Valuation of the Growth Firm Under Inflation and Differential Personal Taxation Revisited

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Abstract

Howe (1988) develops a valuation model of a growth firm in an environment of inflation and differential taxation of interest and equity income. One of the assumptions of his model is that the firm finances with debt a constant fraction of its future growth investment. This paper shows that this assumption implies a declining debt-to-value ratio over time. This paper generalizes Howe's work by assuming a constant debt-to-value ratio, an assumption consistent with much of the literature in this area.

Introduction

Howe's (1988) derivation of a valuation model for a levered firm under inflation and differential personal taxation represents a useful generalization and synthesis of several papers, including Modigliani and Miller (1958, 1961, 1963), Arditti and Pinkerton (1978), and Hochman and Palmon (1985). Howe's Table 1 illustrates how various valuation models in the literature can be generated under different assumptions.[1]

The perceived usefulness of Howe's paper has motivated a reconsideration of his derivation. Implicit in Howe's model is the assumption that the firm's debt-to-value ratio declines over time. Although optimal capital structure is not the focus of his paper, it is difficult to accept that a firm's leverage should decline when leverage-related costs are ignored but nominal interest is tax deductible. It usually is assumed in the literature that the debt-to-value ratio is constant over time. Howe's work is generalized in this note assuming a constant debt-to-value ratio. The generalization also clearly shows the relevance of the appropriate discount rates for the various income streams.

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[1]Howe's model can be extended to incorporate a fixed dividend policy along the lines suggested by Rashid and Amoako-Adu (1987).

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Howe's Model for a Levered Growth Firm

Howe's model for a levered growth firm is based on the following assumptions:

(i) The existing capital assets of the firm produce a perpetual expected real net cash flow of $\bar{x}$;

(ii) The economy experiences a constant rate of expected inflation, denoted by π. This expected inflation is assumed to affect all prices uniformly;

(iii) The firm's existing and future assets are nondepreciating;

(iv) Nominal interest on debt is taxed at t_p, while equity income is taxed at a lower rate, denoted by t_s;

(v) In each year t, a constant fraction, α, of after personal and corporate tax income is reinvested;

(vi) Corporate income is taxed at a constant tax rate, t_c;

(vii) The firm's growth opportunities will earn a constant after corporate tax nominal rate of return, r, in perpetuity;

(viii) A constant fraction, λ, of the growth investment in each period is financed with debt.

Assumptions (i) to (vi) above imply that the after personal and corporate tax income in year t of the unlevered firm without growth opportunities, denoted by Y_t, is:

$$(1) \quad Y_t = \bar{x}\,(1 + \pi)^t\,(1 - t_c)\,(1 - t_s), \qquad t = 1, 2, \ldots.$$

In addition, Howe assumes growth opportunities to the firm may be available either over a finite period T or infinite period. To simplify the exposition, only finite growth opportunities are assumed. Under the above assumptions and finite growth opportunities, Howe's valuation expression as stated in his equation (7a) or (7b) is:[2]

$$(2) \quad V_0 = Y_1/(\rho - \pi) + \{(r/(\rho - \pi)) - 1\}(\alpha Y_1 T/(1 + \rho)) + D_0 G + (\lambda \alpha Y_1 T/(1 + \rho))G$$

[2]See Appendix A for the derivation of equations (2), (4), and (10).

where:

ρ = After personal tax nominal required rate of return on a pure equity stream;

D_0 = Present value of the current debt;

$G = 1 - \{(1-t_s)(R(1-t_c)-\pi)\}/\{(1-t_p^*)R-\pi\}$;

R = Nominal pretax interest rate on bonds;

t_p^* = Marginal personal tax rate on income from bonds.

Other notations are as given in the list of assumptions above.

For the interpretations of the various components of equation (2), see Howe (1988).

