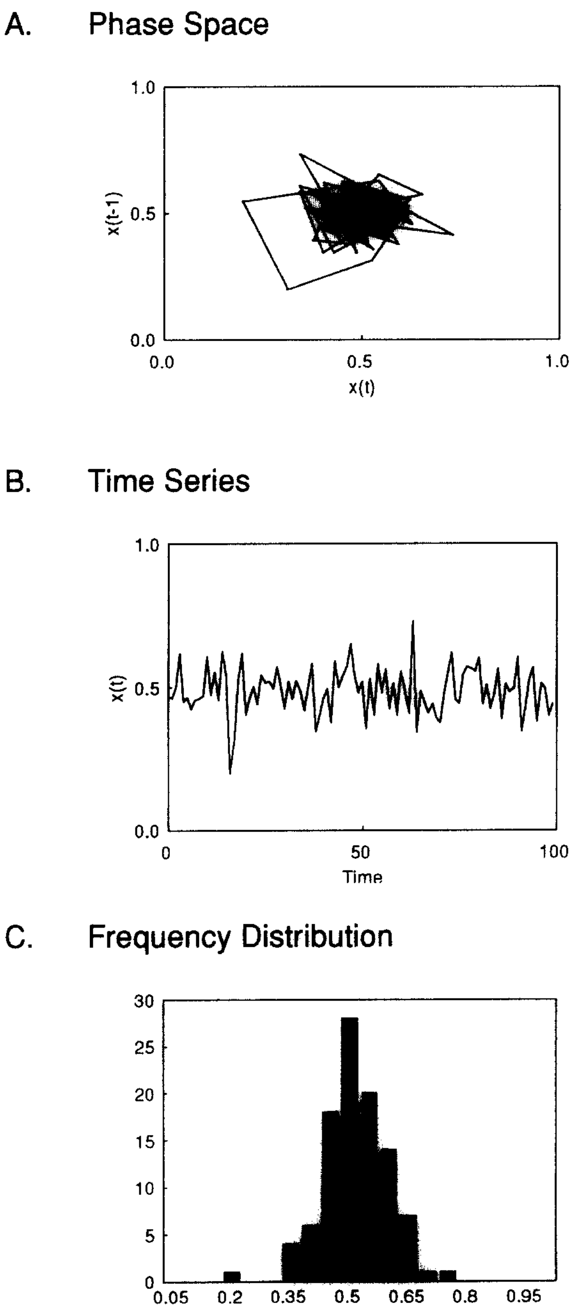
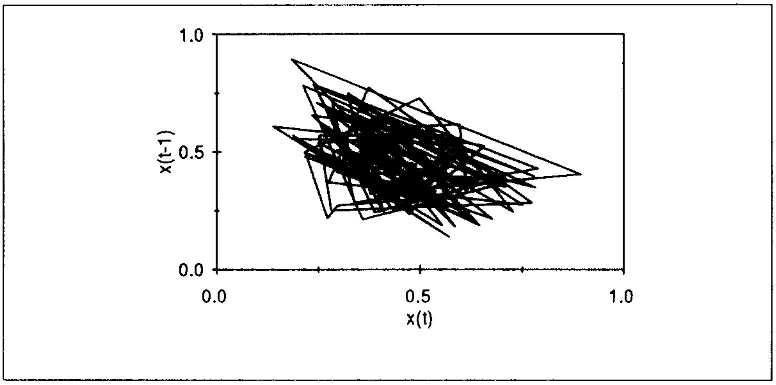
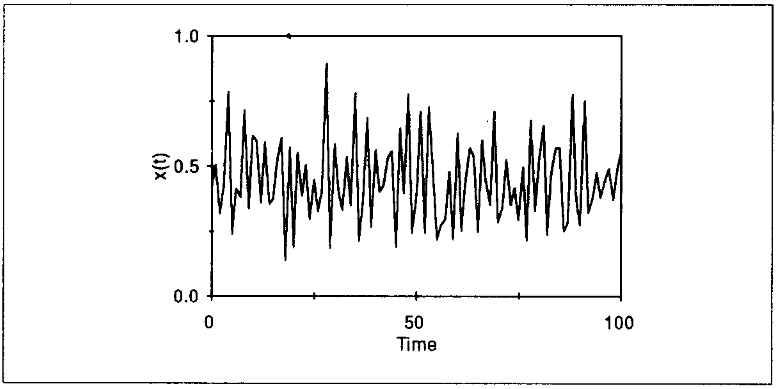


