Risk-neutral versus objective loss distribution and CDO tranche valuation

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Abstract This paper considers the risk-neutral loss distribution as implied by index collateralised debt obligation (CDO) tranche quotes through a 'scenario default rate' model as opposed to the objective measure loss distribution based on historical analysis. The risk-neutral loss distribution turns out to assign greater probability to large realisations of the loss with respect to the objective distribution, thus implying the well-known presence of a risk premium. In the spirit of Elton *et al.* (2001), the paper measures this premium, regressing the tranche excess returns against the three Fama-French factors, augmented by the credit index excess returns. However, on the basis of the limited available sample data, it finds no evidence of a significant risk premium. Following Tarashev and Zhu (2007), the paper quantifies the correlation premium, pricing CDO tranches and indices under two distributions: risk-neutral and objective. Instead of building a time nonhomogeneous default dependency as per Tarashev and Zhu, the paper assumes a time-homogeneous default correlation, as can be found in the S&P CDO Evaluator. *En passant*, the paper analyses the implied risk-neutral default-rate distributions calibrated from April 2004 through April 2006, noting the persistence through time of the distinctive 'bump' in the distribution tail.

Keywords: *default-rate distribution, CDO, CDO tranches, perfect copula, implied copula, transition matrices, rating classes, risk premium, recovery rate*

INTRODUCTION

This paper presents a framework for the consistent pricing of collateralised debt obligation (CDO) tranches across detachments. The risk-neutral loss distribution is derived through a modified 'scenario default rate' approach, inspired by the 'perfect copula' model of Hull and White (2006),[1] which the paper refers to as 'implied copula'. This framework addresses the inconsistency of the Gaussian copula model introduced by Li (2000)[2] (see also McGinty and Ahluwalia, 2004[3]) when it is used in its simplest single-parameter formulation, where a different implied correlation parameter has to be used to price each tranche.

The implied default rate framework and the Gaussian copula model are compared, focusing on the key differences when a common formulation is adopted. In the implied default rate model, loss states are exogenously assigned, conditional on the systemic factor, and the systemic factor probabilities are directly modelled as calibrating parameters. In the Gaussian copula model, both the conditional loss states and the systemic factor probabilities are a by-product of the Gaussian parametric assumption, and calibration must rest on the Gaussian copula parameter(s). In a way, this is reminiscent of non-parametric versus parametric approaches.

The Gaussian copula assumption is not the only one available in the parametric context. Indeed, other examples include the double-T copula[4] and mixture copulas.[5] On this subject, Laurent and Gregory (2005)[6] give an overview of the different possible choices.

This paper aims to estimate the risk premium contained in the CDO tranches. Hull *et al*. (2004)[7] estimate the risk premium in corporate bond spreads of different rating categories by comparing the risk-neutral default probability implied by the corporate bond spread with the average historical default probability of entities with the same rating.

An equivalent exercise for CDO tranches is used here, with more complicated results due to default correlation. As per Hull *et al*.,[7] the objective default probability of the single reference entity is given by the average historical default probabilities of its rating category. To price a CDO tranche, one also needs the average historical correlation matrix. The present study uses a Gaussian copula with the same correlation matrix as in the CDO Evaluator 3.0 used by Standard & Poor's (S&P). This matrix is highly structured and has a much richer economic background than the single-parameter matrix in the Gaussian copula used to quote implied correlation.

In the single-name context, the comparison between risk-neutral and objective measure statistics is well known — for example, see the above-mentioned paper by Hull *et al*.[7] The present paper investigates the analogous issue for aggregate loss distribution. In agreement with single-name default probabilities, the results indicate that, the risk-neutral loss distribution assigns greater probability to large realisations of the loss with respect to the objective distribution, thus implying the well-known presence of a risk premium. This premium is quantified by repricing the market index and tranches under the objective loss distribution and comparing the resulting net present value (NPV) with the risk-neutral one.

IMPLIED COPULA AND IMPLIED DEFAULT-RATE DISTRIBUTION

The study takes the iTraxx Europe index as a reference (125 constituents). It fixes a set of scenarios for the systemic factor M, influencing the default of all the constituent names. It assumes that, conditional on M, defaults of different names are independent. In this way, M is the only variable building dependence across the pool. The market-standard Gaussian single-factor copula makes particular assumptions on the dependence structure across defaults and on M (see appendix). In the implied copula setup, a homogeneous pool assumption is implicitly made, directly postulating default intensities conditional on a finite set of scenarios for the systemic factor M, say:

$$M \in \{\mathrm{m}^0, \mathrm{m}^1, \ldots, \mathrm{m}^{124}\}$$

This effectively assumes that, conditional on M, the stochastic default intensity λ of any name i in the pool, defined as:

$$\text{ProbabilityRiskNeutral}(\text{Name } i \text{ defaults in } [t, t+dt) \,|\, \text{Name } i \text{ has not defaulted by } t) = \lambda \cdot dt$$

further satisfies:

$$\left\{\begin{array}{ccc} \text{Systemic scenario} & \text{Scenario probability} & \text{Conditional intensity} \\ M = \mathrm{m}^0 & \mathrm{p}_0 & \lambda^0 \\ M = \mathrm{m}^1 & \mathrm{p}_1 & \lambda^1 \\ \ldots & \ldots & \ldots \\ M = \mathrm{m}^{124} & \mathrm{p}_{124} & \lambda^{124} \end{array}\right.$$

For a preferred maturity T and for each name $i = 1, 2, \ldots, 125$, this effectively assumes the following default probabilities, conditional on M:

$$\text{ProbabilityRiskNeutral}\left(\text{Name } i \text{ defaults by time } T | M = m^S\right) = 1 - e^{-\lambda^S T}$$

Again, for a preferred maturity T, the pool default rate, conditional on the systemic scenario $M = m^S$, is defined as (in this particular case $N = 125$):

$$DR_N^S(T) = \frac{1}{N}\sum_{i=1}^{N} I\left\{\text{Name } i \text{ defaults before } T | M = m^S\right\}$$

where $I\{\text{condition}\}$ is equal to 1 if the condition is satisfied and 0 otherwise, and where, given the initial assumption, the terms in the summation are independent of each other given M.

