Intrayear Compounding and Fundamental Bond Valuation

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Several articles treat the quoted bond yield as the effective annual interest rate which may lead one to the conclusion that the use of the conventional bond valuation formula results in intolerable errors. This paper derives a new bond valuation formula and shows that the conventional formula does not violate rational market forces, as long as one distinguishes between the quoted bond yield and the effective annual interest rate. It then clarifies the puzzle in valuing par bonds with stated coupon rates that equal their quoted bond yields. Finally, it explores some possible explanations of why the market sticks to the confusing conventional formula.

INTRODUCTION

Since the publication of Chew and Clayton's (1983) article pointing out inconsistent statements of textbook authors concerning the relationship between bond value and the frequency of interest payments, several articles have criticized the conventional bond valuation formula and attempted to demonstrate errors and the resultant effects upon consumers of using the conventional formula. Horvath (1985) contends that the application of the conventional formula leads to understating the bond value because the periodic rate is assumed to be additive. Kochman (1986) follows Horvath's argument and demonstrates the magnitude of errors in mortgages, leases, and wage settlements when the conventional formula is used to compute the periodic cash flows. Lindley, Helms, and Haddad (1987) attempt to identify the source of errors and demonstrate, by extending Horvath's work, the magnitude of errors in yields, given bond prices. In return, Horvath (1988) extends Lindley, Helms, and Haddad's results and shows that the errors in yields, given bond prices, affect bond duration and interest elasticity and, therefore, the effectiveness of standard immunization techniques for bond portfolios. Unfortunately, these authors, like some textbook authors,[1] overlook the fact that although the quoted bond yield is referred to as the *yield to maturity*, the market does not take it literally.

The quoted bond yield is a simple annual interest rate equivalent or an annual percentage rate (APR) which is well-understood by both practitioners and academicians.[2] Nevertheless, for simplicity, the market convention refers to it as the yield to maturity.[3] These

[1] A list of textbooks that treat the quoted bond yield as the effective annual interest rate is available from the authors.

[2] Though the purpose of disclosing the APR is to help consumers make correct decisions when shopping for credit, the APR itself does not achieve this objective. Consumers need to be informed of the method of payment to be able to calculate the real cost of credit. See, for example, Gup (1987, pp. 450-451).

[3] For discussions of this market convention, see Fabozzi (1989, pp. 33-39) and Leibowitz (1975, pp. 234-246).

0747-5535/95/1400-0019 $02.50

authors treat the quoted bond yield as the effective annual interest rate and conclude that errors are large in magnitude when the effective interest rate is high.

This paper derives a variant of the fundamental bond valuation formula and shows that criticisms of the use of the conventional formula do not apply.

FUNDAMENTAL BOND VALUATION

For a bond paying interest j times a year, its quoted bond yield, y, is calculated by multiplying the periodic interest rate, r, by j, such that equation (1) holds. Equation (1) is the so-called conventional formula.[4]

$$(1)\ P_j = PV_I + PV_M$$

$$P_j = \sum_{t=1}^{jn} \frac{\left(\frac{I}{j}\right)}{\left(1 + \frac{y}{j}\right)^t} + \frac{M}{\left(1+\frac{y}{j}\right)^{jn}}$$

where:

P_j	=	Price of the bond;
PV_I	=	Present value of interest stream;
PV_M	=	Present value of maturity value;
I	=	Annual coupon interest (in \$);
M	=	Maturity value (in \$);
y	=	Quoted bond yield;
j	=	Frequency of interest payments per year; and
n	=	Number of years.

In this paper, the term *quoted bond yield* is used to distinguish it from the effective annual interest rate, k, which is obtained by equation (2).

$$(2)\ k = \left(1 + \frac{y}{j}\right)^j - 1$$

Notice that in calculating the periodic interest rate, y/j or r, equation (1) assumes a constant reinvestment rate of cash flows generated by a bond. With an additional assumption that the market correctly prices the bond, the effective annual interest rate is the same as the required annual rate of return. These two terms will be used interchangeably throughout this paper.