A Modification of Howe's Model

Assumption (viii) above implies that the firm's debt-to-value ratio declines over time. To show this, let $D_0/V_0 = \lambda$; by assumption, incremental debt in year t, ΔD_t, is λI_t. To consider growth investment in year 1, let θ_1 be the firm's debt-to-value ratio in year 1. By definition, θ_1 is:

(3) $\theta_1 = (D_0 + \Delta D_0)/(V_0 + \Delta V_1)$

where:

ΔV_1 = Incremental market value of the firm in year 1.

As ΔV_1 is the sum of growth investments, I_1, and the net present value, NPV_1, θ_1 can be written as:

(4) $\theta_1 = \lambda\{1 + (I_1/V_0\}/\{1 + (I_1/V_0) + (NPV_1/V_0)\}$

unless $NPV_1 = 0$, $\theta_1 < \lambda$. Extending this reasoning beyond year 1, it can be shown that the debt-to-value ratio declines over the years. It may be noted here that NPV_t is positive in Howe's model not only due to growth opportunities, but also due to the gain from incremental debt in year t. This point will be elaborated later.

Equation (4) can be explained as follows. If the firm accepts positive NPV projects and finances a constant fraction of the investments with debt, its financial leverage will decline. This result has implications for future interest rates on the bonds of the firm and, therefore, for the required after tax discount rate for the bond interest streams.

Instead, it is assumed in this note that for each year t over the growth horizon, growth investment is financed in such a way that a constant debt-to-value ratio is maintained over time. That is,

$$(5)\quad \lambda = D_0/V_0 = D_1/V_1 = \ldots = D_T/V_T.$$

This assumption requires that for each t, $0 < t \leq T$, debt financing of growth investment is given by:

$$(6)\quad \Delta D_t = \lambda\,(I_t + NPV_t).$$

To derive the valuation equation, the incremental after tax income to equity holders of the firm from growth investment and incremental debt will be identified. This procedure will show the importance of the discount rates required for pure equity streams and the firm's bond interest stream more clearly than does Howe's model.

The current approach will be to obtain the incremental market value of a firm's equity at time t and to use this result to obtain the net present value of growth investment in year t. The present values, at time zero, of all NPVs of growth investments over the growth horizon will be used to derive the market valuation of the growth firm.

Given the assumption of a constant debt-to-value ratio at time t, the incremental after tax nominal income to equity holders of the growth firm in any year j, after t, is:[3]

$$(7)\quad \Delta X_{s,j} = r(1+\pi)^{j-1}\alpha Y_t(1-t_s) - \lambda\{I_t + NPV_t\}(1+\pi)^{j-1}R(1-t_c)\,(1-t_s)$$
$$+\ \pi\lambda\{I_t + NPV_t\}\ (1+\pi)^{j-1}(1-t_s).$$

The first term is the after corporate and personal tax nominal return on growth investment in year j. The second term indicates after corporate tax and personal tax nominal interest cost to the firm arising from the incremental debt at time t. The last term represents the additional debt in year j that is needed to keep the firm's capital structure constant.

[3]This equation is analogous to the after tax income to equity holders in period t of a levered firm with no growth opportunities as shown by Hochman and Palmon (1985).

The derivation of the incremental market value of the firm's equity, denoted by ΔE_t, requires the present values of the three terms of equation (7). To determine these present values requires the identification of the appropriate discount rate for each of these three terms. The first term of equation (7) has the same risk as that of a pure equity stream. Therefore, the relevant discount rate is ρ. For the last two terms of equation (7), the discount rate is $R(1 - t_p^*)$. This is because in year t, Y_t is known, and the uncertainty of the cash flows in these terms is the same as that of the cash flow stream of the firm's bondholders.

In Appendix A, the incremental market value of the firm's equity at time t is shown to be:

$$(8)\quad \Delta E_t = r\alpha Y_t(1 - t_s)/(\rho - \pi) - \lambda(I_t + NPV_t) + \lambda(I_t + NPV_t)G.$$

This equation gives the present value of after tax returns on growth investment, minus incremental debt, ΔD_t, plus the present value of the gain from the incremental debt financing.