Now the infinite (or 'large') pool assumption comes into play. One may assume the number of names N in the pool to be very large. In this case, the law of large numbers implies that (under some mild assumptions) the sample average of the independent and identically distributed random variables:

$$I\left\{\text{Name 1 defaults before } T | M = m^S\right\},\quad I\left\{\text{Name 2 defaults before } T | M = m^S\right\}, \ldots$$

which is equal to $DR_N^S(T)$, given the homogeneity assumption, converges in law to the true mean of each single random variable when N tends to infinity, ie:

$$DR_N^S(T) \underset{\text{law}}{\rightarrow} \text{ExpectationRiskNeutral}\,[I\{\text{Name defaults before}\, T|M = m^S\}] \quad \text{as} \quad N \rightarrow \infty$$

$$\text{ExpectationRiskNeutral}\,[I\{\text{Name defaults before}\, T|M = m^S\}] =$$

$$= \text{ProbabilityRiskNeutral}\{\text{Name defaults before}\, T|M = m^S\} = 1 - e^{-\lambda^S T} := DR^S(T)$$

The set of scenarios $DR^S(T)$ $S = 0, 1 \ldots, 124$, represents the full set of possible default rates for the iTraxx portfolio (ie $S = 0, 1, \ldots, 124$ out of 125 constituents of the portfolio default before maturity T). In turn, given its intuitive meaning, the default rate is also the fraction of names that have defaulted for a given maturity. As such, its natural values would be:

$$DR^S \in \left\{0, \frac{1}{125}, \ldots, \frac{124}{125}\right\} = \left\{DR^0, DR^1, \ldots, DR^{124}\right\}$$

The original perfect copula approach of Hull and White (2005),[1] on which the present methodology draws, considers a more complex spacing in the set of default rates. Default rate scenarios are spaced in such a way that the sums of the net present values of tranches and indexes under each scenario are equally spaced.

In the present case, one may run the natural default rate scenarios backwards to deduce sensible scenarios for the intensities associated with them for the maturity T. This amounts to solving the following equation for λ:

$$1 - e^{-\lambda^S T} = DR^S(T) \in \left\{\frac{0}{125}, \frac{1}{125}, \ldots, \frac{124}{125}\right\}$$

leading to:

$$\lambda^S \in \left\{-\frac{\ln(1-0)}{T}, \frac{\ln(1-\frac{1}{125})}{T}, \ldots, -\frac{\ln(1-\frac{124}{125})}{T}\right\} = \{\lambda^0, \lambda^1, \ldots, \lambda^{124}$$

To sum up, from the set of default rates above, given the particular maturity T of the tranches and the index to be priced, a corresponding set of scenarios for the default intensities can be determined, constituting the parameterisation from which the process started, thus finally closing the loop.

THE INSTRUMENTS PAYOFF

The tranches and the index pay their spread on the dates $t_1, t_2, \ldots t_N$, expressed as year fractions. The start date is defined as $t_0 = 0$. R^S is the recovery rate associated with the scenario S; r_i is the risk-free zero-coupon rate on date t_i.

Given a generic scenario S, the NPV of the premium and default leg of the index will be:

$$\text{NPV}_{\text{Ind}} = \text{Prem}_{\text{Ind}} \cdot \text{Def}_{\text{Ind}}$$
$$\text{Prem}_{\text{Ind}} = \sum \text{Prem}_{\text{Ind}}^S \cdot P_S$$
$$\text{Def}_{\text{Ind}} = \sum \text{Def}_{\text{Ind}}^S \cdot P_S$$
$$\text{Prem}_{Ind}^S = spr_{Ind} \cdot \text{OutNotl}_{Ind}^S$$
$$\text{OutNotl}_{A,B}^S = \sum_{i=1}^{N} (t_i - t_{i-1}) \cdot \exp(-r_i \cdot t_i) \cdot \exp(-\lambda^S \cdot (t_i + t_{i-1})/2)$$
$$\text{Def}_{Ind}^S = (1 - R^S) \cdot \sum_{i=1}^{N} \exp(-r_i \cdot t_i) \cdot \left[\exp(-\lambda^S \cdot t_{i-1}) - \exp(-\lambda^S \cdot t_i)\right]$$

where the notation for the index outstanding notional, the index premium leg, the index spread and the index default leg is self-evident. Given a generic scenario S, the NPV of the premium and default leg of the tranche with attachment A and detachment B

will be:

$$\mathrm{Prem}^S_{A,B} = spr_{A,B} \cdot \mathrm{OutNotl}^S_{A,B}$$

$$\mathrm{OutNotl}^S_{Ind} = \sum_{i=1}^{N} (t_i - t_{i-1}) \cdot \exp(-r_i \cdot t_i) \cdot \left(1 - TrancheLoss\left(S, \frac{t_{i-1}+t_i}{2}\right)\right)$$

$$\mathrm{Def}^S_{A,B} = \sum_{i=1}^{N} \exp(-r_i \cdot t_i) \cdot [TrancheLoss(S, t_i) - TrancheLoss(S, t_{i-1})]$$

$$TrancheLoss(S,t) = \frac{\max(PortfLoss(S,t) - A, 0) - \max(PortfLoss(S,t) - B, 0}{B - A}$$

$$PortfLoss(S,t) = (1 - R^S) \cdot \left(1 - \exp(-\lambda^S \cdot t)\right)$$

where again the notation for the A-B tranche outstanding notional, premium leg, spread, default leg and loss is self-evident.

Note the approximation introduced when computing the integrals involved in determining the average outstanding notional and default leg NPV in each period. For a thorough exposition of CDS pricing and the accuracy of different approximations to the relevant integrals see O'Kane and Turnbull (2003).[8]

IMPLIED DEFAULT-RATE DISTRIBUTION

Consistent with the perfect copula of Hull and White (2005),[1] the numerical problem is finding the weights (ie the scenario probabilities p_s, positive and adding up to 1) to assign to each scenario S so as to reprice the index and the tranches consistently with market quotes. The weights $p_0, p_1, \ldots, p_{124}$ will correspond to the risk-neutral distribution of the default rates. There are 125 scenarios and only six instruments (five tranches plus the index), meaning that the system is under-determined. Indeed, there are too many unknowns (up to 125 — the scenario weights) and too few equations (up to six — the instruments to price).