Previous authors treat the quoted bond yield as the effective annual interest rate and use the following equation to calculate bond prices or determine the required rate of return.

[4] It would be more appropriate to refer to equation (1) as the fundamental bond valuation formula, because other forms of bond valuation formulas are derived either directly or indirectly from equation (1).

$$(3)\ P_j = PV_I + PV_M$$

$$= \sum_{t=1}^{jn} \frac{\left(\frac{I}{j}\right)}{\left(1 + \frac{k}{j}\right)^t} + \frac{M}{\left(1+\frac{k}{j}\right)^{jn}}$$

Lindley, Helms, and Haddad correctly argue that equation (3) violates two axioms which, "a priori, are consistent with market forces" (p. 34) and thus suggest that equation (3) should be replaced by the following formula:

$$(4)\ P_j = \left(\frac{I}{j}\right)\left[\frac{1 - (1 + k)^{-n}}{(1 + k)^{1/j} - 1}\right] + \left[\frac{M}{(1 + k)^n}\right]$$

Equation (4) is merely a simplified version of equation (1), however, if one recognizes that k in equation (4) is an effective annual interest rate, while y in equation (1) is a simple annual interest rate equivalent.[5] The two axioms proposed by Lindley, Helms, and Haddad are quoted here for convenience of discussion.

Axiom 1: The present value of M is unaffected by the number of times the bond pays interest per year.

Axiom 2: If one assumes no risk differences between the maturity value and the interest stream, the effective yearly market rate is the same for discounting the maturity value and the interest stream.

Previous works have focused on the effective-interest-rate quoting and ignored the coupon-rate quoting. But the latter should be as important as the former, as the price of a long-term bond is determined substantially by the timing and scale of interest payments. If the quoted bond yield is a misleading interest rate, the same criticism applies to the stated coupon rate. To see this, let i and M denote the stated coupon rate and the par value of a bond, respectively. Then for an annual bond its coupon payment is equal to I or iM; for a non-annual bond which pays interest j times a year, the coupon payment is I/j per period of I per year. The timing difference in cash flows dictates that the interest streams of the annual bond and that of the non-annual bond must have different futures values, however, given that these two bonds have the same stated coupon rate.[6] With the assumption of a constant reinvestment rate equal to the required rate of return, the effect of the timing of interest streams upon the bond value can be seen by the bond valuation formula derived below. For simplicity, the valuation formula is derived for bond prices at the

[5] Substituting $1 + r$ for $(1 + k)^{1/j}$ and $(1 + r)^{nj}$ for $(1 + k)^n$, equation (4) becomes:

$$P_j = \left(\frac{I}{j}\right)\left[\frac{1 - (1 - r)^{-jn}}{r}\right] + \left[\frac{M}{(1 + r)^{jn}}\right]$$

[6] We consider the future value because the interest payments are assumed to occur at the end of each period.

coupon payment dates. The valuation formula for bond prices between coupon payment dates is shown in Appendix A.

The new bond valuation formula is derived in two steps. The first step is to define the *effective annual coupon payment,* I_j^* as the future value of the periodic coupon payments compounded at the periodic interest rate, r, at the end of each year for a bond paying interest j times a year. That is

$$I_j^* = (I/j) \sum_{t=1}^{j} (1 + r)^{t-1}.$$

By simple algebraic manipulation, I_j^* can be reduced to a product of I and the ratio of the effective annual interest rate to the quoted bond yield.[7] That is,

$$(5)\ I_j^* = I(k/y),$$

where:

$y = j[(1 + k)^{1/j} - 1]$ if j is finite; or

$\ln(1 + k)$, if j approaches infinity.

