Papers

Equity valuation: The effect of market share dynamics on the value of multiple product lines

Received: 10th October, 2008

Srdjan Stojanovic
is Professor of Mathematics in the College of Arts and Sciences at the University of Cincinnati. His research is in financial mathematics and, in particular, partial differential equations methods for pricing and determination of risk premium, with special emphasis on applications in interest rates, equity and foreign exchange.

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, USA
Tel: +1 513 556 4064; E-mail: srdjan@math.uc.edu

Abstract This paper discusses methodological aspects of equity valuation (stock pricing) in the context of continuous-time finance. The special emphasis of this study is to provide a framework and explicit formulas for the balance-sheet itemisation of the product line contributions to the welfare of the firm, and, in particular, the effect of product market share dynamics on the value of the product line, and on the value of the whole firm. The results were derived in the context of the general analytic theory of pricing and hedging of financial contracts in diffusive incomplete markets, and its applications in quantitative equity, both developed by the author in a series of previous works.

Keywords: *quantitative equity, market share dynamics, product line dividend, financial engineering, PDE methods, special functions, accounting, marketing*

INTRODUCTION

According to the famous MM theorem posited by Merton Miller and Franco Modigliani, laureates of the Nobel Prize in Economics, although the value of the firm lies in the present value of the stream of future dividend payments, this value is surprisingly independent of a particular dividend policy.[1] In other words, provided that some very mild conditions are fulfilled, dividend policy has no impact on the value of the firm. This assertion has been challenged philosophically on various occasions, including in the famous paper by Fisher Black,[2] amounting to what has since been loosely termed the 'dividend puzzle': the apparent contradiction between the conclusion of the MM theorem, and the empirically observable fact that investors do care about the dividend policy of the firm.

Stojanovic has made a substantial extension of the quantitative model of Miller and Modigliani, claiming to have quantitatively resolved the above puzzle.[3] He concludes that the sort of valuation formula consistent with the MM theorem is correct only when the firm's costs do not change when the firm's available cash changes. As costs (executive compensation, labour costs, etc) tend to change with the changing fortunes of the company, one would have to apply a quantitative theory and pricing formulas of the sorts introduced by Stojanovic. These conclusions were made utilising a general theory of risk premium, pricing and hedging of financial contracts in incomplete markets modelled by general Itô diffusions, which was previously developed in a series of papers by Stojanovic,[3–7] and which, in turn, can be considered an extension of the Black–Scholes[8] methodology for the pricing and hedging of financial derivatives.

The present paper continues in the quantitative equity direction set forth by Stojanovic.[3] Here, the paper focuses on the case where market share for various products manufactured and marketed by the company are dynamic. The study finds an explicit formula involving special functions (hyper-geometric functions) that lends a particularly convenient methodology (as convenient as the famous Black–Scholes methodology for pricing options[8]) for the valuation of particular product lines (internally on the balance sheet) or the valuation of whole companies, with particular product lines being itemised.

The paper also provides some applications of the developed equity valuation methodology. Indeed, having found a formula for valuating equity which is capable of accounting the market share trends, then such a formula can be used via standard calculus methods to solve optimisation problems in consumer product pricing and marketing, establishing what might be called *marketing engineering* as an upshot of *financial engineering*.

THE MODEL

Consider a company, for example, a consumer products maker, such as P&G, with multiple product lines, each contributing to the balance sheet in terms of income and cost. Moreover, each product line competes on the retail market with other producers. This competition is waged through competitive pricing, advertising, and through the quality of the product, quality of the brand, etc. Thus, the brand management affects the value of the equity and, vice versa, the equity valuation should affect the brand management as well. Thus, the financial engineering yields the marketing engineering and the marketing engineering might also affect the financial engineering.