By definition, $\Delta V_t = \Delta E_t + \Delta D_t$ and, by assumption, $\Delta D_t = \lambda(I_t + NPV_t)$; therefore, ΔV_t can be written as:

$$(9)\quad \Delta V_t = r\alpha Y_t(1 - t_s)/(\rho - \pi) + \lambda(I_t + NPV_t)G.$$

Given that $NPV_t = \Delta V_t - I_t$ and using equation (9), NPV_t can be solved as:[4]

$$(10)\quad NPV_t = \alpha Y_t[r(1 - t_s)/(\rho - \pi) - 1 + \lambda G]/(1 - \lambda G).$$

This equation indicates that even though there may be no growth opportunities (that is, $\rho - \pi = r(1 - t_s)$), $NPV_t > 0$ if $\lambda G > 0$. In other words, the gain from leverage that arises from debt financing of future investments makes projects profitable.

The addition to the current shareholders' wealth is the present value of all NPVs of future growth investments. To calculate the present value of all NPVs over the growth horizon of T periods, each of the terms in equation (10) must be discounted by ρ, as their risk characteristic is the same as that of a pure equity stream. Noting that $Y_t = Y_1 (1 + \pi + \alpha r)^{t-1}$ and discounting each term of equation (10) by ρ yields:

[4] If the firm undertakes projects that are only expansionary (i.e., NPV = 0), then the debt-to-value ratio in each period will be equal to the same constant fraction of investment that is financed by debt. In this situation, Howe's model and the current model coincide. This can be seen by letting $NPV_t = 0$ in equation (9) and by discounting the two terms by ρ.

$$(11)\ \sum_{t=1}^{T} NPV_t/(1+\rho)^t = [\{r(1-t_s)/(\rho-\pi) - 1 + \lambda G\}/(1-\lambda G)]\alpha Y_1 T/(1+\rho).$$

Equation (11) replaces the second term and the last term of equation (2); therefore, the modified equation for the current market value of the firm becomes:

$$(12)\quad V_0 = Y_1/(\rho - \pi) + \{r(1 - t_s)/(\rho - \pi) - 1\}\alpha Y_1 T/\{(1 + \rho)(1 - \lambda G)\}$$
$$D_0 G + (\lambda G/(1 - \lambda G))\alpha Y_1 T/(1 + \rho).$$

The terms $Y_1/(\rho - \pi)$ and D_0G in equations (2) and (12) are identical. The difference in the models lies in other parts of the two equations. The gain from future debt financing cannot be represented in only one term in equation (12), however, so a term by term comparison of the two equations is not possible. Instead, a numerical example is presented to show the difference between the two equations.

Assume $t_s = 0.2$, $t_p^* = 0.4$, $t_c = 0.5$, $\bar{x}$ = \$2.0 million, $\pi = 0.05$, $R = 0.12$, $\rho = 0.14$, T = five years, $r = 0.16$ in Howe's model (and $r(1 - t_s) = 0.16$ in the current model), $\alpha = 0.25$, $\lambda = 0.45$, and D_0 = \$10 million. Then

$Y_1 = (1 - t_s)(1 - t_c)\bar{x}(1 + \pi)$ = \$0.84 million

$G = 1 - [\{(1 - t_s)(R(1 - t_c) - \pi)\}/\{(1 - t_p^*)R - \pi\}]$

$= 0.63636.$

Howe's valuation model (equation (2)) gives the following value:

$V_0 = 9.333 + 0.716 + 6.364 + 0.264$

= \$16.677

On the other hand, the valuation equation of this note (equation (12)) gives:

$V_0 = 9.333 + 1.004 + 6.364 + 0.37$

= \$17.071.

This indicates that Howe's model understates the value of the growth firm.