The bond value can then be determined by discounting the effective annual coupon payments and the par value at the effective annual interest rate. That is,

$$(6)\ P_j = \sum_{t=1}^{n} \frac{I_j^*}{(1 + k)^t} + \frac{M}{(1 + k)^n} \text{ or}$$

$$(7)\ P_j = \left(\frac{I_j^*}{k}\right) [1 - (1 + k)^{-n}] + \frac{M}{(1 + k)^n}$$

where I_j^* is obtained by equation (5); I, k, n, j, and M are defined as before.

Based upon the proof that equations (1) and (6) are equivalent (See Appendix C.), equation (1) is deduced to satisfy the above two axioms through the fact that these axioms are met by equation (6). Equation (6) essentially restates the cash flows of coupon-bearing bonds and their corresponding discount rates so that the effect of the timing and scale of the coupon payments upon the bond value (the coupon effect) is reflected solely by the effective annual coupon payment. More specifically, I_j^* and thus P_j are monotonically increasing functions of j, for all $j \geq 1$.[8] Notice that both the effective annual coupon payment and the par value are discounted at the same required rate of return rather than the same quoted bond yield (i.e., Lindley, Helms, and Haddad's second axiom is satisfied).

[7] See Appendix B.

[8] See Appendix D.

Also, given an annual required rate of return, k, the present value of the par value is unaffected by the frequency of interest compounding or the frequency of interest payments[9] (i.e., Lindley, Helms, and Haddad's first axiom is satisfied), although the quoted bond yield, y, decreases to reflect the increase in the frequency of interest compounding. Moreover, due to the way that the effective annual coupon payment is obtained (See equation (5).), equation (6) allows one to calculate the price of a bond offering a continuously compounded interest yield.[10] The following two examples demonstrate these points.

EXAMPLE 1

Suppose that Bond A, Bond B, Bond C, and Bond D are all 12 percent, ten year noncallable bonds with a par value of $1,000 and have the same tax status on their cash flows. Further, they belong to the same risk class, but differ only in the timing of interest payments. Assume that Bond B, a semi-annual bond, has a market-determined price (P_2) of $1,019.78. The semi-annual interest rate, r, is 5.83 percent and the quoted bond yield, y, is 11.66 percent. In addition, the effective annual interest rate, k, is 12 percent. With the effective annual interest rate and the quoted bond yield, we can obtain the effective annual coupon payment and the price of Bond B by applying equation (5) and equation (6) as follows.

$$\begin{aligned} I_2^* &= \$120\,(12\%/11.66\%) \\ &= \$123.50 \\ P_2 &= \sum_{t=1}^{10} \frac{\$123.5}{(1 + 12\%)^t} + \frac{\$1000}{(1 + 12\%)^{10}} \\ &= \$1{,}019.78 \end{aligned}$$

Assuming that the market correctly prices Bond B and that the reinvestment rate is constant, the required rate of return on Bond B is 12 percent. Then Bond A, Bond C, and Bond D also must have a required rate of return of 12 percent, as they are assumed to have the same risk characteristics and the same tax status of their cash flows. Their quoted bond yields differ, however, because they have different frequencies of interest payments. Further, assuming that no arbitrage opportunities exist, then the prices of Bond A, Bond C, and Bond D must equal the sum of the present values of the effective annual coupon pay-

[9] Here we assume that the frequency of interest payments is equal to the frequency of interest compounding, though they need not be equal. To see this, note that given equation (5) and an effective annual interest rate, k, we may obtain a fixed effective annual coupon payment, I_j^* by adjusting the annual coupon payment, I, and the quoted bond yield, y, simultaneously. Theoretically, there is an infinite number of ways to obtain an effective annual coupon payment.

[10] The fact that interest yields are continuously compounded does not mean that the interest yields need to be paid out continuously. Quoting a continuously compounded yield is simply a way of offering the highest effective annual interest rate, given an annual percentage rate or a quoted bond yield.

ments and the maturity value discounted at the required rate of return, 12 percent. Table 1 summarizes the relevant information about Bond A, Bond B, Bond C, and Bond D.