The equity valuation model

The paper starts with the basic equity model with constant coefficients as introduced by Stojanovic,[3] but in addition to the simple (linear) sources of income and cost as discussed there, income and cost for m different 'product lines' are itemised, obtaining the equity (accounting) model:

$$dY(t) = Y(t)\,(a - \mathrm{D})dt + Y(t)s_0 d\,B_0(t)$$
$$d\mathcal{M}_i(t) = \mathcal{M}_i(t)q_i dt + \mathcal{M}_i(t)\mathrm{s}_i dB_i(t)$$
$$d\mathcal{S}_i(t) = \mathcal{S}_i(t)(1 - \mathcal{S}_i(t))\mathcal{T}_i(p_i, \mathrm{A}_i)dt$$

$$d\mathrm{C}(t) = \left(\sum_{i=1}^{m} \frac{\mathrm{D}_{\mathcal{PL},i}(t)}{\mathcal{N}}\right)dt +$$
$$(\mathrm{i}_0(t) - \mathrm{p}_0(t))dt + (r - \mathrm{p}_1)\mathrm{C}(t)dt - \mathcal{D}(t)dt$$
$$d\mathrm{i}_0(t) = (\mu_{\mathrm{i},0} + \mu_{\mathrm{i},1}\,\mathrm{i}_0(t))dt + (w_{\mathrm{i},0} + w_{\mathrm{i},1}\,\mathrm{i}_0(t))dB_{m+1}(t)$$
$$d\mathrm{p}_0(t) = (\mu_{\mathrm{p}_0,0} + \mu_{\mathrm{p}_0,1}\,\mathrm{p}_0(t))dt + (w_{\mathrm{p}_0,0} + w_{\mathrm{p}_0,1}\,\mathrm{p}_0(t))dB_{m+2}(t)$$

with

$$D_{PL,i}(t) \;=\; M_i(t)S_i(t)(p_i - c_i) - A_i$$

and

$$\mathcal{D}(t) = \alpha_0 + \alpha_1\,\mathrm{i}_0(t) + \alpha_2\,\mathrm{p}_0(t) + \alpha_3\mathrm{C}(t) + \sum_{i=1}^{m} \alpha_{\mathcal{PL},i}\,\frac{\mathrm{D}_{\mathcal{PL},i}(t)}{\mathcal{N}}$$

where:

- $Y(t)$ denotes the price of the hedging tradable, for example DJI or its equivalent, at time t (measured in years), with growth rate a, with dividend rate D, and with volatility s_0.
- $M_i(t)$ denotes the total market size for the ith considered product line, ie for the particular product (such as the total yearly sales of soap-bars in the USA), expressed in total number of units sold by all producers/competitors during a single year, with growth rate q_i, and volatility s_i.
- $S_i(t)$ denotes the market share that belongs to the company considered for valuation, for the ith considered product line, with the value, assured by the postulated dynamics, always to be between 0 and 1 (standing respectively for the 0 per cent and the 100 per cent market share), with $T_i = T_i(p_i, A_i)$ being the 'market share trend', possibly dependent on the product retail price p_i, and also on the product aggregate costs denoted by A_i, ie costs such as advertising expenditure and other product line costs which are independent of the number of units produced. The postulated equation is a deterministic ordinary differential equation and not stochastic due to the desire to find an explicit valuation formula. The more complicated case when it is allowed that the market share is governed by an analogous stochastic differential equation can also be solved but not completely explicitly. One can notice that in this model all the competition is lumped together, taking cumulatively market share equal to $1 - S_i(t)$. More general models, when some of the competition is itemised, can also be considered but this is beyond the scope of the present work.
- $C(t)$ denotes the company's available cash per unit stock, a number usually positive, but possibly also negative if the company is in debt, and where N is the number of stocks outstanding, where $D_{PL}(t)$ is the ith 'product line dividend' in dollars per year (in total, not per stock), depending on the total market size $M_i(t)$, market share $S_i(t)$, product retail price p_i, aggregate costs A_i, and also dependent on the per unit production and/or distribution costs c_i, where r is the (money market) short interest rate (compounded continuously), and where $p_1C(t)$ is the 'surcharge rate cost' (the presence of which is the sole cause for the necessity to adjust the MM theorem as discussed in Stojanovic,[3] and also as shown below), and where $D(t)$ is the dividend (paid to the stock owners) per year per stock.
- $i_0(t)$ denotes the remaining income rate, per year per stock, not accounted for through the itemised product lines above.
- $p_0(t)$ denotes the remaining basic cost rate, per year per stock, not accounted for through the itemised product lines, nor through the surcharge cost rate $p_1C(t)$.