PR and DEF are defined as the matrices with the NPV of the premium and default leg. The rows correspond to the scenarios (remember that scenario S is the scenario where S names out of 125 default before maturity) whereas the columns correspond to the instruments (the index in the first column).

$$\mathrm{PR} = \begin{bmatrix} \mathrm{Prem}^0_{\mathrm{Ind}} & \mathrm{Prem}^0_{0,3} & \cdots & \mathrm{Prem}^0_{12,22} \\ \mathrm{Prem}^1_{\mathrm{Ind}} & \mathrm{Prem}^1_{0,3} & \cdots & \mathrm{Prem}^1_{12,22} \\ \cdots & & & \\ \mathrm{Prem}^{124}_{\mathrm{Ind}} & \mathrm{Prem}^{124}_{0,3} & \cdots & \mathrm{Prem}^{124}_{12,22} \end{bmatrix}$$

$$\mathrm{DEF} = \begin{bmatrix} \mathrm{Def}^0_{\mathrm{Ind}} & \mathrm{Def}^0_{0,3} & \cdots & \mathrm{Def}^0_{12,22} \\ \mathrm{Def}^1_{\mathrm{Ind}} & \mathrm{Def}^1_{0,3} & \cdots & \mathrm{Def}^1_{12,22} \\ \cdots & & & \\ \mathrm{Def}^{124}_{\mathrm{Ind}} & \mathrm{Def}^{124}_{0,3} & \cdots & \mathrm{Def}^{124}_{12,22} \end{bmatrix}$$

If $P = [p_0, \ldots, p_{124}]^T$ the scenario weights vector then the NPV of the instruments will be:

$$\mathrm{NPV} = \begin{bmatrix} \mathrm{NPV}_{Ind} \\ \mathrm{NPV}_{0,3} \\ \cdots \\ \mathrm{NPV}_{12,22} \end{bmatrix} = (\mathrm{PR} - \mathrm{DEF})^T \cdot P$$

To solve the under-determined feature problem, Hull and White (2005)[1] look for the set of weights assigned with the scenarios that best reprices the instruments (minimises $\mathrm{NPV}^T \cdot \mathrm{NPV}$) and that is also as regular as possible. The objective function they select is:

$$\mathrm{NPV}^T \cdot \mathrm{NPV} + c \cdot \sum_{S=1}^{123} \frac{[(p_{S+1} - p_S) - (p_S - p_{S-1})]^2}{1/125}$$

where the summation is a regularisation term. In Hull and White's objective function there is thus a tradeoff between minimisation of the mispricing and regularisation of the scenario distribution.

REGULARISATION THROUGH BID–ASK INFORMATION

Rather than seeking a tradeoff between mispricing and regularisation, the present study aims to find the smoothest distribution that prices all instruments exactly (ie within their bid–ask spreads).

To do this, the problem is split into two steps. The first step minimises the mispricing without considering smoothness of the distribution. The first minimisation is considered successful if all instruments are repriced within the bid–ask spread. This being so, one may proceed with a second optimisation that takes the optimum found in the first step as its starting point in the numerical algorithm. The second step maximises the regularity of the distribution given the constraint that the instruments are priced within the bid–ask spread.

Following these two steps avoids the tradeoff between mispricing and smoothness.

The instruments' NPV can be rewritten as:

$$NPV = \begin{bmatrix} \mathrm{NPV}_{Ind} \\ \mathrm{NPV}_{0,3} \\ \cdots \\ \mathrm{NPV}_{12,22} \end{bmatrix} =$$

$$= \begin{bmatrix} spr_{Ind} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{Ind}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{Ind}^{S} \cdot p_S \\ spr_{Tr}^{0,3} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{0,3}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{0,3}^{S} \cdot p_S \\ spr_{Tr}^{12,22} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{12,22}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{12,22}^{S} \cdot p_S \end{bmatrix}$$

Thus, given a scenario distribution P, the variation in the spreads required to set the NPV of the instruments to 0 is:

$$\Delta spr = \begin{bmatrix} \Delta spr_{Ind} \\ \Delta spr_{0,3} \\ \cdots \\ \Delta spr_{12,22} \end{bmatrix} =$$

$$= \begin{bmatrix} -\mathrm{NPV}_{Ind} / \sum_{S=0}^{124} \mathrm{OutNotl}_{Ind}^{S} \cdot p_S \\ -\mathrm{NPV}_{0,3} / \sum_{S=0}^{124} \mathrm{OutNotl}_{0,3}^{S} \cdot p_S \\ -\mathrm{NPV}_{12,22} / \sum_{S=0}^{124} \mathrm{OutNotl}_{12,22}^{S} \cdot p_S \end{bmatrix}$$

If the absolute value of this vector's components is larger than half the bid–ask spread then one cannot use the scenario distribution P to price the instruments within the bid–ask spread.

A CONVENIENT RELATIONSHIP BETWEEN DEFAULT RATES AND RECOVERY RATES

For their estimations in all scenarios, Hull and White (2005)[1] first use a flat recovery rate of 40 per cent:

$$R^S = 40\%;\ S = 0, 1, \ldots, 124$$

They report that it is possible to fit market data in the first half of 2004 by using a flat recovery of 40 per cent. To fit more recent data (as of November 2005), however, they incorporate the following best fit relationship (following Hamilton *et al.*, 2005),[9] expressing recovery as a function of the default rate:

$$R^S = \max[0\%, 52\% - 6.9 \cdot (1 - \exp(-\lambda^S))];\ S = 0, 1, \ldots, 124$$

This means that in correspondence of

default rates above 7.53 per cent, the recovery is null. The present approach is instead to fit a nonlinear relationship between recovery rates and annualised default rates as in Altman *et al.* (2002):[10]

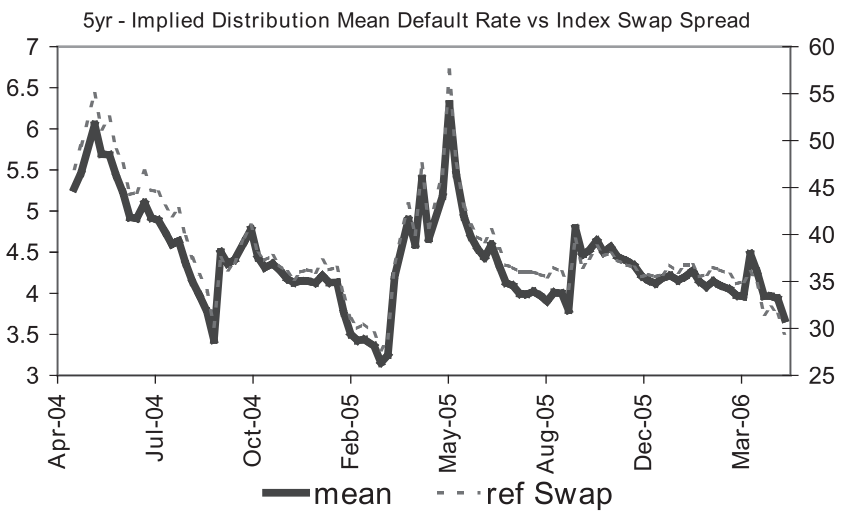
$$[1] \quad R^S = \begin{cases} 0 \ , S = 0 \\ -0.10563 \cdot \ln\left(1 - \exp\left(-\lambda^S\right)\right), \ S = 1, 2, \ldots, 124 \end{cases}$$

The relationship implemented here does not have an intercept, thus a recovery close to 0 can be obtained in case almost all entities (124 out of 125) default.