Combining the second and the last terms of equation (12) gives the implicit cost of capital for investments. For a future investment to increase the value of the corporation, the following must hold:

$$(13)\quad \alpha Y_1 T/(1+\rho)\ \{r(1-t_s)/(\rho-\pi)-1+\lambda G\}/(1-\lambda G) > 0.$$

This implies that

$$\{r(1-t_s)/(\rho-\pi)-1+\lambda G\} > 0.$$

Therefore, the cutoff rate is:

$$(14)\quad r^*(1-t_s) > (\rho-\pi)\ (1-\lambda G\}.$$

This is the same rate as that derived by Howe. Howe's r* is only after both corporate and personal taxes, however, while the current r* is only after corporate tax.[5]

Finally, it is shown in Appendix B that equation (12) is consistent with many valuation models contained in the literature. Equation (12) does not reduce to the Arditti and Pinkerton valuation model, however. Like Howe, Arditti and Pinkerton assume that the firm's debt financing of growth investments is based on a constant fraction of investments which implies an intertemporal declining debt-to-value ratio. In contrast, the model of this paper assumes that the firm maintains a constant debt-to-value ratio.

Conclusion

This paper shows that Howe's (1988) valuation model is based on the assumption of a declining debt-to-value ratio. The model is rederived assuming a constant debt-to-value ratio. Unlike Howe's model, the derivation of the market value of the firm in this paper clearly shows the relevance of various discount rates for different income streams. The two valuation models were compared numerically, and the results show that Howe's model understates the market value of the growth firm. Finally, it is shown that the implied cost of capital remains the same as in Howe's model.

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[5] r is defined as the after corporate tax nominal rate of return on investments to emphasize the fact that although αY_t is $\alpha\bar{x}(1+\pi)^t(1-t_c)(1-t_s)$, the firm's shareholders have to pay personal taxes again from $r(1+\pi)^{j-1}\alpha Y_t$ for $j > t$. In other words, $(1-t_s)$ in Y_t does not mean that there is no need to pay personal taxes from the return series: $r(1+\pi)^{j-1}\alpha Y_t$ with $j > t$, for $j = 1, 2, \ldots, \infty$.

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Appendix A

1. Derivation of Equation (2)

The derivation of this equation is elaborated by Howe. For convenience, only a brief presentation is given below. This equation is based on the sum of four separate components. Each of these components is treated separately. Notation used is as defined in the next.

- The present value of the unlevered and no growth component:

$$= \sum_{t=1}^{\infty} \bar{x}\,(1+\pi)^t\,(1-t_c)\,(1-t_s)/(1+\rho)^t.$$

$$= \bar{x}\,(1-t_c)\,(1-t_s)\sum_{t=1}^{\infty} \{(1+\pi)/(1+\rho)\}^t$$

$$= \bar{x}\,(1+\pi)\,(1-t_c)\,(1-t_s)/(\rho-\pi) = Y_1/(\rho-\pi)$$

- The present value of the growth opportunities over T periods, assuming no financial leverage:

$$= \sum_{t=1}^{T} \{\ \sum_{s=1}^{\infty} r\alpha Y_t(1+\pi)^{s-1}/(1+\rho)^s - \alpha Y_t\}/(1+\rho)^t$$

$$= \sum_{t=1}^{T} \alpha Y_t\,\{r \sum_{s=1}^{\infty} (1+\pi)^{s-1}/(1+\rho)^s - 1\}/(1+\rho)^t$$

$$= \sum_{t=1}^{T} \alpha Y_t\,\{r/(\rho-\pi) - 1\}/(1+\rho)^t$$

Because $Y_t = Y_1\,(1+\pi+\alpha r)^{t-1}$, this present value becomes:

$$= \{r/(\rho-\pi) - 1\}\,\alpha Y_1 \sum_{t=1}^{T} (1+\pi+\alpha r)^{t-1}/(1+\rho)^t\ .$$