One can observe that, even though the cost to the company due to the dividend payoff is equal to $D(t)$ per year per share, the benefit to the stock owner is reduced, due to the dividend tax at the rate T, to $(1 - T)D(t)$ per year per stock. One can observe that $T = 1$ would correspond to the 100 per cent dividend tax rate.

Of course, $B_i(t)$ for $0 \leq i \leq m+2$ denotes $m+3$ standard Brownian motions, possibly correlated, with instantaneous correlation coefficients to be denoted by ρ, j, i for $0 \leq i < j \leq m+2$.

The product line valuation model

Sometimes, possibly for internal business accounting and management purposes, it may be convenient to be able to value a designated product line rather than the whole company. This can be achieved as a simpler problem than the one just described. Indeed, in such a case it suffices to consider:

$$
\begin{aligned}
dY(t) &= Y(t)(a - D)dt + Y(t)s_0 dB_0(t) \\
dM_i(t) &= M_i(t)q_i dt + M_i(t)s_i dB_i(t) \\
dS_i(t) &= S_i(t)(1 - S_i(t))T_i(p_i, A_i)dt
\end{aligned}
$$

with the product line dividend ('paid' to the company) in the amount of

$$D(t) = M_i(t)S_i(t)(p_i - c_i) - A_i$$

using the notation analogous to the one used above. Both problems will be solved by means of explicit mathematical formulas, and the simpler problem — the valuation of the single product line — will be solved as a special case of the problem of valuation of the whole firm.

THE SOLUTION FORMULAS

The paper will now present the explicit mathematical formulas to solve the posed problems. The methodology of pricing is the one detailed by Stojanovic.[3,5,6] Generally speaking, the methodology consists of solving two partial differential equations. The first to be solved is the risk premium (or market price of risk) partial differential equation, the solution of which depends on the value of the (market) *relative risk aversion parameter*. The second step, after finding the (risk aversion dependent) risk premium, is to solve the pricing partial differential equation. In an incomplete market setting, such a solution often depends on the value of the risk aversion parameter, as does the solution of the risk premium partial differential equation.

In the present example, however, the pricing formula (to be specified below) is independent of the (relative) risk aversion parameter, ie the fair price is unique, in spite of the fact that pricing is being undertaken in an incomplete market setting. This is atypical. Usually in the incomplete market setting the fair prices depend on the risk aversion, ie on the chosen or determined risk premium, ie market price of risk. The risk premium in this example, ie the market price of non-hedgeable risk, is equal to zero, as the hedging tradable is assumed to have a price obeying very simple stochastic differential equation dynamics (log-normal). See Stojanovic[3] for more details on such a phenomenon when pricing in an incomplete market setting.

The actual mathematical derivation is a bit technical, and hence beyond the scope of this paper, but details of the derivation are available from the author.

Equity valuation formula

Utilising the general analytic theory of pricing of financial contracts in incomplete multidimensional diffusive markets[5,6] and finding an entire solution[3] of the pricing partial differential equation, the following explicit formula for the value of the company with the balance sheet described as above can be derived. The value, ie the fair price of a single stock, is equal to:

$$V(t, Y, \mathcal{M}_1, \ldots, \mathcal{M}_m, \mathcal{S}_1, \ldots, \mathcal{S}_m, \mathrm{C}, \mathrm{i}_0, \mathrm{p}_0)$$
$$= V(\mathcal{M}_1, \ldots, \mathcal{M}_m, \mathcal{S}_1, \ldots, \mathcal{S}_m, \mathrm{C}, \mathrm{i}_0, \mathrm{p}_0)$$
$$= \mathrm{B}_0 + \mathrm{C}\,\mathrm{B}_\mathrm{C} + \mathrm{B}_{\mathrm{i}_0}\,\mathrm{i}_0 + \mathrm{B}_{\mathrm{p}_0}\,\mathrm{p}_0 + \sum_{i=1}^{m} \mathcal{M}_i \mathcal{V}_\mathcal{S}, i(\mathcal{S}_i)$$

where

$$\mathrm{B}_0 = \frac{1-\mathcal{T}}{r}\left(\alpha_0 - \frac{\alpha_3\alpha_0}{\mathrm{p}_1+\alpha_3} - \frac{\alpha_3 \sum_{i=1}^m \mathrm{A}_i}{\mathcal{N}(\mathrm{p}_1+\alpha_3} - \right.$$

$$\frac{\mathrm{p}_1}{\mathcal{N}(\mathrm{p}_1+\alpha_3)} \sum_{i=1}^{m} \mathrm{A}_i \alpha_{\mathcal{PL},i}$$

$$+\frac{\mathrm{p}_1\alpha_1+\alpha_3}{\mathrm{p}_1+\alpha_3}\left(\frac{\mu_{\mathrm{i},0} + (r-\mu_{\mathrm{i},1})\frac{w_{\mathrm{i},0}}{w_{\mathrm{i},1}}}{r-\mu_{\mathrm{i},1}+\frac{\mathrm{a}-r}{\mathrm{s}_0} w_{\mathrm{i},1}\rho_{m+1,0}} - \frac{w_{\mathrm{i},0}}{w_{\mathrm{i},1}}\right)$$

$$\left.+\frac{\mathrm{p}_1\alpha_2-\alpha_3}{\mathrm{p}_1+\alpha_3}\left(\frac{\mu_{\mathrm{p}_0,0} + (r-\mu_{\mathrm{p}_0,1})\frac{w_{\mathrm{p}_0,0}}{w_{\mathrm{p}_0,1}}}{r-\mu_{\mathrm{p}_0,1}+\frac{\mathrm{a}-r}{\mathrm{s}_0} w_{\mathrm{p}_0,1}\rho_{m+2,0}} - \frac{w_{\mathrm{p}_0,0}}{w_{\mathrm{p}_0,1}}\right)\right)$$

$$\mathrm{B}_\mathrm{C} = \frac{(1-\mathcal{T})\alpha_3}{\mathrm{p}_1+\alpha_3}$$

$$\mathrm{B}_{\mathrm{i}_0} = (1-\mathcal{T})\frac{\mathrm{p}_1\alpha_2-\alpha_3}{\mathrm{p}_1+\alpha_3}\,\frac{1}{r-\mu_{\mathrm{i},1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{i},1}\rho_{m+1,0}}$$

$$\mathrm{B}_{\mathrm{p}0} = (1-\mathcal{T})\frac{\mathrm{p}_1\alpha_2-\alpha_3}{\mathrm{p}_1+\alpha_3}\,\frac{1}{r-\mu_{\mathrm{p}0,1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{p}0,1}\rho_{m+2,0}}$$

$$\mathcal{V}_{\mathcal{S},i}(\mathcal{S}_i) = \begin{cases} \dfrac{\mathrm{d}_i\mathcal{S}_{i2}F_1(1, 1-\frac{\mathrm{b}_i}{\mathcal{T}_i}; 2-\frac{\mathrm{b}_i}{\mathcal{T}_i}; \frac{\mathcal{S}_i}{\mathcal{S}_i-1})}{(1-\mathcal{S}_i)(\mathrm{b}_i-\mathcal{T}_i)} & \text{if } \mathcal{T}_i < 0 \\ \dfrac{\mathrm{d}_i\mathcal{S}_i}{\mathrm{b}_i} & \text{if } \mathcal{T}_i = 0 \\ \dfrac{\mathrm{d}_{i2}F_1(1, \frac{\mathrm{b}_i}{\mathcal{T}_i}; \frac{\mathcal{T}_i+\mathrm{b}_i}{\mathcal{T}_i}; \frac{\mathcal{S}_i-1}{\mathcal{S}_i}}{\mathrm{b}_i} & \text{if } \mathcal{T}_i > 0 \end{cases}$$