IMPLIED DEFAULT-RATE DISTRIBUTION THROUGH TIME

The methodology outlined above is now applied to the market credit default swap (CDS) index and standardised CDO tranche spreads from April 2004 to April 2005. A risk-neutral default-rate density is implied.

Figure 1 shows that the reference index CDS spread (right-hand scale) and the average calibrated default rate (left-hand scale), as expected, are strongly correlated. Figure 2, meanwhile, shows



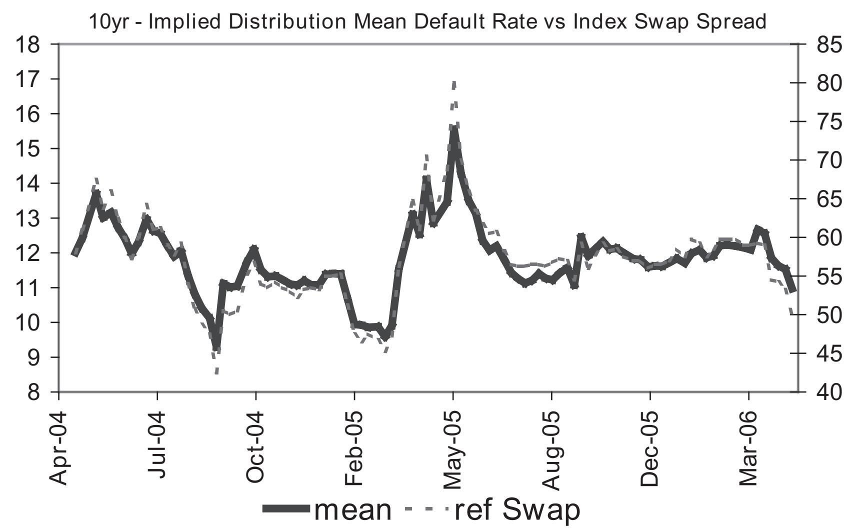


Figure 1: Five and ten-year mean default rate and reference swap

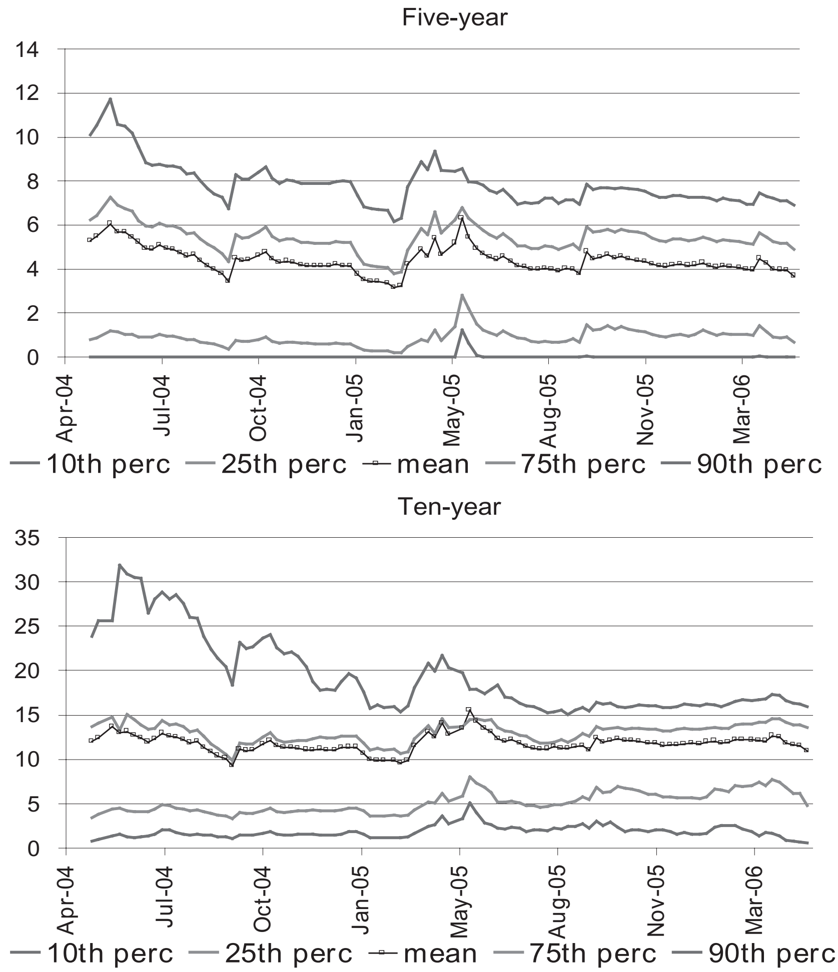


Figure 2: Five and ten-year risk-neutral default-rate distribution percentiles (number of defaults)

that compared with the second half of 2005 and the beginning of 2006, the implied default-rate density (as measured by the difference between the 75th and 25th percentile) had much larger dispersion in 2004. This feature is especially pronounced for the ten-year maturity. One should also note that, until April 2005, the dispersion of the distribution was positively correlated with the reference spread movements, as proxied by the average default rate: the dispersion decreases (increases) when the index decreases (increases).

The figures also illustrate the impact of the Ford-GMAC crisis. Around April 2005, the average spread increased while the distribution simultaneously narrowed. The dispersion narrowed, and on 16th May, 2005, the implied average default rate on the ten-year maturity was above the 75th percentile.

FACTOR MODEL FOR TRANCHE RETURNS

To measure the difference in the risk premium contained in standardised CDO tranches with respect to credit indices, this paper uses a series of excess returns over the risk-free rate for the five and ten-year CDX North America investment grade indices (CDX.NA.IG being the most widespread corporate investment grade CDS index) and for its five and ten-year standardised tranches (0–3, 3–7, 7–10, 10–15, 15–30).

The authors calculated the excess return over the risk-free rate for a credit-linked note (CLN) on the index and the tranches. A CLN is basically a portfolio comprising a risk-free note plus a short protection position on a credit derivative. In this case, the credit derivative is given by the index and its tranches (the authors also imagined rolling the short protection position every six months in the on-the-run series). The total excess return over the risk-free rate in-between any two dates will be the accrual of the protection premium plus the variation in the mark to market of the protection sold minus the loss given default of the relevant credit dervative.