Figure 6—Mixed Firm Portfolio Generated From Five Different Equations

A. Phase Space



B. Time Series



C. Frequency Distribution

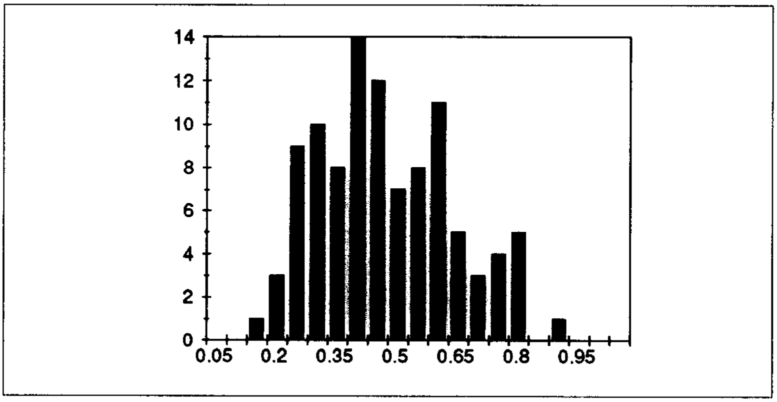


Table 3–Analysis of Simulated Portfolios

	M = 2		M = 5		M = 10		Largest
	α_M	BDS	α_M	BDS	α_M	BDS	Lyapunov
One Firms (ε= .348)	1.15	59.7	2.66	50.4	5.34	44.3	11.62
Two Firms (ε = .252)	1.00	-.576	2.53	-.404	6.05	-2.001	9.72
Ten Firms (ε= .162)	.65	.219	1.67	-0.32	4.19	-1.534	8.68
20 Firms (ε= .130)	.50	1.56	1.29	.259	2.57	.144	5.04
Random (ε= .146)	.61	-.830	1.57	-.805	2.93	.297	5.99

Table 4–Analysis of Market Portfolios

	M = 2		M = 5		M = 10		Largest
	α_M	BDS	α_M	BDS	α_M	BDS	Lyapunov
NYSE (ε = .1772)	.789	2.82	2.00	1.38	4.02	.750	7.712
S&P 500 (ε = .1761)	.787	2.79	2.01	1.51	4.00	.741	9.353
Dow Jones (ε = .1570)	.668	2.40	1.81	2.42	3.63	1.18	7.702
Wilshire (ε = .1763)	.783	2.30	2.01	1.48	3.68	.251	7.734

increased. The decrease in the Lyapunov exponent is what should be noted, not the magnitude. In sample sizes this small, as pointed out by Eckmann and Ruelle (1992) and Brock and Baek (1991), the precision is low.[6]