with

$$\mathrm{b}_i = r - q_i + \frac{\mathrm{a}-r}{\mathrm{s}_0}\,\mathrm{s}_i\rho_{i,0}$$

$$\mathrm{d}_i = \frac{(1-\mathcal{T})(p_i-c_i)}{\mathcal{N}}\,\frac{\alpha_3+\mathrm{p}_1\alpha_{\mathcal{PL},i}}{\mathrm{p}_1+\alpha_3}$$

and where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty}(a)_k(b)_k/(c)_k x^k/k!$$
$$= 1 + \frac{a\,b\,x}{c} + \frac{a(a+1)b(b+1)x^2}{2c(c+1)} +$$
$$\frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{6c(c+1)(c+2)} + O(x^4)$$

is the hyper-geometric function, a well-known special function.

The 'Miller–Modigliani version' of this valuation formula, ie the 'no-surcharge costs version', is obtained simply by setting $p_1 = 0$ and simplifying. Thus:

$$\mathrm{B}_0 = \frac{1-\mathcal{T}}{r}\left(-\frac{1}{\mathcal{N}}\sum_{i=1}^{m}\mathrm{A}_i + \right.$$

$$\frac{\mu_{i,0} + (r-\mu_{i,1})\frac{w_{i,0}}{w_{\mathrm{i},1}}}{r-\mu_{\mathrm{i},1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{i},1}\,\rho_{m+1,0}} - \frac{w_{i,0}}{w_{\mathrm{i},1}}$$

$$-\frac{\mu_{\mathrm{p}_0,0} + (r-\mu_{\mathrm{p}_0,1})\frac{w_{\mathrm{p}_0,0}}{w_{\mathrm{p}_0,1}}}{r-\mu_{\mathrm{p}_0,1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{p}_0,1}\,\rho_{m+2,0}} + \frac{w_{\mathrm{p}_0,0}}{w_{\mathrm{p}_0,1}}$$

$$\mathrm{B}_\mathrm{C} = 1-\mathcal{T}$$

$$\mathrm{B}_{\mathrm{i}0} = \frac{1-\mathcal{T}}{r-\mu_{\mathrm{i},1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{i},1}\,\rho_{m+1,0}}$$

$$\mathrm{B}_{\mathrm{p}0} = -\frac{1-\mathcal{T}}{r-\mu_{\mathrm{p}0,1}+\frac{\mathrm{a}-r}{\mathrm{s}_0}w_{\mathrm{p}0,1}\,\rho_{m+2,0}}$$

and $V_{S,\,i}(S_i)$ is evaluated using:

$$\mathrm{d}_i = \frac{(1-\mathcal{T})(p_i-c_i)}{\mathcal{N}}$$

So, as expected, if there are no surcharge costs, the value of the firm does not depend on a particular dividend policy (although under some mild technical conditions where the dividend is not identically equal to zero, the value of the company would be equal to zero, no matter what). On the other hand, if surcharge costs are anticipated, ie if $p_1 \neq 0$, then the previous, more elaborate formula needs to be applied.

Product line valuation formula

Furthermore, if in the above one sets:

$$\mathcal{T} = 0, \alpha\mathrm{i} = 0, N = 1, \alpha_{PL,i} = 1, \alpha_{PL,j} = 0$$

for $j \neq i$, one obtains:

$$\mathrm{B}_0 = -\frac{\mathrm{A}_i}{r}, \mathrm{B}_{\mathrm{p}_0} = \mathrm{B}_{\mathrm{i}_0} = \mathrm{B}_\mathrm{C} = 0$$

and the valuation of a single (designated) product line formula becomes simply:

$$\mathcal{V}(\mathcal{M}_i, \mathcal{S}_i) = \mathcal{M}_i \mathcal{V}_{\mathcal{S},i}(\mathcal{S}_i) - \frac{\mathrm{A}_i}{r}$$

where $V_{S,i}(S_i)$ is evaluated as before but using

$$d_i = p_i - c_i$$

The hedging formula

In addition to stock pricing, in some instances it may be useful to have a formula for (the most conservative) hedging as well — for example, if one has an ownership position in the considered company, one may wish to hedge against a possible stock price decline.