Elton *et al*. (1999)[11] show that, by using corporate bond data from 1987 to 1997, the cross-section of corporate bond returns can be explained by the same factors explaining the cross-section of equity returns. Unfortunately, due to the lack of historical data, it is not possible to perform the same analysis for the liquid market for CDO tranches.

The sample was limited to data from September 2004 to April 2007. In this way it was possible to limit the sample to a period over which all instruments, indices and tranches, over both maturities, five- and ten-year displayed positive information ratios (annualised excess returns over the risk-free rate divided by annualised return volatility), as shown in Table 1a.

The limited sample shown in Table 1b indeed indicates that CDX index returns, for both the five- and ten-year maturity, can be explained by the same factors explaining the cross-section of equity returns, as noted by Elton *et al*.:[11]

$$[1] \quad r_t^{Ind,5y} = \alpha^{Ind,5y} + \beta_1^{5y} \cdot Rm_t + \beta_2^{5y} \cdot HML_t + \beta_3^{5y} \cdot SMB_t + \varepsilon_t^{Ind,5y}$$

$$[2] \quad r_t^{Ind,10y} = \alpha^{Ind,10y} + \beta_t^{10y} \cdot Rm_t + \beta_2^{10y} \cdot HML_t + \beta_3^{10y} \cdot SMB_t + \varepsilon_t^{Ind,10y}$$

From Table 1b, one can see that the significance of the equity market factor (*Rm*) excess returns is substantial for both five- and ten-year index maturities. One can also see that the value factor HML (returns of the high minus low book-to-market portfolios) and the SMB (returns of the small minus big market cap portfolios) do not appear to be statistically significant, although this could be due to the limited sample size. Most importantly, the credit premium does not seem to be

Table 1b: Significance (t-stat) of the three Fama-French factors to explain the returns of the five and ten-year CDX.NA.IG indices (see Equations 1 and 2)

	Five-year index	Ten-year index
(Intercept)	0.17	–0.40
Equity	6.11	5.27
SMB	1.34	1.13
HML	1.84	1.82

HML, returns of the high minus low book-to-market portfolios; SMB, returns of the small minus big market cap portfolios

Table 1a: Information ratios (from September 2004 to April 2007) of the CDX.NA.IG index and tranche

	Main	0–3	3–7	7–10	10–15	15–30
Five-year	0.9	1.0	1.1	1.2	1.0	0.8
Ten-year	0.6	0.8	0.4	0.9	0.9	0.9

significant for the small sample considered — for both regressions and maturities the alphas do not appear to be significant. Here, the credit risk premium can be explained by the three Fama-French factors.

The next stage is to check whether it is possible for the five and ten-year CDO tranches to have a premium that the three Fama-French factors cannot significantly explain statistically. The factors used to explain tranche returns are the same as in Equations 1 and 2, with the addition of the excess return of the index with the same maturity as the tranche under consideration.

For example, for the ten-year maturity 10–15 tranche, the following regression is estimated:

$$[3] \quad r_t^{10-15,10y} = \alpha^{10-15,10y} + \beta_1^{10-15,10y} \cdot Rm_t + \beta_2^{10-15,10y} \cdot HML_t + \beta_3^{10-15,10y} \cdot SMB_t + \beta_{Ind}^{10-15,10y} \cdot r_t^{Ind,10y} + \varepsilon_t^{10-15,10y}$$

Table 1c shows how the tranche excess return regression yields no positive alphas. The tranche premiums can all be explained by the excess index return for the same maturity.

MULTI-FACTOR GAUSSIAN COPULA FOR ASSET RETURNS

Being conscious of the limitations of an analysis of the kind outlined in the previous section, and given the limited sample available, the premium contained in CDO tranches needs to be measured from another perspective.

Recently, Tarashev and Zhu (2007) have shown how standardised CDO tranches may in fact contain a risk premium.[12] They define the correlation risk premium as the average of the time series difference between the risk-neutral correlation and the homogenised physical correlation.

On a given day, the risk-neutral correlation is thus defined as the flat pairwise correlation that, when plugged into a (single-factor) Gaussian copula, minimises the mispricing, expressed as a percentage of the traded spread, of all tranches (actually only the first four — they do not take into consideration the senior AAA 15–30 per cent tranche) on that day.

The homogenised physical correlation would instead be the flat pairwise correlation that fits the 'predicted' tranche spread as closely as possible. The

Table 1c: Significance (t-stat) of the three Fama-French factors plus the relevant index to explain the returns of the five and ten-year tranches (see Equation 3)

	0–3	3–7	7–10	10–15	15–30
Five-year					
(Intercept)	0.40	0.74	1.14	0.76	−0.17
Equity	−0.27	1.62	2.30	1.59	0.94
SMB	−0.25	0.08	−0.59	−1.14	−1.62
HML	1.68	1.56	0.54	0.02	2.14
Ind five-year	13.7	8.62	4.85	7.63	9.13
Ten-year					
(Intercept)	0.80	−0.50	1.12	1.12	1.10
Equity	0.60	1.16	0.94	2.20	1.44
SMB	−0.44	−1.11	0.93	−1.43	−0.62
HML	0.20	1.66	−0.19	−1.11	−0.54
Ind ten-year	9.26	9.30	5.29	7.05	5.88

HML, returns of the high minus low book-to-market portfolios; SMB, returns of the small minus big market cap portfolios predicted tranche spread would be obtained from a multifactor Gaussian copula where the pairwise correlations are set to the historical correlations between the distance to defaults. On any date, the distance to defaults of a single reference entity would be implied from the quoted CDS spreads via a structural model for corporate defaults as in Merton (1973).[13]

Following Tarashev and Zhu,[12] the correlation premium pricing CDO tranches and indices is quantified under the two distributions: risk-neutral and objective. Instead of building a time nonhomogeneous default dependency implicit in the definition of the homogenised physical correlation as per Tarashev and Zhu, the present study assumes a time-homogeneous default correlation, as can be found in the S&P CDO Evaluator.

The approach will be to use an estimate of the asset returns correlation such as the one underlying the multifactor Gaussian copulas of the S&P Evaluator, Moody's CDOROM or Fitch Vector 3.2 models, which are all based on estimates of the asset correlation between corporates, and all agree on a multi-factor structure based upon countries and sectors.