MARKET ANALYSES

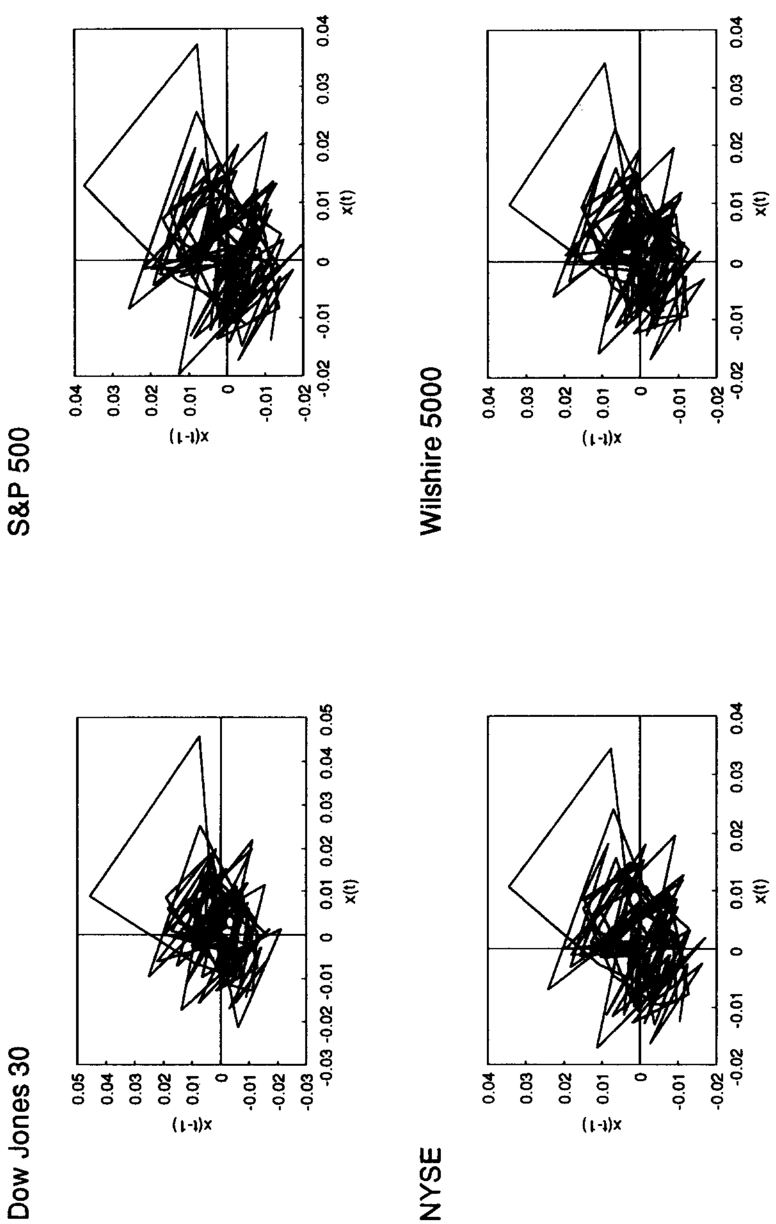
Similar results are found for all four return series (Table 4 and Figure 7). Correlation dimensions are low and most nearly approximate those of the ten and 20 firm simulated portfolios. Only at the lowest embedding dimension (M = 2) do the BDS statistics lend support for rejection of the independent and identically distributed returns hypothesis. The largest Lyapunov exponents for each index series are similar to those found in our earlier simulations. Together these results corroborate our expectations of finding little or no chaotic behavior in market portfolio *indices*.

CONCLUSIONS

The results of simulation analyses presented in this paper suggest researchers seeking evidence of chaotic behavior in the financial markets should look to the returns on individual securities rather than to aggregate market indexes. Beginning with known, individual chaotic series, we create portfolios that appear indistinguishable from those resulting from a sequence of random numbers. The most favorable condition for producing chaotic return indices is if all the stocks in a portfolio follow the same chaotic generator, yet even in this case a portfolio's chaotic properties decrease as more firms are added. This decline is swift—almost all chaotic properties seem to have disappeared in a portfolio of 20 firms. Our findings illustrate that chaotic equations are non-additive, but these findings have been ignored by researchers searching for evidence in empirical data.

[6] A sample size of 100 represents eight years of monthly returns or approximately five months of observed daily returns.

Figure 7—Phase Diagrams of Actual Security Indices



Failure to find evidence of chaos is comforting. We have no *a priori* reason for believing that returns series of all firms in a given index are generated by similar processes and that initial conditions are identical for all firms. In a similar vein, firms themselves are composed of portfolios of projects and lines of business. A strict application of the disappearing chaos hypothesis would maintain that no visible signs of chaotic behavior will be found in single firm returns.

Our results neither support nor reject the efficient markets hypothesis. It may behoove investors to *behave* as if portfolio returns follow a random walk because, as we have shown, it is entirely possible for a portfolio to behave in a stochastic fashion while its underlying series are governed by deterministic chaos. Further investigation of the behavior of individual security return series seems warranted.

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