As this exercise is set in an incomplete market framework, unless the perfect correlation exists between the governing Brownian motions, one cannot hope to be able to hedge perfectly, like in the Black–Scholes[8] complete market case, but rather the success of hedging will depend on the level of correlation — the more correlated the factors are with the tradable, the more perfect hedging is possible. The notion of the 'most conservative hedging' was introduced by Stojanovic.[5,6,9] Using the formula (2.48) from Stojanovic,[6] and using the pricing formula above, one can easily compute the (most conservative) hedging formula for the present problem to be:

$$\mathcal{H}(\mathcal{M}_1, \ldots, \mathcal{M}_m, \mathcal{S}_1, \ldots, \mathcal{S}_m, \mathrm{i}_0, \mathrm{p}_0)$$
$$= -\frac{1}{\mathrm{s}_0}\Big(\mathrm{B}_{\mathrm{i}_0}(w_{\mathrm{i},0} + \mathrm{i}_0 w_{\mathrm{i},1})\,\rho_{m+1,0}$$
$$+\mathrm{B}_{\mathrm{p}_0}(w_{\mathrm{p}_0,0} + \mathrm{p}_0 w_{\mathrm{p}_0,1})\,\rho_{m+2,0} + \sum_{i=1}^{m} \mathcal{M}_i \mathrm{s}_i\,\rho_{i,0}\,\mathcal{V}_{\mathcal{S},i}(\mathcal{S}_i)\Big)$$

The quantity $H(M_1, \ldots, M_m, S_1, \ldots, S_m, i_0, p_0)$ represents the dollar value of the investment in the tradable with price $Y(t)$ to hedge a single stock with price $V(M_1, \ldots, M_m, S_1, \ldots, S_m, \mathrm{C}, i_0, p_0)$. Less conservative (more speculative) optimal hedging is possible but it is much more complicated, and beyond the scope of the present paper.

DISCUSSION OF THE PRODUCT LINE VALUATION FORMULA

Stojanovic has presented a comprehensive study of equity valuation in the case with no product lines, and without taking market share gain or loss into account.[3] As such, the present paper discusses only the new part — the product line valuation formula above, and the effect of market share gain or loss. The above stated product line valuation formula can be rewritten by putting all the pieces together as:

$$V(\mathcal{M}_i,\mathcal{S}_i)=\begin{cases}-\dfrac{\mathrm{A}_i}{r}+\dfrac{(p_i-c_i)\mathcal{M}_i\frac{\mathcal{S}_i}{1-\mathcal{S}_i}{}_2F_1\left(1,1-\frac{r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}}{\mathcal{T}_i};2-\frac{r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}}{\mathcal{T}_i};\frac{\mathcal{S}_i}{\mathcal{S}_i-1}\right)}{r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}-\mathcal{T}_i} & \text{if } \mathcal{T}_i<0\\ -\dfrac{\mathrm{A}_i}{r}+\dfrac{(p_i-c_i)\,\mathcal{M}_i\mathcal{S}_i}{r-q_i+\frac{\mathrm{a}-r}{\mathrm{s}_0}\mathrm{s}_i\,\rho_{i,0}} & \text{if } \mathcal{T}_i=0\\ -\dfrac{\mathrm{A}_i}{r}+\dfrac{(p_i-c_i)\mathcal{M}_{i2}F_1\left(1,\frac{r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}}{\mathcal{T}_i};\frac{\mathcal{T}_i+r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}}{\mathcal{T}_i};\frac{\mathcal{S}_i-1}{\mathcal{S}_i}\right)}{r-q_i+\frac{\mathrm{a}-r}{\mathrm{S}_0}\mathrm{s}_i\rho_{i,0}} & \text{if } \mathcal{T}_i>0\end{cases}$$