Table 1—Stated Coupon Rates and Effective Annual Interest Rates

Bond	i	j	k	r	y	I_j^*	PV_I	PV_I	P_j
A	12%	1	12%	12.00%	12.00%	120.00	678.03	321.97	1,000.00
B	12%	2	12%	5.83%	11.66%	123.50	697.81	321.97	1,019.78
C	12%	4	12%	2.87%	11.49%	125.33	708.15	321.97	1,030.12
D	12%	∞	12%	0^+%	11.33%	127.10	718.15	321.97	1,040.12

Table 1 indicates that

- Given a required rate of return equal to the stated coupon rate, the quoted bond yield decreases as j increases;
- The present value of the par values (PV_M) of annual, semi-annual, quarterly, and continuous bonds, which is equal to $321.97, is unaffected by the value of j; and
- The present values of the interest streams (PV_I) and thus the bond price (P_j) increase as j increases because I_j^* is an increasing function of j.

Therefore, if the stated coupon rate is equal to the effective annual interest rate, only an annual bond ($j = 1$) will be selling at par, because for an annual bond I_j^* reduces to I because $k = y$. But a non-annual bond ($j > 1$) will be selling at prices greater than the par value because $I_j^* > I$ for all $j > 1$.

According to equation (6), two bonds with the same maturity and par value would have the same price only if they provide both the same effective annual interest rate and the same effective annual coupon payment. Without being aware of this, one would confront the puzzle raised by Lindley, Helms, and Haddad (p. 37) that both an annual bond and a non-annual bond would be selling at par if the quoted bond yield is equal to the stated coupon rate if equation (1) is used, a seemingly inconsistent statement. The puzzle in valuing par bonds arises from mistreating the quoted bond yield as the effective annual interest rate. It should be noted that, given the stated coupon rate being equal to the quoted bond yield, both annual and non-annual bonds always will sell at par. This is because a decrease in the present value of the par value due to an increase in the effective annual interest rate[11] is offset exactly by a corresponding increase in the present value of interest streams (and vice versa). Example 2 demonstrates this point.

EXAMPLE 2

Suppose that Bond E, Bond F, Bond G, and Bond H are also 12 percent, ten year, noncallable bonds with a par value of $1000. They are priced to offer a quoted bond yield of 12 percent, but have different timing in their interest streams. Table 2 summarizes the relevant information about these four bonds.

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[11] The increase in the effective annual interest rate is, in turn, due to an increase in the frequency of interest payments, given a quoted bond yield.

Table 2—Stated Coupon Rates and Quoted Bond Yields

Bond	i	j	k	r	y	I_j^*	PV_I	PV_1	P_j
E	12%	1	12.00%	12.00%	12%	120.00	678.03	321.97	1,000
F	12%	2	12.36%	6.00%	12%	123.60	688.19	311.80	1,000
G	12%	4	12.55%	3.00%	12%	125.50	693.42	306.58	1,000
H	12%	∞	12.75%	0^+%	12%	127.50	698.81	301.19	1,000

Table 2 indicates that given a quoted bond yield, the effective annual interest rate and the effective annual coupon payment increase as the frequency of interest payments increases. The value of the bond, P_j, however, is unaffected by the frequency of interest payments. In other words, if the quoted bond yield is equal to the stated coupon rate, bonds will sell at par regardless of the frequency of interest payments. A change in the present value of the par value due to an increase in the effective annual interest rate is offset by a corresponding change in the present value of interest streams. For example, the present values of the par value and the interest stream are, respectively, \$321.97 and \$687.03 for Bond E and \$301.19 and \$698.81 for Bond H. The difference in the present value of the par values between Bond E and Bond H is the same as the difference in the present values of their interest streams, except that they have opposite signs.