In particular, the study measures the risk premium contained in CDO tranches as the portion of the default leg NPV of the index and the tranches that is not explained by the objective valuation framework. In this framework the scenario probabilities $P = [p_0, \ldots, p_{124}]$ are the result of a multifactor Gaussian copula where the probability of default of the single names is mapped from the historical default frequency via the single reference entity rating and the correlation structure is assumed to be the same as in the S&P CDO Evaluator.

DEFAULT RATE SIMULATION UNDER THE OBJECTIVE MEASURE

The study uses a set of N reference obligations associated to a set of K rating classes. Each rating class is assigned a probability of default equal to the average historical default rate of that rating class. To value CDO tranches, one also needs a measure of the dependency between the defaults of the N reference obligations in the portfolio.

This dependency is modelled via a multi-factor Gaussian copula:

- first, a vector X of N jointly standardised normal random variables with correlation matrix Σ is randomly drawn;
- the cumulative normal $U_i = N(X_i)$ for all components X_i of X is then computed, where $N(x)$ is the cumulative standard normal;
- finally, the study computes the simulated default time for each name i as the inverse of the historically measured survival function $\tau_i = S_i^{-1}(U_i)$, $S_i(T) = \exp\left(-\int_0^T \lambda_i(s)ds\right)$ where $\lambda_i(s)$ is here the deterministic time-dependent default intensity of name i.

If the simulated default time τ is less than T then the associated reference obligation in the current simulation will default before the maturity of the CDO. The loss incurred by the portfolio will be the notional invested in the name (for the iTraxx it will be 0.8 per cent = 1/125) times 1 minus the recovery associated with that name.

In the risk-neutral framework, the way the correlation is implied from market quotes assumes the correlation matrix Σ to be flat in the sense that the off-diagonal terms are all equal to a single parameter p: the compound correlation. Given the market spread of a tranche, the corresponding compound correlation would be the p value to be put into a Gaussian copula linking the default of different names that sets the NPV of the tranche to 0, given the risk-neutral default intensities $\lambda_{Ind}(s)$ stripped by the CDS term structure of the index constituents. Clearly, this is a quoting mechanism rather than a model, and to achieve consistency, the related quotes of different tranches need to be calibrated with a single model, such as the perfect copula model above, where the notion of correlation matrix is somehow lost apart from its quoting mechanism interpretation.

The correlation matrix used to obtain the default and loss-rate distributions under the objective measure is block diagonal, as per the S&P CDO Evaluator. The correlation between any two names will be 15 per cent if both names belong to the same global or regional sector and 5 per cent otherwise.

Note: in determining the block diagonal structure of the pool of obligors, the S&P CDO Evaluator differentiates between local, regional and global sectors. A local sector is only affected by the macroeconomic forces within the country where the asset resides (eg 'building and development'). A regional sector is affected by the macroeconomic forces of the region (eg 'rail industries'). A global sector assumes that the same economic forces affect all companies in that sector, regardless of location (eg 'oil and gas').

RISK PREMIUM EVIDENCE IN THE INDEX SWAP AND THE ITRAXX TRANCHES

Given the dollar value of the default leg of an instrument (index CDS or CDO tranche) one might ask how much of this dollar value is justified in terms of default risk. In other words, what is the difference between the NPV of the default leg under the risk-neutral and objective measures?

A different question to consider is how much more the premium leg would have paid in excess of the default leg had the default distribution been the historical one? This corresponds to the difference between the NPV of the premium and default leg under the objective measure. Both differences can be considered as related to a compensation for the risk of default in the instrument notional.

Given the market quotes of a set of instruments as of 27th January, 2006 (Table 2a), in order to be remunerated for the risk of default of the notional amount, one would expect both differences to be positive. In both cases, one subtracts the NPV of the default leg under the objective measure (Table 2b). In the former definition, it is subtracted from the NPV of the default leg under the risk-neutral measure (Table 2c) and in the latter definition from the NPV of the premium leg under the objective measure (Table 2d). One can see that the two differences are indeed positive for all instruments and maturities (Table 2e–f).

One can observe that for all maturities, roughly half (40–60 per cent) of the NPV of the equity tranche is justified in terms of default risk. One can also see that only a small portion of the mezzanine (3–6) ten-year tranche is justified in terms of default risk.

Of course, the quantification of the remuneration for risk for each instrument (index and tranche) depends

Table 2a: Market quotes (27th January, 2006)

	3	5	7	10
Index	0.1875%	0.3550%	0.4725%	0.5750%
0–3	5.0000%	27.7500%	48.2500%	57.6250%
3–6	0.0850%	0.7900%	1.8750%	5.3000%
6–9	0.0300%	0.2700%	0.4800%	1.0050%
9–12	—	0.1250%	0.2700%	0.4450%
12–22	—	0.0563%	0.1200%	0.2250%

Table 2b: NPV default leg under the objective measure

	3	5	7	10
Index	0.3093%	0.5902%	0.8991%	1.3776%
0–3	10.2967%	19.4756%	28.9749%	41.5994%
3–6	0.0132%	0.1949%	0.9632%	3.9975%
6–9	0.0000%	0.0027%	0.0312%	0.2979%
9–12	0.0000%	0.0000%	0.0020%	0.0224%
12–22	0.0000%	0.0000%	0.0001%	0.0011%

Table 2c: NPV default leg under the risk-neutral measure

	3	5	7	10
Index	0.5158%	1.5784%	2.8203%	4.5794%
0–3	17.5643%	44.6892%	66.5971%	76.2986%
3–6	0.2350%	3.5302%	11.1250%	38.1894%
6–9	0.0830%	1.2144%	2.9118%	8.1438%
9–12	0.0000%	0.5636%	1.6477%	3.6493%
12–22	0.0000%	0.2538%	0.7335%	1.8616%

Table 2d: NPV premium leg under the objective measure

	3	5	7	10
Index	0.5179%	1.5996%	2.8889%	4.7598%
0–3	18.1851%	48.3559%	74.8461%	90.7758%
3–6	0.2355%	3.5807%	11.5529%	44.1433%
6–9	0.0831%	1.2242%	2.9629%	8.4422%
9–12	0.0000%	0.5668%	1.6667%	3.7397%
12–22	0.0000%	0.2550%	0.7407%	1.8909%

Table 2e: NPV premium leg under the objective measure *minus* NPV default leg under the objective measure

	3	5	7	10
Index	0.2086%	1.0094%	1.9898%	3.3821%
0–3	7.8884%	28.8803%	45.8712%	49.1764%
3–6	0.2224%	3.3858%	10.5897%	40.1458%
6–9	0.0831%	1.2215%	2.9316%	8.1443%
9–12	0.0000%	0.5668%	1.6647%	3.7173%
12–22	0.0000%	0.2550%	0.7407%	1.8898%

Table 2f: NPV default leg under the risk-neutral measure *minus* NPV premium leg under the objective measure

	3	5	7	10
Index	0.2065%	0.9882%	1.9212%	3.2018%
0–3	7.2675%	25.2136%	37.6222%	34.6992%
3–6	0.2218%	3.3353%	10.1619%	34.1919%
6–9	0.0830%	1.2118%	2.8805%	7.8459%
9–12	0.0000%	0.5636%	1.6457%	3.6268%
12–22	0.0000%	0.2538%	0.7334%	1.8605%

heavily on the simulation engine outlined above. If, for example, one was to use a heterogeneous rating transition matrices or a different copula, one would get different NPVs of the default leg under the objective measure and thus different numbers in Table 2b, e and f.