where it is necessary to assume that

$$r>0,\, r-q_i+\frac{\mathrm{a}-r}{\mathrm{s}_0}\mathrm{s}_i\,\rho_{i,0}>0$$

(so far, the paper has not specified mathematical assumptions under which all of the preceding formulas hold true). The interest rate has to be positive in order that the present value of future fixed costs is finite. The second condition assures that the present value of the future income is finite.

Figure 1 presents a plot of the product line equity value and the corresponding most conservative hedging. The higher value (price) corresponds to the case when the market share trend is positive, the straight line to the case when the market share trend is zero, and the lowest value corresponds to the case when the market share trend is negative. The corresponding hedges are in the same order. The product line value may happen to be negative (when the fixed costs, eg advertising expenditure, are too high for the given business outlook).

AN APPLICATION: CALCULATING AN OPTIMAL RETAIL PRICE AND ADVERTISING EXPENDITURE

So far the paper has not discussed a possible structure of the market share trend. For example, the following form for the market share trend would make sense:

$$\mathcal{T}=\alpha(p_2-p_1)^3+\beta(\sqrt{A_1}-\sqrt{A_2})+\delta$$

where p_2 is the competing retail price, while p_1 is the considered company's product retail price; A_2 and A_1 are aggregate advertising expenditure for the two competing product lines. It is assumed that $\alpha\geq 0$, $\beta\geq 0$; finally, the constant term δ corresponds to the brand and product quality superiority or inferiority (if negative). The cubic term

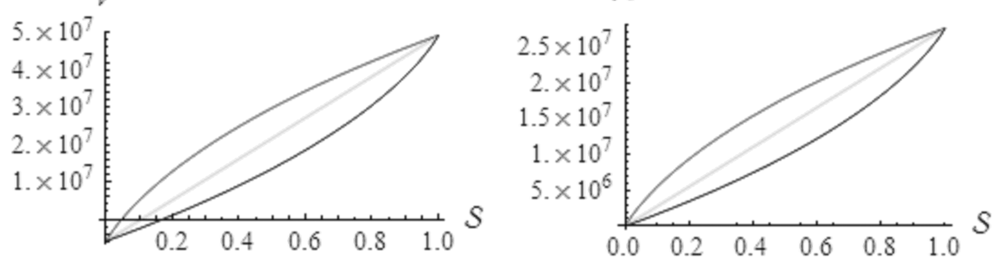

Figure 1: The product line equity value and the corresponding most conservative hedging

$(p_2 - p_1)^3$ reflects on the plausible fact that small price differences are not very consequential, while the large retail price differences are very much so. The opposite seems reasonable in regard to the advertising expenditure $\sqrt{A_1} - \sqrt{A_2}$: small advertising expenditure is much better than no expenditure at all, which is the property of the square-root function, while the effects of different yet already large advertising expenditures are not that much pronounced. Of course, in practice the parameters α, β and δ would have to be estimated statistically.

Once the above form of the market share trend is substituted in the already stated formula for the equity value, either for the whole company, or for a single product line, one can switch the nature of the parameters vs variables, and consider (say, product line) equity value as a function of the product retail price and advertising expenditure, and thereby solve an optimisation problem: finding the (optimal) product retail price and advertising expenditure so that the corresponding equity value is maximised. This is not much more than a multivariable calculus problem now when the explicit formula for the equity value has been found. Indeed one can use, for example, the steepest ascent (Newton's) method, and compute iteratively the approximations of the optimal (maximal) product line value. Figure 2 presents an example of such an iteration superimposed on the plot of the product line value (as a function of p_1, A_1).