POSSIBLE EXPLANATIONS FOR THE MARKET CONVENTION

The practice of quoting the simple annual interest rate equivalent instead of the effective annual interest rate may have arisen because of the difficulty in calculating the geometric mean. But why has the market continued to quote this simplified and perhaps misleading interest rate when calculators or computers are readily available? Some possible explanations follow. First, the current practice may be the most efficient method of yield quoting that provides investors the most relevant information on bonds in simple and easily understood figures. Given the method of interest payment, a bond's coupon rate and quoted bond yield can be obtained without much difficulty.

The second explanation is that market participants are responsible for analyzing and assimilating quoted bond yields before using them in the investment decision making process. This market convention is not unique in bond yield quoting. Corporate management is required by law to present fair financial reports according to the Generally Accepted Accounting Principles (GAAP). It is the financial analysts' responsibility to analyze and adjust accounting data, however, before these GAAP-consistent numbers can be used. The evidence of market efficiency ensures that bonds would not be mispriced persistently. The semi-strong form of market efficiency requires that security prices reflect all publicly available information. For traded bonds the relevant information refers primarily to the default risk. Because major rating agencies publish bond ratings periodically, the default risk of bonds should be impounded in their prices. In other words, bond prices are determined by discounting the par value and the interest stream at the required rates of return rather than the quoted bond yield. The quoted bond yield must be converted into the effective rate before it enters a decision maker's pricing formulation.

The third explanation is that the current practice of bond yield quoting allows financial institutions to quote continuously compounded yields. For example, some savings and loans have employed continuously compounded rates to attract customers during the time that government-prescribed interest rate ceilings (Regulation Q) were in effect. Given a quoted annual percentage rate, a continuously compounded rate renders the highest effective annual interest rate, other things being equal. On the other hand, because of its mathematical simplicity, the continuously compounded yields are used widely by academicians as well. For example, in deriving the pricing models for options and futures contracts, interest rates often are assumed to be compounded continuously. The continuously compounded rates would be meaningful, however, only if the quoted bond yields are defined in terms of equation (2). To see this, we note that

$$1 + k = \left(1 + \frac{y}{j}\right)^j.$$

As j approaches infinity,

$$\lim_{j \to \infty} \left(1 + \frac{y}{j}\right)^j = e^y.$$

Therefore, a continuously compounded yield given an effective annual interest rate is $y = \ln(1 + k)$.

SUMMARY

Bond prices can be expressed correctly in various forms, such as equations (1), (4), (6), and (7). The confusion associated with bond valuation arises from the misconception of both the quoted bond yield and the stated coupon rate. This confusion would vanish, however, if one were to look at the issue from the perspective of equation (6). Equation (6) unravels several interesting points. First, given a required rate of return being equal to the stated coupon rate, a bond's price is an increasing function of the frequency of interest payments. This is reflected by the fact that the effective annual coupon payment is an increasing function of the frequency of interest payments. The present value of par value is unaffected by the frequency of interest payments, however, though the quoted bond yield decreases as the frequency of interest payments increases. Second, equation (6) discounts both the interest stream and the par value of a bond at the same required annual rate of return, whereas equation (1) discounts them at the periodic interest rate. Neither equation (6) nor equation (1) discounts future cash flows at the quoted bond yield directly. Rather, both equations (1) and (6) discount the interest stream and the par value of a bond at the same required rate of return. Third, if the stated coupon rate is equal to the quoted bond yield, bonds always will sell at par regardless of the frequency of interest payments because if the quoted bond yield is fixed, the effect of j upon the present value of the par value is offset completely by that upon the present value of the interest stream. Finally, we note that equation (6) has an advantage over the other equations in pricing bonds offering continuously compounded rates through the use of the effective annual coupon payment.

A change in the convention of yield quoting (such as quoting the periodic interest rate and the periodic coupon rate for non-annual bonds) could help mitigate the confusion. But the professional market participants might not bother to do this. Prudent market participants always will covert the quoted yield to maturity into the effective annual interest rate or the periodic interest rate in determining prices of capital or money market instruments and in calculating interest expenses or revenues. Hence, the errors and the resultant adverse effects upon consumers caused by using the equation (1) would not exist. Furthermore, the existence of persistent errors that can be translated into abnormal returns would contradict the evidence of market efficiency.