Letting $x = (1 + \pi + \alpha r)/(1 + \rho)$ and expanding x^T around 1 and dropping all the terms except the first two of the Taylor series, yields:

$$= \{r/(\rho - \pi) - 1\}\ \alpha Y_1 T/(1 + \rho).$$

- The present value of the gains from the current leverage:

$$= D_0 - D_0 \sum_{t=1}^{\infty} \{R(1 - t_c) - \pi\}(1 + \pi)^{t-1}\ (1 - t_s)/\{1 + R(1 - t_p^*)\}^t$$

$$= D_0[1 - (1 - t_s)\{R(1 - t_c) - \pi\} \sum_{t=1}^{\infty} (1 + \pi)^{t-1}/\{1 + R(1 - t_p^*)\}^t$$

$$= D_0[1 - (1 - t_s)\ \{R(1 - t_c) - \pi\}/\{R(1 - t_p^* - \pi\}]$$

$$= D_0 G$$

- The present value of the gains from debt financing of growth investments over T periods:

In year t, $0 < t \leq T$, debt financing of growth investment amounts to $\lambda I_t = \lambda \alpha Y_t$. The present value of the gains from this incremental leverage as of time t is $\lambda \alpha Y_t G$. The present value of the gains of debt financing of all growth investments as of time zero is:

$$= \lambda \alpha G \sum_{t=1}^{T} Y_t/(1 + \rho)^t$$

$$= \lambda \alpha G Y_1 \sum_{t=1}^{T} (1 + \pi + \alpha r)^{t-1}/(1 + \rho)^t$$

$$\doteq \lambda \alpha Y_1 G T/(1 + \rho).$$

Summing over the above four present values gives equation (2) in the text.

2. Derivation of Equation (4)
By definition, the debt-to-value ratio in year 1, denoted by θ_1, is:

$$\theta_1 = (D_0 + \Delta D_1)/(V_0 + \Delta V_1).$$

Noting that $\Delta D_1 = \lambda I_1$ and $\Delta V_1 = I_1 + NPV_1$, θ_1 can be written as:

$$\theta_1 = (D_0 + \lambda I_1)/(V_0 + I_1 + NPV_1).$$

Dividing each term in the numerator and denominator by V_0,

$$\theta_1 = \{D_0/V_0 + \lambda I_1/V_0\}/\{1 + I_1/V_0 + NPV_1/V_0\}.$$

Because $D_0/V_0 = \lambda$, θ_1 is:

$$\theta_1 = \lambda\{1 + I_1/V_0\}/\{1 + I_1/V_0 + NPV_1/V_0\}.$$

3. Derivation of Equation (8)
The derivation of the incremental market value of the firm's equity as of year t, denoted by ΔE_t, requires the present values of the three components of equation (7).

- The present value of the first term:

$$= \sum_{j=1}^{\infty} r(1 + \pi)^{j-1} \alpha Y_t (1 - t_s)/(1 + \rho)^j$$

$$= r\alpha Y_t(1 - t_s) \sum_{j=1}^{\infty} (1 + \pi)^{j-1}/(1 + \rho)^j$$

$$= r\alpha Y_t(1 - t_s)/(\rho - \pi)$$

- The present value of the sum of the second and third terms of equation (7):

Summing over the last two terms of equation (7) yields:

$$= -\lambda(I_t + NPV_t)\,(1 + \pi)^{j-1}\{R(1 - t_c) - \pi\}(1 - t_s).$$

Discounting this at $R(1 - t_p^*)$, the present value as of time t is:

$$= -\lambda(I_t + NPV_t)\{R(1 - t_c) - \pi\}(1 - t_s) \sum_{j=1}^{\infty} (1 + \pi)^{j-1}/\{1 + R(1 - t_p^*)\}^j$$

$$= -\lambda(I_t + NPV_t)\{R(1 - t_c) - \pi\}(1 - t_s)/\{R(1 - t_p^*) - \pi\}$$

$$= -\lambda(I_t + NPV_t)\ (1 - G).$$

Thus, ΔE_t is:

$$\Delta E_t = r\alpha Y_t(1 - t_s)/(\rho - \pi) - \lambda(I_t + NPV_t) + \lambda(I_t + NPV_t)G.$$