CONCLUSIONS

Whether it is for internal firm management purposes (including marketing), or for the outside value-investment considerations, new and more comprehensive equity valuation methodologies, consistent with the classical ones, yet based on the new incomplete-market pricing and hedging theory, should find important applications. This paper has considered a particular aspect of the fair equity value determination: how does (or should) the market share dynamics outlook, market share gain or market share loss affect the valuation of particular product lines, or a value of the whole company. Such a problem has nonlinear underlying (market share) dynamics, and yet it has

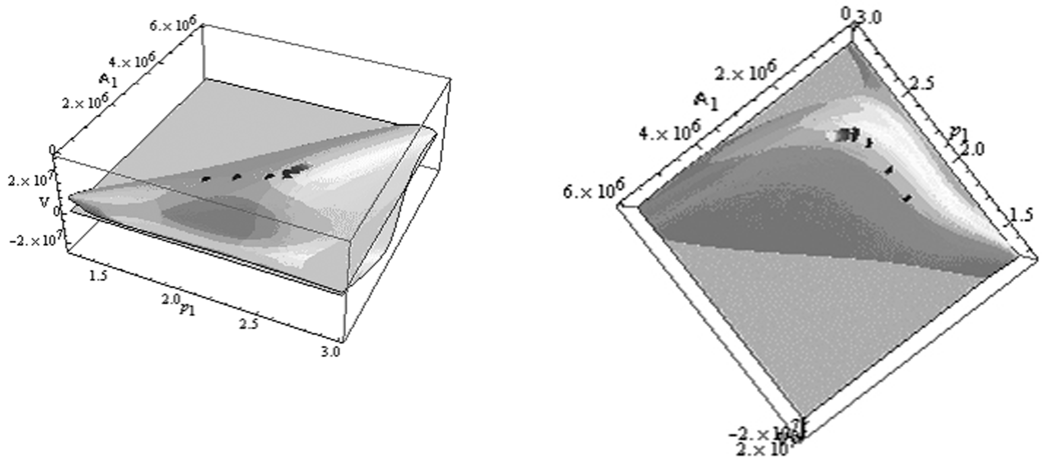

Figure 2: Optimal marketing and retail pricing: Maximising equity value

allowed a derivation of an explicit valuation (and hedging) formula, and thus can be applied as conveniently as other much simpler and less realistic equity models.

References

1 Miller, M. H. and Modigliani, F. (1961) 'Dividend policy, growth, and the valuation of shares' *Journal of Business*, Vol. 34, October, pp. 411–433.
2 Black, F. (1976) 'The dividend puzzle', *Journal of Portfolio Management*, Vol. 2, No. 2, pp. 5–8.
3 Stojanovic, S. (2006) 'The dividend puzzle unpuzzled', available at SSRN: http://ssrn.com/abstract = 879514.
4 Stojanovic, S. (2005) 'Risk premium and fair option prices under stochastic volatility: the HARA solution', *Les Comptes Rendus Série Mathematique*, Vol. 340, No. 7, pp. 551–556.
5 Stojanovic, S. (2005) 'Higher dimensional fair option pricing and hedging under HARA and CARA utilities', working paper, available at: http://ssrn.com/abstract = 912763.
6 Stojanovic, S. (2006) 'Pricing and hedging of multi type contracts under multidimensional risks in incomplete markets modelled by general Itô SDE systems', *Asia Pacific Financial Markets*, Vol. 13, No. 4, pp. 345–372.
7 Stojanovic, S. (2007) 'Risk premium, pricing and hedging for variance swaps', in Nelken, I. (ed.) 'Volatility as an Asset Class', Risk Books, Incisive Media Financial Publishing, London, pp. 259–285.
8 Black, F. and Scholes, M. (1973) 'The pricing of options and corporate liabilities', *Journal of Political Economy*, Vol. 81, No. 3, pp. 637–654.
9 Stojanovic, S. (2003) 'Computational Financial Mathematics using Mathematica®', Birkhäuser, Boston, MA.