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APPENDIX A—ACCRUED INTEREST

Using the valuation formula for bond prices between coupon payment dates following Whitmore's (1985) approach, the exact accrued interest is:

$$A_f^* = (I_j^*/k)\,[(1 + k)^f - 1]$$

where

A_f^* = Accrued interest;
f = Fraction of time that has elapsed (in years);
I_j^* = Effective annual coupon payment; and
k = Effective annual interest rate.

Assume that there are $N - 1$ (an integer) years to maturity in addition to the odd year, $1 - f$. Then the total time to maturity, T, is equal to $N - f$. The bond price at the instant before year 1 is:

$$P_1 = I_j^* + \left(\frac{I_i^*}{k}\right) [(1 - (1 + k)^{-N+1}] + \frac{M}{(1 + k)^{N-1}}$$

The current quoted bond price, net of accrued interest, is:

$$P_1 = \frac{P_1}{(1 + k)^{1-f}} - A_f^*$$

or

$$P = \left\{ \left(\frac{I_i^*}{k}\right) [1 - (1 + k)^{-N+1}] + \frac{M}{(1 + k)^{N-1}} \right\} \left\{ \frac{1}{(1 + K)^{1-f}} \right\}$$

$$= \left(\frac{I_i^*}{k}\right) [(1 + k)^{f-1} - (1 + k)^{-T}] + \frac{M}{(1 + k)^{T}} \; .$$

The invoice price is the sum of the bond price and the accrued interest.

APPENDIX B—DERIVATION OF EQUATION (5)

Using the fact that the sum of the finite geometric series, $1 + x + x^2 + x^3 + \ldots + x^{n-1}$ is

$$\frac{x^n - 1}{x - 1},$$

then we have

$$= (I / J) \sum_{t=1}^{j} (1 + r)^{t-1}$$

$$= (I / J) \left[\frac{(1 + r)^j - 1}{(1 + r) - 1}\right]$$

$$(8)\ I_j^* = (I / J) \left[\frac{k}{r}\right]$$

$$= (I / J) \left[\frac{k}{(Y / j)}\right]$$

$$= I(k/Y)$$

Q.E.D.

APPENDIX C—PROOFS OF EQUIVALENCE OF EQUATIONS (1) AND (6)

The equality of the present values of par values of equations (1) and (6) is obvious because $(1 + r)^j = 1 + k$. What we need to show is the equality of the present values of the interest streams of equations (1) and (6). The present value of the interest stream of equation (6) is

$$\sum_{t=1}^{n} \frac{I_j^*}{(1 + k)^t}.$$

Using equation (8) of Appendix B, we have

$$\sum_{t=1}^{n} \frac{I_j^*}{(1 + k)^t} = \sum_{t=1}^{n} \frac{(I / j)\left[\frac{(1 + r)^j - 1}{r}\right]}{(1 + r)^{jt}}$$

$$= \left(\frac{I}{j}\right) \left[\frac{(1 + r)^j - 1}{r}\right] \sum_{t=1}^{n} \frac{1}{(1 + r)^{jt}}$$

$$= \left(\frac{I}{j}\right) \left[\frac{(1+r)^j - 1}{r}\right] \left[\frac{1}{(1+r)^{jt}}\right] + \ldots + \frac{1}{(1+r)^{(n-1)j}}\Bigg]$$

$$= \left(\frac{I}{j}\right) \left[\frac{(1+r)^j - 1}{r}\right] \left[\frac{1}{(1+r)^{jt}}\right] \left[\frac{\frac{1}{(1+r)^{nj}} - 1}{\frac{1}{(1+r)^j} - 1}\right]$$