COMPARISON BETWEEN THE DEFAULT-RATE DENSITY UNDER THE OBJECTIVE AND RISK-NEUTRAL MEASURE

Figure 3 plots the default-rate density under the objective and risk-neutral measure (the abscissas correspond to the number of defaulted obligors in the iTraxx CDO pool: 125 names). The risk-neutral density is obtained from the market quotes in Table 2e (27th January, 2006) using the methodology described in the section entitled 'Implied default rate distribution'. The objective measure density is instead obtained from a simulation following the methodology outlined in the section entitled 'Default rate simulation under the objective measure'.

One can immediately notice the different centres of the densities corresponding to the same maturity between the objective and risk-neutral measure. The risk-neutral densities shift to the right. This corresponds to the default risk premium documented in Hull *et al*. (2006).

Figure 4 now zooms in on two features, corresponding to different areas of the abscissa (number of defaulted obligors at maturity) of the density under the risk-neutral measure. In the left plot, one can observe a series of bumps of increasing size, and shifting

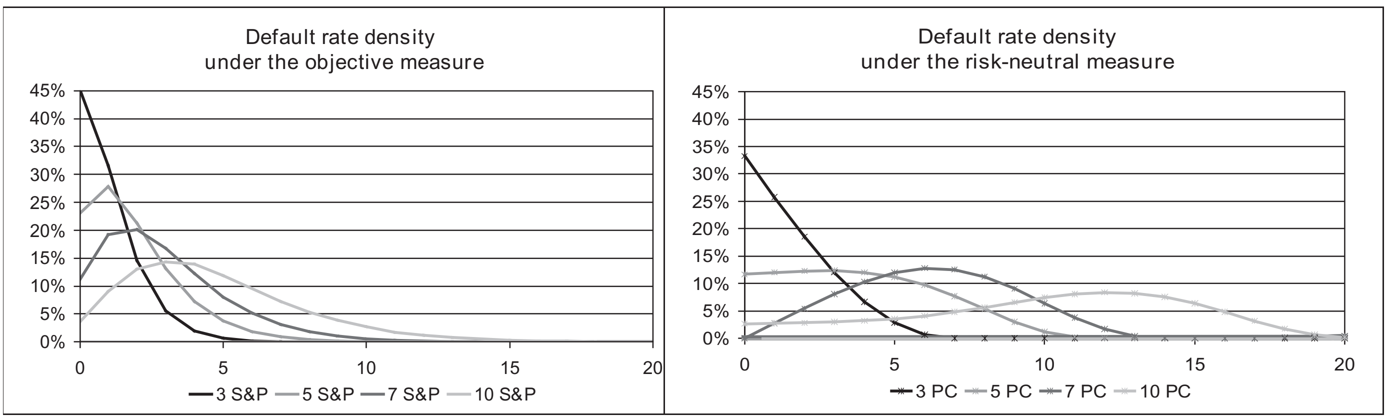


Figure 3: Default-rate density under the objective measure vs under the risk-neutral measure

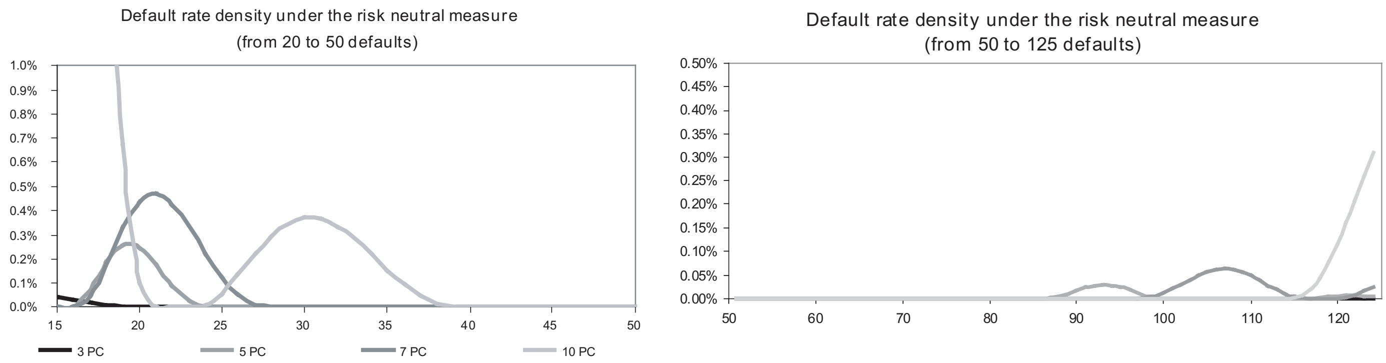


Figure 4: Default-rate density under the objective measure vs under the risk-neutral measure

further to the right, as maturity increases, in the range of 15–50 defaults, a scenario of extremely severe loss in the pool of CDO obligors. In the right plot, one can observe the far-end tail, referring to 50–125 defaults out of 125 obligors in the iTraxx pool. One can also observe a small bump increasing in size as this catastrophic scenario matures.

Figure 5 shows the implied default-rate distribution on the iTraxx tranches calibrated at different times for the five- and ten-year maturity. The authors note the persistence of the above-mentioned bumps in terms of size (probability mass) and location (range of default numbers).