4. Derivation of Equation (10)

Subtracting I_t from both sides of equation (9) and noting that $NPV_t = \Delta V_t - I_t$ and $I_t = \alpha Y_t$, obtains:

$$NPV_t = r\alpha Y_t(1 - t_s)/(\rho - \pi) + \lambda(\alpha Y_t + NPV_t)G - \alpha Y_t$$

$$NPV_t(1 - \lambda G) = \alpha Y_t\{r(1 - t_s)/(\rho - \pi) - 1 + \lambda G\}$$

$$NPV_t = \alpha Y_t\{r(1 - t_s)/(\rho - \pi) - 1 + \lambda G\}/(1 - \lambda G).$$

5. Derivation of Equation (11)

The present value as of time zero of the net present values of all growth investments is:

$$\sum_{t=1}^{T} NPV_t/(1+\rho)^t = \sum_{t=1}^{T}(1/(1-\lambda G))\{r(1-t_s)/(\rho-\pi)-1+\lambda G\}\alpha \sum_{t=1}^{T} Y_t/(1+\rho)^t$$

Noting that $Y_t = Y_1\ (1 + \pi + \alpha r)^{t-1}$ and expanding $\{(1 + \pi + \alpha r)/(1 + \rho)\}^T$ around 1 and dropping all terms except the first two of the Taylor series:

$$= (1/(1 - \lambda G))\{r(1 - t_s)/(\rho - \pi) - 1 + \lambda G\}\alpha Y_1 T/(1 + \rho).$$

6. Derivation of Equation (12)

The current market value of the firm in this paper is:

$$V_0 = Y_1/(\rho - \pi) + D_0G + \sum_{t=1}^{T} NPV_t/(1 + \rho)^t .$$

Replacing the last term of this equation from equation (11) gives:

$$V_0 = Y_1/(\rho - \pi) + D_0G + \{r(1 - t_s)/(\rho - \pi) - 1\}\alpha Y_1T/\{(1 + \rho)(1 - \lambda G)\}$$
$$+ (\lambda G/(1 - \lambda G))\alpha Y_1T/(1 + \rho).$$

Appendix B

1. If $\pi = t_p^* = t_s = t_c = 0$ and $\alpha = 0$, equation (12) reduces to:

$$V_0 = \bar{x}/\rho.$$

This is the well-known Modigliani and Miller (1958) result.

2. Letting $\pi = t_p^* = t_s = 0$ and $\alpha = 0$, equation (12) becomes:

$$V_0 = \bar{x}.(1 - t_c)/\rho) + t_c D_0.$$

This is the same equation as derived by Modigliani and Miller (1963).

3. If π, t_s, t_p^*, and t_c are zero, equation (12) yields:

$$V_0 = \bar{x}/\rho \ \{1 + \frac{\alpha T}{1 + \rho} (r - \rho)\}.$$

This is the same as equation (22b) of Miller and Modigliani (1961, p. 422).

4. Letting π, t_p^*, and $t_s = 0$ in equation (12),

$$V_0 = (\bar{x}\ (1 - t_c)/\rho) + t_c D_0 + \{(r/\rho) - 1\}\alpha Y_1 T/\{(1 + \rho)\ (1 - \lambda t_c)\}$$
$$+ (\lambda t_c/1 - \lambda t_c))\alpha Y_1 T/(1 + \rho).$$

Although the variables used in this model are those of Arditti & Pinkerton (1978, p. 69), this valuation equation is not the same as their equation (14). The reason for the discrepancy is that, like Howe, Arditti & Pinkerton also assumed that debt financing constitutes a constant fraction of growth investments. In contrast, equation (12) is based on the assumption of a constant debt-to-value ratio over the growth horizon.

5. Letting $\alpha = 0$ in equation (12),

$$V_0 = (\bar{x}\ (1 + \pi)\ (1 - t_c)\ (1 - t_s)/(\rho - \pi) + D_0 G.$$

This provides the Hochman and Palmon (1985) valuation equation.

6. Letting $\pi = 0$ and $\alpha = 0$, equation (12) reduces to Miller's model (1977):

$$V_0 = \bar{x}\,(1 - t_c)\,(1 - t_s)/\rho + D_0[1 - \{(1 - t_s)\,(1 - t_c)/(1 - t_p^*)\}].$$