$$= \left(\frac{I}{j}\right) \left[\frac{(1+r)^j - 1}{r}\right] \left[\frac{1}{(1+r)^{jt}}\right] \left[\frac{1-(1+r)^{nj}}{(1+r)^{nj}}\right] \left[\frac{(1+r)^j}{1-(1+r)^j}\right]$$

$$= \left(\frac{I}{j}\right) \left[\frac{(1+r)^{nj} - 1}{r(1+r)^{nj}}\right]$$

$$= \left(\frac{I}{j}\right) \left(\frac{1}{1+r}\right) \left[\frac{(1+r)^{nj} - 1}{r(1+r)^{nj}}\right] \left(\frac{1+r}{1}\right)$$

$$= \left(\frac{I}{j}\right) \left(\frac{1}{1+r}\right) \left[1 - \frac{1}{(1+r)^{nj}}\right] \left[\frac{1}{1 - \frac{1}{1+r}}\right]$$

$$= \left(\frac{I}{j}\right) \left[\frac{1}{1+r}\right] \left[\frac{\frac{1}{(1+r)^{nj}} - 1}{\frac{1}{(1+r)} - 1}\right]$$

$$= \left(\frac{I}{j}\right) \left(\frac{1}{1+r}\right) \left[1 + \frac{1}{(1+r)} + \cdots + \frac{1}{(1+r)^{nj-1}}\right]$$

$$= \left(\frac{I}{j}\right) \left[\frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots \frac{1}{(1+r)^{nj}}\right]$$

$$= \left(\frac{I}{j}\right) \sum_{t=1}^{jn} \left[\frac{1}{(1+r)^t}\right]$$

$$= \sum_{t=1}^{jn} \left[\frac{(I/j)}{(1+r)^t}\right]$$

Q.E.D.

APPENDIX D—PROOF OF I_j^* BEING A MONOTONICALLY INCREASING FUNCTION OF J

Note that $y(j) = j[1 + k)^{1/j} - 1]$ and that $y(j)$ is a continuous function of j, for $0 < j < \infty$.[12] From Appendix A we see that $I_j^* = I(k/y)$, where I and k are constants. To show that I_j^* is a monotonically increasing function of j, we simply show that y(j) is a monotonically decreasing function of j or show that

$$\frac{dy(j)}{dj} = y'(j) < 0 \text{ for all } j > 0.$$

Taking the derivative of y(j) with respect to j, we have

$$(9)\ y'(j) = [(1 + k)^{1/j} - 1]+j[\ln (1 + k)](1 + k)^{1/j}(-1/j^2)$$

$$= (1 + k)^{1/j}[1+(1/j)\ln (1 + k)]-1.$$

From equation (9) we see that $\lim j \to y'(j) = 0$. That is, y(j) has its extreme value, $\ln (1 + k)$, as j approaches infinity. Now, what is left to be shown is that y'(j) is a monotonically increasing function of j or

$$\frac{d^2y(j)}{dj^2} = y''(j) > 0 \text{ for all } 0 < j < \infty.$$

Taking the derivative of y'(j), we have

$$\begin{aligned} y'(j) &= (1 + k)^{1/j}[\ln(1 + k)](-1 / j^2)[1 - \ln(1 + k)^{1/j}] \\ &= -(1 + k)^{1/j}[\ln(1 + k)]\left(-\frac{1}{j}\right) \\ &= (1 + k)^{1/j}(1 / j^2)[\ln(1 + k)][\ln(1 + k)1 / j] \\ &= (1 + j^3)(1 + k)^{1/j}[\ln(1 + k)]^2 \\ &> 0 \end{aligned}$$

because $j > 0$ and $k > 0$. Therefore, $y'(j) < 0$ for all $0 < j < \infty$. Therefore, y(j) decreases at an increasing rate as j increases and thus I_j^* increases at a decreasing rate as j increases, for $0 < j < \infty$.

Q.E.D.

[12] The result of the proof is also true for the discrete case.