As this feature persists, it may be appropriate to look for more complex dynamical loss models that can produce a bump feature in the tail. A related dynamic loss model that can be consistently calibrated to tranche and index data for different maturities is the generalised Poisson loss model of Brigo *et al.* (2007).[14]

Brigo *et al.* (2007)[15] also address consistency with single-name data and

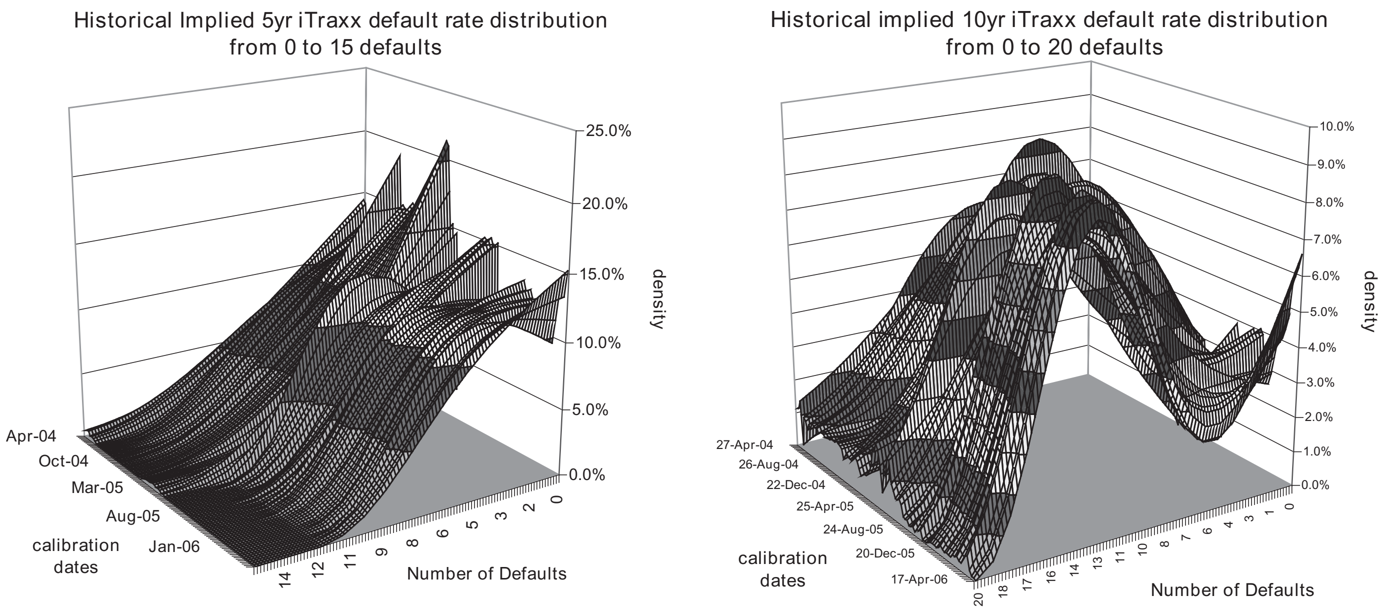


Figure 5: Five- and ten-year implied default-rate distribution calibrated to the iTraxx tranches

default clusters, leading to a top-down approach known as the generalised Poisson cluster loss (GPCL) model.

Walker (2007)[16] and Brigo *et al.* (2006)[17] provide different approaches for extracting market information from standardised CDO tranches that are consistent across maturities.

CONCLUSIONS AND FURTHER RESEARCH

This paper has consistently priced CDO tranches across detachments. The risk-neutral loss distribution has been derived through a modified 'scenario default rate' approach. It has then priced CDO tranches under the objective measure by using the Gaussian copula with the same historical correlation matrix used in the S&P CDO Evaluator 3.0 and historical default probabilities deduced by rating classes. The study has found that the risk-neutral loss distribution assigns a greater probability of large realisations of the loss with respect to the objective distribution, thus implying the well-known presence of a risk premium. This risk premium has been quantified using infinite pool limit results to simplify the approach. It would be interesting to explore what happens in a finite pool setup. This would allow the introduction of more structure in the initial area of the loss distribution.

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APPENDIX

The Gaussian single-factor copula models assume a Gaussian copula structure driving the exponential random variables, generating jumps in the related default processes of the names in the pool. This results in the default probability for each name, conditional on the Gaussian systemic factor M, to be given by:

[2] ProbabilityRiskNeutral{Name defaults before $T|M\} = m\} =$

$$N\left(\frac{N^{-1}(Pd(T)) - m\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

where $Pd(T)$ is the risk-neutral probability that any name defaults by time T.

As before, defaults are independent given M, and this allows the joint default probabilities for the whole pool to be computed by multiplying the conditional risk-neutral probabilities in Equation 2 and integrating over M under its Gaussian density. Under the large pool assumption, the above probability is also the pool default rate by T given M. In the single-factor Gaussian copula, the systemic factor is a continuous random variable. If one discretises the domain of the systemic factor M in intervals of length dm, the Gaussian copula can be summed as:

One factor Gaussian Copula

$$\begin{bmatrix} \text{Systemic scenario} & \text{Scenario probability} & \text{Conditional default rate} \\ \cdots & \cdots & \cdots \\ \mathrm{M} \in [\mathrm{m}; \mathrm{m}+\mathrm{dm}) & \mathrm{N}(\mathrm{m}+\mathrm{dm}) - \mathrm{N}(\mathrm{m}) & \mathrm{N}\left(\frac{N^{-1}(Pd(T)) - \mathrm{m}\sqrt{\rho}}{\sqrt{1-\rho}}\right) \\ \cdots & \cdots & \cdots \end{bmatrix}$$

It is now possible to compare this with the implied copula assumption used in this paper. In the present case, Equation 2 for the default probability was replaced with the more natural, intensity-based equation:

ProbabilityRiskNeutral$\{$Namedefaults before $T|M = m^S\} = 1 - e^{-\lambda^S T}$

leading to:

$$\begin{bmatrix} \text{Systemic} & \text{Scenario} & \text{Conditional} \\ \text{scenario} & \text{probability} & \text{default rate} \\ \mathrm{M} = \mathrm{m}^0 & p^0 & 1 - e^{-\lambda^0 T} \\ \mathrm{M} = \mathrm{m}^1 & p^1 & 1 - e^{-\lambda^1 T} \\ \cdots & \cdots & \cdots \\ \mathrm{M} = \mathrm{m}^{124} & p^{124} & 1 - e^{-\lambda^{124} T} \end{bmatrix}$$

Implied Copula

As far as pricing is concerned, the end result of these approaches is the default-rate distribution. This distribution has little flexibility in the Gaussian single-factor copula case, in that one can play only with the single copula parameter ρ, the compound correlation, scenario probabilities being fixed by the Gaussian assumption. If one is to price a set of instruments (eg CDO tranches) with a single model specification, having just one parameter can be unrealistic. In the implied copula approach, one can instead play with the scenario probabilities so as to obtain a rich variety of possible default-rate distributions, which can help in pricing a set of instruments with a single model specification.

